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004

STABILITY OF CATENOIDS AND HELICOIDS IN HYPERBOLIC SPACE*

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Abstract. In this paper, we study the stability of catenoids and helicoids in 3-dimensional hyperbolic space. We will prove the following results (Theorems 1.4 and 1.5).

1) For a family of spherical minimal catenoids $\{\mathcal{C}_a\}_{a>0}$ in hyperbolic 3-space (see §3 for detailed definitions), there exist a constant $a_l > 0$ such that $\mathcal{C}_a$ is a least area minimal surface (see §2.1 for the definition) if $a \geqslant a_l$.
2) For a family of minimal helicoids $\{\mathcal{H}_{\bar{a}}\}_{\bar{a}\geqslant 0}$ in hyperbolic 3-space (see §2.4 for detailed definitions), there exists a constant $\bar{a}_c > 0$ such that
   - $\mathcal{H}_{\bar{a}}$ is a globally stable minimal surface if $0 \leqslant \bar{a} \leqslant \bar{a}_c$, and
   - $\mathcal{H}_{\bar{a}}$ is an unstable minimal surface with Morse index infinity if $\bar{a} > \bar{a}_c$.

Key words. Hyperbolic spaces, minimal surfaces, catenoids, helicoids, stability.

Mathematics Subject Classification. 53A10.

1. Introduction. The study of the catenoid and the helicoid in 3-dimensional Euclidean space $\mathbb{R}^3$ can be traced back to Leonhard Euler and Jean Baptiste Meusnier in the 18th century. Since then mathematicians have found many properties of the catenoid and the helicoid in $\mathbb{R}^3$. The first property is that the catenoid and the helicoid in $\mathbb{R}^3$ are both unstable. Actually do Carmo and Peng [dCP79] proved that the plane is the *unique* stable complete minimal surface in $\mathbb{R}^3$. Let's list some other properties here (all minimal surfaces are in $\mathbb{R}^3$): the plane and the catenoid are the only minimal surfaces of revolution in $\mathbb{R}^3$ (Bonnet in 1860, see [MP12, §2.5]); the catenoid is the unique embedded complete minimal surface in $\mathbb{R}^3$ with finite topology and with two ends [Sch83]; the catenoid and the Enneper's surface are the only orientable complete minimal surfaces in $\mathbb{R}^3$ with Morse index equal to one [LR89]; the plane and the catenoid are the only embedded complete minimal surfaces of finite total curvature and genus zero in $\mathbb{R}^3$ [LR91]; the plane and the helicoid are the only ruled minimal surfaces in $\mathbb{R}^3$ (Catalan in 1842, see [MP12, §2.5] or [FT91, pp.34–35]); the helicoid in $\mathbb{R}^3$ has genus zero, one end, infinite total curvature [MP12, §2.5] and infinite Morse index [Tuz93, p.199]; the plane and the helicoid are the unique simply-connected, complete, embedded minimal surfaces in $\mathbb{R}^3$ [MR05]. The reader can see the survey [MP11] and the book [MP12] for more properties of the catenoid and the helicoid, and the references cited therein.

In this paper we will study the stability of the *spherical* catenoids and the helicoids in hyperbolic 3-space. There are three models of the hyperbolic n-space, but we may use the notation $\mathbb{H}^n$ to denote the hyperbolic n-space without emphasizing the model. We list some properties of the catenoids and helicoids in $\mathbb{H}^3$: each catenoid (hyperbolic, parabolic or spherical) is a complete embedded minimal surface in $\mathbb{H}^3$ (see [dCD83, Theorem (3.26)]); each spherical catenoid in $\mathbb{H}^3$ has finite total curvature (see the computation in [dCD83, p.708]); each helicoid in $\mathbb{H}^3$ is a complete embedded ruled minimal surface (see [Mor82, Theorem 1] and [Tuz93, pp.221–222]); each helicoid in $\mathbb{H}^3$ has infinite total curvature (see the computation in [Mor82, p.60]). The stability of the spherical catenoids and helicoids in $\mathbb{H}^3$ is characterized by Theorem 1.2 and

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Theorem 1.5 respectively. The reader can see the paper [Tuz93] for more properties of the catenoids and the helicoids in $\mathbb{H}^3$.

It was Mori [Mor81] who studied the spherical catenoids in the hyperboloid model of the hyperbolic 3-space at first. Then Do Carmo and Dajczer [dCD83] studied three types of rotationally symmetric minimal hypersurfaces in the hyperboloid model of the hyperbolic $(n+1)$-space. A rotationally symmetric minimal hypersurface in $\mathbb{H}^{n+1}$ is called a *spherical* catenoid if it is foliated by spheres, a *hyperbolic* catenoid if it is foliated by totally geodesic hyperplanes, or a *parabolic* catenoid if it is foliated by horospheres. Do Carmo and Dajczer proved that the hyperbolic and parabolic catenoids in $\mathbb{H}^3$ are globally stable (see [dCD83, Theorem 5.5]), then Candel proved that the hyperbolic and parabolic catenoids in $\mathbb{H}^3$ are least area minimal surfaces (see [Can07, p.3574]), so all of them are globally stable.

Compared with the hyperbolic and parabolic catenoids, the spherical catenoids in $\mathbb{H}^3$ are more complicated. The behavior of the spherical catenoids in $\mathbb{H}^3$ are influenced by their behaviour at infinity, Levitt and Rosenberg [LR85, Theorem 3.2] proved that a connected minimal surface $\mathcal{C} \subset \mathbb{H}^3$ whose asymptotic boundary consists of two disjoint round circles $C_1, C_2 \subset S^2_\infty$ is a spherical catenoid if it is regular at infinity (see [LR85, Theorem 3.2] and [dCGT86, Theorem 3]). On the other hand, for any catenoid $\mathcal{C}$ asymptotic to $C_1 \cup C_2$ in $\mathbb{H}^3$, Gomes [Gom87] (see also Theorem 3.2) proved that the *distance* between C_1 and C_2 has a uniform upper bound ≈ 1.00229.

On the stability of catenoids, let $\mathcal{C}_a$ be a spherical catenoid, where $a > 0$ is the hyperbolic distance between $\mathcal{C}_a$ and its rotation axis, Mori, Do Carmo and Dajczer, Bérard and Sa Earp, and Seo proved the following result (see [Mor81, dCD83, BSE10, Seo11]): There exist two constants $A_1 \approx 0.46288$ and $A_2 \approx 0.5915$ such that $\mathcal{C}_a$ is unstable if $0 < a < A_1$, and $\mathcal{C}_a$ is globally stable if $a > A_2$.

Remark 1.1. The constants A_1 and A_2 were given by Seo in [Seo11, Corollary 4.2] and by Bérard and Sa Earp in [BSE10, Lemma 4.4] respectively. A few years ago, Do Carmo and Dajczer showed that $\mathcal{C}_a$ is unstable if $a \lessapprox 0.42315$ in [dCD83], and Mori showed that $\mathcal{C}_a$ is stable if $a > \cosh^{-1}(3) \approx 1.76275$ in [Mor81].

According to the numerical computation, Bérard and Sa Earp claimed that A_1 should be the same as A_2 (see [BSE10, Proposition 4.10] or [BGS87, Theorem 3.14]). More precisely, we have the following theorem.

Theorem 1.2 (Bérard and Sa Earp). *There exists a constant $a_c \approx 0.49577$ such that the following statements are true:*

1) *$\mathcal{C}_a$ is an unstable minimal surface with Morse index one if $0 < a < a_c$;*
2) *$\mathcal{C}_a$ is a globally stable minimal surface if $a \geqslant a_c$.*

Remark 1.3. As we will see, the constant a_c is the unique critical number of the function $\varrho(a)$ given by (3.13).

Sketch of proof of Theorem 1.2. For any $a > 0$, let σ_a be the catenary given by (3.9) and let $\mathcal{C}_a$ be the minimal surface of revolution whose generating curve is the catenary σ_a. Let $E(a)$ be the function defined by [BSE10, Equation (4.13)]. Bérard and Sa Earp [BSE10, Theorem 4.7 (1)] proved the following result: The catenoid $\mathcal{C}_a$ is globally stable if $E(a) \leqslant 0$, and unstable with index 1 if $E(a) > 0$.

Moreover Bérard and Sa Earp [BSE10, p.3665] found the identity $E(a) = \varrho'(a)/\sqrt{2}$, where $\varrho'(a)$ is defined by (3.15). Now according to Lemma 3.3, the function ϱ' has a unique zero $a_c \in (0, \infty)$ such that $\varrho'(a) < 0$ if $a > a_c$ and $\varrho'(a) > 0$ if $0 < a < a_c$, hence Theorem 1.2 follows. □

Similar to the case of hyperbolic and parabolic catenoids, we want to know whether the globally stable spherical catenoids are least area minimal surfaces. In this paper, we prove that there exists a positive number a_l given by (4.3) such that $\mathcal{C}_a$ is a least area minimal surface if $a \geqslant a_l$. More precisely, we shall prove the following result.

THEOREM 1.4. *There exists a constant* $a_l \approx 1.10055$ *defined by* (4.3) *such that for any* $a \geqslant a_l$ *the catenoid* $\mathcal{C}_a$ *is a least area minimal surface.*

Similar to Theorem 3.2, we have Corollary 4.4 of Theorem 1.4: if the distance between two circles on S^2_∞ is $\leqslant 0.729183$, then there exists a least area minimal catenoid asymptotic to these two circles.

