

THE EMBEDDED HOMOLOGY OF HYPERGRAPHS AND APPLICATIONS*

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Abstract. Hypergraphs are mathematical models for many problems in data sciences. In recent decades, the topological properties of hypergraphs have been studied and various kinds of (co)homologies have been constructed (cf. [4, 7, 19]). In this paper, generalising the usual homology of simplicial complexes, we define the embedded homology of hypergraphs as well as the persistent embedded homology of sequences of hypergraphs. As a generalisation of the Mayer-Vietoris sequence for the homology of simplicial complexes, we give a Mayer-Vietoris sequence for the embedded homology of hypergraphs. Moreover, as applications of the embedded homology, we study acyclic hypergraphs and construct some indices for the data analysis of hyper-networks.

Key words. Hypergraph, acyclic hypergraph, homology, persistent homology, Mayer-Vietoris sequence, hyper-network.

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1. Introduction. Given a set V , we define its **power set** $\Delta[V]$ as the collection of all the non-empty subsets of V . Throughout this paper, we assume that V is a finite and totally ordered set. For any subset $\{v_0, \dots, v_n\}$ of V , we assume $v_0 \prec \dots \prec v_n$ if there is no extra claim, where \prec is the total order of V .

In topology, simplicial complexes are classical models for triangulated topological spaces. An **(abstract) simplicial complex** K is a pair (V_K, K) where V_K is a set and K is a subset of $\Delta[V_K]$ satisfying the following conditions (cf. [14, p. 107]):

- (i). for any $v \in V_K$, the single-point set $\{v\}$ is in K ;
- (ii). for any $\sigma \in K$ and any non-empty subset $\tau \subseteq \sigma$, τ is in K .

The elements of V_K are called **vertices**, and the elements of K are called **simplices**. A simplex consisting of $n + 1$ elements in V , $n \geq 0$, is called an **n -simplex**. By (i), we can identify the 0-simplices of K with the vertices. Hence the simplicial complex (V_K, K) can be simply denoted as K . For any $n \geq 0$, the collection of all n -simplices in K are denoted as K_n . The **dimension** of a simplicial complex K is the largest integer n such that K_n is non-empty. For any $n \geq 1$, a **$(n - 1)$ -face** of an n -simplex is an $(n - 1)$ -simplex obtained by removing one vertex in the n -simplex.

Let G be an abelian group. Given a non-empty finite set S , we use $G(S)$ to denote the collection of all formal linear combinations of the elements in S with coefficients in G . In particular, $\mathbb{Z}(S)$ is the free \mathbb{Z} -module generated by S .

EXAMPLE 1.1 ([14, pp. 103 - 105]). A *standard n -simplex* is denoted as

$$\Delta^n = \{v_0, \dots, v_n\}.$$

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The $(n - 1)$ -faces of the standard n -simplex Δ^n are denoted as

$$\Delta_i^{n-1} = \{v_0, \dots, \hat{v}_i, \dots, v_n\}, \quad 0 \leq i \leq n.$$

We have the face maps d_i sending Δ^n to Δ_i^{n-1} , for $0 \leq i \leq n$. And we have the boundary maps

$$\partial_n : G(\Delta^n) \longrightarrow G(\Delta_0^{n-1}, \dots, \Delta_n^{n-1})$$

given by $\partial_n = \sum_{i=0}^n (-1)^i d_i$, which extends linearly over G . The power set of Δ^n is denoted as $\Delta[n]$, called *the standard simplicial complex spanned by $n + 1$ vertices*.

Hypergraph was invented as a model for hyper-networks by data scientists. Mathematically, the hypergraph is a generalisation of the notion of the simplicial complex. A **hypergraph** is a pair $(V_{\mathcal{H}}, \mathcal{H})$ where $V_{\mathcal{H}}$ is a set and \mathcal{H} is a subset of $\Delta[V_{\mathcal{H}}]$ (cf. [1, 19]). An element of $V_{\mathcal{H}}$ is called a **vertex** and an element of \mathcal{H} is called a **hyperedge**. For any $n \geq 0$, we call a hyperedge consisting of $n + 1$ vertices an **n -hyperedge**, and we denote \mathcal{H}_n as the collection of all the n -hyperedges of \mathcal{H} . We define the **dimension** of \mathcal{H} as the largest integer n such that \mathcal{H}_n is non-empty. In this paper, we assume that $V_{\mathcal{H}}$ is the union of all the vertices of the hyperedges of \mathcal{H} and we simply denote the hypergraph $(V_{\mathcal{H}}, \mathcal{H})$ as \mathcal{H} .

Given a hypergraph \mathcal{H} , if for any hyperedge $\sigma \in \mathcal{H}$, any non-empty subsets of σ are hyperedges of \mathcal{H} , then \mathcal{H} is a simplicial complex, and the hyperedges of \mathcal{H} are simplices.

In recent decades, various (co)homology theories of hypergraphs have been intensively studied. In 1991, A.D. Parks and S.L. Lipscomb [19] defined the associated simplicial complex of a hypergraph, that is, the minimal simplicial complex that a hypergraph can be embedded in. They also studied the homology of the associated simplicial complex. In 1992, F.R.K. Chung and R.L. Graham [4] constructed certain cohomology for hypergraphs, with mod 2 coefficients, in a combinatorial way. In 2009, E. Emtander [7] constructed certain simplicial complexes for hypergraphs (called the independence simplicial complexes) and studied the homology of these simplicial complexes; and J. Johnson [15] applied the topology of hypergraphs to study hyper-networks of complex systems.

A graph is a hypergraph whose hyperedges consist of at most two vertices. And a directed graph, or simply called a digraph, is the geometric object obtained by associating a direction with each edge of a graph. Since 2012, the homology theory of graphs and digraphs has attracted the attention of A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau [8, 9, 10, 11, 12].

In this paper, by generalising [9] and using the associated simplicial complexes defined in [19], we construct the embedded homology of hypergraphs and study the persistent embedded homology of sequences of hypergraphs. In particular, if the hypergraph is a simplicial complex, then the embedded homology coincides with the usual homology. Moreover, generalising the Mayer-Vietoris sequence for the homology of simplicial complexes (cf. [14, pp. 149 - 153]), we give a Mayer-Vietoris sequence for the embedded homology as well as a persistent version of Mayer-Vietoris sequence for the persistent embedded homology of hypergraphs. Furthermore, we apply the associated simplicial complexes and the embedded homology to characterise an important family of hypergraphs: the acyclic hypergraphs. Finally, as applications of the embedded homology of hypergraphs in data sciences, we construct some topological indices for the data analysis of hyper-networks.

The paper is organised as follows. In Section 2, by generalising the definition of the homology of (directed) graphs given by A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau [9, Section 3.3], we construct certain homology for graded groups embedded in chain complexes. In Section 3, we define the embedded homology of hypergraphs using the associated simplicial complexes defined in [19]. Moreover, under certain conditions, we give a Mayer-Vietoris sequence for the embedded homology of hypergraphs in Theorem 3.10. In Section 4, we study the persistent embedded homology of sequences hypergraphs and give a persistence version of the Mayer-Vietoris sequence in Theorem 4.1. In Section 5, we use the associated simplicial complexes and the embedded homology to study acyclic hypergraphs. We give some characterisations for the associated simplicial complexes of acyclic hypergraphs in Theorem 5.2. And we study the embedded homology of a particular family of acyclic hypergraphs in Subsection 5.2. In Section 6, we apply the embedded homology of hypergraphs and construct some indices to measure the connectivity of hyper-networks, the differentiation of hyper-networks with respect to a function on the vertices, and the correlation of two functions on the vertices of hyper-networks.

2. Homology of graded groups embedded in chain complexes. In this section, we construct the homology of graded groups which are embedded in chain complexes.

Firstly, we review the homology of chain complexes. Let $C_n, n = 0, 1, 2, \dots$, be a sequence of abelian groups such that there exists a sequence of homomorphisms (called **boundary maps**)

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \tag{2.1}$$

with $\partial_n \circ \partial_{n+1} = 0$ for each n . Such a sequence (2.1) of abelian groups and homomorphisms is called a **chain complex** (cf. [14, p. 106]). Both the intersection and the direct sum of a family of chain complexes are still chain complexes. The **n -th homology** of the chain complex (2.1) is defined as the quotient group¹ (cf. [14, p. 106])

$$H_n(\{C_*, \partial_*\}) = \text{Ker}\partial_n / \text{Im}\partial_{n+1}.$$

Secondly, we consider graded groups embedded in chain complexes and define the infimum chain complexes and the supremum chain complexes. For each $n \geq 0$, let D_n be a subgroup of C_n . In particular, if for each $n \geq 1, \partial D_n \subseteq D_{n-1}$, then we call the sequence D_* a **subchain complex** of C_* . Given a subchain complex of a chain complex, the sequence of quotient groups, equipped with the respective quotient maps, is still a chain complex.

DEFINITION 1. Given a graded group D_* embedded in a chain complex C_* , the **infimum chain complex** $\text{Inf}_*(D_*, C_*)$ of the sequence $\{D_*, C_*\}$ is the chain complex

$$\text{Inf}_n(D_*, C_*) = \sum \{C'_n \mid C'_* \text{ is a subchain complex of } C_* \text{ and } C'_n \subseteq D_n\}.$$

It follows immediately from Definition 1 that if D_* is a sub-chain complex, then $\text{Inf}_*(D_*, C_*) = D_*$.

¹We use $(\cdot)_*$ to denote a sequence of objects, where $*$ = 0, 1, 2, \dots .

In [9, Section 3.3], A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau constructed the chain complex $D_n \cap \partial_n^{-1}(D_{n-1})$ where ∂_n^{-1} denotes the pre-image of ∂_n . In the next proposition, we show that our infimum chain complex in Definition 1 coincides with the explicit formula given in [9, Section 3.3].

