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006

DIRAC-HARMONIC MAPS BETWEEN RIEMANN SURFACES*

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Abstract. In this paper, we consider the existence and structure of Dirac-harmonic maps between closed Riemann surfaces. Utilizing the Riemann-Roch formula, we compute the dimension of harmonic spinors along a map, based on which we prove an existence theorem for Dirac-harmonic maps between closed Riemann surfaces. We also obtain a structure theorem for Dirac-harmonic maps between two surfaces if their genera and the degree of the map satisfy a certain relation.

Key words. Dirac-harmonic maps, Riemann surfaces, Riemann-Roch formula.

Mathematics Subject Classification. 53C43, 53C27.

1. Introduction. Dirac-harmonic maps have been introduced in [4, 3]. They were motivated by the supersymmetric σ-model of quantum field theory. They replace the anticommuting spinor field of that model, which takes values in a Grassmannian algebra and makes the model supersymmetric, by a commuting field. Nevertheless, they preserve important symmetries, in particular conformal invariance. Mathematically, they can be seen as an extension of the harmonic map problem as they couple a harmonic map type field with a spinor field. Since all the fields are ordinary, commuting variables, we may apply the methods of the geometric calculus of variations. A technical difficulty, however, arises from the fact that the underlying action functional is not bounded from below, in contrast to standard harmonic maps where it is nonnegative.

We now present the mathematical definitions. (M, g) is a Riemann surface with a conformal metric g and a fixed spin structure, and ΣM the spinor bundle over M, on which we chose a Hermitian metric $\langle \cdot, \cdot \rangle$. The classical connection ∇ on ΣM induced from the Levi-Civita connection on TM is compatible with $\langle \cdot, \cdot \rangle$. Let (N, h) be a Riemannian manifold (subsequently, it will likewise be of dimension 2, that is, a Riemann surface with a conformal metric), Φ a map from M to N, and $\Phi^{-1}TN$ the pull-back bundle of TN by Φ. We also denote the metric induced from the metrics on ΣM and $\Phi^{-1}TN$ on the twisted bundle $\Sigma M \otimes \Phi^{-1}TN$ by $\langle \cdot, \cdot \rangle$. Likewise, we also denote the connection on $\Sigma M \otimes \Phi^{-1}TN$ induced from those on ΣM and $\Phi^{-1}TN$ by ∇.

A cross-section Ψ of $\Sigma M \otimes \Phi^{-1}TN$ can be locally written as $\Psi = \psi^\alpha \otimes \theta_\alpha$, where $\{\psi^\alpha\}$ are local cross-sections of ΣM, $\{\theta_\alpha\}$ are local cross-sections of $\Phi^{-1}TN$. We always use the standard summation convention.

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The *Dirac operator along the map* Φ is

$$\begin{aligned} \not{D}\Psi &:= e_i \cdot \nabla_{e_i} \Psi \\ &= \not{\partial}\psi^\alpha \otimes \theta_\alpha + e_i \cdot \psi^\alpha \otimes \nabla_{e_i} \theta_\alpha, \end{aligned}$$

where $\{e_i\}$ is a local orthonormal frame on M, $\not{\partial} := e_i \cdot \nabla_{e_i}$ is the Dirac operator on ΣM and $X\cdot$ is the Clifford multiplication by the vector field X on M.

The action functional of the theory is

$$L(\Phi, \Psi) = \frac{1}{2} \int_M \left(\|\mathrm{d}\Phi\|^2 + \langle \Psi, \not{D}\Psi \rangle \right),$$

and as mentioned, it couples the harmonic map type field Φ with the spinor field Ψ, because the Dirac operator $\not{D}$ depends on Φ. We see this coupling also from the Euler-Lagrange equations for $L(\Phi, \Psi)$ that critical points (Φ, Ψ) have to satisfy (c.f. [3]):

$$\begin{cases} \tau(\Phi) = \dfrac{1}{2} \left\langle \psi^\alpha, e_i \cdot \psi^\beta \right\rangle R^N(\theta_\alpha, \theta_\beta) \Phi_*(e_i), \\ \not{D}\Psi = 0, \end{cases} \tag{1.1}$$

where $R^N(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, $\forall X, Y \in \Gamma(TN)$ stands for the curvature operator of N, and $\tau(\Phi)$ is the tension field of Φ. Therefore, solutions of (1.1) are called *Dirac-harmonic maps from M to N*.

Not every solution of (1.1) needs to be coupled, however, as either component could be trivial. When Φ is constant, Ψ satisfies the ordinary Dirac equation, and when Ψ vanishes, Φ is a harmonic map. We therefore say that a Dirac-harmonic map is uncoupled if the underlying map is harmonic. From our perspective, such solutions are trivial. Ammann-Ginoux [1] analyzed the space of Dirac-harmonic maps by using tools from index theory, and the existence of uncoupled solutions was proved. A question that we shall address in this paper is when such Dirac-harmonic maps are necessarily uncoupled.

In this paper, we will consider Dirac-harmonic maps between Riemann surfaces M and N. For that purpose, we shall now analyze the relevant geometry of M. The spinor bundle ΣM can be identified with

$$\Sigma M = K_M^{1/2} \oplus \Lambda^{0,1} K_M^{1/2},$$

where $K_M^{1/2}$ is a square root of the canonical line bundle K_M of M and $\Lambda^{0,1} K_M^{1/2} = \Lambda^{0,1} T^* M \otimes K_M^{1/2}$ is the vector bundle of $(0,1)$-forms valued in $K_M^{1/2}$. Choose a local conformal parameter $z = x + \sqrt{-1}y$ of M and denote the metric of M locally by $\lambda(z) |\mathrm{d}z|^2$. Then every spinor ψ on M locally can be written as

$$\psi = fs + g\mathrm{d}\bar{z} \otimes \frac{s}{|s|^2},$$

where s is a local holomorphic section of $K_M^{1/2}$ and f, g are local complex functions. From now on, we fixed a nontrivial holomorphic section s of $K_M^{1/2}$ with possibly singularities. For convenience, we simplify the expression of ψ by

$$\psi = f + g\mathrm{d}\bar{z}.$$

Let N be a Riemann surface and $\Phi : M \longrightarrow N$ be a smooth map. Choose a local conformal parameter $\phi = u + \sqrt{-1}v$ of N and denote the metric of N locally by $\rho(\phi)\,|\mathrm{d}\phi|^2$. Denote the local representation of Φ by ϕ. Then the Dirac bundle $\Sigma M \otimes \Phi^{-1}TN$ can be split as follows:

$$\Sigma M \otimes \Phi^{-1}TN = \left(K_M^{1/2} \otimes \Phi^{-1}T_{1,0}N\right) \oplus \left(\Lambda^{0,1}K_M^{1/2} \otimes \Phi^{-1}T_{1,0}N\right) \oplus \left(K_M^{1/2} \otimes \Phi^{-1}T_{0,1}N\right) \oplus \left(\Lambda^{0,1}K_M^{1/2} \otimes \Phi^{-1}T_{0,1}N\right)$$

and we can rewrite the spinor Ψ as follows:

$$\Psi = f\partial_\phi + \mathrm{d}\bar{z} \otimes g\partial_\phi + \bar{p}\partial_{\bar{\phi}} + \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}}.$$

Introduce a global $(1,0)$-form Θ defined locally by

$$\Theta = (f\bar{g} - \bar{p}q)\,\rho\mathrm{d}z.$$

Suppose (Φ, Ψ) is Dirac-harmonic, then Φ is harmonic if $\Theta = 0$ (see Lemma 2.1).