Furthermore, Mori [Mor82] and Do Carmo and Dajczer [dCD83] also studied the helicoid in the hyperboloid model of the hyperbolic 3-space (see §2.4). Mori [Mor82] studied the stability of the helicoids $\mathcal{H}_{\bar{a}}$ for $\bar{a} \geqslant 0$ in hyperbolic 3-space $\mathbb{H}^3$. He showed that $\mathcal{H}_{\bar{a}}$ is globally stable if $\bar{a} \leqslant 3\sqrt{2}/4 \approx 1.06$, and it is unstable if $\bar{a} \geqslant \sqrt{105\pi}/8 \approx 2.27$. In this paper we shall prove the following result.

THEOREM 1.5. *For a family of minimal helicoids* $\{\mathcal{H}_{\bar{a}}\}_{\bar{a}\geqslant 0}$ *in hyperbolic* 3*-space* $\mathbf{H}^3$ *that is defined by* (2.7), *there exist a constant* $\bar{a}_c = \coth(a_c) \approx 2.17968$ *such that the following statements are true:*

1) $\mathcal{H}_{\bar{a}}$ *is a globally stable minimal surface if* $0 \leqslant \bar{a} \leqslant \bar{a}_c$, *and*
2) $\mathcal{H}_{\bar{a}}$ *is an unstable minimal surface with index infinity if* $\bar{a} > \bar{a}_c$.

Plan of the paper. This paper is organized as follows. In §2 we introduce three types of catenoids in the hyperbloid model $\mathbf{H}^3$ of the hyperbolic 3-space, and the helicoids in three models $\mathbf{H}^3$, $\mathbb{B}^3$ and $\mathbb{U}^3$ of the hyperbolic 3-space. In §3 we define the spherical catenoids in $\mathbb{B}^3$, then we prove a theorem of Gomes (Theorem 3.2). In §4 we prove Theorem 1.4. In §5 we prove Theorem1.5. In §6, we prove technical lemmas which are used to prove Theorems 1.4 and 1.5.

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2. Preliminaries.

2.1. Basic theory of minimal surfaces. Let Σ be a surface immersed in a three dimensional Riemannian manifold M^3. We pick up a local orthonormal frame field $\{e_1, e_2, e_3\}$ for M^3 such that, restricted to Σ, the vectors $\{e_1, e_2\}$ are tangent to Σ and the vector e_3 is perpendicular to Σ. Let $A = (h_{ij})_{2\times 2}$ be the second fundamental form of Σ, whose entries h_{ij} are represented by $h_{ij} = \langle \nabla_{e_i} e_3\, , e_j\rangle$ for $i, j = 1, 2$, where ∇ is the covariant derivative in M^3, and $\langle\cdot\, ,\cdot\rangle$ is the metric of M^3. An immersed surface $\Sigma \subset M^3$ is called a *minimal surface* if its *mean curvature* $H = h_{11} + h_{22} \equiv 0$.

For any immersed minimal surface Σ in M^3, the *Jacobi operator* on Σ is defined by

$$\mathcal{L} = \Delta_\Sigma + (|A|^2 + \mathrm{Ric}(e_3))\ , \tag{2.1}$$

where Δ_Σ is the Lapalican defined on Σ, $|A|^2 = \sum_{i,j=1}^2 h_{ij}^2$ is the square of the length of the second fundamental form on Σ and $\mathrm{Ric}(e_3)$ is the Ricci curvature of M^3 in the direction e_3.

Suppose that Σ is a complete minimal surface immersed in a complete Riemannian 3-manifold M^3, for any compact connected subdomain Ω of Σ, its first eigenvalue is defined by

$$\lambda_1(\Omega) = \inf \left\{ -\int_\Omega f\mathcal{L}f \;\middle|\; f \in C_0^\infty(\Omega) \text{ and } \int_\Omega f^2 = 1 \right\} . \tag{2.2}$$

We say that Ω is *stable* if $\lambda_1(\Omega) > 0$, *unstable* if $\lambda_1(\Omega) < 0$ and *maximally weakly stable* if $\lambda_1(\Omega) = 0$. Suppose that Ω_1 and Ω_2 are connected subdomains of Σ with $\Omega_1 \subset \Omega_2$, then $\lambda_1(\Omega_1) \geqslant \lambda_1(\Omega_2)$. If $\Omega_2 \setminus \overline{\Omega}_1 \neq \emptyset$, then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$. If $\Omega \subset \Sigma$ is maximally weakly stable, then for any compact connected subdomains $\Omega_1, \Omega_2 \subset \Sigma$ satisfying $\Omega_1 \subsetneq \Omega \subsetneq \Omega_2$, obviously Ω_1 is stable whereas Ω_2 is unstable. Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ be an exhaustion of Σ, then the first eigenvalue of Σ is defined by

$$\lambda_1(\Sigma) = \lim_{n\to\infty} \lambda_1(\Omega_n) . \tag{2.3}$$

This definition is independent of the choice of the exhaustion. We say that Σ is *globally stable* or *stable* if $\lambda_1(\Sigma) > 0$ and *unstable* if $\lambda_1(\Sigma) < 0$.

The following theorem was proved by Fischer-Colbrie and Schoen in [FCS80, Theorem 1] (see also [CM11, Proposition 1.39]).

THEOREM 2.1 (Fischer-Colbrie and Schoen). *Let Σ be a complete two-sided minimal surface in a Riemannian 3-manifold M^3, then Σ is stable if and only if there exists a positive function $\phi : \Sigma \to \mathbb{R}$ such that $\mathcal{L}\phi = 0$.*

The *Morse index* of a compact connected subdomain Ω of Σ is the number of negative eigenvalues of the Jacobi operator $\mathcal{L}$ (counting with multiplicity) acting on the space of smooth sections of the normal bundle that vanish on $\partial\Omega$. The *Morse index* of Σ is the supremum of the Morse indices of compact subdomains of Σ. The following proposition of Fischer-Colbrie [FC85, Proposition 1] can be applied to show that some unstable minimal surface has infinite Morse index.

THEOREM 2.2 (Fischer-Colbrie). *Let Σ be a complete two-sided minimal surface in a Riemannian 3-manifold M^3. If Σ has finite Morse index then there is a compact set K in Σ so that $\Sigma \setminus K$ is stable and there exists a positive function ϕ on Σ so that $\mathcal{L}\phi = 0$ on $\Sigma \setminus K$.*

In this paper we consider the minimal surfaces in hyperbolic 3-space. From now on, suppose that M^3 is the hyperbolic 3-space $\mathbb{H}^3$ (see §2.2 for the definition).

Let D be a compact disk-type minimal surface immersed in $\mathbb{H}^3$, then D is called a *least area minimal surface* in $\mathbb{H}^3$ if $\operatorname{Area}(D) \leqslant \operatorname{Area}(D')$ for any compact disk $D' \subset \mathbb{H}^3$ with $\partial D' = \partial D$, where $\operatorname{Area}(\cdot)$ denotes the area of the surfaces in $\mathbb{H}^3$. A complete disk $\Sigma \subset \mathbb{H}^3$ asymptotic to a Jordan curve in S^2_∞ is called a *least area minimal surface* in $\mathbb{H}^3$ if any compact disk-type subdomain of Σ is a least area minimal surface in $\mathbb{H}^3$.

Let S be a compact annulus-type minimal surface immersed in $\mathbb{H}^3$, whose boundary is the union of two disjoint simple closed curves C_1, C_2, then S is called a *least area minimal surface* in $\mathbb{H}^3$ (see [MY82, p.412]) if

- $\operatorname{Area}(S) \leqslant \operatorname{Area}(S')$ for each annulus S' in $\mathbb{H}^3$ with $\partial S' = \partial S$, and
- $\operatorname{Area}(S) < \operatorname{Area}(D_1) + \operatorname{Area}(D_2)$, where $D_1, D_2 \subset \mathbb{H}^3$ are the least area minimal disks spanned by C_1, C_2 respectively.

A complete immersed annulus-type minimal surface $\Sigma \subset \mathbb{H}^3$ whose asymptotic boundary consists of two disjoint Jordan curves in S^2_∞ is called a *least area minimal surface* in $\mathbb{H}^3$ if any compact annulus-type subdomain of Σ, which is homotopically equivalent to Σ, is a least area minimal surface.

2.2. Models of the hyperbolic 3-space. We consider three models of the hyperbolic 3-space $\mathbb{H}^3$ (see [BP92, §A.1]).

Let $\mathbb{L}^4$ denote the Lorentzian 4-space, i.e. a vector space $\mathbb{R}^4$ with the Lorentzian inner product $\langle x\,,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, where $x, y \in \mathbb{R}^4$. The orientation preserving isometry group of $\mathbb{L}^4$ is $\mathsf{SO}^+(1,3)$. The *hyperboloid model* of $\mathbb{H}^3$ is the unit sphere of $\mathbb{L}^4$, i.e., $\mathbf{H}^3 = \{x \in \mathbb{L}^4 \mid \langle x\,,x\rangle = -1,\, x_1 \geqslant 1\}$ with the induced metric.

The *Poincaré ball model* of $\mathbb{H}^3$ is the open unit ball $\mathbb{B}^3 = \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 < 1\}$ equipped with the hyperbolic metric $ds^2 = 4(du^2 + dv^2 + dw^2)/(1-r^2)^2$, where $r = \sqrt{u^2+v^2+w^2}$. The orientation preserving isometry group of $\mathbb{B}^3$ is denoted by $\mathsf{M\ddot{o}b}(\mathbb{B}^3)$, which consists of Möbius transformations that preserve the unit ball $\mathbb{B}^3$ (see [MT98, Theorem 1.7]). The hyperbolic space $\mathbb{B}^3$ has a natural compactification: $\overline{\mathbb{B}^3} = \mathbb{B}^3 \cup S^2_\infty$, where $S^2_\infty \cong \mathbb{C} \cup \{\infty\}$ is called the *Riemann sphere*.