PROPOSITION 2.1. *Given a chain complex C_* and a sequence of subgroups D_* , the infimum chain complex is given by*

$$\text{Inf}_n(D_*, C_*) = D_n \cap \partial_n^{-1}(D_{n-1}). \tag{2.2}$$

Proof. In order to prove (2.2), we need to show that $\{D_n \cap \partial_n^{-1}(D_{n-1})\}_{n \geq 0}$ is the largest subchain complex of C_* that are contained in D_* as a graded abelian group. Let $\alpha \in D_n \cap \partial_n^{-1}(D_{n-1})$. Then $\alpha \in D_n$ and $\partial_n \alpha \in D_{n-1}$. Since $\partial_{n-1} \partial_n \alpha = 0$,

$$\partial_n \alpha \in D_{n-1} \cap \partial_{n-1}^{-1}(D_{n-2}).$$

Thus $\{D_n \cap \partial_n^{-1}(D_{n-1})\}_{n \geq 0}$ is a subchain complex of C_* . Let $C'_* \subseteq D_*$ be a subchain complex of C_* . Let $\alpha' \in C'_n$. Then $\alpha' \in D_n$ and $\partial_n \alpha' \in D_{n-1}$. Hence

$$\alpha' \in D_n \cap \partial_n^{-1}(D_{n-1}).$$

Thus for each $n \geq 0$,

$$C'_n \subseteq D_n \cap \partial_n^{-1}(D_{n-1}).$$

Therefore, $\{D_n \cap \partial_n^{-1}(D_{n-1})\}_{n \geq 0}$ is the largest subchain complex of C_* that are contained in D_* as a graded abelian group. \square

REMARK 1. In (2.2), when $n = 0$, we let $\partial_0 = 0$ and $D_{-1} = 0$.

DEFINITION 2. Given a graded group D_* embedded in a chain complex C_* , the **supremum chain complex** $\text{Sup}_*(D_*, C_*)$ of the sequence $\{D_*, C_*\}$ is the chain complex

$$\text{Sup}_n(D_*, C_*) = \bigcap \{C'_n \mid C'_* \text{ is a subchain complex of } C_* \text{ and } D_n \subseteq C'_n\}.$$

It follows immediately from Definition 2 that if D_* is a sub-chain complex, then $\text{Sup}_n(D_*, C_*) = D_*$.

PROPOSITION 2.2. *Given a chain complex C_* and a sequence of subgroups D_* , the supremum chain complex is given by*

$$\text{Sup}_n(D_*, C_*) = D_n + \partial_{n+1} D_{n+1}. \tag{2.3}$$

Proof. In order to prove (2.3), we need to show that $\{D_n + \partial_{n+1} D_{n+1}\}_{n \geq 0}$ is the smallest subchain complex of C_* that contains D_* as a graded abelian group. Let $\bar{\alpha} \in D_n + \partial_{n+1} D_{n+1}$. Then

$$\bar{\alpha} = \beta_n + \partial_{n+1} \beta_{n+1}$$

where $\beta_* \in D_*$. Since $\partial_n \partial_{n+1} \beta_{n+1} = 0$, we have $\partial_n \bar{\alpha} = \partial_n \beta_n$. Hence

$$\partial_n \bar{\alpha} \in D_{n-1} + \partial_n D_n.$$

Hence $\{D_n + \partial_{n+1}D_{n+1}\}_{n \geq 0}$ is a subchain complex of C_* . Let $C'_* \supseteq D_*$ be a subchain complex of C_* . Then $\beta_n \in C'_n$ and $\partial_{n+1}\beta_{n+1} \in C'_n$. Thus $\bar{\alpha} \in C'_n$. Thus for each $n \geq 0$,

$$D_n + \partial_{n+1}D_{n+1} \subseteq C'_n.$$

Therefore, $\{D_n + \partial_{n+1}D_{n+1}\}_{n \geq 0}$ is the smallest subchain complex of C_* that contains D_* as a graded abelian group. \square

Given two graded subgroups D_* and D'_* , by a straight-forward computation, we have some basic properties of the infimum chain complexes and the supremum chain complexes, in next proposition.

PROPOSITION 2.3. *Let C_* be a chain complex and D_*, D'_* be graded subgroups of C_* . Then*

$$\begin{aligned} \text{Inf}_n(D_* \cap D'_*, C_*) &= \text{Inf}_n(D_*, C_*) \cap \text{Inf}_n(D'_*, C_*), \\ \text{Inf}_n(D_* + D'_*, C_*) &\supseteq \text{Inf}_n(D_*, C_*) + \text{Inf}_n(D'_*, C_*), \\ \text{Sup}_n(D_* \cap D'_*, C_*) &\subseteq \text{Sup}_n(D_*, C_*) \cap \text{Sup}_n(D'_*, C_*), \\ \text{Sup}_n(D_* + D'_*, C_*) &= \text{Sup}_n(D_*, C_*) + \text{Sup}_n(D'_*, C_*). \end{aligned}$$

Thirdly, we study the homology of the infimum chain complexes and the supremum chain complexes. It follows from Proposition 2.1 and [9, Proposition 3.13] that

$$\begin{aligned} H_n(\text{Inf}_*(D_*, C_*)) &= \text{Ker}(\partial_n|_{D_n \cap \partial_n^{-1}(D_{n-1})}) / \text{Im}(\partial_{n+1}|_{D_{n+1} \cap \partial_{n+1}^{-1}(D_n)}) \\ &= \text{Ker}(\partial_n|_{D_n}) / (D_n \cap \partial_{n+1}D_{n+1}). \end{aligned}$$

Moreover, we have the next proposition.

PROPOSITION 2.4. *The homology of $\text{Inf}_*(D_*, C_*)$ and the homology of $\text{Sup}_*(D_*, C_*)$ are isomorphic.*

Proof. With the help of the isomorphism theorem of groups, we have

$$\begin{aligned} H_n(\text{Sup}_*(D_*, C_*)) &= \text{Ker}(\partial_n|_{D_n + \partial_{n+1}D_{n+1}}) / \text{Im}(\partial_{n+1}|_{D_{n+1} + \partial_{n+2}D_{n+2}}) \\ &= (\partial_{n+1}D_{n+1} + \text{Ker}(\partial_n|_{D_n})) / \partial_{n+1}D_{n+1} \\ &\cong \text{Ker}(\partial_n|_{D_n}) / \text{Ker}(\partial_n|_{D_n}) \cap \partial_{n+1}D_{n+1} \\ &= \text{Ker}(\partial_n|_{D_n}) / (D_n \cap \partial_{n+1}D_{n+1}) \\ &= H_n(\text{Inf}_*(D_*, C_*)). \end{aligned}$$

The assertion follows. \square

3. The embedded homology of hypergraphs. In this section, we study the embedded homology of hypergraphs. In Subsection 3.1, we study the associated simplicial complex of hypergraphs. In Subsection 3.2, we define the embedded homology of hypergraphs and prove some basic properties of the embedded homology. In Subsection 3.3, we give a Mayer-Vietoris sequence for the embedded homology of hypergraphs.

3.1. The associated simplicial complex of hypergraphs. Given a hypergraph \mathcal{H} , A.D. Parks and S.L. Lipscomb [19] defined its **associated simplicial complex** $K_{\mathcal{H}}$ to be the smallest simplicial complex such that the hyperedges of \mathcal{H} is a subset of the simplices of $K_{\mathcal{H}}$. Precisely, the set of all simplices of $K_{\mathcal{H}}$ consists of all the non-empty subsets $\tau \subseteq \sigma$, for all $\sigma \in \mathcal{H}$ (cf. [19, Lemma 8]). All the hyperedges in $K_{\mathcal{H}} \setminus \mathcal{H}$ forms a hypergraph, which will be called the **complement hypergraph** of \mathcal{H} and denoted as \mathcal{H}^c , in this paper.

Firstly, we give a functor from the category of hypergraphs to the category of simplicial complexes, sending \mathcal{H} to $K_{\mathcal{H}}$. A simplicial map from a simplicial complex K to a simplicial complex K' is a map f sending a vertex of K to a vertex of K' such that for any simplex $\sigma = \{v_0, \dots, v_n\}$ of K , $f(\sigma) = \{f(v_0), \dots, f(v_n)\}$ is a simplex of K' . A morphism of hypergraphs from a hypergraph \mathcal{H} to a hypergraph \mathcal{H}' is a map f sending a vertex of \mathcal{H} to a vertex of \mathcal{H}' such that whenever $\sigma = \{v_0, \dots, v_n\}$ is a hyperedge of \mathcal{H} , $f(\sigma) = \{f(v_0), \dots, f(v_n)\}$ is a hyperedge of \mathcal{H}' . Given a morphism of hypergraphs $f : \mathcal{H} \rightarrow \mathcal{H}'$, we have a simplicial map $\tilde{f} : K_{\mathcal{H}} \rightarrow K_{\mathcal{H}'}$ sending a simplex $\{v_0, v_1, \dots, v_n\}$ in $K_{\mathcal{H}}$ to the simplex $\{f(v_0), f(v_1), \dots, f(v_n)\}$ in $K_{\mathcal{H}'}$. Consequently, we have a functor \mathcal{F} from the category of hypergraphs to the category of simplicial complexes, sending a hypergraph \mathcal{H} to its associated simplicial complex $K_{\mathcal{H}}$ and sending a morphism f of hypergraphs to its induced simplicial map \tilde{f} of the corresponding associated simplicial complexes. The functor \mathcal{F} is the adjoint functor of the forgetful functor from the category of simplicial complexes to the category of hypergraphs.

Secondly, we prove some basic properties of the associated simplicial complexes, in the remaining part of this section. The next proposition gives some basic properties of the associated simplicial complexes of hypergraphs.

PROPOSITION 3.1. *Let \mathcal{H} and \mathcal{H}' be hypergraphs. Then*

$$K_{\mathcal{H} \cup \mathcal{H}'} = K_{\mathcal{H}} \cup K_{\mathcal{H}'}, \tag{3.1}$$

$$K_{\mathcal{H} \cap \mathcal{H}'} \subseteq K_{\mathcal{H}} \cap K_{\mathcal{H}'}. \tag{3.2}$$

Moreover, the equality in (3.2) holds if for any $\sigma \in \mathcal{H}$ and any $\sigma' \in \mathcal{H}'$, either $\sigma \cap \sigma'$ is empty or $\sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}'$.