Our first main result is the following:

THEOREM 1.1. *Let (M, g) be a closed Riemann surface with a fixed spin structure, (N, h) be a closed Riemann surface and $\Phi : M \longrightarrow N$ be a smooth map. Then there is a complex vector space V with complex dimension $4\,|\deg(\Phi)(g_N - 1)|$ such that every $\Psi \in V$ is a harmonic spinor along the map Φ with the associated $(1,0)$-form $\Theta = 0$. As a consequence, if Φ is harmonic, then*

$$\dim_{\mathbb{C}} \{(\Phi, \Psi) \text{ is uncoupled Dirac-harmonic}\} \geq 4\,|\deg(\Phi)(g_N - 1)|\,.$$

REMARK 1.1. It is well known [7, 6, 8] that every continuous map ϕ from a closed Riemann surface M to another closed Riemann surface with an arbitrary Riemmanian metric has a harmonic representation if $g_N > 0$. In the case of $g_N = 0$, for every continuous map $\phi : M \longrightarrow N$, there are metrics on M and N such that ϕ has a harmonic representation except in the following case

$$g_M = 1, g_N = 0, |\deg(\phi)| = 1.$$

Combing the classical existence results for harmonic maps between closed Riemann surfaces and Theorem 1.1, one can derive existence results for uncoupled Dirac-harmonic maps between closed Riemann surfaces with the map part being homotopic to a given continuous map.

REMARK 1.2. Eells-Wood [9] and Lemaire [16] proved that there is no harmonic map from the 2-torus to the 2-sphere with degree ± 1 whatever the metrics. Moreover, L.Yang [20] proved that there is no coupled Dirac-harmonic map from the 2-torus to the 2-sphere with nontrivial degree. Hence there is no Dirac-harmonic map when $g_M = 1, g_N = 0$ and $|\deg(\Phi)| = 1$ (c.f. Branding [2]).

We recall that by a topological theorem of H. Kneser (c.f. [9, 13]), for every continuous map $\Phi : M \longrightarrow N$, if $\deg(\Phi) \neq 0$ and $g_N \geq 2$, then $g_M - 1 \geq |\deg(\Phi)|\,(g_N - 1)$. In particular, $g_M \geq g_N$.

Our next main result yields a formula for the dimension of the harmonic spinor spaces along a fixed map under the following condition:

$$g_M - 1 < 2\left|\deg(\Phi)(g_N - 1)\right|. \tag{1.2}$$

THEOREM 1.2. *Let (M, g) be a closed Riemann surface with a fixed spin structure, (N, h) be a closed Riemann surface and $\Phi : M \longrightarrow N$ be a smooth map. Suppose* (1.2) *holds, then the space of harmonic spinors along the map Φ is a $4\left|\deg(\Phi)(g_N - 1)\right|$ dimensional complex linear vector space.*

The first non-trivial Dirac-harmonic map was given in [4] for $M = N = S^2$, based on an explicit construction involving a harmonic map and a twistor-spinor on the domain manifold. More precisely, given a harmonic map $\Phi : S^2 \longrightarrow S^2$ and a twistor spinor $\eta \in \Sigma S^2$, we construct a spinor Ψ along the map Φ as

$$\Psi = e_1 \cdot \eta \otimes \Phi_*(e_1) + e_2 \cdot \eta \otimes \Phi_*(e_2),$$

where $\{e_1, e_2\}$ is a local orthonormal frame of S^2. Then (Φ, Ψ) is a Dirac-harmonic map. In [20], L.Yang proved that every Dirac-harmonic map between 2-spheres can be constructed in this way with η possibly having isolated singularities, i.e., η is smooth except possibly at finitely many points and satisfies the following twistor equation except at those points

$$\nabla_{e_i}\eta + \frac{1}{2} e_i \cdot \partial\!\!\!/ \eta = 0.$$

For the reader's convenience, the existence of twistor spinors with possibly isolated singularities on a closed Riemann surface with fixed spin structure will be given in Lemma 3.8.

The following table lists what is known about the existence of Dirac-harmonic maps.

Here, we shall derive a structure theorem when the target is the 2-sphere and the domain satisfies some topological assumption, for example,

$$g_M - 1 < \left|\deg(\Phi)(g_N - 1)\right|.$$

The case of $g_M = g_N = 0$ was consider by L.Yang. If $g_N = 1$, then $g_M = 0$, and a result of [2] (see also Proposition 3.6) claims that every Dirac-harmonic map (Φ, Ψ) is trivial. If $g_N \geq 2$, a topological theorem of Kneser says that Φ is trivial in the sense $\deg(\Phi) = 0$. Moreover, as we have explained in Remark 1.2, there is no Dirac-harmonic map from the 2-torus to the 2-sphere for which the degree of the map part is ± 1. (See table 1.) For the remaining cases, we have the following

THEOREM 1.3. *Let (M, g) be a closed Riemann surface of genus g_M and with a metric g locally given by $\lambda(z)\left|\mathrm{d}z\right|^2$ and let $N = S^2$ with an arbitrary metric. Suppose the degree of a map $\Phi : M \longrightarrow N$ satisfying $\left|\deg(\Phi)\right| > 1$ and*

$$1 \leq g_M < \left|\deg(\Phi)\right| + 1, \tag{1.3}$$

and let (Φ, Ψ) be a Dirac-harmonic map from M to N. Then either

Table 1
Relationship between g_M, g_N and $\deg(\Phi)$

<table>
<tr><td colspan="2">Given a smooth map $\Phi : M \longrightarrow N$ (unless $g_M = 1, g_N = 0, |\deg(\Phi)| = 1$)</td><td>Existence of uncoupled Dirac-harmonic maps, see Theorem 1.1 and Remark 1.1.</td></tr>
<tr><td colspan="2">$g_M - 1 < 2\,|\deg(\Phi)(g_N - 1)|$</td><td>$\Phi$ is harmonic (c.f. [20]). Formula for the dimension of the space of harmonic spinor along the map Φ, see Theorem 1.2.</td></tr>
<tr><td colspan="3">$g_M - 1 < |\deg(\Phi)(g_N - 1)|$</td></tr>
<tr><td colspan="2">$g_N \geq 2$</td><td>$\deg(\Phi) = 0$. (Kneser's topological theorem)</td></tr>
<tr><td colspan="2">$g_N = 1$</td><td>$g_M = 0$ and (Φ, Ψ) is trivial (c.f. [2], see also Proposition 3.6).</td></tr>
<tr><td rowspan="3">$g_N = 0$</td><td>$\deg\Phi = 0$</td><td>trivial solution</td></tr>
<tr><td>$\deg(\Phi)$=1</td><td>$g_M = 0$ (c.f. [2].) Structure theorem of L. Yang [20].</td></tr>
<tr><td>$|\deg(\Phi)| > 1$</td><td>$1 \leq g_M < |\deg(\Phi)| + 1$, structure Theorem 1.3</td></tr>
</table>

(1) Φ *is holomorphic and*

$$\Psi = \lambda^{-1}\left(\partial_{\bar{z}} \cdot \eta \otimes \partial\Phi(\partial_z) + \partial_z \cdot \eta \otimes \bar{\partial}\bar{\Phi}(\partial_{\bar{z}})\right),$$

where η is a twistor spinor on M possibly with isolated singularities, or

(2) Φ *is anti-holomorphic and*

$$\Psi = \lambda^{-1}\left(\partial_{\bar{z}} \cdot \eta \otimes \partial\bar{\Phi}(\partial_z) + \partial_z \cdot \eta \otimes \bar{\partial}\Phi(\partial_{\bar{z}})\right),$$

where η is a twistor spinor on M possibly with isolated singularities.

2. The complex form of the Dirac-harmonic map equation. Let M be a Riemann surface and N a Riemannian manifold. Let Φ be a map from M to N and Ψ a spinor along the map Φ, i.e., a cross-section of the Dirac bundle $\Sigma M \otimes \Phi^{-1}TN$. Every Riemann surface is oriented and spin. The spin structures are in one-to-one correspondence with holomorphic square roots of the canonical bundle of M ([12] or c.f. [15]). Hence, there are exactly 2^{2g} distinct spin structures on M with genus g. For the construction of spin structures, we refer to [15, 12].