The *upper half space model* of $\mathbb{H}^3$ is a three dimensional space $\mathbb{U}^3 = \{z + tj \mid z \in \mathbb{C} \text{ and } t > 0\}$ equipped with the (hyperbolic) metric $ds^2 = (|dz|^2 + dt^2)/t^2$, where $z = x + iy$ for $x, y \in \mathbb{R}$ and j is the unit vector along the (vertical) t-axis. The orientation preserving isometry group of $\mathbb{U}^3$ is denoted by $\mathsf{PSL}_2(\mathbb{C})$, which consists of linear fractional transformations.

2.3. Catenoids in the hyperboloid model $\mathbf{H}^3$. There are three types of catenoids in $\mathbf{H}^3$ (see [dCD83, pp.698–699]): spherical catenoids $\mathscr{C}_1(\tilde{a})$, hyperbolic catenoids $\mathscr{C}_{-1}(\tilde{a})$, and parabolic catenoids $\mathscr{C}_0$.

For any subspace P of the Lorentzian 4-space $\mathbb{L}^4$, let $\mathsf{O}(P)$ be the subgroup of $\mathsf{SO}^+(1,3)$ that leaves P pointwise fixed. Let $\{e_1, \ldots, e_4\}$ be an orthonormal basis of $\mathbb{L}^4$ (it might not be the standard orthonormal basis). Suppose that $P^2 = \mathrm{span}\{e_3, e_4\}$, $P^3 = \mathrm{span}\{e_1, e_3, e_4\}$ and $P^3 \cap \mathbf{H}^3 \neq \emptyset$. Let C be a regular curve in $P^3 \cap \mathbf{H}^3 = \mathbf{H}^2$ that does not meet P^2. The orbit of C under the action of $\mathsf{O}(P^2)$ is called a *rotation surface generated by C around P^2*. If a rotation surface has mean curvature zero, then it's called a *catenoid*.

Here we only give the definition of spherical catenoid in the hyperboloid model $\mathbf{H}^3$. Let $\{e_1, \ldots, e_4\}$ be an orthonormal basis of $\mathbb{L}^4$ such that $\langle e_4\,, e_4\rangle = -1$. Suppose that P^2, P^3 and C are the same as those defined as above. For any point $x \in \mathbb{L}^4$, write $x = \sum x_k e_k$. If the curve C is parameterized by

$$x_1(s) = \sqrt{\tilde{a}\cosh(2s) - 1/2}\,, \quad \tilde{a} > 1/2 \tag{2.4}$$

and

$$x_3(s) = \sqrt{x_1^2(s) + 1}\,\sinh(\phi(s))\,,\ x_4(s) = \sqrt{x_1^2(s)+1}\,\cosh(\phi(s))\,, \tag{2.5}$$

where

$$\phi(s) = \int_0^s \frac{\sqrt{\tilde{a}^2 - 1/4}}{(\tilde{a}\cosh(2\sigma) + 1/2)\sqrt{\tilde{a}\cosh(2\sigma) - 1/2}}\,d\sigma\,, \tag{2.6}$$

then the rotation surface is called a *spherical catenoid*, and it's denoted by $\mathscr{C}_1(\tilde{a})$.

2.4. Helicoids in hyperbolic 3-space. In this subsection, we introduce the equations of helicoids in $\mathbf{H}^3$, $\mathbb{B}^3$ and $\mathbb{U}^3$.

The helicoid $\mathcal{H}_{\bar{a}}$ in the hyperboloid model $\mathbf{H}^3$ is the surface parameterized by the (u,v)-plane as follows (see [dCD83, p.699]):

$$\mathcal{H}_{\bar{a}} = \left\{ x \in \mathbf{H}^3 \;\middle|\; \begin{matrix} x_1 = \cosh u \cosh v, & x_2 = \cosh u \sinh v \\ x_3 = \sinh u \cos(\bar{a}v), & x_4 = \sinh u \sin(\bar{a}v) \end{matrix} \right\}, \tag{2.7}$$

where $-\infty < u, v < \infty$.

For any constant $\bar{a} \geqslant 0$, the helicoid $\mathcal{H}_{\bar{a}} \subset \mathbf{H}^3$ is an embedded minimal surface (see [Mor82]). The axis of the helicoid $\mathcal{H}_{\bar{a}}$ in $\mathbf{H}^3$ is given by $(\cosh v, \sinh v, 0, 0)$ for $-\infty < v < \infty$, which is the intersection of the x_1x_2-plane of $\mathbb{L}^4$ and $\mathbf{H}^3$.

The helicoid $\mathcal{H}_{\bar{a}}$ in $\mathbb{B}^3$ is parameterized as follows:

$$\mathcal{H}_{\bar{a}} = \left\{ (x,y,z) \in \mathbb{B}^3 \;\middle|\; \begin{matrix} x = \dfrac{\sinh u \cos(\bar{a}v)}{1+\cosh u \cosh v}, \\ y = \dfrac{\sinh u \sin(\bar{a}v)}{1+\cosh u \cosh v}, \\ z = \dfrac{\cosh u \sinh v}{1+\cosh u \cosh v} \end{matrix} \right\}, \tag{2.8}$$

where $-\infty < u, v < \infty$ (see Figure 1).

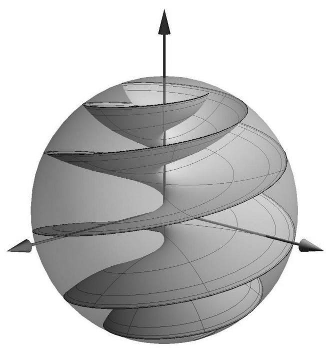

FIG. 1. *The helicoid $\mathcal{H}_{\bar{a}}$ with $\bar{a} = 5$ in $\mathbb{B}^3$.*

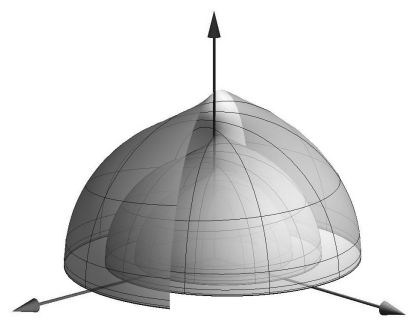

FIG. 2. *The helicoid $\mathcal{H}_{\bar{a}}$ with $\bar{a} = 5$ in $\mathbb{U}^3$.*

The helicoid $\mathcal{H}_{\bar{a}}$ in $\mathbb{U}^3$ is parameterized as follows:

$$\mathcal{H}_{\bar{a}} = \left\{ (z,t) \in \mathbb{U}^3 \mid z = e^{v+\sqrt{-1}\,\bar{a}v} \tanh u \text{ and } t = e^v \operatorname{sech} u \right\}, \tag{2.9}$$

where $-\infty < u, v < \infty$ (see Figure 2), and the axis of $\mathcal{H}_{\bar{a}}$ is the t-axis.

From the equation (2.9), we can see that each helicoid $\mathcal{H}_{\bar{a}}$ is invariant under the one-parameter group $G_{\bar{a}} \subset \mathsf{PSL}_2(\mathbb{C})$ consisting of loxodromic transformations which fix the same t-axis , i.e., $G_{\bar{a}} = \left\{ z \mapsto \exp\left(v + \sqrt{-1}\,\bar{a}v\right) z \mid -\infty < v < \infty \right\}$. When $v = 0$, we get a semi unit circle whose center is the origin in the xt-plane from (2.9). So a helicoid $\mathcal{H}_{\bar{a}}$ in $\mathbb{U}^3$ could be obtained by rotating the (upper) semi unit circle with center the origin along the t-axis about angle $\bar{a}v$ and translating it along the t-axis about *hyperbolic distance* v for all $v \in \mathbb{R}$.

3. The proof of a theorem of Gomes. In this section, we will prove a theorem of Gomes, which plays an important role in the paper [HW15] for constructing barrier surfaces. Moreover we will describe the shape of the curve (see Figure 6) defined by (3.13′).

At first we follow Hsiang [Hsi82, BdCH09] to introduce the minimal spherical catenoids in the Poincaré model $\mathbb{B}^3$ of the hyperbolic 3-space. Let X be a subset of $\mathbb{B}^3$, we define the *asymptotic boundary* of X by

$$\partial_\infty X = \overline{X} \cap S^2_\infty \ , \tag{3.1}$$

where $\overline{X}$ is the closure of X in $\overline{\mathbb{B}^3}$. Using the above notation, we have $\partial_\infty \mathbb{B}^3 = S^2_\infty$. If P is a geodesic plane in $\mathbb{B}^3$, then P is perpendicular to S^2_∞ and $C \overset{\text{def}}{=} \partial_\infty P$ is an Euclidean circle on S^2_∞. We also say that P is *asymptotic to* C.

Suppose that $G \cong \mathsf{SO}(2)$ is a subgroup of $\mathsf{M\ddot{o}b}(\mathbb{B}^3)$ that leaves a geodesic $\gamma \subset \mathbb{B}^3$ pointwise fixed. We call G the *spherical group of* $\mathbb{B}^3$ *associated with the geodesic* γ and γ the *rotation axis* of G. A surface in $\mathbb{B}^3$ which is invariant under the spherical group G is called a *spherical surface* or a *surface of revolution.* For two circles C_1 and C_2 in $\mathbb{B}^3$, if there is a geodesic γ such that each of C_1 and C_2 is invariant under the group of rotations that fixes γ pointwise, then C_1 and C_2 are said to be *coaxial*, and γ is called the *rotation axis* of C_1 and C_2.

Suppose that G is the spherical group of $\mathbb{B}^3$ associated with the geodesic

$$\gamma_0 = \{(u, 0, 0) \in \mathbb{B}^3 \mid -1 < u < 1\} \ , \tag{3.2}$$

then $\mathbb{B}^3/G \cong \mathbb{B}^2_+$, where

$$\mathbb{B}^2_+ := \{(u, v) \in \mathbb{B}^2 \mid v \geqslant 0\} \ . \tag{3.3}$$

We shall equip the half space $\mathbb{B}^2_+$ with a warped product metric.