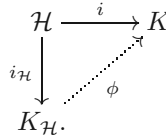
Proof. To verify (3.1), we notice that $\{v_0, \dots, v_n\} \in K(\mathcal{H} \cup \mathcal{H}')$ if and only if $v_0, \dots, v_n \in V_{\mathcal{H}} \cup V_{\mathcal{H}'}$ and there exists a hyperedge $\sigma \in \mathcal{H} \cup \mathcal{H}'$ such that $\{v_0, \dots, v_n\} \subseteq \sigma$. That is, either σ is a hyperedge of \mathcal{H} and $\{v_0, \dots, v_n\} \subseteq \sigma$, or σ is a hyperedge of \mathcal{H}' and $\{v_0, \dots, v_n\} \subseteq \sigma$. Hence $\{v_0, \dots, v_n\} \in K(\mathcal{H} \cup \mathcal{H}')$ if and only if either $\{v_0, \dots, v_n\} \in K_{\mathcal{H}}$ or $\{v_0, \dots, v_n\} \in K_{\mathcal{H}'}$.

To verify (3.2), we notice that $\{v_0, \dots, v_n\} \in K(\mathcal{H} \cap \mathcal{H}')$ if and only if $v_0, \dots, v_n \in V_{\mathcal{H}} \cap V_{\mathcal{H}'}$ and there exists a hyperedge $\sigma \in \mathcal{H} \cap \mathcal{H}'$ such that $\{v_0, \dots, v_n\} \subseteq \sigma$. On the other hand, $\{v_0, \dots, v_n\} \in K_{\mathcal{H}} \cap K_{\mathcal{H}'}$ if and only if there exist a hyperedge $\sigma \in \mathcal{H}$ and a hyperedge $\sigma' \in \mathcal{H}'$ such that $\{v_0, \dots, v_n\} \subseteq \sigma$ and $\{v_0, \dots, v_n\} \subseteq \sigma'$. Hence $\{v_0, \dots, v_n\} \in K(\mathcal{H} \cap \mathcal{H}')$ implies $\{v_0, \dots, v_n\} \in K_{\mathcal{H}} \cap K_{\mathcal{H}'}$. In particular, the equality of (3.2) holds under our additional the assumption. \square

The next proposition gives a universal property of the associated simplicial complexes of hypergraphs.

PROPOSITION 3.2. *Let \mathcal{H} be a hypergraph and let $i_{\mathcal{H}} : \mathcal{H} \rightarrow K_{\mathcal{H}}$ be the canonical embedding of \mathcal{H} into the associated simplicial complex. If there is a simplicial complex*

K and an injective map $i : \mathcal{H} \rightarrow K$, then there exists an injective simplicial map ϕ such that $\phi \circ i_{\mathcal{H}} = i$, making the following diagram commute:



Proof. Let $\{v_0, \dots, v_n\}$ be a hyperedge of \mathcal{H} . Then, since i is injective, $\{i(v_0), \dots, i(v_n)\}$ is an n -simplex of K . For any subset $\{v_{j_0}, \dots, v_{j_m}\}$ of $\{v_0, \dots, v_n\}$, since K is a simplicial complex, we have that $\{i(v_{j_0}), \dots, i(v_{j_m})\}$ is a simplex of K . On the other hand, by the definition, any simplex of $K_{\mathcal{H}}$ is of the form $\{v_{j_0}, \dots, v_{j_m}\}$ (here we regard $i_{\mathcal{H}}$ as the canonical inclusion and do not distinguish a hyperedge of \mathcal{H} with its image in $K_{\mathcal{H}}$). Therefore, we obtain a simplicial map $\phi : K_{\mathcal{H}} \rightarrow K$ sending the simplex $\{v_{j_0}, \dots, v_{j_m}\}$ to the simplex $\{i(v_{j_0}), \dots, i(v_{j_m})\}$. It follows that ϕ is injective and $\phi \circ i_{\mathcal{H}} = i$. \square

The next proposition is a consequence of Proposition 3.2.

PROPOSITION 3.3. *Let \mathcal{H} be a hypergraph and let K be a simplicial complex such that there is an injective map $i : \mathcal{H} \rightarrow K$. Then*

$$\text{Inf}_*(G(\mathcal{H}_*), G(K_*)) = \text{Inf}_*(G(\mathcal{H}_*), G((K_{\mathcal{H}})_*)) \tag{3.3}$$

and

$$\text{Sup}_*(G(\mathcal{H}_*), G(K_*)) = \text{Sup}_*(G(\mathcal{H}_*), G((K_{\mathcal{H}})_*)). \tag{3.4}$$

For simplicity, we denote the simplicial complex (3.3) as $\text{Inf}_*(\mathcal{H})$ and denote the simplicial complex (3.4) as $\text{Sup}_*(\mathcal{H})$. By Proposition 3.3, both $\text{Inf}_*(\mathcal{H})$ and $\text{Sup}_*(\mathcal{H})$ do not depend on the choice of the ambient simplicial complex K that \mathcal{H} is embedded in.

3.2. The embedded homology of hypergraphs. Given a hypergraph \mathcal{H} and an abelian group G , we define the n -th **embedded homology** of \mathcal{H} (with coefficients in G) as

$$H_n(\mathcal{H}) = H_n(\text{Inf}_*(\mathcal{H})).$$

In particular, if \mathcal{H} is a simplicial complex, then the embedded homology $H_*(\mathcal{H})$ coincides with the usual simplicial homology of \mathcal{H} . In this subsection, we give some basic properties of the embedded homology of hypergraphs. Then we give an example illustrating the computation.

The next proposition follows from Proposition 2.4 and Proposition 3.3.

PROPOSITION 3.4. *Let G be an abelian group. Let \mathcal{H} be a hypergraph, K be a simplicial complex that \mathcal{H} can be embedded in and ∂_* be the boundary map of K . Then*

$$\begin{aligned}
 H_n(\mathcal{H}) &\cong H_n(\text{Sup}_*(\mathcal{H})) \\
 &\cong \text{Ker}(\partial_n|_{G(\mathcal{H}_n)}) / ((G(\mathcal{H}_n)) \cap \partial_{n+1}(G(\mathcal{H}_{n+1}))).
 \end{aligned}$$

The next proposition gives a geometric interpretation of the 0-th embedded homology.

PROPOSITION 3.5. *Let \mathcal{H} be a hypergraph such that all the vertices are hyperedges. Then $\mathcal{H}_0 \cup \mathcal{H}_1$ is a simplicial complex. Moreover, with integral coefficients, $H_0(\mathcal{H}) = \mathbb{Z}^{\oplus k}$ where k is the number of connected components of $\mathcal{H}_0 \cup \mathcal{H}_1$.*

Proof. Since all the vertices are hyperedges, we see that $\mathcal{H}_0 \cup \mathcal{H}_1$ is a simplicial complex. Moreover,

$$\begin{aligned} H_0(\mathcal{H}) &= \text{Ker}(\partial_0|_{\mathbb{Z}(\mathcal{H}_0)})/\partial_1(\mathbb{Z}(\mathcal{H}_1)) \cap \mathbb{Z}(\mathcal{H}_0) \\ &= \mathbb{Z}(\mathcal{H}_0)/\partial_1(\mathbb{Z}(\mathcal{H}_1)) \\ &= H_0(\mathcal{H}_0 \cup \mathcal{H}_1). \end{aligned}$$

Since $H_0(\mathcal{H}_0 \cup \mathcal{H}_1) = \mathbb{Z}^k$ where k is the number of connected components of $\mathcal{H}_0 \cup \mathcal{H}_1$, the assertion follows. \square

The next proposition gives the top embedded homology of a hypergraph.

PROPOSITION 3.6. *Let \mathcal{H} be a hypergraph of dimension n , $n \geq 0$. Then $H_n(\mathcal{H}) = H_n(K_{\mathcal{H}})$.*

Proof. Since the dimension of \mathcal{H} is n , \mathcal{H}_i is empty for any $i \geq n + 1$. Hence $K_{\mathcal{H}}$ is a simplicial complex of dimension n and $\mathcal{H}_n = (K_{\mathcal{H}})_n$. Therefore, if we let ∂_* be the boundary map of $K_{\mathcal{H}}$ and take integral coefficients for convenience, then both $H_n(\mathcal{H})$ and $H_n(K_{\mathcal{H}})$ are $\text{Ker}\partial_n$. The assertion follows. \square

The next proposition gives the naturality property of the embedded homology.

PROPOSITION 3.7. *Let $f : \mathcal{H} \rightarrow \mathcal{H}'$ be a morphism of hypergraphs. Then f induces a homomorphism of graded groups $f_* : H_*(\mathcal{H}) \rightarrow H_*(\mathcal{H}')$.*

Proof. The functor \mathcal{F} sends f to a simplicial map $\tilde{f} : K_{\mathcal{H}} \rightarrow K_{\mathcal{H}'}$. Let $\{v_0, \dots, v_n\}$ be a simplex of $K_{\mathcal{H}}$. Given an abelian group G , we let \tilde{f}_G be the map sending $\{v_0, \dots, v_n\}$ to $\{f(v_0), f(v_1), \dots, f(v_n)\}$ if $f(v_0), f(v_1), \dots, f(v_n)$ are distinct, and sending $\{v_0, \dots, v_n\}$ to 0 (the identity element of G) otherwise. Then by extending \tilde{f}_G linearly over G , we obtain a chain map (still denoted as \tilde{f}_G) from the chain complex $G(K_{\mathcal{H}*})$ to the chain complex $G(K_{\mathcal{H}'*})$. Moreover, restricting the chain map \tilde{f}_G to the infimum chain complex, we obtain a chain map from $\text{Inf}_*(\mathcal{H})$ to $\text{Inf}_*(\mathcal{H}')$. This induces a homomorphism of graded groups $f_* : H_*(\mathcal{H}) \rightarrow H_*(\mathcal{H}')$. \square

The next example illustrates how to compute the embedded homology of hypergraphs.