Here we give a short description of the spin structure and spin bundle on M. We use standard notations and formulae from classical Riemann surface theory and spin geometry (c.f. [14, 15, 12, 11]). Choose a local conformal parameter $z = x + \sqrt{-1}y$ of M and denote the metric of M locally by $\lambda(z)\,|\mathrm{d}z|^2$. Introduce

$$\partial_z = \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right),$$

and

$$\mathrm{d}z = \mathrm{d}x + \sqrt{-1}\mathrm{d}y, \quad \mathrm{d}\bar{z} = \mathrm{d}x - \sqrt{-1}\mathrm{d}y.$$

Since $\lambda\,|\mathrm{d}z|^2 = \lambda\left(|\mathrm{d}x|^2 + |\mathrm{d}y|^2\right)$, we have

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle = \lambda,$$

whence

$$\langle \partial_z, \partial_z \rangle = \langle \partial_{\bar z}, \partial_{\bar z} \rangle = \frac{\lambda}{2}, \quad \langle \mathrm{d}z, \mathrm{d}z \rangle = \langle \mathrm{d}\bar z, \mathrm{d}\bar z \rangle = \frac{2}{\lambda}.$$

Now the Laplace operator Δ is

$$\Delta = \frac{4}{\lambda}\frac{\partial^2}{\partial z \partial \bar z}.$$

Decompose the spinor bundle as $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ with (c.f. [12])

$$\Sigma^+ M := \left\{\psi \in \Sigma M : \partial_{\bar z} \cdot \psi = 0\right\}, \quad \Sigma^- M := \left\{\psi \in \Sigma M : \partial_z \cdot \psi = 0\right\}.$$

Now we identify the spinor bundle ΣM with $L \oplus \Lambda^{0,1} L$, where L is either holomorphic square root of the canonical bundle K_M of M, such that $\Sigma^+ M = L$ and $\Sigma^- M = \Lambda^{0,1} L$. Then every half spinor in $\Sigma^+ M$ locally can be written as (c.f. [12, 11]):

$$\psi^+ = fs,$$

while every half spinor in $\Sigma^- M$ locally can be written as

$$\psi^- = g\mathrm{d}\bar z \otimes \frac{s}{|s|^2},$$

where s is a local holomorphic section of L and f, g are local complex functions. For convenience, we choose a nontrivial global holomorphic section s of L with possibly isolated singularities, and we omit the symbol s and simply write the spinor ψ^+ as

$$\psi^+ = f$$

and ψ^- as

$$\psi^- = g\mathrm{d}\bar z.$$

Here the Clifford multiplication is defined by (c.f. [12])

$$X \cdot \psi := \sqrt{2}\left((X^{1,0})^\flat \wedge \psi - \iota_{X^{0,1}} \psi \right),$$

where $(X^{1,0})^\flat$ is the dual of $X^{1,0}$ and the operator ι is the dual of the wedge operator $\wedge$, i.e.,

$$\partial_z \cdot f = \frac{\lambda}{\sqrt{2}} f\mathrm{d}\bar z, \quad \partial_{\bar z} \cdot f = 0, \quad \partial_z \cdot (g\mathrm{d}\bar z) = 0, \quad \partial_{\bar z} \cdot (g\mathrm{d}\bar z) = -\sqrt{2} g. \tag{2.1}$$

Then the Dirac operator is

$$\partial\!\!\!/ = \frac{2}{\lambda}\left(\partial_z \cdot \nabla_{\partial \bar z} + \partial_{\bar z} \cdot \nabla_{\partial z} \right),$$

where ∇ is the covariant derivative of the holomorphic line bundle L. Since s is holomorphic, i.e., $\nabla_{\partial\bar{z}} s = 0$, we have $|s|^{-2} s$ is anti-holomorphic, i.e., $\nabla_{\partial z}(|s|^{-2} s) = 0$. A direct computation yields

$$\partial\!\!\!/(f \otimes s) = \sqrt{2}\bar{\partial} f \otimes s, \quad \partial\!\!\!/(g\mathrm{d}\bar{z} \otimes |s|^{-2} s) = \sqrt{2}\bar{\partial}^*(g\mathrm{d}\bar{z}) \otimes |s|^{-2} s.$$

In other words, we have

$$\partial\!\!\!/|_{\Sigma^+ M} = \sqrt{2}\bar{\partial}, \quad \partial\!\!\!/|_{\Sigma^- M} = \sqrt{2}\bar{\partial}^*.$$

Here we omit the symbol s for convenience.

Next we consider the twistor bundle $\Sigma M \otimes \Phi^{-1}TN$ and split this bundle as

$$\Sigma M \otimes \Phi^{-1}TN = \left(\Sigma^+ M \otimes \Phi^{-1}TN\right) \bigoplus \left(\Sigma^- M \otimes \Phi^{-1}TN\right).$$

Thus every spinor Ψ along the map Φ locally has the form

$$\Psi = f^\alpha \otimes \theta_\alpha + g^\alpha \mathrm{d}\bar{z} \otimes \theta_\alpha,$$

where f^α, g^α are local sections of L respectively and θ_α are local sections of $\Phi^{-1}TN$. Recall the Euler-Lagrange equations for Dirac-harmonic maps

$$\begin{cases} \tau(\Phi) = \mathcal{R}(\Phi, \Psi) := \dfrac{1}{2}\left(\Psi^\alpha, e_i \cdot \Psi^\beta\right) R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i), \\ D\!\!\!\!/\,\Psi = 0. \end{cases}$$

Definition 2.1. We say that a Dirac-harmonic map is uncoupled if the map part is harmonic and is coupled otherwise.

First we write the curvature term $\mathcal{R}(\Phi, \Psi)$ as follows:

$$\begin{aligned}
&\mathcal{R}(\Phi, \Psi) \\
=&\frac{1}{\lambda} \operatorname{Re}\left\{\left\langle \Psi^\alpha, \partial_z \cdot \Psi^\beta \right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_z) + \left\langle \Psi^\alpha, \partial_{\bar{z}} \cdot \Psi^\beta \right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_{\bar{z}})\right\} \\
=&\frac{1}{\lambda} \operatorname{Re}\left\{\left\langle g^\alpha \mathrm{d}\bar{z}, \frac{1}{\sqrt{2}\lambda} f^\alpha \mathrm{d}\bar{z} \right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_z) + \left\langle f^\alpha, -\sqrt{2} g^\alpha \right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_{\bar{z}})\right\} \\
=&\frac{\sqrt{2}}{\lambda} \operatorname{Re}\left\{g^\alpha \overline{f^\beta} R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_z) - f^\alpha \overline{g^\beta} R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_{\bar{z}})\right\} \\
=&-\frac{2\sqrt{2}}{\lambda} \operatorname{Re}\left\{R^N(f, \bar{g})\Phi_*(\partial_{\bar{z}})\right\},
\end{aligned}$$

where we simply denote

$$\Psi = f + \mathrm{d}\bar{z} \otimes g = f^\alpha \otimes \theta_\alpha + g^\alpha \mathrm{d}\bar{z} \otimes \theta_\alpha.$$

Introduce

$$\Theta := f^\alpha \bar{g}^\beta \mathrm{d}z \otimes \theta_\alpha \wedge \theta_\beta = \mathrm{d}z \otimes f \wedge \bar{g}.$$

Then Θ is global defined, i.e., $\Theta \in \Gamma(T^*M \otimes \Phi^{-1}(TN \wedge TN))$ and

$$\mathcal{R}(\Phi, \Psi) = -\frac{2\sqrt{2}}{\lambda} \operatorname{Re}\left\{R^N(\Theta(\partial z))\Phi_*(\partial_{\bar{z}})\right\}.$$

Now the Euler-Lagrange equations for Dirac-harmonic maps are of the following form.

$$\begin{cases} \Phi^\alpha_{z\bar z} + \Gamma^\alpha_{\beta\gamma}(\Phi)\Phi^\beta_z\Phi^\gamma_{\bar z} + \frac{\sqrt{2}}{4}R^\alpha_{\beta\gamma\delta}(\Phi)\left(\Phi^\beta_{\bar z} f^\gamma \bar g^\delta - \Phi^\beta_z g^\gamma \bar f^\delta\right) = 0, \\ f^\alpha_{\bar z} + \Gamma^\alpha_{\beta\gamma}(\Phi)\Phi^\beta_{\bar z} f^\gamma = 0, \\ g^\alpha_z + \Gamma^\alpha_{\beta\gamma}(\Phi)\Phi^\beta_z g^\gamma = 0. \end{cases}$$

The following Lemma is obvious.