For any point $p = (u, v) \in \mathbb{B}^2_+$, there is a unique geodesic segment γ' passing through p that is perpendicular to γ_0 at q. Let $x = \operatorname{dist}(O, q)$ and $y = \operatorname{dist}(p, q) = \operatorname{dist}(p, \gamma_0)$ (see Figure 3), where $\operatorname{dist}(\cdot, \cdot)$ denotes the hyperbolic distance, then by [Bea95, Theorem 7.11.2] (see also [BSE10]), we have

$$\tanh x = \frac{2u}{1 + (u^2 + v^2)} \quad \text{and} \quad \sinh y = \frac{2v}{1 - (u^2 + v^2)} \ . \tag{3.4}$$

Equivalently, we also have

$$u = \frac{\sinh x \cosh y}{1 + \cosh x \cosh y} \quad \text{and} \quad v = \frac{\sinh y}{1 + \cosh x \cosh y} \ . \tag{3.5}$$

It's well known that $\mathbb{B}^2_+$ can be equipped with the *metric of warped product* in terms of the parameters x and y as follows:

$$ds^2 = \cosh^2 y \cdot dx^2 + dy^2 \ , \tag{3.6}$$

where dx represents the hyperbolic metric on the geodesic γ_0 in (3.2).

If $\mathcal{C}$ is a minimal surface of revolution in $\mathbb{B}^3$ with respect to the axis γ_0 defined by the equation (3.2), then it is called a *catenoid* and the curve $\sigma = \mathcal{C} \cap \mathbb{B}^2_+$ is called the *generating curve* or a *catenary* of $\mathcal{C}$.

Let $\sigma \subset \mathbb{B}^2_+$ be the generating curve of a minimal catnoid $\mathcal{C}$. Suppose that the parametric equations of σ are given by: $x = x(s)$ and $y = y(s)$, where $s \in (-\infty, \infty)$

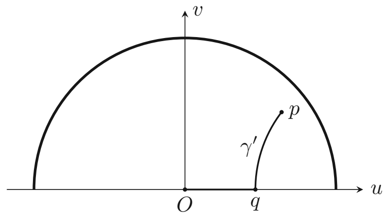

FIG. 3. $x = \mathrm{dist}(O, q)$ *and* $y = \mathrm{dist}(p, q)$.

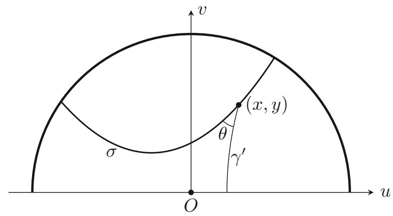

FIG. 4. θ *is the angle between* σ *and* γ'.

is an arc length parameter of σ. By the argument in [Hsi82, pp. 486–488], the curve σ satisfies the following equations

$$\frac{2\pi \sinh y \cdot \cosh^2 y}{\sqrt{\cosh^2 y + (y')^2}} = 2\pi \sinh y \cdot \cosh y \cdot \sin\theta = k \text{ (constant)} , \tag{3.7}$$

where $y' = dy/dx$ and θ is the angle between the tangent vector of σ and the vector $e_y = \partial/\partial y$ at the point $(x(s), y(s))$ (see Figure 4).

By the argument in [Gom87, pp.54–58]), up to isometry, we assume that the curve σ is only symmetric about the v-axis and intersects the v-axis orthogonally at $y_0 = y(0)$, and so $y'(0) = 0$. Substitute these into (3.7), we get $k = 2\pi \sinh(y_0)\cosh(y_0)$, and then we have the following equation

$$\sin\theta = \frac{\sinh(y_0)\cosh(y_0)}{\sinh(y)\cosh(y)} = \frac{\sinh(2y_0)}{\sinh(2y)} . \tag{3.8}$$

Now we solve for dx/dy in terms of y in (3.7) and integrate dx/dy from y_0 to y for any $y \geqslant y_0$, then we have the following equality

$$x(y) = \int_{y_0}^{y} \frac{\sinh(2y_0)}{\cosh\tau} \frac{d\tau}{\sqrt{\sinh^2(2\tau) - \sinh^2(2y_0)}} . \tag{3.9}$$

Let $y \to \infty$ in (3.9), we get

$$x(\infty) = \int_{y_0}^{\infty} \frac{\sinh(2y_0)}{\cosh\tau} \frac{d\tau}{\sqrt{\sinh^2(2\tau) - \sinh^2(2y_0)}} . \tag{3.10}$$

We can see that the hyperbolic distance between two totally geodesic planes bounded by the boundary of the catenoid $\mathcal{C}$, whose generating curve is σ, is equal to the double of $x(\infty)$. We will see that this distance can not be too large (see Theorem 3.2).

Replacing the initial data y_0 by a parameter $a \in (0, \infty)$ in (3.9), we set

$$\rho(a, t) = \int_{a}^{t} \frac{\sinh(2a)}{\cosh\tau} \frac{d\tau}{\sqrt{\sinh^2(2\tau) - \sinh^2(2a)}} , \quad t \geqslant a . \tag{3.11}$$

Let $\sigma_a \subset \mathbb{B}^2_+$ be the catenary determined by (3.11) or (3.9) for $y_0 = a$, which is symmetric about the v-axis and whose initial data is a (actually the hyperbolic distance between σ_a and the origin of $\mathbb{B}^2_+$ is equal to a).

DEFINITION 3.1. For $0 < a < \infty$, the surface of revolution around the axis γ_0 in (3.2) generated by the catenary σ_a is called a *catenoid*, and is denoted by $\mathcal{C}_a$.

Obviously the asymptotic boundary of any spherical catenoid $\mathcal{C}_a$ is the union of two circles (see also [Gom87, Proposition 3.1]). It's important for us to determine whether there exists a minimal spherical catenoid asymptotic to any given pair of disjoint round circles on S^2_∞, since in [HW15] we construct quasi-Fuchsian 3-manifolds that contain arbitrarily many incompressible minimal surface by using the minimal spherical catenoids as the barrier surfaces.

If C_1 and C_2 are two disjoint circles on S^2_∞, then they are always coaxial. In fact, let P_1 and P_2 be the geodesic planes asymptotic to C_1 and C_2 respectively, there always exists a unique geodesic γ such that γ is perpendicular to both P_1 and P_2. So C_1 and C_2 are coaxial with respect to γ. We define the distance between C_1 and C_2 by

$$d(C_1, C_2) = \mathrm{dist}(P_1, P_2)\ . \tag{3.12}$$

In order to prove Theorem 3.2, we define a function $\varrho(a)$ of the parameter a:

$$\varrho(a) = \rho(a, \infty) = \int_a^\infty \frac{\sinh(2a)}{\cosh t} \frac{dt}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}}\ . \tag{3.13}$$

Using the substitution $t \mapsto t + a$, the above function (3.13) can be written as

$$\varrho(a) = \int_0^\infty \frac{\sinh(2a)}{\cosh(a+t)} \frac{dt}{\sqrt{\sinh^2(2a+2t) - \sinh^2(2a)}}\ . \tag{3.13$'$}$$

The following theorem is a quantitative version of Proposition 3.2 in [Gom87].

THEOREM 3.2 (Gomes). *There exists a constant $a_c \approx 0.49577$ such that for two disjoint circles $C_1, C_2 \subset S^2_\infty$, if*

$$d(C_1, C_2) \leqslant 2\varrho(a_c) \approx 1.00229\ , \tag{3.14}$$

then there exist a spherical minimal catenoid $\mathcal{C}$ which is asymptotic to $C_1 \cup C_2$, where $\varrho(a)$ is the function defined by (3.13).

LEMMA 3.3. *The derivative function $\varrho'(a)$ satisfies the following conditions:*
- *$\varrho'(a) \to \infty$ as $a \to 0^+$ and $\varrho'(a) < 0$ on $[A_3, \infty)$, and*
- *$\varrho'(a)$ is decreasing on $(0, A_4)$,*

where $A_3 \approx 0.530638$ and $A_4 \approx 0.715548$ are constants defined in (6.1) *and* (6.3).

Proof. By direct computation, we obtain the derivative of ϱ as follows:

$$\varrho'(a) = \int_0^\infty \frac{\sinh(a+t)(5\cosh^2(a+t) - \cosh^2(3a+t))}{\cosh^2(a+t)\sqrt{\sinh(2t)\sinh^3(4a+2t)}}\, dt\ . \tag{3.15}$$

It's easy to verify that $\varrho'(a)$ is well defined for $a > 0$, and then

$$\lim_{a \to 0^+} \varrho'(a) = \int_0^\infty \frac{1}{\sinh t \cosh^2 t}\, dt = \left[\log\left(\frac{\cosh t - 1}{\cosh t + 1}\right) + \frac{1}{\cosh t}\right]_{t=0}^{t=\infty} = \infty\ .$$

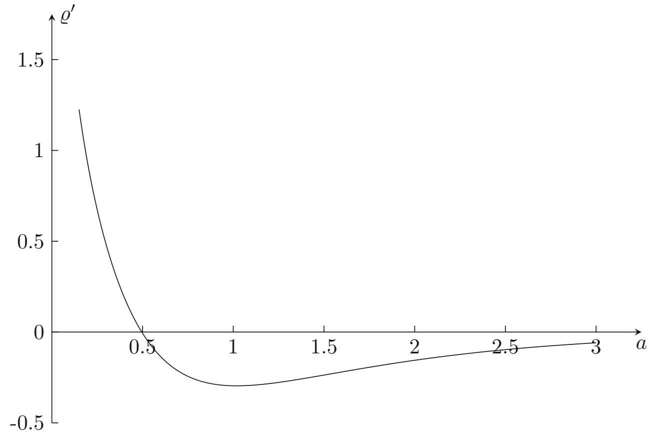

FIG. 5. *The derivative of the function $\varrho(a)$ for $a \in (0,3]$.*

By Lemma 6.1, $\varrho'(a) < 0$ on (A_3, ∞).