EXAMPLE 3.8. Let the hypergraph $\mathcal{H} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_1, v_2\}\}$. Then its associated simplicial complex is

$$K_{\mathcal{H}} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}, \{v_0, v_1, v_2\}\}.$$

The following picture illustrates \mathcal{H} and $K_{\mathcal{H}}$:



Moreover, if we let ∂_n be the boundary map of $K_{\mathcal{H}}$ and take integral coefficients, then

$$\begin{aligned} H_0(\mathcal{H}) &= \text{Ker}(\partial_0|_{\mathbb{Z}(\{v_0\},\{v_1\},\{v_2\})})/\partial_1(\mathbb{Z}(\{v_0, v_1\})) \cap \mathbb{Z}(\{v_0\}, \{v_1\}, \{v_2\}) \\ &= \mathbb{Z}(\{v_0\}, \{v_1\}, \{v_2\})/\mathbb{Z}(\{v_1\} - \{v_0\}) \cap \mathbb{Z}(\{v_0\}, \{v_1\}, \{v_2\}) \\ &= \mathbb{Z} \oplus \mathbb{Z}, \\ H_1(\mathcal{H}) &= \text{Ker}(\partial_1|_{\mathbb{Z}(\{v_0, v_1\})})/\partial_2(\mathbb{Z}(\{v_0, v_1, v_2\})) \cap \mathbb{Z}(\{v_0, v_1\}) \\ &= 0, \\ H_2(\mathcal{H}) &= \text{Ker}(\partial_2|_{\mathbb{Z}(\{v_0, v_1, v_2\})})/0 \cap \mathbb{Z}(\{v_0, v_1, v_2\}) \\ &= 0. \end{aligned}$$

3.3. The Mayer-Vietoris sequence for the embedded homology of hypergraphs. In the homology theory of simplicial complexes, the Mayer-Vietoris sequence is an algebraic tool to compute the homology groups of simplicial complexes. With the Mayer-Vietoris sequence, we are able to reduce the homology of a complicated simplicial complex K into the homology of simpler simplicial complexes $K_1, K_2 \subseteq K$ such that K is the union of K_1 and K_2 . In this subsection, we give a Mayer-Vietoris sequence for the embedded homology of hypergraphs in Theorem 3.10. The Mayer-Vietoris sequence for the embedded homology of hypergraphs allows us to reduce the embedded homology of a complicated hypergraph to the embedded homology of simpler hypergraphs.

Let \mathcal{H} and \mathcal{H}' be two hypergraphs. With the help of Proposition 2.3, we have the following short exact sequences

$$\begin{aligned} 0 \longrightarrow \text{Inf}_*(\mathcal{H}) \cap \text{Inf}_*(\mathcal{H}') \longrightarrow \text{Inf}_*(\mathcal{H}) \oplus \text{Inf}_*(\mathcal{H}') \\ \longrightarrow \text{Inf}_*(\mathcal{H}) + \text{Inf}_*(\mathcal{H}') \longrightarrow 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 \longrightarrow \text{Inf}_*(\mathcal{H}) + \text{Inf}_*(\mathcal{H}') \longrightarrow \text{Inf}_*(\mathcal{H} \cup \mathcal{H}') \\ \longrightarrow \text{Inf}_*(\mathcal{H} \cup \mathcal{H}')/(\text{Inf}_*(\mathcal{H}) + \text{Inf}_*(\mathcal{H}')) \longrightarrow 0. \end{aligned} \tag{3.6}$$

Moreover, we have the next proposition.

PROPOSITION 3.9. *Let \mathcal{H} and \mathcal{H}' be two hypergraphs such that for any $\sigma \in \mathcal{H}$ and any $\sigma' \in \mathcal{H}'$, either $\sigma \cap \sigma'$ is empty or $\sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}'$. Then we have a short exact sequence*

$$0 \longrightarrow \text{Inf}_*(\mathcal{H}) \cap \text{Inf}_*(\mathcal{H}') \longrightarrow \text{Inf}_*(\mathcal{H}) \oplus \text{Inf}_*(\mathcal{H}') \longrightarrow \text{Inf}_*(\mathcal{H} \cup \mathcal{H}') \longrightarrow 0.$$

Proof. We choose $K_{\mathcal{H} \cup \mathcal{H}'}$ as the ambient simplicial complex in Proposition 3.3 to construct $\text{Inf}_*(\mathcal{H})$, $\text{Inf}_*(\mathcal{H}')$, $\text{Inf}_*(\mathcal{H} \cap \mathcal{H}')$ and $\text{Inf}_*(\mathcal{H} \cup \mathcal{H}')$. Let $\alpha \in \text{Inf}_n(\mathcal{H} \cup \mathcal{H}')$. Then

$$\alpha \in G(\mathcal{H}_n \cup \mathcal{H}'_n) \tag{3.7}$$

and

$$\partial_n \alpha \in G(\mathcal{H}_{n-1} \cup \mathcal{H}'_{n-1}). \tag{3.8}$$

Moreover, for any non-negative integer i ,

$$G(\mathcal{H}_i \cup \mathcal{H}'_i) = G(\mathcal{H}_i) + G(\mathcal{H}'_i). \tag{3.9}$$

From (3.7) and (3.9), we can assume that $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in G(\mathcal{H}_n)$ and $\alpha_2 \in G(\mathcal{H}'_n \setminus \mathcal{H}_n)$. We can also assume that

$$\partial_n \alpha_1 = \beta_1 + \gamma_1 \tag{3.10}$$

where $\beta_1 \in G(\mathcal{H}_{n-1})$, $\gamma_1 \in G(\mathcal{H}_{n-1}^c)$; and

$$\partial_n \alpha_2 = \beta_2 + \gamma_2 \tag{3.11}$$

where $\beta_2 \in G(\mathcal{H}'_{n-1})$, $\gamma_2 \in G(\mathcal{H}'_{n-1}^c)$. From (3.8) - (3.11), we have

$$\gamma_1 + \gamma_2 \in G(\mathcal{H}_{n-1}) + G(\mathcal{H}'_{n-1}).$$

It follows that

$$\gamma_1 + \gamma_2 = 0. \tag{3.12}$$

Suppose

$$\gamma_1 = \sum_t k_t d_i \sigma_t, \tag{3.13}$$

$$\gamma_2 = \sum_s h_s d_i \sigma'_s, \tag{3.14}$$

where $k_t, h_s \in G$, $\sigma_t \in \mathcal{H}_n$, $\sigma'_s \in \mathcal{H}'_n$, d_i 's are the face maps of $K_{\mathcal{H} \cup \mathcal{H}'}$, $d_i \sigma_t \in \mathcal{H}_{n-1}^c$ and $d_i \sigma'_s \in \mathcal{H}'_{n-1}$. We claim that for any summand $k_t d_i \sigma_t$ of (3.13) and any summand $h_s d_i \sigma'_s$ of (3.14),

$$d_i \sigma_t \neq d_j \sigma'_s. \tag{3.15}$$

To prove (3.15), we suppose to the contrary that $d_i \sigma_t = d_j \sigma'_s$ for some i, j, t and s . Then we have the next two cases.

CASE 1. $\sigma_t = \sigma'_s$.

By multiplying certain coefficient in G , σ_t is a summand of α_1 . Hence $\sigma_t \in \mathcal{H}$. And by multiplying another coefficient in G , σ'_s is a summand of α_2 . Hence $\sigma'_s \in \mathcal{H}' \setminus \mathcal{H}$. This contradicts $\sigma_t = \sigma'_s$.

CASE 2. $\sigma_t \neq \sigma'_s$.

Then $d_i \sigma_t = d_j \sigma'_s = \sigma_t \cap \sigma'_s$, which is non-empty. Moreover, since $\sigma_t \in \mathcal{H}$ and $\sigma'_s \in \mathcal{H}'$, we have $\sigma_t \cap \sigma'_s \in \mathcal{H} \cap \mathcal{H}'$. Therefore, $d_i \sigma_t \in \mathcal{H}$ and $d_j \sigma'_s \in \mathcal{H}'$. This contradicts the assumption on γ_1 and γ_2 .

Summarising Case 1 and Case 2, we have (3.15). It follows from (3.12) - (3.15) that $\gamma_1 = \gamma_2 = 0$. Therefore, the canonical homomorphism from $\text{Inf}_*(\mathcal{H}) \oplus \text{Inf}_*(\mathcal{H}')$ to $\text{Inf}_*(\mathcal{H} \cup \mathcal{H}')$ is an epimorphism. Consequently, the short exact sequence (3.6) is trivial, and the short exact sequence (3.5) implies the assertion. \square

The following theorem is an immediate consequence of Proposition 3.9.

THEOREM 3.10. *Let \mathcal{H} and \mathcal{H}' be two hypergraphs such that for any $\sigma \in \mathcal{H}$ and any $\sigma' \in \mathcal{H}'$, either $\sigma \cap \sigma'$ is empty or $\sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}'$. Then we have a long exact sequence of homology*

$$\dots \longrightarrow H_n(\mathcal{H} \cap \mathcal{H}') \longrightarrow H_n(\mathcal{H}) \oplus H_n(\mathcal{H}') \longrightarrow H_n(\mathcal{H} \cup \mathcal{H}') \longrightarrow H_{n-1}(\mathcal{H} \cap \mathcal{H}') \longrightarrow \dots$$

The long exact sequence given in Theorem 3.10 is called the **Mayer-Vietoris sequence** for the embedded homology of hypergraphs.

REMARK 2. Generally, let \mathcal{H} and \mathcal{H}' be two arbitrary hypergraphs. By (3.5) and (3.6), we have two long exact sequences of homology

$$\begin{aligned} & \cdots \longrightarrow H_n(\mathcal{H} \cap \mathcal{H}') \longrightarrow H_n(\mathcal{H}) \oplus H_n(\mathcal{H}') \\ & \longrightarrow H_n(\text{Inf}_*(\mathcal{H}) + \text{Inf}_*(\mathcal{H}')) \longrightarrow H_{n-1}(\mathcal{H} \cap \mathcal{H}') \longrightarrow \cdots, \\ & \cdots \longrightarrow H_n(\text{Inf}_*(\mathcal{H}) + \text{Inf}_*(\mathcal{H}')) \longrightarrow H_n(\mathcal{H} \cup \mathcal{H}') \\ & \longrightarrow H_n(\mathcal{H} \cup \mathcal{H}' / (\text{Inf}_*(\mathcal{H}) + \text{Inf}_*(\mathcal{H}'))) \longrightarrow H_{n-1}(\text{Inf}_*(\mathcal{H}) + \text{Inf}_*(\mathcal{H}')) \longrightarrow \cdots. \end{aligned}$$

The next example illustrates how to use the Mayer-Vietoris sequence to simplify the computation of the embedded homology of hypergraphs.