LEMMA 2.1. *Any Dirac-harmonic map is uncoupled if the associated* Θ *is trivial.*

Moreover, we have

LEMMA 2.2. *If* Ψ *is harmonic along the map* Φ*, i.e.,* $\not{D}\Psi = 0$*, then* Θ *is harmonic.*

Proof. Choose θ_α such that $\nabla\theta_\alpha = 0$ at a considered point. Then at that point $\Gamma^\alpha_{\beta\gamma} = 0$. Moreover, since Ψ is harmonic, we have

$$f^\alpha_{\bar z} = 0 = g^\alpha_z.S$$

Therefore the differential of Θ is

$$D\Theta := \mathrm{d}z \wedge \nabla_{\partial_z}\Theta + \mathrm{d}\bar z \wedge \nabla_{\partial_{\bar z}}\Theta = (f^\alpha \bar g^\beta)_{\bar z}(\mathrm{d}\bar z \wedge \mathrm{d}z) \otimes (\theta_\alpha \wedge \theta_\beta) = 0,$$

and the codifferential of Θ is

$$D^*\Theta := -\frac{2}{\lambda}\left(\iota_{\partial_{\bar z}}\nabla_{\partial_z}\Theta + \iota_{\partial_z}\nabla_{\partial_{\bar z}}\Theta\right) = -\frac{2}{\lambda}(f^\alpha \bar g^\beta)_{\bar z} \otimes (\theta_\alpha \wedge \theta_\beta) = 0.$$

Hence Θ is harmonic in the sense that $D\Theta = 0, D^*\Theta = 0$. □

From now on, we assume that N is also a Riemann surface and choose a local conformal parameter $\phi = u + \sqrt{-1}v$ of N and the metric of N is given by $\rho(\phi)\,|\mathrm{d}\phi|^2$. Decompose $\mathrm{d}\Phi$ as follows:

$$\mathrm{d}\Phi = \partial\Phi + \bar\partial\Phi + \partial\bar\Phi + \bar\partial\bar\Phi,$$

where

$$\partial\Phi = \phi_z \mathrm{d}z \otimes \partial_\phi, \quad \bar\partial\Phi = \phi_{\bar z}\mathrm{d}\bar z \otimes \partial_\phi,$$

and

$$\partial\bar\Phi = \bar\phi_z \mathrm{d}z \otimes \partial_{\bar\phi}, \quad \bar\partial\bar\Phi = \bar\phi_{\bar z}\mathrm{d}\bar z \otimes \partial_{\bar\phi}.$$

It is clear that $\overline{\partial\Phi} = \bar\partial\bar\Phi, \overline{\bar\partial\Phi} = \partial\bar\Phi$. Moreover

$$\|\mathrm{d}\Phi\|^2 = 2\|\partial\Phi\|^2 + 2\left\|\bar\partial\Phi\right\|^2, \quad J(\Phi) = \|\partial\Phi\|^2 - \left\|\bar\partial\Phi\right\|^2.$$

Here $J(\Phi)$ is the Jacobian of Φ. If e is a local unit tangent vector field of M, then

$$\partial\Phi = \frac{1}{4}\left(\mathrm{Id} - \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} - \sqrt{-1}J^M\right),$$
$$\bar\partial\Phi = \frac{1}{4}\left(\mathrm{Id} - \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} + \sqrt{-1}J^M\right),$$

and

$$\partial\bar{\Phi} = \frac{1}{4}\left(\mathrm{Id} + \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} - \sqrt{-1}J^M\right),$$
$$\bar{\partial}\bar{\Phi} = \frac{1}{4}\left(\mathrm{Id} + \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} + \sqrt{-1}J^M\right).$$

We get that following formulae

$$\|\partial\Phi\|^2 = \frac{1}{4}\|\mathrm{d}\Phi\|^2 + \frac{1}{2}J(\Phi), \quad \|\bar{\partial}\Phi\|^2 = \frac{1}{4}\|\mathrm{d}\Phi\|^2 - \frac{1}{2}J(\Phi),$$
$$J(\Phi) = \left\langle J^N(\mathrm{d}\Phi(e)), \mathrm{d}\Phi(J^M(e))\right\rangle.$$

In this special case, if we use the canonical isomorphism

$$TN = T_{1,0}N \oplus T_{0,1}N = K_N^{-1} \oplus \bar{K}_N^{-1},$$

where K_N is the canonical line bundle, we can split the twsitor bundle $\Sigma M \otimes \Phi^{-1}TN$ as follows:

$$\Sigma M \otimes \Phi^{-1}TN = \left(K_M^{1/2} \otimes \Phi^{-1}T_{1,0}N\right) \oplus \left(\Lambda^{0,1}K_M^{1/2} \otimes \Phi^{-1}T_{1,0}N\right)$$
$$\oplus \left(K_M^{1/2} \otimes \Phi^{-1}T_{0,1}N\right) \oplus \left(\Lambda^{0,1}K_M^{1/2} \otimes \Phi^{-1}T_{0,1}N\right)$$

and rewrite the spinor Ψ locally as follows:

$$\Psi = s \otimes f\partial_\phi + \frac{s}{|s|^2} \otimes \mathrm{d}\bar{z} \otimes g\partial_\phi + s \otimes \bar{p}\partial_{\bar{\phi}} + \frac{s}{|s|^2} \otimes \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}}.$$

Here s is a local holomorphic section of L and f, g, p, q are local complex functions. For convenience, we omit the symbol s and simply denote Ψ by

$$\Psi = f\partial_\phi + \mathrm{d}\bar{z} \otimes g\partial_\phi + \bar{p}\partial_{\bar{\phi}} + \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}}.$$

LEMMA 2.3. *Ψ is harmonic if and only if $f\partial_\phi, q\partial_\phi$ are holomorphic and $g\partial_\phi, p\partial_\phi$ are anti-holomorphic, i.e.,*

$$f_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} f = 0, \quad q_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} q = 0,$$

and

$$g_z + (\log\rho)_\phi \phi_z g = 0, \quad p_z + (\log\rho)_\phi \phi_z p = 0.$$

Proof. A direct computation. For convenience, we provide a detailed calculation. Locally, the Dirac operator $\not{D}$ has the following form

$$\not{D} = \frac{2}{\lambda}\left(\partial_z \cdot \nabla_{\partial_{\bar{z}}} + \partial_{\bar{z}} \cdot \nabla_{\partial_z}\right).$$

Denote the spinor Ψ as

$$\Psi = f\partial_\phi + \mathrm{d}\bar{z} \otimes g\partial_\phi + \bar{p}\partial_{\bar{\phi}} + \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}},$$

i.e.,

$$\Psi = s \otimes f\partial_\phi + \frac{s}{|s|^2} \otimes \mathrm{d}\bar{z} \otimes g\partial_\phi + s \otimes \bar{p}\partial_{\bar{\phi}} + \frac{s}{|s|^2} \otimes \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}}.$$

Here s is a local holomorphic section of L, i.e., $\nabla_{\partial_{\bar{z}}} s = 0$. Since $\left\langle s, |s|^{-2} s \right\rangle = 1$ and L is a line bundle, we know that $\nabla_{\partial_z}(|s|^{-2} s) = 0$, i.e., $|s|^{-2} s$ is anti-holomorphic. From the definition of Clifford multiplication (2.1),

$$\partial_z \cdot 1 = \frac{\lambda}{\sqrt{2}}\mathrm{d}\bar{z}, \quad \partial_{\bar{z}} \cdot 1 = 0, \quad \partial_z \cdot \mathrm{d}\bar{z} = 0, \quad \partial_{\bar{z}} \cdot \mathrm{d}\bar{z} = -\sqrt{2}.$$