Next, we calculate the second derivative of $\varrho(a)$ as follows:

$$\varrho''(a) = \int_0^\infty \frac{\psi(a,t)}{16\cosh^3(a+t)\sqrt{\sinh(2t)\sinh^5(4a+2t)}}\,dt\ , \tag{3.16}$$

where $\psi(a,t)$ is the function defined by (6.2). According to Lemma 6.2, $\varrho''(a) < 0$ for $a \in (0, A_4)$, thus $\varrho'(a)$ is decreasing on $(0, A_4)$. See Figure 5. ◻

Proof of Theorem 3.2. Let $\varrho(a)$ be the function defined by (3.13) or (3.13′). We claim that $\varrho(0) = 0$, and as a increases, $\varrho(a)$ increases monotonically, reaches a maximum, and then decreases asymptotically to zero as a goes to infinity (see Figure 6).

It's easy to show $\varrho(a) \to 0$ as $a \to \infty$. In fact, we have

$$\begin{aligned}
\varrho(a) &= \int_0^\infty \frac{\sinh(2a)}{\cosh(a+t)} \frac{dt}{\sqrt{\sinh^2(2a+2t) - \sinh^2(2a)}} \\
&= \int_0^\infty \frac{1}{\cosh(t+a)} \frac{dt}{\sqrt{\left(\dfrac{\sinh(2a+2t)}{\sinh(2a)}\right)^2 - 1}} \\
&< \int_0^\infty \frac{1}{\cosh a} \frac{dt}{\sqrt{(\sinh(2t)+\cosh(2t))^2 - 1}}\ .
\end{aligned}$$

Since $\sinh(2t) + \cosh(2t) = e^{2t}$, we have

$$\begin{aligned}
\varrho(a) &< \frac{1}{\cosh a}\int_0^\infty \frac{dt}{\sqrt{e^{4t}-1}} = \frac{1}{\cosh a}\int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-4t}}}\,dt \\
&= \frac{1}{\cosh a}\cdot\frac{\pi}{4} \longrightarrow 0 \quad \text{as } a \to \infty\ .
\end{aligned} \tag{3.17}$$

Moreover, since $\lim_{a\to 0^+} \varrho(a) = 0$, $\varrho(a) > 0$ for $a \in (0,\infty)$ and $\varrho(a) \to 0$ as $a \to \infty$, it must have at least one maximum value in $(0, \infty)$.

By Lemma 3.3, we know that $\varrho'(a)$ has a unique zero a_c such that $\varrho'(a) > 0$ if $0 < a < a_c$ and $\varrho'(a) < 0$ if $a > a_c$, hence the proof of the claim is complete.

According to the numerical computation: the function $\varrho(a)$ achieves its maximum value ≈ 0.501143 when $a = a_c \approx 0.49577$, and so $2\varrho(a_c) \approx 1.00229$. □

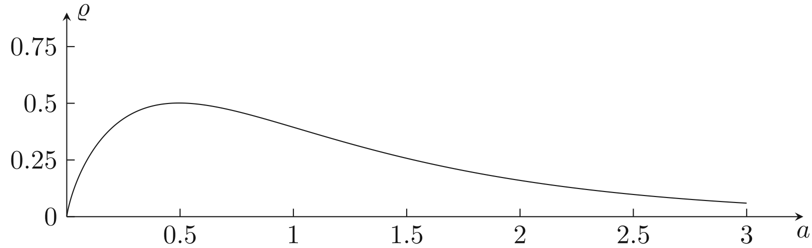

FIG. 6. *The graph of the function $\varrho(a)$ defined by* (3.13) *for $a \in [0, 3]$.*

REMARK 3.4. In [dOS98, p. 402], de Oliveria and Soret show that for any two congruent circles (in $\partial_\infty \mathbb{U}^3 = \mathbb{R}^2 \times \{0\}$) of Euclidean diameter d and disjoint from each other by the Euclidean distance D, there exists *two* catenoids bounding the two circles if and only if $D/d \leqslant \delta$ for some $\delta > 0$. Direct computation shows that $\delta = \cosh(\varrho(a_c)) - 1 \approx 0.12763$, where $\varrho(a)$ is the function defined by (3.13) and a_c is the unique critical point of the function $\varrho(a)$.

4. Least Area Spherical Catenoids. In this section, we will prove Theorem 1.4. The following estimate is crucial to the proof of Theorem 1.4.

LEMMA 4.1. *For all real numbers $a > 0$, consider the functions*

$$f(a) = \int_a^\infty \sinh t \cdot \left(\frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}} - 1 \right) dt \ , \tag{4.1}$$

and $g(a) = \cosh a - 1$, then we have the following results:

1) *$f(a) \geqslant 0$ is well defined for each $a \in (0, \infty)$.*
2) *$f(a) < g(a)$ for sufficiently large a.*

Proof. (1) Using the substitution $t \to t + a$, we have

$$f(a) = \int_0^\infty \sinh(a+t) \left(\frac{\sinh(2a+2t)}{\sqrt{\sinh^2(2a+2t) - \sinh^2(2a)}} - 1 \right) dt \ .$$

We claim that $f(a) < K \cosh a$, where $K < 1$ is a constant. Actually, let

$$\Phi(a,t) = \frac{\sinh(2a+2t)}{\sqrt{\sinh^2(2a+2t) - \sinh^2(2a)}} \ ,$$

then for any fixed $t \in [0, \infty)$, it's easy to verify that $\Phi(a, t)$ is increasing on $[0, \infty)$ with respect to a. Therefore we have the estimate

$$\begin{aligned}\Phi(a,t) &\leqslant \lim_{a\to\infty} \frac{\sinh(2a+2t)}{\sqrt{\sinh^2(2a+2t)-\sinh^2(2a)}} = \frac{\sinh(2t)+\cosh(2t)}{\sqrt{(\sinh(2t)+\cosh(2t))^2-1}}\\ &= \frac{e^{2t}}{\sqrt{e^{4t}-1}} = \frac{1}{\sqrt{1-e^{-4t}}}\ .\end{aligned}$$

Since $\sinh(a+t) < (\sinh t + \cosh t)\cosh a = e^t \cosh a$, we have the estimate

$$\begin{aligned}f(a) &< \cosh a \int_0^\infty e^t \left(\frac{e^{2t}}{\sqrt{e^{4t}-1}} - 1 \right) dt = \cosh a \int_0^\infty e^t \left(\frac{1}{\sqrt{1-e^{-4t}}} - 1 \right) dt\\ &= \cosh a \int_0^1 \frac{1}{x^2} \left(\frac{1}{\sqrt{1-x^4}} - 1 \right) dx \quad (t \mapsto x = e^{-t})\end{aligned}$$

Since $x^2 + 1 \geqslant 1$, we have

$$K = \int_0^1 \frac{1}{x^2} \left(\frac{1}{\sqrt{1-x^4}} - 1 \right) dx < \int_0^1 \frac{1}{x^2} \left(\frac{1}{\sqrt{1-x^2}} - 1 \right) dx = 1\ , \tag{4.2}$$

where we use the substitution $x \to \sin x$ to evaluate the above integral.

(2) We have proved that $f(a) < K \cosh a$ for any $a \in [0, \infty)$. Let

$$a_l = \cosh^{-1} \left(\frac{1}{1-K} \right) \approx 1.10055\ , \tag{4.3}$$

then $f(a) < g(a)$ if $a \geqslant a_l$. □

We need the coarea formula that will be used in the proof of Theorem 1.4. The proof of (4.4) in Lemma 4.2 can be found in [Wan12].

Lemma 4.2 (Calegari and Gabai [CG06, §1]). *Suppose Σ is a surface in hyperbolic 3-space $\mathbb{B}^3$. Let $\gamma \subset \mathbb{B}^3$ be a geodesic, for any point $q \in \Sigma$, define $\theta(q)$ to be the angle between the tangent space to Σ at q, and the radial geodesic that is through q (emanating from γ) and is perpendicular to γ. Then*

$$\mathrm{Area}(\Sigma \cap \mathscr{N}_s(\gamma)) = \int_0^s \int_{\Sigma\cap\partial\mathscr{N}_t(\gamma)} \frac{1}{\cos\theta}\, dldt\ , \tag{4.4}$$

where $\mathscr{N}_s(\gamma)$ is the hyperbolic s-neighborhood of the geodesic γ.

Now we are able to prove Theorem 1.4.

Theorem 1.4. *There exists a constant $a_l \approx 1.10055$ defined by (4.3) such that for any $a \geqslant a_l$ the catenoid $\mathcal{C}_a$ is a least area minimal surface.*

Proof. First of all, let $a \geqslant a_l$ be an arbitrary constant. Suppose that $\partial_\infty \mathcal{C}_a = C_1 \cup C_2$, and let P_i be the geodesic plane asymptotic to C_i $(i = 1, 2)$. Let $\sigma_a = \mathcal{C}_a \cap \mathbb{B}^2_+$ be the generating curve of the catenoid $\mathcal{C}_a$.

For $x \in (-\varrho(a), \varrho(a))$, where $\varrho(a)$ is defined by (3.13), let $P(x)$ be the geodesic plane perpendicular to the u-axis such that $\mathrm{dist}(O, P(x)) = |x|$. Now let

$$\Sigma = \bigcup_{|x|\leqslant x_1} (\mathcal{C}_a \cap P(x))\ , \tag{4.5}$$

for some $0 < x_1 < \varrho(a)$. Obviously Let $\partial\Sigma = C_+ \cup C_-$. Note that C_+ and C_- are coaxial with respect to the u-axis.