EXAMPLE 3.11. (i). Let \mathcal{H} and \mathcal{H}' be hypergraphs such that (a). for any $\sigma \in \mathcal{H}$ and any $\sigma' \in \mathcal{H}'$, either $\sigma \cap \sigma'$ is empty or $\sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}'$; (b). the intersection of \mathcal{H} and \mathcal{H}' is a disjoint union of standard simplicial complexes

$$\mathcal{H} \cap \mathcal{H}' = \coprod_{i=1}^k \Delta[n_i].$$

Then since each $\Delta[n_i]$ is contractible, for any $n \geq 1$,

$$H_n(\mathcal{H} \cap \mathcal{H}') = 0.$$

Hence by Theorem 3.10, for any $n \geq 2$,

$$H_n(\mathcal{H} \cup \mathcal{H}') \cong H_n(\mathcal{H}) \oplus H_n(\mathcal{H}').$$

(ii). Suppose $\mathcal{H}^j, 1 \leq j \leq m$, is a sequence of hypergraphs such that for any $j_1 \neq j_2$, (a). for any $\sigma \in \mathcal{H}^{j_1}$ and any $\sigma' \in \mathcal{H}^{j_2}$, either $\sigma \cap \sigma'$ is empty or $\sigma \cap \sigma' \in \mathcal{H}^{j_1} \cap \mathcal{H}^{j_2}$; (b). $\mathcal{H}^{j_1} \cap \mathcal{H}^{j_2}$ is a disjoint union of standard simplicial complexes. Then by (i) and an induction on m , for any $n \geq 2$,

$$H_n(\cup_{j=1}^m \mathcal{H}^j) \cong \oplus_{j=1}^m H_n(\mathcal{H}^j).$$

(iii). A concrete example of (ii) is as follows. For each $j = 0, 1, 2, 3$, let

$$\mathcal{H}^j = \Delta[v_0, v_1, v_2, v_3] \cup \{\{v_0, \dots, \hat{v}_j, \dots, v_3, w_j\}\}$$

where \hat{v}_j stands for removing v_j . Then by (ii), with integral coefficients,

$$H_2(\cup_{j=1}^3 \mathcal{H}^j) \cong \oplus_{j=1}^3 H_2(\mathcal{H}^j) \cong 0.$$

4. The persistent embedded homology of hypergraphs and Mayer-Vietoris sequence. Persistent homology has been used to study the invariant topological structures of sequences of topological objects. Some algorithms to compute persistent homology of simplicial complexes have been given by A. Zomorodian and G. Carlsson [21]. In this section, we study the persistent embedded homology of sequences of hypergraphs and give a persistent version of Mayer-Vietoris sequence for the persistent embedded homology of hypergraphs, in Theorem 4.1.

Before studying the persistent embedded homology of hypergraphs, we review some definitions. A **persistence complex** $\mathcal{C} = \{C_*^i, \varphi_i\}_{i \geq 0}$ is a family of chain complexes $\{C_*^i\}_{i \geq 0}$, together with chain maps $\varphi_i : C_*^i \rightarrow C_*^{i+1}$ (cf. [21, Definition 3.1]). A **persistence abelian group** $\mathcal{G} = \{G^i, \psi_i\}_{i \geq 0}$ is a family of abelian groups G^i , together with group homomorphisms $\psi_i : G^i \rightarrow G^{i+1}$. A persistence complex \mathcal{C} (resp. a persistence abelian group \mathcal{G}) is called of **finite type** if each component complex C^i (resp. G^i) is a finitely generated abelian group and there exists an integer N such that the maps φ_i (resp. ψ_i) are isomorphisms for all $i \geq N$ (cf. [21, Definition 3.3]).

Now we turn to the persistent embedded homology of hypergraphs. We consider a sequence of hypergraphs with morphisms

$$\mathcal{H}^0 \xrightarrow{f_0} \mathcal{H}^1 \xrightarrow{f_1} \mathcal{H}^2 \xrightarrow{f_2} \dots \xrightarrow{f_{i-1}} \mathcal{H}^i \xrightarrow{f_i} \dots \tag{4.1}$$

Given an abelian group G , the sequence (4.1) induces persistence complexes

$$\{\text{Inf}_*(G(\mathcal{H}_*^i), G((K_{\mathcal{H}^i})_*)), \tilde{f}_{i,G}\}_{i \geq 0}, \tag{4.2}$$

$$\{\text{Sup}_*(G(\mathcal{H}_*^i), G((K_{\mathcal{H}^i})_*)), \tilde{f}_{i,G}\}_{i \geq 0}. \tag{4.3}$$

Here $\tilde{f}_{i,G}$ are the maps given in the proof of Proposition 3.7. Moreover, both (4.2) and (4.3) induce a persistence abelian group

$$H_*(\mathcal{H}^0) \xrightarrow{(f_0)_*} H_*(\mathcal{H}^1) \xrightarrow{(f_1)_*} H_*(\mathcal{H}^2) \xrightarrow{(f_2)_*} \dots \tag{4.4}$$

We call the sequence (4.4) the **persistent embedded homology** of the sequence (4.1).

A **filtration** of hypergraphs is a sequence (4.1) such that each $f_i, i \geq 0$, is injective. Given a hypergraph \mathcal{H} , there are canonical ways to construct a filtration of hypergraphs. We give one way to construct a filtration as follows.

Let d be a distance function on the set of vertices $V_{\mathcal{H}}$ of a hypergraph \mathcal{H} . For any $r > 0$, we let $K(\mathcal{H}, r)$ be the simplicial complex consisting of all simplices of the form

$$\{v_0, \dots, v_n \mid n \geq 0, v_0, \dots, v_n \in V_{\mathcal{H}} \text{ and } d(v_i, v_j) < r\}$$

and let the hypergraph

$$\mathcal{H}(r) = \mathcal{H} \cap K(\mathcal{H}, r).$$

Then for any increasing sequence of numbers $0 < r_1 < r_2 < \dots < r_k < \dots$, we get a filtration of hypergraphs

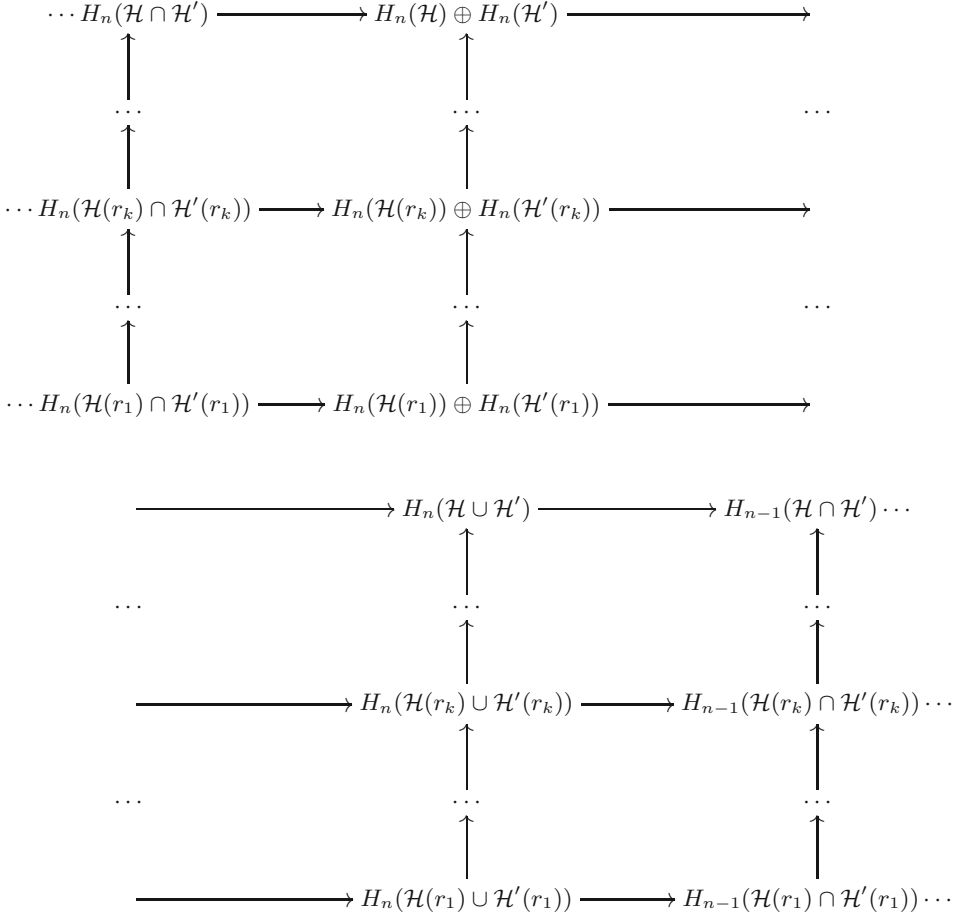
$$\mathcal{H}(r_1) \subseteq \mathcal{H}(r_2) \subseteq \dots \subseteq \mathcal{H}(r_k) \subseteq \dots \subseteq \mathcal{H}. \tag{4.5}$$

Since all hypergraphs are assumed to have finite vertices, there exists a positive R such that for any $r > R, \mathcal{H}(r) = \mathcal{H}(R)$. Therefore, the persistent homology of the filtration (4.5) is of finite type. Moreover, if $\lim_{k \rightarrow \infty} r_k = \infty$, then $\mathcal{H}(r_k)$ converges to \mathcal{H} . In this case, the persistent embedded homology $H_*(\mathcal{H}(r_k))$ converges to $H_*(\mathcal{H})$.