Therefore,

$$\begin{aligned} D\!\!\!/\Psi =& \partial\!\!\!/(fs) \otimes \partial_\phi + \sqrt{2}(\log\rho)_\phi \phi_{\bar{z}} fs\mathrm{d}\bar{z} \otimes \partial_\phi \\ &+ \partial\!\!\!/(g\,|s|^{-2}\, s\mathrm{d}\bar{z}) \otimes \partial_{\bar{\phi}} - \frac{2\sqrt{2}}{\lambda}(\log\rho)_\phi \phi_z g\,|s|^{-2}\, s \otimes \partial_{\bar{\phi}} \\ &+ \partial\!\!\!/(\bar{p}s) \otimes \partial_\phi + \sqrt{2}(\log\rho)_\phi \phi_{\bar{z}} \bar{p}s\mathrm{d}\bar{z} \otimes \partial_\phi \\ &+ \partial\!\!\!/(\bar{q}\,|s|^{-2}\, s\mathrm{d}\bar{z}) \otimes \partial_{\bar{\phi}} - \frac{2\sqrt{2}}{\lambda}(\log\rho)_\phi \phi_z \bar{q}\,|s|^{-2}\, s \otimes \partial_{\bar{\phi}}. \end{aligned}$$

Here the Dirac operator $\partial\!\!\!/$ defined on the spinor bundle ΣM can be calculated as follows

$$\partial\!\!\!/|_{\Sigma^+ M} = \sqrt{2}\bar{\partial}, \quad \partial\!\!\!/|_{\Sigma^- M} = \sqrt{2}\bar{\partial}^*.$$

Since $\bar{\partial}^*(g\mathrm{d}\bar{z}) = -\dfrac{2}{\lambda} g_z$, we get

$$\begin{aligned} D\!\!\!/\Psi =& \sqrt{2}s \otimes \mathrm{d}\bar{z} \otimes \left\{ \left(f_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} f\right)\partial_\phi + \left(\bar{p}_{\bar{z}} + (\log\rho)_{\bar{\phi}} \bar{\phi}_{\bar{z}} \bar{p}\right)\partial_{\bar{\phi}} \right\} \\ &- \frac{2\sqrt{2}}{\lambda}\frac{s}{|s|^2} \otimes \left\{ \left(g_z + (\log\rho)_\phi \phi_z g\right)\partial_\phi + \left(\bar{q}_z + (\log\rho)_{\bar{\phi}} \bar{\phi}_z \bar{q}\right)\partial_{\bar{\phi}} \right\}. \end{aligned}$$

□

Set

$$\Theta := (f\bar{g} - \bar{p}q)\,\rho(\phi)\mathrm{d}z, \tag{2.2}$$

then Θ is a global defined holomorphic $(1,0)$-form on M. In fact,

Lemma 2.4. *If Ψ is harmonic, then $f\bar{g}\rho\mathrm{d}z$ and $\bar{p}q\rho\mathrm{d}z$ are both holomorphic.*

Proof. It is a consequence of Lemma 2.2. Here we give another direct proof. We only prove that $f\bar{g}\rho\mathrm{d}z$ is a holomorphic $(1,0)$-form. Applying Lemma 2.3, suppose $f\bar{g} \neq 0$, then

$$\begin{aligned} (\log f)_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} =& 0, \\ (\log\bar{g})_{\bar{z}} + (\log\bar{g})_{\bar{\phi}} \bar{\phi}_{\bar{z}} =& 0. \end{aligned}$$

Therefore,

$$(\log(f\bar{g}\rho))_{\bar{z}} = (\log f)_{\bar{z}} + (\log\bar{g})_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} + (\log\rho)_{\bar{\phi}} \bar{\phi}_{\bar{z}} = 0.$$

This equality implies that $f\bar{g}\rho$ is a local holomorphic function which means that $f\bar{g}\rho\mathrm{d}z$ is holomorphic. □

As a direct application, we give a new proof of the following result due to L. Yang [20].

THEOREM 2.5. *There is no coupled Dirac-harmonic map from the 2-sphere equipped with an arbitrary metric to any Riemann surface.*

Proof. Since there is no nontrivial holomorphic 1-form on the 2-sphere equipped with any metric, we can apply Lemma 2.4 together with Lemma 2.1 to complete the proof of this theorem. □

Now we can state the following

PROPOSITION 2.6. *In our complex notation, the Euler-Lagrange equations become*

$$\begin{cases} \phi_{z\bar{z}} + (\log\rho)_\phi \phi_z \phi_{\bar{z}} + \dfrac{\sqrt{2}}{4}\kappa^N(\phi)\left(\phi_{\bar{z}}\left(f\bar{g} - \bar{p}q\right) - \phi_z\left(\bar{f}g - p\bar{q}\right)\right)\rho = 0 \\ f_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} f = 0, \\ q_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} q = 0, \\ g_z + (\log\rho)_\phi \phi_z g = 0, \\ p_z + (\log\rho)_\phi \phi_z p = 0. \end{cases}$$

Proof. We rewrite the functional L as

$$L(\Phi,\Psi) = \frac{1}{2}\int_M \|\mathrm{d}\Phi\|^2 + \langle \not{D}\Psi,\Psi\rangle = 2\int_M \left\|\bar{\partial}\Phi\right\|^2 + \frac{1}{2}\int_M \langle \not{D}\Psi,\Psi\rangle + \int_M J(\Phi).$$

Let $\Phi = \phi + t\eta$ and fix the coefficients of Ψ, i.e.,

$$\Psi = f\partial_\Phi + \bar{p}\partial_{\bar{\Phi}} + \mathrm{d}\bar{z}\otimes g\partial_\Phi + \mathrm{d}\bar{z}\otimes \bar{q}\partial_{\bar{\Phi}}.$$

Moreover, suppose Ψ is harmonic along the map ϕ. Then

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L(\Phi,\Psi) \\ &= 2\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\int_M \left\|\bar{\partial}\phi + t\bar{\partial}\eta\right\|^2 + \frac{1}{2}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\int_M \langle \not{D}\Psi,\Psi\rangle \\ &= \frac{2}{i}\int_M \mathrm{Re}\left\{\phi_{\bar{z}}\left(\bar{\eta}_z + (\log\rho)_{\bar{\phi}}\bar{\phi}_z\bar{\eta}\right)\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar{z} + \frac{1}{2}\int_M \left\langle \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\not{D}\Psi,\Psi\right\rangle \\ &= -\frac{2}{i}\int_M \mathrm{Re}\left\{\left(\phi_{z\bar{z}} + (\log\rho)_\phi\phi_z\phi_{\bar{z}}\right)\bar{\eta}\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar{z} + \frac{1}{2}\int_M \left\langle \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\not{D}\Psi,\Psi\right\rangle. \end{aligned}$$

Since Ψ is harmonic, then $\rho f\bar{g}$ and $\rho\bar{p}q$ are holomorphic. By Lemma 2.3