Claim 1. Area$(\Sigma) <$ Area(P_+) + Area(P_-), *where $P_\pm$ are the compact subdomains of $P(\pm x_1)$ that are bounded by $C_\pm$ respectively.*

Proof of Claim 1. Recall that $P_\pm$ are two (totally) geodesic disks with hyperbolic radius y_1, so the area of $P_\pm$ is given by

$$\text{Area}(P_+) = \text{Area}(P_-) = 4\pi \sinh^2\left(\frac{y_1}{2}\right) = 2\pi(\cosh y_1 - 1) \ , \tag{4.6}$$

where $(x_1, y_1) \in \sigma_a$ satisfies the equation (3.9) in the case when $y_0 = a$.

Recall that Area$(\Sigma) =$ Area$(\Sigma \cap \mathscr{N}_{y_1}(\gamma_0))$, by the co-area formula we have

$$\text{Area}(\Sigma) = \int_a^{y_1} \left(\text{Length}(\Sigma \cap \partial\mathscr{N}_t(\gamma_0)) \cdot \frac{1}{\cos\theta}\right) dt \ , \tag{4.7}$$

where the angle θ is given by (3.8), hence

$$\text{Area}(\Sigma) = \int_a^{y_1} \left(4\pi \sinh t \cdot \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}}\right) dt \ . \tag{4.8}$$

By Lemma 4.1, for any $a \geqslant a_l$ we have

$$\int_a^{\infty} \sinh t \cdot \left(\frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}} - 1\right) dt < \cosh a - 1 \ ,$$

therefore for any $y_1 \in (a, \infty)$ we have

$$4\pi \int_a^{y_1} \sinh t \cdot \left(\frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}} - 1\right) dt < 4\pi(\cosh a - 1) \ ,$$

and then Area$(\Sigma) <$ Area(P_+) + Area(P_-). $\square$

Claim 2. *There is no annulus-type minimal surface with the same boundary as that of Σ which has smaller area than that of Σ.*

Proof of Claim 2. Recall that $a \geqslant a_l$. We need two notations:

- Ω denotes the subregion of $\mathbb{B}^3$ bounded by $P(-x_1)$ and $P(x_1)$, and
- $\mathbf{T}_a$ denotes the simply connected subregion of $\mathbb{B}^3$ bounded by $\mathcal{C}_a$.

Assume that Σ' is a least area annulus whose boundary is the same as that of Σ, and Area$(\Sigma') <$ Area(Σ). Since Σ' is a least area annulus, it must be a minimal surface. By the maximum principle, Σ' must be contained in Ω. Furthermore, $\{\mathcal{C}_\alpha\}_{\alpha \geqslant a_c}$ locally foliates $\Omega \subset \mathbb{B}^3$ by Proposition 4.8(1) in [BSE10], therefore Σ' must be contained in $\mathbf{T}_a \cap \Omega$ by the maximum principle. It's easy to verify that the boundary of $\mathbf{T}_a \cap \Omega$ is given by $\partial(\mathbf{T}_a \cap \Omega) = \Sigma \cup P_+ \cup P_-$.

Now we claim that Σ' is symmetric about any geodesic plane that passes through the u-axis, i.e., Σ' is a surface of revolution. Otherwise, using the reflection about the geodesic planes that pass through the u-axis, we can find another annulus Σ'' with $\partial\Sigma'' = \partial\Sigma'$ such that either Area$(\Sigma'') <$ Area(Σ') or Σ'' contains *folding curves* so that we can find an annulus with smaller area by the argument in [MY82, pp.418–419]. Similarly, Σ' is symmetric about the vw-plane.

Let $\sigma' = \Sigma' \cap \mathbb{B}^2_+$, then σ' must satisfy (3.7) for some constant $a' > 0$, therefore Σ' is a compact subdomain of some catenoid $\mathcal{C}_{a'}$ (see Figure 7). Obviously $\mathcal{C}_{a'} \cap \mathcal{C}_a = C_+ \cup C_-$. Since $\Sigma' \subset \mathbf{T}_a \cap \Omega$, we have $a' \leqslant a$ by Lemma 4.9 in [BSE10]. We claim that if $a' < a$, then Σ' must be unstable. In fact, if $a' < a$, since $\varrho(a') < \varrho(a) < \varrho(a_c)$, we have $a' < a_c$ by Lemma 4.9 in [BSE10], hence $\mathcal{C}_{a'}$ is unstable by Theorem 1.2. Furthermore, according to Proposition 4.8 in [BSE10], Σ' contains the maximal weakly stable domain of $\mathcal{C}_{a'}$ (which is contained in $\mathbf{T}_\alpha$ for $\alpha = a_c$), so Σ' is unstable, and then it can not be a least area minimal surface unless $\Sigma' \equiv \Sigma$.

Therefore any compact annulus of the form (4.5) is a least area minimal surface. □

Now let S be any compact domain of $\mathcal{C}_a$ that is homotopic to $\mathcal{C}_a$, then we always can find a compact annulus Σ of the form (4.5) such that $S \subset \Sigma$. If S is not a least area minimal surface, then we can use the cutting and pasting technique to show that Σ is not a least area minimal surface. This is contradicted to the above argument.

Therefore if $a \geqslant a_l$, then $\mathcal{C}_a$ is a least area minimal surface. □

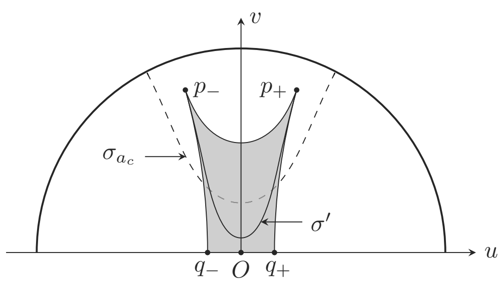

FIG. 7. *The shaded region is equal to* $(\mathbf{T}_a \cap \Omega) \cap \mathbb{B}^2_+$.

REMARK 4.3. In the proof of *Claim 2* in Theorem 1.4, if Σ' is an annulus type minimal surface but it is not a least area minimal surface, then it might not be a surface of revolution (see [Lóp00, p. 234]).

COROLLARY 4.4. *There exists a finite constant* $a_l \approx 1.10055$ *such that for two disjoint circles* $C_1, C_2 \subset S^2_\infty$, *if* $d(C_1, C_2) \leqslant 2\varrho(a_l) \approx 0.72918$, *then there exist a least area spherical minimal catenoid* $\mathcal{C}$ *which is asymptotic to* $C_1 \cup C_2$, *where the function* $\varrho(a)$ *is given by* (3.13).

5. Stability of Helicoids. In this section, we will prove Theorem 1.5. One of the key points in the proof is that two conjugate minimal surfaces are either both stable or both unstable.

5.1. Conjugate minimal surface. Let $M^3(c)$ be the 3-dimensional space form whose sectional curvature is equal to a constant c.

DEFINITION 5.1 ([dCD83, pp. 699-700]). Let $f : \Sigma \to M^3(c)$ be a minimal surface in isothermal parameters (σ, t). Denote by $\mathrm{I} = E(d\sigma^2 + dt^2)$ and $\mathrm{II} = \beta_{11} d\sigma^2 + 2\beta_{12} d\sigma dt + \beta_{22} dt^2$ the first and second fundamental forms of f, respectively.

Set $\psi = \beta_{11} - i\beta_{12}$ and define a family of quadratic forms depending on a parameter θ, $0 \leqslant \theta \leqslant 2\pi$, by $\beta_{11}(\theta) = \mathrm{Re}(e^{i\theta}\psi)$, $\beta_{22}(\theta) = -\mathrm{Re}(e^{i\theta}\psi)$ and $\beta_{12}(\theta) = \mathrm{Im}(e^{i\theta}\psi)$.

Then the forms $\mathrm{I}_\theta = \mathrm{I}$ and $\mathrm{II}_\theta = \beta_{11}(\theta)d\sigma^2 + 2\beta_{12}(\theta)d\sigma dt + \beta_{22}(\theta)dt^2$ give rise to an isometry family $f_\theta : \widetilde{\Sigma} \to M^3(c)$ of minimal immersions, where $\widetilde{\Sigma}$ is the universal covering of Σ. The immersion $f_{\pi/2}$ is called the *conjugate immersion* to $f_0 = f$.

LEMMA 5.2. *Let $f : \Sigma \to M^3(c)$ be an immersed minimal surface, and let $f_{\pi/2} : \widetilde{\Sigma} \to M^3(c)$ be its conjugate minimal surface, where $\widetilde{\Sigma}$ is the universal covering of Σ. Then the minimal immersion f is globally stable if and only if its conjugate immersion $f_{\pi/2}$ is globally stable.*

Proof. Let $\tilde{f}$ be the universal lifting of f, then $\tilde{f} : \widetilde{\Sigma} \to M^3(c)$ is a minimal immersion. It's well known that the global stability of f implies the global stability of $\tilde{f}$. Actually the minimal surfaces Σ and $\widetilde{\Sigma}$ share the same Jacobi operator defined by (2.1). If Σ is globally stable, there exists a positive Jacobi field on Σ according to Theorem 2.1, which implies that $\widetilde{\Sigma}$ is also globally stable, since the corresponding positive Jacobi field on $\widetilde{\Sigma}$ is given by composing.

Next we claim that $\tilde{f}$ and $f_{\pi/2}$ share the same Jacobi operator. In fact, since the Laplacian depends only on the first fundamental form, $\tilde{f}$ and $f_{\pi/2}$ have the same Laplacian. Furthermore according to the definition of the conjugate minimal immersion, $\tilde{f}$ and $f_{\pi/2}$ have the same square norm of the second fundamental form, i.e. $|A|^2 = (\beta_{11}^2 + 2\beta_{12}^2 + \beta_{22}^2)/E^2$, where we used the notations in Definition 5.1.