Generalising the Mayer-Vietoris sequence of the embedded homology to the persistent embedded homology, the next theorem follows from Proposition 3.10.

THEOREM 4.1. *Let \mathcal{H} and \mathcal{H}' be two hypergraphs such that for any $\sigma \in \mathcal{H}$ and any $\sigma' \in \mathcal{H}'$, either $\sigma \cap \sigma'$ is empty or $\sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}'$. Let d be a distance function*

on $\mathcal{H} \cup \mathcal{H}'$. Then for any $0 < r_1 < r_2 < \dots < r_k < \dots$, we have long exact sequences of embedded homology in each row of the following commutative diagram



Proof. We first claim that for each $r > 0$, $\sigma \in \mathcal{H}(r)$ and $\sigma' \in \mathcal{H}'(r)$ imply that either $\sigma \cap \sigma'$ is empty or $\sigma \cap \sigma' \in \mathcal{H}(r) \cap \mathcal{H}'(r)$. To prove the claim, we choose any $\sigma \in \mathcal{H}(r)$ and any $\sigma' \in \mathcal{H}'(r)$ such that $\sigma \cap \sigma'$ is non-empty. Then $\sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}'$. Moreover, for any vertices $v, w \in \sigma \cap \sigma'$, we have $d(v, w) < r$. Hence $\sigma \cap \sigma' \in K(\mathcal{H} \cup \mathcal{H}', r)$. Since

$$\begin{aligned}
 \mathcal{H}(r) \cap \mathcal{H}'(r) &= (K(\mathcal{H}, r) \cap \mathcal{H}) \cap (K(\mathcal{H}', r) \cap \mathcal{H}') \\
 &= (K(\mathcal{H} \cup \mathcal{H}', r) \cap \mathcal{H}) \cap (K(\mathcal{H} \cup \mathcal{H}', r) \cap \mathcal{H}') \\
 &= K(\mathcal{H} \cup \mathcal{H}', r) \cap (\mathcal{H} \cap \mathcal{H}'),
 \end{aligned}$$

we see that $\sigma \cap \sigma' \in \mathcal{H}(r) \cap \mathcal{H}'(r)$. The claim is obtained. By Proposition 3.10, each row of the above commutative diagram is a long exact sequence. With the help of the naturality property of the embedded homology given in Proposition 3.7, the assertion follows. \square

5. Applications of the associated simplicial complex and the embedded homology in Acyclic Hypergraphs. Acyclic Hypergraphs is an important family

of hypergraphs. A hypergraph \mathcal{H} is said to be acyclic if \mathcal{H} can be reduced to an empty set by repeatedly applying the following two operations:

- (O1). if v is a vertex that belongs to only one hyperedge, then delete v from the hyperedge containing it;
- (O2). if $\sigma \subsetneq \sigma'$ are two hyperedges, then delete σ from \mathcal{H} .

The notion of acyclic hypergraphs was firstly introduced as an analogue of trees in graphs (cf. [1]). It is the mathematical model for database schemas in database theory (cf. [3]). In this section, we study acyclic hypergraphs by using the associated simplicial complexes and the embedded homology.

5.1. The associated simplicial complex of Acyclic Hypergraphs. In this subsection, we strengthen [19, Theorem 9] and give a necessary condition and a sufficient condition on the associated simplicial complexes for the acyclic property of hypergraphs, in Theorem 5.2. Then we give some examples.

Characterising the associated simplicial complexes of acyclic hypergraphs, the following theorem is proved in [19].

THEOREM 5.1 ([19, Theorem 9]). *Let \mathcal{H} be a connected acyclic hypergraph. Then with integral coefficients, $H_n(K_{\mathcal{H}})$ is zero when $n \geq 1$ and is \mathbb{Z} when $n = 0$.*

REMARK 3. For a general acyclic hypergraph \mathcal{H} , it follows from Theorem 5.1 that $H_n(K_{\mathcal{H}})$ is zero when $n \geq 1$ and is $\mathbb{Z}^{\oplus k}$ when $n = 0$ where k is the number of connected components of $K_{\mathcal{H}}$.

To characterise acyclic hypergraphs more precisely, we strengthen Theorem 5.1 and prove the next theorem.

THEOREM 5.2. *Let \mathcal{H} be a hypergraph.*

- (a). *If \mathcal{H} is acyclic, then $K_{\mathcal{H}}$ is acyclic as well, and $K_{\mathcal{H}}$ has the homotopy type of finite discrete points.*
- (b). *If $K_{\mathcal{H}}$ is the associated simplicial complex of a finite disjoint union of simplices whose sets of vertices are mutually non-intersecting, then \mathcal{H} is acyclic.*

The following corollary is a consequence of Theorem 5.2 (a).

COROLLARY 5.3. *Let \mathcal{H} be an acyclic hypergraph. If $n \geq 1$ is the dimension of \mathcal{H} , then $H_n(\mathcal{H}) = 0$.*

Proof. By Proposition 3.6, we have $H_n(\mathcal{H}) = H_n(K_{\mathcal{H}})$. Since $n \geq 1$, by Theorem 5.2 (a), we have $H_n(K_{\mathcal{H}}) = 0$. The assertion follows. \square

The following corollary is a particular case of Theorem 5.2 (b).

COROLLARY 5.4. *Let \mathcal{H} be a hypergraph such that $K_{\mathcal{H}}$ is the associated simplicial complex of a simplex. Then \mathcal{H} is acyclic.*

In order to prove Theorem 5.2, we consider the following operation

- (O1)'. if v is a vertex that belongs to only one hyperedge consisting of at least two vertices, then delete v from the hyperedge containing it.

We prove the next lemmas.

LEMMA 5.5. *A hypergraph \mathcal{H} is acyclic if and only if \mathcal{H} can be reduced to finite discrete points by finite steps of the operations (O1)' and (O2).*

Proof. The assertion follows by a simple observation. \square

LEMMA 5.6. *Given a hypergraph \mathcal{H} , let \mathcal{H}' be a hypergraph obtained from \mathcal{H} by arbitrary finite steps of the operations (O1)' and (O2). Then $K_{\mathcal{H}}$ and $K_{\mathcal{H}'}$ are homotopy equivalent.*

Proof. The assertion follows from a geometric observation that the operation (O2) on \mathcal{H} does not change $K_{\mathcal{H}}$. And the operation (O1)' on \mathcal{H} gives a deformation retract of $K_{\mathcal{H}}$, hence it does not change the homotopy type of $K_{\mathcal{H}}$. \square

Now we turn to prove Theorem 5.2.

Proof of Theorem 5.2 (a). Suppose \mathcal{H} is acyclic. Then \mathcal{H} can be reduced to the empty set after finite steps of the operations (O1) and (O2). Since $K_{\mathcal{H}}$ is obtained by adding all the non-empty subsets $\tau \subsetneq \sigma$ for all $\sigma \in \mathcal{H}$ as hyperedges, we see that after finite steps of (O2), $K_{\mathcal{H}}$ can be reduced to \mathcal{H} . Hence $K_{\mathcal{H}}$ can be reduced to the empty set after finite steps of the operations (O1) and (O2). This implies $K_{\mathcal{H}}$ is acyclic. Moreover, it follows from Lemma 5.5 and Lemma 5.6 that $K_{\mathcal{H}}$ has the homotopy type of finite discrete points. \square

Proof of Theorem 5.2 (b). We divide the proof into two steps.

STEP 1. We assume that $K_{\mathcal{H}}$ is the associated simplicial complex of an n -simplex. Then there exists exactly one n -simplex in $K_{\mathcal{H}}$, denoted as Δ^n . Since $K_{\mathcal{H}}$ is the smallest simplicial complex containing \mathcal{H} , we see that Δ^n is the exactly one hyperedge in \mathcal{H}_n . Since Δ^n consists of all the vertices of $K_{\mathcal{H}}$, it consists of all the vertices of \mathcal{H} . Hence for any hyperedge $\tau \in \mathcal{H}$, we have $\tau \subseteq \Delta^n$. By applying (O2) repeatedly, all the hyperedges $\tau \subsetneq \Delta^n$ of \mathcal{H} can be deleted and \mathcal{H} can be reduced to the hypergraph consisting of only one hyperedge Δ^n . Thus \mathcal{H} is acyclic.

STEP 2. We assume that $K_{\mathcal{H}}$ is the associated simplicial complex of a disjoint union of simplices $\coprod_{j=1}^m \Delta^{n_j}$, where for any distinct i and j , their sets of vertices $V_{\Delta^{n_i}}$ and $V_{\Delta^{n_j}}$ are non-intersecting. Then following the notation of Example 1.1 and with the help of Proposition 3.1,

$$K_{\mathcal{H}} = \coprod_{j=1}^m \Delta[n_j].$$

For each $j = 1, \dots, m$, we let

$$\mathcal{H}(j) = \mathcal{H} \cap \Delta[n_j].$$

Then $\Delta[n_j]$ is the associated simplicial complex of $\mathcal{H}(j)$. By STEP 1, we see that $\mathcal{H}(j)$ is acyclic. Since \mathcal{H} is the disjoint union of $\mathcal{H}(j)$'s, we see that \mathcal{H} is acyclic as well. \square

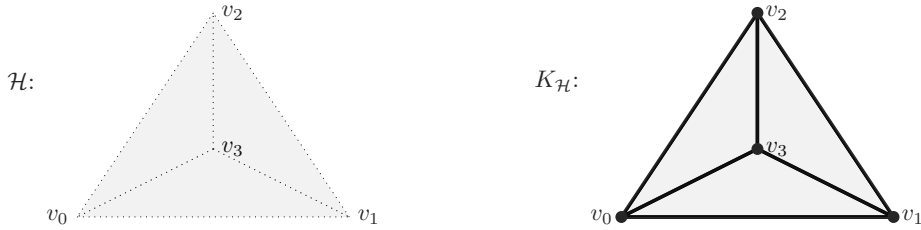
The converse of Theorem 5.2 (a) is not true. We give such examples as follows.