$$\begin{aligned} \int_M \langle \not{D}\Psi,\Psi\rangle =& \frac{\sqrt{2}}{i}\int_M \rho\left(\left(f_{\bar{z}} + (\log\rho)_\phi\phi_{\bar{z}}f\right)\bar{g} + \left(\bar{p}_{\bar{z}} + (\log\rho)_{\bar{\phi}}\bar{\phi}_{\bar{z}}\bar{p}\right)q\right)\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ &- \frac{\sqrt{2}}{i}\int_M \rho\left(\left(g_z + (\log\rho)_\phi\phi_z g\right)\bar{f} + \left(\bar{q}_z + (\log\rho)_{\bar{\phi}}\bar{\phi}_z\bar{q}\right)p\right)\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ =& \frac{2\sqrt{2}}{i}\int_M \mathrm{Re}\left\{\left(\left(f_{\bar{z}} + (\log\rho)_\phi\phi_{\bar{z}}f\right)\bar{g} + \left(\bar{p}_{\bar{z}} + (\log\rho)_{\bar{\phi}}\bar{\phi}_{\bar{z}}\bar{p}\right)q\right)\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar{z}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_M \left\langle \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\not{D}\Psi, \Psi \right\rangle \\
=&\frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{ \left(\left((\log\rho)_{\phi\phi}\eta + (\log\rho)_{\phi\bar{\phi}}\bar{\eta} \right)\phi_{\bar{z}} + (\log\rho)_{\phi}\eta_{\bar{z}} \right) f\bar{g} \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z} \\
&+ \frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{ \left(\left((\log\rho)_{\bar{\phi}\phi}\eta + (\log\rho)_{\bar{\phi}\bar{\phi}}\bar{\eta} \right)\bar{\phi}_{\bar{z}} + (\log\rho)_{\bar{\phi}}\bar{\eta}_{\bar{z}} \right) \bar{p}q \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z} \\
=&\frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{ (\log\rho)_{\phi\bar{\phi}} \left(\phi_{\bar{z}}\bar{\eta} - \bar{\phi}_{\bar{z}}\eta \right) f\bar{g} \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z} \\
&+ \frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{ (\log\rho)_{\phi\bar{\phi}} \left(\bar{\phi}_{\bar{z}}\eta - \phi_{\bar{z}}\bar{\eta} \right) \bar{p}q \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z} \\
=&-\frac{\sqrt{2}}{2i}\int_M \rho\kappa^N \mathrm{Re}\left\{ \left(\phi_{\bar{z}} f\bar{g} - \phi_z \bar{f} g \right)\bar{\eta} \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z} \\
&- \frac{\sqrt{2}}{2i}\int_M \rho\kappa^N \mathrm{Re}\left\{ \left(\phi_z p\bar{q} - \phi_{\bar{z}}\bar{p}q \right)\bar{\eta} \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z} \\
=&-\frac{\sqrt{2}}{2i}\int_M \rho\kappa^N \mathrm{Re}\left\{ \phi_{\bar{z}} \left(f\bar{g} - \bar{p}q \right)\bar{\eta} \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z} \\
&+ \frac{\sqrt{2}}{2i}\int_M \rho\kappa^N \mathrm{Re}\left\{ \phi_z \left(\bar{f} g - p\bar{q} \right)\bar{\eta} \right\} \rho \mathrm{d}z\wedge \mathrm{d}\bar{z}.
\end{aligned}$$

The rest of the proof is obvious. □

3. Dirac-harmonic map between closed Riemann surfaces. In this section, we let M, N be closed Riemann surfaces and L be either fixed holomorphic square root of the canonical bundle of M. As mentioned before, split the twsitor bundle $\Sigma M\otimes\Phi^{-1}TN$ as follows:

$$\begin{aligned}
\Sigma M\otimes\Phi^{-1}TN = &\left(K_M^{1/2}\otimes\Phi^{-1}T_{1,0}N \right) \oplus \left(\Lambda^{0,1}K_M^{1/2}\otimes\Phi^{-1}T_{1,0}N \right) \\
&\oplus \left(K_M^{1/2}\otimes\Phi^{-1}T_{0,1}N \right) \oplus \left(\Lambda^{0,1}K_M^{1/2}\otimes\Phi^{-1}T_{0,1}N \right)
\end{aligned}$$

and rewrite the spinor Ψ locally as follows:

$$\Psi = s\otimes f\partial_\phi + \frac{s}{|s|^2}\otimes \mathrm{d}\bar{z}\otimes g\partial_\phi + s\otimes\bar{p}\partial_{\bar{\phi}} + \frac{s}{|s|^2}\otimes \mathrm{d}\bar{z}\otimes\bar{q}\partial_{\bar{\phi}}.$$

Here s is a local holomorphic section of L (i.e., harmonic spinor).

Denote

$$\begin{aligned}
h(L\otimes\Phi^{-1}T_{1,0}N) &:= \dim_{\mathbb{C}}\left\{ \Psi\in\Gamma(L\otimes\Phi^{-1}T_{1,0}N) \text{ is harmonic} \right\}, \\
h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{1,0}N) &:= \dim_{\mathbb{C}}\left\{ \Psi\in\Gamma(\Lambda^{0,1}L\otimes\Phi^{-1}T_{1,0}N) \ \text{ is harmonic} \right\}, \\
h(L\otimes\Phi^{-1}T_{0,1}N) &:= \dim_{\mathbb{C}}\left\{ \Psi\in\Gamma(L\otimes\Phi^{-1}T_{0,1}N) \text{ is harmonic} \right\}, \\
h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N) &:= \dim_{\mathbb{C}}\left\{ \Psi\in\Gamma(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N) \ \text{ is harmonic} \right\}.
\end{aligned}$$

Then we have

LEMMA 3.1. *Suppose $\Phi : M \longrightarrow N$ is a smooth map. Then we have the following isomorphisms.*

$$\begin{aligned}\Lambda^{0,1}L \otimes \Phi^{-1}T_{0,1}N &\cong (L \otimes \Phi^{-1}K_N^{-1})^*,\\ \Lambda^{0,1}L \otimes \Phi^{-1}T_{1,0}N &\cong (L \otimes \Phi^{-1}K_N)^*,\\ L \otimes \Phi^{-1}T_{0,1}N &\cong L \otimes \Phi^{-1}K_N,\end{aligned}$$

and hence

$$\begin{aligned}h(\Lambda^{0,1}L \otimes \Phi^{-1}T_{0,1}N) &= h(L \otimes \Phi^{-1}K_N^{-1}) = l(L \otimes \Phi^{-1}K_N^{-1}),\\ h(\Lambda^{0,1}L \otimes \Phi^{-1}T_{1,0}N) &= h(L \otimes \Phi^{-1}\bar{K}_N^{-1}) = l(L \otimes \Phi^{-1}K_N),\end{aligned}$$

where K_N is the canonical line bundle of N and

$$l(D) \coloneqq \dim_{\mathbb{C}} \{\, f \text{ is a meromorphic function on } M : (f) + D \geq 0\} .$$

Proof. By using the canonical isomorphism $T_{1,0}N \cong K_N^{-1}$, we note that

$$\begin{aligned}\Lambda^{0,1}L \otimes \Phi^{-1}T_{0,1}N &\cong \Lambda^{0,1}L \otimes \Phi^{-1}\bar{K}_N^{-1}\\ &\cong \bar{K}_M \otimes L \otimes \Phi^{-1}(K_N^*)^{-1}\\ &\cong K_M^* \otimes L \otimes \Phi^{-1}(K_N^{-1})^*\\ &\cong L^* \otimes L^* \otimes L \otimes \Phi^{-1}(K_N^{-1})^*\\ &\cong L^* \otimes \Phi^{-1}(K_N^{-1})^*\\ &\cong (L \otimes \Phi^{-1}K_N^{-1})^*.\end{aligned}$$

The other two isomorphisms can be obtained in a similar way and the dimension formulae follow from divisor and line bundle theory (e.g., [10, 14]). We only prove the identity

$$h(L \otimes \Phi^{-1}T_{1,0}N) = l(L \otimes \Phi^{-1}K_N^{-1}),$$

since the other identities can be proved similarly. Choose nontrivial holomorphic sections s and η with possibly isolated singularities of L and $\Phi^{-1}T_{1,0}N$ respectively. Then for every $\Psi \in \Gamma(L \otimes \Phi^{-1}T_{1,0}N)$, there is a unique complex function f on M such that

$$\Psi = fs \otimes \eta.$$

Now Ψ is harmonic if and only if f is a meromorphic function on M. The smoothness of Ψ is then equivalent to see that $(f)+(s)+(\eta) > 0$, where (f) is the divisor generated by f. Since $(s) + (\eta)$ can be viewed as a canonical divisor of $L \otimes \Phi^{-1}T_{1,0}N$, we get the desired equality. □

LEMMA 3.2. *Suppose $\Phi : M \longrightarrow N$ is a smooth map. Then*

$$\begin{aligned}\deg(L \otimes \Phi^{-1}K_N^{-1}) &= g_M - 1 - \deg(\Phi)(2g_N - 2),\\ \deg(L \otimes \Phi^{-1}K_N) &= g_M - 1 + \deg(\Phi)(2g_N - 2),\end{aligned}$$

where g_M, g_N is the genus of M, N respectively.