By (2.1) and Theorem 2.1, the proof of the lemma is complete. □

5.2. Proof of Theorem 1.5. For hyperbolic and parabolic catenoids in $\mathbf{H}^3$ do Carmo and Dajczer proved that they are globally stable (see [dCD83, Theorem (5.5)]). Then Candel proved that the both hyperbolic and parabolic catenoids are least area minimal surfaces (see [Can07, p.3574]).

The following result will be used for proving Theorem 1.5. The identity (5.1) can be found in [BSE09, p.34].

LEMMA 5.3 (Bérard and Sa Earp). *The spherical catenoid $\mathscr{C}_1(\tilde{a})$ in $\mathbf{H}^3$ is isometric to the spherical catenoid $\mathcal{C}_a$ in $\mathbb{B}^3$ defined in §3 if and only if*

$$2\tilde{a} = \cosh(2a) \ . \tag{5.1}$$

Proof. The spherical catenoid $\mathcal{C}_a$ is obtained by rotating the generating curve σ_a along the axis γ_0 in (3.2). The distance between σ_a and γ_0 is a.

The spherical catenoid $\mathscr{C}_1(\tilde{a})$ in $\mathbf{H}^3$ can be obtained by rotating the generating curve C given by (2.4) and (2.5) along the geodesic $P^2 \cap \mathbf{H}^3$. The distance between C and $P^2 \cap \mathbf{H}^3$ is $\sinh^{-1}\left(\sqrt{\tilde{a} - 1/2}\right)$.

Therefore the catenoid $\mathscr{C}_1(\tilde{a})$ in $\mathbf{H}^3$ is isometric to the catenoid $\mathcal{C}_a$ in $\mathbb{B}^3$ if and only if $a = \sinh^{-1}\left(\sqrt{\tilde{a} - 1/2}\right)$, which implies (5.1). □

The following result can be found in [dCD83, Theorem (3.31)].

THEOREM 5.4 (do Carmo-Dajczer). *Let $f : \mathscr{C} \to \mathbf{H}^3$ be a minimal catenoid defined in §2.3. Its conjugate minimal surface is the geodesically-ruled minimal surface $\mathcal{H}_{\bar{a}}$ given by (2.7) where*

$$\begin{cases} \bar{a} = \sqrt{(\tilde{a} + 1/2)/(\tilde{a} - 1/2)} \ , & \textit{if } \mathscr{C} = \mathscr{C}_1(\tilde{a}) \textit{ is spherical} \ , \\ \bar{a} = \sqrt{(\tilde{a} - 1/2)/(\tilde{a} + 1/2)} \ , & \textit{if } \mathscr{C} = \mathscr{C}_{-1}(\tilde{a}) \textit{ is hyperbolic} \ , \\ \bar{a} = 1 \ , & \textit{if } \mathscr{C} = \mathscr{C}_0 \textit{ is parabolic} \ . \end{cases} \tag{5.2}$$

Now we are able to prove the theorem.

THEOREM 1.5. For a family of minimal helicoids $\{\mathcal{H}_{\bar{a}}\}_{\bar{a}\geqslant 0}$ in hyperbolic 3-space given by (2.7), there exist a constant $\bar{a}_c = \coth(a_c) \approx 2.17968$ such that the following statements are true:

1) $\mathcal{H}_{\bar{a}}$ is a globally stable minimal surface if $0 \leqslant \bar{a} \leqslant \bar{a}_c$, and
2) $\mathcal{H}_{\bar{a}}$ is an unstable minimal surface with Morse index one if $\bar{a} > \bar{a}_c$.

Proof. When $\bar{a} = 0$, $\mathcal{H}_{\bar{a}}$ is a hyperbolic plane, so it is globally stable.

According to Theorem 5.4, when $0 < \bar{a} < 1$, $\mathcal{H}_{\bar{a}}$ is conjugate to the hyperbolic catenoid $\mathscr{C}_{-1}(\tilde{a})$, where $\bar{a} = \sqrt{(\tilde{a}-1/2)/(\tilde{a}+1/2)}$ by (5.2), and when $\bar{a} = 1$, $\mathcal{H}_{\bar{a}}$ is conjugate to the parabolic catenoid $\mathscr{C}_0$ in $\mathbf{H}^3$. Therefore when $0 < \bar{a} \leqslant 1$, the helicoid $\mathcal{H}_{\bar{a}}$ is globally stable by [dCD83, Theorem (5.5)].

When $\bar{a} > 1$, $\mathcal{H}_{\bar{a}}$ is conjugate to the spherical catenoid $\mathscr{C}_1(\tilde{a})$ in $\mathbf{H}^3$ by Theorem 5.4, which is isometric to the spherical catenoid $\mathcal{C}_a$ in $\mathbb{B}^3$, where $2\tilde{a} = \cosh(2a)$ by (5.1) and $\bar{a} = \sqrt{(\tilde{a}+1/2)/(\tilde{a}-1/2)}$ by (5.2). Therefore $\mathcal{H}_{\bar{a}}$ is conjugate to the spherical catenoid $\mathcal{C}_a$ in $\mathbb{B}^3$, where $\bar{a} = \coth(a)$. By Theorem 1.2, the catenoid $\mathcal{C}_a$ is globally stable if $a \geqslant a_c \approx 0.49577$, therefore $\mathcal{H}_{\bar{a}}$ is globally stable when $1 < \bar{a} \leqslant \bar{a}_c = \coth(a_c) \approx 2.17968$. On the other hand, if $0 < a < a_c$, then the catenoid $\mathcal{C}_a$ is unstable (with Morse index one) by Theorem 1.2, therefore $\mathcal{H}_{\bar{a}}$ is unstable when $\bar{a} > \bar{a}_c$.

At last we will show that the Morse index of the helicoid $\mathcal{H}_{\bar{a}}$ for $\bar{a} > \bar{a}_c$ is infinite. Otherwise according to Theorem 2.2, there is a compact subdomain K of $\mathcal{H}_{\bar{a}}$ such that $\mathcal{H}_{\bar{a}} \setminus K$ is stable. But this is impossible, since we always can find a (noncompact) subdomain $\mathcal{U}$ of $\mathcal{H}_{\bar{a}} \setminus K$ as follows: Consider the parametric equation (2.8) of $\mathcal{H}_{\bar{a}}$ in the Poincaré ball model $\mathbb{B}^3$ of the hyperbolic 3-space, and write $\mathcal{U}$ by

$$\mathcal{U} = \left\{(x,y,z) \in \mathcal{H}_{\bar{a}} \mid u \in \mathbb{R} \text{ and } v_0 \leqslant v \leqslant v_0 + 2\pi \sinh(\coth^{-1}(\bar{a}))\right\} ,$$

where $v_0 > 0$ is sufficiently large such that K is underneath the geodesic of $\mathcal{H}_{\bar{a}} \subset \mathbb{B}^3$ which passes through the point $(0,0,\tanh(v_0/2)) \in \mathcal{H}_{\bar{a}}$. It's easy to verify that $\mathcal{U}$ is disjoint from K, i.e. $\mathcal{U} \subset \mathcal{H}_{\bar{a}} \setminus K$, and $\mathcal{U}$ is isometric to the fundamental domain of the universal cover of the catenoid $\mathcal{C}_a$, where $a = \coth^{-1}(\bar{a}) < a_c$. Therefore $\mathcal{U}$ is unstable, and so is $\mathcal{H}_{\bar{a}} \setminus K$. □

6. Technical lemmas.

In this section we will prove Lemma 6.1 and Lemma 6.2.

LEMMA 6.1. *Let $\phi(a,t) = \sqrt{5}\cosh(a+t) - \cosh(3a+t)$, then $\phi(a,t) \leqslant 0$ for $(a,t) \in [A_3,\infty) \times [0,\infty)$, where the constant A_3 is defined by*

$$A_3 = \cosh^{-1}\left(\frac{\sqrt{3+\sqrt{5}}}{2}\right) \approx 0.530638 . \tag{6.1}$$

Proof. It's easy to verify that $\phi(a,t) \leqslant 0$ is equivalent to

$$\cosh(3a) - \sqrt{5}\cosh a + \tanh t \cdot (\sinh(3a) - \sqrt{5}\sinh a) \geqslant 0 .$$

Since $\tanh t \geqslant 0$ for $t \geqslant 0$ and $\sinh(3a) - \sqrt{5}\sinh a \geqslant 0$ for $a \geqslant 0$, we need solve the inequality $\cosh(3a) - \sqrt{5}\cosh a \geqslant 0$. Let A_3 be the solution of the equation

$$0 = \sqrt{5}\cosh a - \cosh(3a) = (\sqrt{5} - (4\cosh^2 a - 3))\cosh a ,$$

then $\phi(a,t) \leqslant 0$ if $a \geqslant A_3$ and $t \geqslant 0$. □

LEMMA 6.2. *Let $\psi(a,t)$ be the function defined by*

$$\begin{aligned}\psi(a,t) = &76\sinh(2a) - 22\sinh(2t) + 29\sinh(4a+2t) + \sinh(8a+2t)\\ &- 26\sinh(6a+4t) - 6\sinh(10a+4t)\\ &- 25\sinh(8a+6t) + \sinh(12a+6t)\ .\end{aligned} \tag{6.2}$$

Then $\psi(a,t) < 0$ for all $(a,t) \in [0, A_4] \times [0,\infty)$, where the constant

$$A_4 = \frac{1}{4}\cosh^{-1}\left(\frac{35+\sqrt{1241}}{8}\right) \approx 0.715548 \tag{6.3}$$

is the solution of the equation $4\cosh^2(4a) - 35\cosh(4a) - 1 = 0$.