EXAMPLE 5.7. Let $n \geq 3$. Then following the notations in Example 1.1, we have that for any $0 \leq i \leq n$, the hypergraph

$$\mathcal{H} = \{\Delta_j^{n-1} \mid 0 \leq j \leq n, j \neq i\} \tag{5.1}$$

is not acyclic, while $K_{\mathcal{H}}$ has the homotopy type of a single point.

The hypergraph \mathcal{H} in Example 5.7 is the collection of $(n-1)$ -faces of Δ^n excluding the i -th $(n-1)$ -face. The following picture shows the case $n = 3, i = 3$ in Example 5.7.



Proof of Example 5.7. Given a fixed i , by (5.1), we see that v_i belongs to n hyperedges of \mathcal{H} and for any $j \neq i$, v_j belongs to $n - 1$ hyperedges of \mathcal{H} . Since $n \geq 3$, each vertex of \mathcal{H} belongs to at least two hyperedges. Hence \mathcal{H} cannot be reduced by (O1). On the other hand, it is clear that for any distinct j and l , Δ_j^{n-1} is not contained in Δ_l^{n-1} . Hence \mathcal{H} cannot be reduced by (O2). Therefore, \mathcal{H} cannot be reduced by either (O1) or (O2). Hence \mathcal{H} is not acyclic. \square

The converse of Theorem 5.2 (b) is also not true. The following is such an example.

EXAMPLE 5.8. Let $\mathcal{H} = \{\{v_0, v_1, v_2\}, \{v_1, v_2, v_3\}\}$. Then \mathcal{H} is acyclic, while

$$K_{\mathcal{H}} = \{\{v_0, v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}\}$$

is connected and is not the associated simplicial complex of any single simplex.

5.2. Hypergraphs whose associated simplicial complex is $\Delta[n]$. A particular family of acyclic hypergraphs is the hypergraphs whose associated simplicial complexes are $\Delta[n]$. In this section, we study the embedded homology of this family of acyclic hypergraphs.

Let \mathcal{H} be a hypergraph whose associated simplicial complex is $\Delta[n]$, $n \geq 2$. Then \mathcal{H} is acyclic. Besides the triviality of $H_n(\mathcal{H})$ given in Corollary 5.3, the following proposition shows the triviality of $H_{n-1}(\mathcal{H})$.

PROPOSITION 5.9. *Let \mathcal{H} be a hypergraph of dimension n such that the associated simplicial complex of \mathcal{H} is $\Delta[n]$, $n \geq 2$. Then $H_{n-1}(\mathcal{H})$ is zero.*

Proof. For simplicity, we take integral coefficients. Let ∂_k , $k = 0, 1, \dots, n$, be the boundary maps of $\Delta[n]$. Then for any $k = 0, 1, \dots, n$,

$$H_k(\mathcal{H}) = \text{Ker}(\partial_k|_{\mathbb{Z}(\mathcal{H}_k)}) / (\mathbb{Z}(\mathcal{H}_k) \cap \partial_{k+1}\mathbb{Z}(\mathcal{H}_{k+1})). \tag{5.2}$$

Since the associated simplicial complex of \mathcal{H} is $\Delta[n]$, we have $\mathcal{H}_n = \{\Delta^n\}$ and

$$\partial_n \Delta^n = \sum_{j=0}^n (-1)^j \Delta_j^{n-1}. \tag{5.3}$$

Since $n \geq 2$, we have

$$\text{Ker} \partial_{n-1} = \mathbb{Z} \left(\sum_{j=0}^n \Delta_j^{n-1} \right). \tag{5.4}$$

Moreover, since $\mathcal{H}_{n-1} \subseteq \{\Delta_j^{n-1} \mid j = 0, 1, \dots, n\}$, we consider the next two cases.

CASE 1. $\mathcal{H}_{n-1} \subsetneq \{\Delta_j^{n-1} \mid j = 0, 1, \dots, n\}$.

Then by (5.4), we have

$$\text{Ker}(\partial_{n-1}|_{\mathbb{Z}(\mathcal{H}_{n-1})}) = 0.$$

It follows from (5.2) that $H_{n-1}(\mathcal{H}) = 0$.

CASE 2. $\mathcal{H}_{n-1} = \{\Delta_j^{n-1} \mid j = 0, 1, \dots, n\}$.

Then by (5.4), we have

$$\text{Ker}(\partial_{n-1}|_{\mathbb{Z}(\mathcal{H}_{n-1})}) = \mathbb{Z}(\sum_{j=0}^n \Delta_j^{n-1}). \tag{5.5}$$

Moreover, with the help of (5.3), we have

$$\partial_n(\mathbb{Z}(\mathcal{H}_n)) = \mathbb{Z}(\sum_{j=0}^n \Delta_j^{n-1}), \tag{5.6}$$

which is a submodule of $\mathbb{Z}(\mathcal{H}_{n-1})$. It follows from (5.2), (5.5) and (5.6) that $H_{n-1}(\mathcal{H}) = 0$.

Summarising CASE 1 and CASE 2, we have $H_{n-1}(\mathcal{H}) = 0$. The assertion follows. \square

For any hypergraph \mathcal{H} , we will construct an acyclic hypergraph \mathcal{H}' which contains \mathcal{H} and has the same embedded homology with \mathcal{H} , in the next theorem.

THEOREM 5.10. *For any hypergraph \mathcal{H} , there exists an acyclic hypergraph \mathcal{H}' such that*

- (i). $K_{\mathcal{H}'} = \Delta[n]$ for some $n \geq 2$;
- (ii). $\mathcal{H} \subsetneq \mathcal{H}'$;
- (iii). $H_*(\mathcal{H}') \cong H_*(\mathcal{H})$.

Proof. Given a hypergraph \mathcal{H} , by adding extra vertices x, y to $V_{\mathcal{H}}$ and letting $\sigma = V_{\mathcal{H}} \cup \{x, y\}$, we have a hypergraph $\mathcal{H}' = \mathcal{H} \cup \{\sigma\}$, which satisfies (ii). Any hyperedge of \mathcal{H}' is a subset of σ . Hence if σ is an n -hyperedge, then $K_{\mathcal{H}'} = \Delta[n]$. We obtain (i). Moreover, $\mathcal{H}'_i = \mathcal{H}_i$ for any $0 \leq i \leq n - 2$, \mathcal{H}'_{n-1} is empty and $\mathcal{H}'_n = \{\sigma\}$. Therefore, $H_i(\mathcal{H}') \cong H_i(\mathcal{H})$ for any $0 \leq i \leq n - 2$, while both $H_{n-1}(\mathcal{H}')$ and $H_n(\mathcal{H}')$ are zero. We obtain (iii). \square

The embedded homology of hypergraphs gives richer information than the associated simplicial complex. An acyclic hypergraph whose associated simplicial complex is $\Delta[n]$ may have highly non-trivial embedded homology, as shown in the next theorem.

THEOREM 5.11. *For any $m \geq 1$ and any finitely-generated abelian groups G_1, \dots, G_m , there exists an acyclic hypergraph \mathcal{H} such that*

- (i). $K_{\mathcal{H}} = \Delta[n]$, where n is the dimension of \mathcal{H} ;
- (ii). n is less than or equal to $m + 3$;
- (iii). $H_i(\mathcal{H}) = G_i$ for $1 \leq i \leq m$;
- (iv). $\mathcal{H}_0 \cup \mathcal{H}_1$ is connected.

Proof. For any finitely-generated abelian groups G_1, \dots, G_m , we let $M(G_i, i)$ be the Moore space (cf. [14, p. 143]) such that

- $H_i(M(G_i, i)) = G_i$;
- $H_j(M(G_i, i)) = 0$ for any $j \neq i, j \geq 1$;
- $M(G_i, i)$ is connected;
- $M(G_i, i)$ has cells only in dimension 0, i and $i + 1$.

Let K be a simplicial complex model for the wedge sum of $M(G_i, i)$, $1 \leq i \leq m$. Then K satisfies $H_i(K) = G_i$ for $1 \leq i \leq m$, and K is a connected simplicial complex whose dimension is less than or equal to $m + 1$. By Theorem 5.10, there exists an acyclic hypergraph \mathcal{H} containing K such that $K_{\mathcal{H}} = \Delta[n]$ where n is the dimension of \mathcal{H} , n is greater than the dimension of K by 2, and $H_i(\mathcal{H}) \cong H_i(K)$ for any $i \geq 0$. With the help of Proposition 3.5, $\mathcal{H}_0 \cup \mathcal{H}_1$ is connected. The assertion follows. \square

REMARK 4. By Proposition 5.9, we see that if the abelian group G_m is non-trivial, then the dimension of \mathcal{H} in Theorem 5.11 is greater than or equal to $m + 2$. Hence by Theorem 5.11 (iii), the dimension of \mathcal{H} is either $m + 2$ or $m + 3$.

6. Applications of the embedded homology in data analysis of hyper-networks. Hypergraphs are mathematical models of hyper-networks, which has significant applications in the data analysis of engineering, technology, economics, marketing, etc. A hyper-network is a system consisting of players/items as well as relations among the players/items. For example, if we take all the google users in the world as players, and assign a relation among the google users whenever they are in a google group, then we get a hyper-network. If we use a vertex to represent a player/item and use a hyperedge to represent a relation among the players/items, then we get a hypergraph model for a hyper-network.

In this section, by applying the embedded homology of hypergraphs, we construct the following indices: a hyper-network connectivity index to measure the connectivity of the vertices of a hyper-network, a hyper-network differentiation index to measure the differentiation of the vertices of a hyper-network with respect to certain functions on the vertices, and a hyper-network correlation index to measure the correlation between two functions on the vertices of a hyper-network. This section is speculation-based and contains no mathematical results. Nevertheless, the indices constructed in this section are possible to have potential applications in data analysis of hyper-networks.