Proof. The first formula follows from

$$\deg(L\otimes\Phi^{-1}K_N^{-1}) = \deg(L) - \deg(\Phi)\deg(K_N) = g_M - 1 - \deg(\Phi)(2g_N - 2).$$

The last equality has used the fact that L is a holomorphic square root of K_M and $\deg(K_M) = 2g_M - 2$. The second formula can be obtained similarly. See [20] for similar results. □

By using Lemma 3.1 and Lemma 3.2, we can reprove a result of Yang [20]:

THEOREM 3.3. *Suppose* (1.2) *holds, then every Dirac-harmonic map* (Φ, Ψ) *is uncoupled.*

Proof. Under the assumption (1.2), we get either $g_M - 1 - \deg(\Phi)(2g_N - 2) < 0$ and hence

$$h(L\otimes\Phi^{-1}T_{1,0}N) = h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N) = 0,$$

or $g_M - 1 + \deg(\Phi)(2g_N - 2) < 0$ and hence

$$h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{1,0}N) = h(L\otimes\Phi^{-1}T_{0,1}N) = 0.$$

In either case, by the construction of the holomorphic $(1,0)$-form Θ (see (2.2)), we know that Θ must be trivial and Φ then is harmonic. □

COROLLARY 3.4. *Suppose* (1.3) *holds, then every Dirac-harmonic map must be a holomorphic or anti-holomorphic map coupled with a harmonic spinor along this map.*

Proof. Note that (1.2) holds if (1.3) is valid. Then using the theory for harmonic map [9, 19, 13] or Theorem A.1, we get this corollary. □

Using the Riemann-Roch formula (c.f. [14]), we have the following

PROPOSITION 3.5.

$$\begin{aligned} h(L\otimes\Phi^{-1}T_{1,0}N) - h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{1,0}N) &= -2\deg(\Phi)(g_N - 1),\\ h(L\otimes\Phi^{-1}T_{0,1}N) - h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N) &= 2\deg(\Phi)(g_N - 1). \end{aligned}$$

Proof. The Riemann Roch formula says that for every divisor D,

$$l(D) = \deg(D) - g_M + 1 + l(K_M\otimes D^{-1}).$$

Applying Lemma 3.1 and Lemma 3.2, we know that

$$\begin{aligned} h(L\otimes\Phi^{-1}T_{1,0}N) &= l(L\otimes\Phi^{-1}K_N^{-1})\\ &= \deg(L\otimes\Phi^{-1}K_N^{-1}) - g_M + 1 + l(L\otimes\Phi^{-1}K_N)\\ &= -2\deg(\Phi)(g_N - 1) + l(L\otimes\Phi^{-1}K_N)\\ &= -2\deg(\Phi)(g_N - 1) + h(\Lambda^{0,1}L\otimes\Phi^{-1}K_N^{-1}). \end{aligned}$$

The second identity can be proved similarly. □

Now we can prove the existence Theorem 1.1 for Dirac-harmonic maps.

Proof of Theorem 1.1. By using Proposition 3.5, we know that the space of harmonic spinors along the map Φ with the associated form $\Theta = 0$ is a complex linear space with complex dimension at least $4\,|\deg(\Phi)(g_N-1)|$. To see this, if $\deg(\Phi)(g_N-1)\le 0$, then by Proposition 3.5, we have $h(L\otimes\Phi^{-1}T_{1,0}N)\ge 2\,|\deg(\Phi)(g_N-1)|$ and $h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N)\ge 2\,|\deg(\Phi)(g_N-1)|$, if we choose harmonic spinors $\Psi\in\Gamma(\Sigma M\otimes\Phi^{-1}TN)$ with local form

$$\Psi = f\partial_\phi + \mathrm{d}\bar z\otimes \bar q\partial_{\bar\phi},$$

then we see that such Ψ's form a complex vector space with dimension at least $4\,|\deg(\Phi)(g_N-1)|$.

Similarly, if $\deg(\Phi)(g_N-1)>0$, then we can choose

$$\Psi = \bar p\partial_{\bar\phi} + \mathrm{d}\bar z\otimes g\partial_\phi,$$

and such harmonic spinors also form a complex vector space with dimension at least $4\,|\deg(\Phi)(g_N-1)|$. According to the definition of the associated form Θ (c.f. (2.2))

$$\Theta = (f\bar g - \bar p q)\rho\mathrm{d}z,$$

we know that $\Theta=0$ in both cases. In particular, such (Φ,Ψ)'s must be uncoupled Dirac-harmonic maps. □

Proof of Theorem 1.2. We first consider the case

$$\deg(\Phi)(g_N-1)\ge 0.$$

Then, according to Proposition 3.5, we know that

$$h(L\otimes\Phi^{-1}T_{0,1}N) = 2\deg(\Phi)(g_N-1) + h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N).$$

Lemma 3.2 together with (1.2) implies that

$$\deg(L\otimes\Phi^{-1}K_N^{-1}) = g_M - 1 - 2\deg(\Phi)(g_N-1) < 0.$$

Therefore, Lemma 3.1 implies that

$$h(L\otimes\Phi^{-1}T_{1,0}N) = h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N) = l(L\otimes\Phi^{-1}K_N^{-1}) = 0.$$

Thus,

$$h(L\otimes\Phi^{-1}T_{0,1}N) = h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{1,0}N) = 2\deg(\Phi)(g_N-1).$$

Hence the space of harmonic spinors along the map Φ is a complex linear space with dimension $4|\deg(\Phi)(g_N-1)|$.

The case of $\deg(\Phi)(g_N-1)<0$ can be handled in a similar way. To see this, we note that

$$\deg(L\otimes\Phi^{-1}K_N) = g_M - 1 + 2\deg(\Phi)(g_N-1) < 0.$$

and hence the following hold

$$h(L\otimes\Phi^{-1}T_{0,1}N) = h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{1,0}N) = l(L\otimes\Phi^{-1}K_N) = 0.$$
$$h(L\otimes\Phi^{-1}T_{1,0}N) = h(\Lambda^{0,1}L\otimes\Phi^{-1}T_{0,1}N) = -2\deg(\Phi)(g_N-1) > 0.$$

Again, the space of harmonic spinors along the map Φ is a complex linear space with dimension $4|\deg(\Phi)(g_N - 1)|$. □

PROPOSITION 3.6. *There is no nontrivial Dirac-harmonic map from the 2-sphere to the 2-torus.*

REMARK 3.1. Branding [2] proved that there is no nontrivial Dirac-harmonic map from S^2 to T^n (both equipped with standard metrics) by using the Bochner method.

Proof of Proprosition 3.6. Suppose (Φ, Ψ) is a Dirac-harmonic map from the 2-sphere to the 2-torus, then we can apply Theorem 1.2 since (1.2) holds ($g_M = 0, g_N = 1$). In particular, Ψ must be trivial. Then applying the theory of harmonic maps[9, 13, 19] or Theorem A.1, we know that Φ is holomorphic or anti-holomorphic. The Riemann-Hurwitz formula (c.f. [10, 5, 14]) says that if Φ is a non-constant (anti-)holomorphic map, then

$$|\deg(\Phi)|\,\chi(N) = \chi(M) + r, \quad r \geq 0,$$

which contradicts the assumption $\chi(M) = 2$ and $\chi(N) = 0$. As a consequence, Φ must be a constant. □

The following Proposition is a consequence of Theorem 3.3 and a result of Schoen and Yau [18] and Sampson [17].

PROPOSITION 3.7. *Let M, N be two closed Riemann surfaces of the same genus and assume that the metric of N has negative Gauss curvature. Suppose (Φ, Ψ) is a Dirac-harmonic map from M to N and $\deg(\Phi) = 1$. Then Φ is a diffeomorphism.*

Proof. By known results about harmonic maps [9, 13, 19] or Theorem A.2, we need only to prove that Φ is harmonic. That is a direct consequence of Theorem 3.3 since $g_M = g_N \geq 2$. □

In the rest of this section, we shall give a proof of Theorem 1.3. As a preparation, we provide a short remark on the existence of twistor spinors with possibly isolated singularities on closed Riemann surfaces.