Proof. Expand each term in $\psi(a,t)$ with the form $\sinh(ma+nt)$, then we may write $\psi(a,t) = \psi_1(a,t) + \psi_2(a,t)$, where

$$\begin{aligned}\psi_1(a,t) = &-22\sinh(2t) + 29\sinh(2t)\cosh(4a) + \sinh(2t)\cosh(8a)\\ &- 26\sinh(4t)\cosh(6a) - 6\sinh(4t)\cosh(10a)\\ &- 25\sinh(6t)\cosh(8a) + \sinh(6t)\cosh(12a))\ ,\end{aligned}$$

and

$$\begin{aligned}\psi_2(a,t) = &76\sinh(2a) + 29\cosh(2t)\sinh(4a) + \cosh(2t)\sinh(8a)\\ &- 26\cosh(4t)\sinh(6a) - 6\cosh(4t)\sinh(10a)\\ &- 25\cosh(6t)\sinh(8a) + \cosh(6t)\sinh(12a)\ .\end{aligned}$$

CLAIM. *$\psi_1(a,t) \leqslant 0$ and $\psi_2(a,t) \leqslant 0$ for $(a,t) \in [0, A_4] \times [0,\infty)$.*

Proof of Claim. First of all, we will show that $\psi_1(a,\cdot) \leqslant 0$ for $a \in [0, A_4]$. Since $\cosh(2t) \geqslant 1$ for any $t \in [0,\infty)$, we have the estimate

$$\begin{aligned}\psi_1(a,t) = &-\sinh(2t)(22 - 29\cosh(4a) - \cosh(8a)\\ &\qquad + 52\cosh(2t)\cosh(6a) + 12\cosh(2t)\cosh(10a))\\ &-\sinh(6t)(25\cosh(8a) - \cosh(12a))\\ \leqslant &-\sinh(2t)(22 - 29\cosh(4a) - \cosh(8a)\\ &\qquad + 52\cosh(6a) + 12\cosh(10a))\\ &-\sinh(6t)(25\cosh(8a) - \cosh(12a))\ .\end{aligned}$$

Since $52\cosh(6a) - 29\cosh(4a) > 0$ and $12\cosh(10a) - \cosh(8a) > 0$ for $0 \leqslant a < \infty$ and $25\cosh(8a) - \cosh(12a) > 0$ for $0 \leqslant a \leqslant A_4$, we have $\psi_1(a,\cdot) < 0$ for $0 \leqslant a \leqslant A_4$.

Secondly, since $\sinh((m+n)a) = \sinh(ma+na) \geqslant \sinh(ma) + \sinh(na)$, where $a \geqslant 0$ and m, n are positive integers, we have the following inequalities

- $\sinh(6a) \geqslant \sinh(4a) + \sinh(2a)$,
- $\sinh(8a) \geqslant 4\sinh(2a)$, and
- $\sinh(10a) \geqslant \begin{cases}\sinh(8a) + \sinh(2a)\\ \sinh(4a) + 3\sinh(2a)\\ 5\sinh(2a)\end{cases}$,

which can imply the estimate

$$\begin{aligned}\psi_2(a,t) \leqslant & -46\sinh(2a)(\cosh(4t)-1) - 30\sinh(2a)(\cosh(6t)-1)\\ & -29\sinh(4a)(\cosh(4t)-\cosh(2t))\\ & -\sinh(8a)(\cosh(4t)-\cosh(2t))\\ & -\cosh(6t)\left(\frac{35}{2}\sinh(8a)-\sinh(12a)\right).\end{aligned}$$

As $\frac{35}{2}\sinh(8a)-\sinh(12a) = \sinh(4a)(1+35\cosh(4a)-4\cosh^2(4a)) \geqslant 0$ if $0 \leqslant a \leqslant A_4$ and the fact $\cosh(6t) \geqslant \cosh(4t) \geqslant \cosh(2t) \geqslant 1$ for $0 \leqslant t < \infty$, we have $\psi_2(a,t) \leqslant 0$ for $(a,t) \in [0, A_4] \times [0, \infty)$. □

Therefore $\psi(a,t) < 0$ for $(a,t) \in [0, A_4] \times [0,\infty)$. □

REFERENCES

[BdCH09] A. Back, M. do Carmo, and W.-Y. Hsiang, *On some fundamental equations of equivariant Riemannian geometry*, Tamkang J. Math., 40:4 (2009), pp. 343–376.

[Bea95] A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1995, Corrected reprint of the 1983 original.

[BGS87] J. L. Barbosa, J. de Miranda Gomes, and A. M. Silveira, *Foliation of* 3*-dimensional space forms by surfaces with constant mean curvature*, Bol. Soc. Brasil. Mat., 18:2 (1987), pp. 1–12.

[BP92] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992.

[BSE09] P. Bérard and R. Sa Earp, *Lindelöf's theorem for catenoids revisited*, arXiv:0907.4294v1 (2009).

[BSE10] ———, *Lindelöf's theorem for hyperbolic catenoids*, Proc. Amer. Math. Soc., 138:10 (2010), pp. 3657–3669.

[Can07] A. Candel, *Eigenvalue estimates for minimal surfaces in hyperbolic space*, Trans. Amer. Math. Soc., 359:8 (2007), pp. 3567–3575 (electronic).

[CG06] D. Calegari and D. Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc., 19:2 (2006), pp. 385–446.

[CM11] T. H. Colding and W. P. Minicozzi, II, *A course in minimal surfaces*, Graduate Studies in Mathematics, vol. 121, American Mathematical Society, Providence, RI, 2011.

[dCD83] M. do Carmo and M. Dajczer, *Rotation hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc., 277:2 (1983), pp. 685–709.

[dCGT86] M. do Carmo, J. de Miranda Gomes, and G. Thorbergsson, *The influence of the boundary behaviour on hypersurfaces with constant mean curvature in H^{n+1}*, Comment. Math. Helv., 61:3 (1986), pp. 429–441.

[dCP79] M. do Carmo and C. K. Peng, *Stable complete minimal surfaces in $\mathbf{R}^3$ are planes*, Bull. Amer. Math. Soc. (N.S.), 1:6 (1979), pp. 903–906.

[dOS98] G. de Oliveira and M. Soret, *Complete minimal surfaces in hyperbolic space*, Math. Ann., 311:3 (1998), pp. 397–419.

[FC85] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math., 82:1 (1985), pp. 121–132.

[FCS80] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math., 33:2 (1980), pp. 199–211.

[FT91] A. T. Fomenko and A. A. Tuzhilin, *Elements of the geometry and topology of minimal surfaces in three-dimensional space*, Translations of Mathematical Monographs, vol. 93, American Mathematical Society, Providence, RI, 1991.

[Gom87] J. de Miranda Gomes, *Spherical surfaces with constant mean curvature in hyperbolic space*, Bol. Soc. Brasil. Mat., 18:2 (1987), pp. 49–73.

[Hsi82] W.-Y. Hsiang, *On generalization of theorems of A. D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature*, Duke Math. J., 49:3 (1982), pp. 485–496.

[HW15] Z. Huang and B. Wang, *Counting minimal surfaces in quasi-Fuchsian three-manifolds*, Trans. Amer. Math. Soc., 367:9 (2015), pp. 6063–6083.

[Lóp00] R. López, *Hypersurfaces with constant mean curvature in hyperbolic space*, Hokkaido Math. J., 29:2 (2000), pp. 229–245.

[LR85] G. Levitt and H. Rosenberg, *Symmetry of constant mean curvature hypersurfaces in hyperbolic space*, Duke Math. J., 52:1 (1985), pp. 53–59.

[LR89] F. J. López and A. Ros, *Complete minimal surfaces with index one and stable constant mean curvature surfaces*, Comment. Math. Helv., 64:1 (1989), pp. 34–43.

[LR91] ———, *On embedded complete minimal surfaces of genus zero*, J. Differential Geom., 33:1 (1991), pp. 293–300.

[Mor81] H. Mori, *Minimal surfaces of revolution in H^3 and their global stability*, Indiana Univ. Math. J., 30:5 (1981), pp. 787–794.

[Mor82] ———, *On surfaces of right helicoid type in H^3*, Bol. Soc. Brasil. Mat., 13:2 (1982), pp. 57–62.

[MP11] W. H. Meeks, III and J. Pérez, *The classical theory of minimal surfaces*, Bull. Amer. Math. Soc. (N.S.), 48:3 (2011), pp. 325–407.

[MP12] ———, *A survey on classical minimal surface theory*, University Lecture Series, vol. 60, American Mathematical Society, Providence, RI, 2012.

[MR05] W. H. Meeks, III and H. Rosenberg, *The uniqueness of the helicoid*, Ann. of Math. (2), 161:2 (2005), pp. 727–758.

[MT98] K. Matsuzaki and M. Taniguchi, *Hyperbolic manifolds and Kleinian groups*, Oxford Mathematical Monographs, The Oxford University Press, New York, 1998.

[MY82] W. W. Meeks, III and S.-T. Yau, *The classical Plateau problem and the topology of three-dimensional manifolds*, Topology, 21:4 (1982), pp. 409–442.

[Sch83] R. M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom., 18:4 (1983), pp. 791–809 (1984).

[Seo11] K. Seo, *Stable minimal hypersurfaces in the hyperbolic space*, J. Korean Math. Soc., 48:2 (2011), pp. 253–266.

[Tuz93] A. A. Tuzhilin, *Global properties of minimal surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ and their Morse type indices*, Minimal surfaces, Adv. Soviet Math., vol. 15, Amer. Math. Soc., Providence, RI, 1993, pp. 193–233.

[Wan12] B. Wang, *Minimal surfaces in quasi-Fuchsian 3-manifolds*, Math. Ann., 354:3 (2012), pp. 955–966.