6.1. The hyper-network connectivity index. Let \mathcal{H} be a hypergraph. The connectivity of \mathcal{H} measures how intimately the vertices of \mathcal{H} are connected with each other by the hyperedges in \mathcal{H} . In recent years, the connectivity of \mathcal{H} has been investigated from various aspects, for example, [5, 6, 13, 16]. Among these references, the connectivity is characterized from a homological aspect in [5]. We apply the 0-th embedded homology of hypergraphs and give a connectivity index to measure the connectivity of \mathcal{H} .

Let $\mathcal{H}^0 = \mathcal{H}$. For any $k \geq 1$, by an induction on k , we define a sequence of operations (Rk) and a sequence of hypergraphs \mathcal{H}^k as follows:

(Rk). For a vertex v of \mathcal{H}^{k-1} , if there exist exactly k hyperedges $\sigma_1, \dots, \sigma_k$ in \mathcal{H}^{k-1} such that v is a vertex of each σ_i , $i = 1, \dots, k$, then we remove v from each of $\sigma_1, \dots, \sigma_k$.

(\mathcal{H}^k). By taking the operation (Rk) on the hypergraph \mathcal{H}^{k-1} repeatedly until the hypergraph cannot be reduced by (Rk) anymore, we obtain a hypergraph \mathcal{H}^k .

We define the **hyper-network connectivity index** of \mathcal{H} to be the number

$$\text{Conn}(\mathcal{H}) = \sum_{k \geq 0} \frac{\dim H_0(\mathcal{H}^k; \mathbb{Q})}{2^{k+1} \cdot |V_{\mathcal{H}^k}|},$$

where $|\cdot|$ denotes the cardinality of a set. Then $\text{Conn}(\mathcal{H})$ is a positive number smaller than or equal to 1, which reflects how intimately the vertices of \mathcal{H} are connected by the hyperedges. As $\text{Conn}(\mathcal{H})$ increases, the connectivity of the vertices of \mathcal{H} becomes less significant. In particular, if $\text{Conn}(\mathcal{H}) = 1$, then $V_{\mathcal{H}}$ is totally discrete and there is no hyperedge in \mathcal{H} containing more than one point.

EXAMPLE 6.1. Let each point represents a person. Let G be the graph constructed by connecting two points whenever the corresponding two persons are in parents-children relations or in spouse relations. Then $\text{Conn}(G)$ can be used to measure how integrated the community is. If $\text{Conn}(G)$ becomes smaller/larger than before, then we conclude that the community becomes more/less integrated.

6.2. The hyper-network differentiation index. Let \mathcal{H} be a hypergraph and let $\varphi : V_{\mathcal{H}} \rightarrow [0, 1]$ be a function on the vertices of \mathcal{H} . The problem how $V_{\mathcal{H}}$ is differentiated with respect to φ and the relations given as hyperedges of \mathcal{H} is investigated in [20] from a perspective of dynamical systems. And the differentiation phenomenon of stocks in the financial market network is studied in [18]. We give a hyper-network differentiation index to measure how $V_{\mathcal{H}}$ is differentiated by using the embedded homology of hypergraphs.

For any $t \in [0, 1]$, let $\mathcal{H}(t)$ be the hypergraph consisting of all the hyperedges of \mathcal{H} whose vertices v satisfy $\varphi(v) \geq t$. That is,

$$\mathcal{H}(t) = \Delta[v \in V_{\mathcal{H}} \mid \varphi(v) \geq t] \cap \mathcal{H}.$$

For any $i \geq 0$ and $n \geq 1$, the sequence of the dimension of the embedded homology

$$\dim H_i(\mathcal{H}(k/n); \mathbb{Q}), \quad k = 0, 1, \dots, n \tag{6.1}$$

is a barcode (that is, a step function of one variable which is a finite sum of constant functions on intervals). Letting $n \rightarrow \infty$, since $V_{\mathcal{H}}$ is finite, the barcode (6.1) stabilises for n sufficiently large. We denote the limit barcode as the following function

$$f_{i,\varphi} : [0, 1] \rightarrow \mathbb{R}_{\geq 0}.$$

Let γ be a random variable in the function space

$$W_{\varphi} = \{ \gamma : V_{\mathcal{H}} \rightarrow [0, 1] \mid \text{for any } t \in [0, 1], \text{ the number of vertices } v \in V_{\mathcal{H}} \text{ such that } \varphi(v) \geq t \text{ equals to the number of vertices } v \in V_{\mathcal{H}} \text{ such that } \gamma(v) \geq t \}.$$

We denote E as the expectation and define $f_{i,\gamma}$ in the same way as $f_{i,\varphi}$ by only substituting φ with γ . Then the degree of fitness

$$\text{Fit}(f_{i,\varphi}, E(f_{i,\gamma} \mid \gamma \in W_{\varphi}))$$

reflects the differentiation of $V_{\mathcal{H}}$ with respect to φ and \mathcal{H} . Moreover, we let

$$\text{Diff}(\varphi, \mathcal{H}) = \sum_{i \geq 0} \frac{\text{Fit}(f_{i,\varphi}, E(f_{i,\gamma} \mid \gamma \in W_{\varphi}))}{2^{i+1}}.$$

We call $\text{Diff}(\varphi, \mathcal{H})$ the **hyper-network differentiation index** of φ with respect to \mathcal{H} . It is a number between 0 and 1. As the number $\text{Diff}(\varphi, \mathcal{H})$ increases, the differentiation of $V_{\mathcal{H}}$ with respect to φ and \mathcal{H} becomes more significant.

REMARK 5. Given two non-negative measurable functions β_1, β_2 on a measure space X such that

$$0 < \|\beta_1\|_2, \|\beta_2\|_2 < \infty$$

where $\|\cdot\|_2$ is the L^2 -norm of a function, we define the **degree of fitness** between β_1 and β_2 as

$$\text{Fit}(\beta_1, \beta_2) = \frac{\|\beta_1 - \beta_2\|_2}{\|\beta_1\|_2 + \|\beta_2\|_2}.$$

The degree of fitness is a number between 0 and 1. Smaller $\text{Fit}(\beta_1, \beta_2)$ means more significance of the fitness between β_1 and β_2 .

EXAMPLE 6.2. Let G be the graph given in Example 6.1. For any $n \geq 0$, whenever the points v_0, \dots, v_n of G form a loop (a single point is regarded as a trivial loop), we give a hyperedge $\{v_0, \dots, v_n\}$ in \mathcal{H} . Let φ be a function with value in $[0, 1]$ measuring the annual income/social status/personal property/education level of people. Then our hyper-network differentiation index $\text{Diff}(\varphi, \mathcal{H})$ can measure the social differentiation and mobility. If $\text{Diff}(\varphi, \mathcal{H})$ becomes smaller/larger than before, then we can conclude that the social differentiation decreases/increases.

6.3. The hyper-network correlation index. Correlation analysis can be conducted on networks, for example, [2, 17]. Let \mathcal{H} be a hypergraph and let $\varphi, \psi : V_{\mathcal{H}} \rightarrow [0, 1]$ be two functions on the vertices of \mathcal{H} . To measure the correlation between φ and ψ with respect to the relations on $V_{\mathcal{H}}$ given as hyperedges of \mathcal{H} , we give a hyper-network correlation index by using the embedded homology of hypergraphs.

For any $0 \leq t, s \leq 1$, let $\mathcal{H}(t, s)$ be the hypergraph consisting of all the hyperedges of \mathcal{H} whose vertices v satisfy $\varphi(v) \geq t$ and $\psi(v) \geq s$. That is,

$$\mathcal{H}(t, s) = \Delta[v \in V_{\mathcal{H}} \mid \varphi(v) \geq t \text{ and } \psi(v) \geq s] \cap \mathcal{H}.$$

For any $i \geq 0$ and $n \geq 1$, the sequence of the dimension of the embedded homology

$$\dim H_i(\mathcal{H}(k/n, l/n); \mathbb{Q}), \quad k, l = 0, 1, \dots, n \tag{6.2}$$

is a 2-dimensional barcode (that is, a step function of two variables which is a finite sum of constant functions on squares). Letting $n \rightarrow \infty$, since $V_{\mathcal{H}}$ is finite, the 2-dimensional barcode (6.2) stabilises when n is sufficiently large. We denote the limit barcode as the following function

$$g_{i, \varphi, \psi} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_{\geq 0}.$$

Let $\gamma_1 \in W_{\varphi}$ and $\gamma_2 \in W_{\psi}$ be independent random variables. We define $g_{i, \gamma_1, \gamma_2}$ in the same way as $g_{i, \varphi, \psi}$ by only substituting φ with γ_1 and substituting ψ with γ_2 . Then the degree of fitness of 2-variable functions

$$\text{Fit}(g_{i, \varphi, \psi}, E(g_{i, \gamma_1, \gamma_2} \mid \gamma_1 \in W_{\varphi}, \gamma_2 \in W_{\psi}))$$

reflects the correlation of φ and ψ on $V_{\mathcal{H}}$ with respect to the hyperedges in \mathcal{H} . Moreover, we let

$$\text{Corr}(\varphi, \psi, \mathcal{H}) = \sum_{i \geq 0} \frac{\text{Fit}(g_{i, \varphi, \psi}, E(g_{i, \gamma_1, \gamma_2} \mid \gamma_1 \in W_{\varphi}, \gamma_2 \in W_{\psi}))}{2^{i+1}}.$$

We call $\text{Corr}(\varphi, \psi, \mathcal{H})$ the **hyper-network correlation index** of φ and ψ with respect to \mathcal{H} . It is a number between 0 and 1. As the number $\text{Corr}(\varphi, \psi, \mathcal{H})$ increases, the correlation between φ and ψ with respect to \mathcal{H} becomes more significant.

EXAMPLE 6.3. Let \mathcal{H} be the hypergraph given in Example 6.2. Let φ be a function with value in $[0, 1]$ measuring the education level of a person. Let ψ be a function with value in $[0, 1]$ measuring the annual income of a person. Then our hyper-network correlation index $\text{Corr}(\varphi, \psi, \mathcal{H})$ can measure the correlation between the education level and the annual income with the consideration of social relations. If $\text{Corr}(\varphi, \psi, \mathcal{H})$ becomes smaller/larger than before, then we can conclude that education level becomes less/more related to annual income.

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