As before, we also suppose M is a closed Riemann surface with genus g_M and Euler characteristic $\chi(M) = 2 - 2g_M$. Choose a local conformal parameter $z = x + \sqrt{-1}y$ and denote the metric locally by $\lambda(z)\,|dz|^2$. Let L be either fixed holomorphic square root of the canonical line bundle K_M of M. The spin bundle ΣM of M can be viewed as $L \oplus \Lambda^{0,1}L$. Every spinor ψ on M can be written locally as

$$\psi = f \otimes \frac{s}{|s|^2} + g\frac{d\bar{z}}{|d\bar{z}|^2} \otimes s, \tag{3.1}$$

where s is a local holomorphic section of L and f, g are local complex functions. Recall the Clifford multiplication

$$\partial_z \cdot 1 = \frac{\lambda}{\sqrt{2}}d\bar{z}, \quad \partial_{\bar{z}} \cdot 1 = 0, \quad \partial_z \cdot d\bar{z} = 0, \quad \partial_{\bar{z}} \cdot d\bar{z} = -\sqrt{2}.$$

Now we say that a spinor ψ is a twistor spinor if it satisfies the following twistor function on M,

$$\nabla_{\partial_z}\psi + \frac{1}{2}\partial_z \cdot \partial\!\!\!/\psi = 0, \quad \nabla_{\partial_{\bar{z}}}\psi + \frac{1}{2}\partial_{\bar{z}} \cdot \partial\!\!\!/\psi = 0, \tag{3.2}$$

or equivalently

$$\partial_z \cdot \nabla_{\partial_z}\psi = \partial_{\bar z} \cdot \nabla_{\partial_{\bar z}}\psi = 0.$$

If (3.2) holds on M except at finitely many points, we say that ψ is a *twistor spinor, possibly with isolated singularities*. Under the local expression (3.1), the twistor equation (3.2) is equivalent to

$$f_z = g_{\bar z} = 0.$$

Therefore, ψ is a twistor spinor if and only if $f \otimes |s|^{-2} s$ and $g\mathrm{d}\bar z \otimes s$ both are twistor spinors. Choose a nontrivial harmonic spinor s, possibly with isolated singularities, and a nonzero anti-meromorphic $(0,1)$-form η, i.e., $\nabla_{\partial_{\bar z}} s = 0, \nabla_{\partial_z}\eta = 0$ except some finite points. Then we can write ψ as

$$\psi = f\frac{s}{|s|^2} + g\frac{\eta}{|\eta|^2} \otimes s.$$

By using the fact $K_M = 2L$, we get

$$\begin{aligned}
\dim_{\mathbb C}\{\psi \in \Gamma(L) \text{ is a twistor spinor}\} =& \dim_{\mathbb C}\left\{\bar f \text{ is meromorphic } : (f) - (s) \ge 0\right\}\\
=& \dim_{\mathbb C}\{f \text{ is meromorphic } : (f) - (s) \ge 0\}\\
=& l(-L).\\
\dim_{\mathbb C}\left\{\psi \in \Gamma(\Lambda^{0,1}L) \text{ is a twistor spinor}\right\} =& \dim_{\mathbb C}\{g \text{ is meromorphic } : (g) - (\eta) + (s) \ge 0\}\\
=& l(-L).
\end{aligned}$$

Here

$$l(D) := \dim_{\mathbb C}\{f \text{ is meromorphic on M} : (f) + D \ge 0\}.$$

For every divisor D, according to Riemann-Roch formula, we have

$$l(-L+D) = \deg(-L+D) + 1 - g_M + l(3L-D) = \deg(D) + 2 - 2g_M + l(3L-D).$$

As a consequence, the twistor spinors on the 2-sphere form a 4-dimensional complex linear space while on the 2-torus with trivial spin structure, they form a 2-dimensional complex linear space (c.f. [12]). Also, there is no nontrivial twistor spinor on closed Riemann surfaces with higher genus ($g_M \ge 2$). However, one can claim the following

LEMMA 3.8. *Suppose* $\deg(D) + \chi(M) > 0$*, then there is a nontrivial twistor spinor that may have isolated singularities.*

Now we give a

Proof of Theorem 1.3. According to Corollary 3.4, we know that Φ is (anti-)holomorphic. Without loss of generality, assume Φ is holomorphic, then the spinor Ψ along the map Φ locally can be written as

$$\Psi = f\partial_\phi + \mathrm{d}\bar z \otimes \bar q \partial_{\bar\phi}.$$

By assumption (1.3), Φ is not constant and hence the zeros of $\partial\Phi$ are isolated.

$$\begin{aligned}
\Psi =& -\frac{\sqrt{2}f}{2\phi_z}\partial_{\bar z} \cdot \mathrm{d}\bar z \otimes \partial\Phi(\partial_z) + \frac{\sqrt{2}\bar q}{\lambda\bar\phi_{\bar z}}\partial_z \cdot 1 \otimes \bar\partial\bar\Phi(\partial_{\bar z})\\
=& \frac{2}{\lambda}\left(\partial_{\bar z} \cdot \eta \otimes \partial\Phi(\partial_z) + \partial_z \cdot \eta \otimes \bar\partial\bar\Phi(\partial_{\bar z})\right),
\end{aligned}$$

where

$$\eta = \frac{\sqrt{2}\bar{q}}{2\bar{\phi}_{\bar{z}}} - \frac{\sqrt{2}\lambda f}{4\phi_z} d\bar{z}$$

is a globally defined spinor on M with possibly isolated singularities.

Notice that

$$f_{\bar{z}} = 0 = q_{\bar{z}}$$

since $\phi_{\bar{z}} = 0$ and Ψ is harmonic.

$$\nabla_{\partial_z}\eta = -\left(\frac{\sqrt{2}\lambda f}{4\phi_z}\right)_z d\bar{z}, \quad \nabla_{\partial_{\bar{z}}}\eta = \left(\frac{\sqrt{2}\bar{q}}{2\bar{\phi}_{\bar{z}}}\right)_{\bar{z}},$$

and

$$\partial\!\!\!/\eta = \frac{2}{\lambda}\left(\left(\frac{\lambda f}{2\phi_z}\right)_z + \frac{\lambda}{2}\left(\frac{\bar{q}}{\bar{\phi}_{\bar{z}}}\right)_{\bar{z}} d\bar{z}\right).$$

Here we have used the relationship

$$\partial_z \cdot 1 = \frac{\sqrt{2}\lambda}{2} d\bar{z}, \quad \partial_{\bar{z}} \cdot d\bar{z} = -\sqrt{2}.$$

Noting that $\partial_z \cdot d\bar{z} = 0 = \partial_{\bar{z}} \cdot 1$, we get

$$\nabla_{\partial_z}\eta + \frac{1}{2}\partial_z \cdot \partial\!\!\!/\eta = 0, \quad \nabla_{\partial_{\bar{z}}}\eta + \frac{1}{2}\partial_{\bar{z}} \cdot \partial\!\!\!/\eta = 0$$

which means that η is a twistor spinor. □

Appendix A. Some known results about harmonic maps. In this appendix, we list some known results about harmonic maps between closed Riemann surfaces. All results can be found in [9, 13, 19]. Let M and N be two closed Riemann surfaces with local conformal parameter $z = x+\sqrt{-1}y$ and $\phi = u+\sqrt{-1}v$ respectively. Denote the metric of M and N locally by $\lambda(z)\,|dz|^2$ and $\rho(\phi)\,|d\phi|^2$ respectively. Then a smooth map $\Phi : M \longrightarrow N$ is harmonic if and only if

$$\phi_{z\bar{z}} + (\log\rho)_\phi \phi_z \phi_{\bar{z}} = 0.$$

Based on this equation, we get the following Bochner foumulae

$$\Delta_M \log\|\partial\Phi\| = \kappa^M - \kappa^N J(\phi),$$
$$\Delta_M \log\|\bar{\partial}\Phi\| = \kappa^M + \kappa^N J(\phi).$$

Therefore, if Φ is a harmonic map, then $\partial\Phi$ has an isolated zero set if $\partial\Phi$ is not identically zero, while $\bar{\partial}\Phi$ has an isolated zero set if $\bar{\partial}\Phi$ is not identically zero. Hence, according to these Bochner formulae, we get

THEOREM A.1 ([9]). *Suppose Φ is harmonic and*

$$g_M - 1 < |\deg(\Phi)(g_N - 1)|,$$

then Φ is either holomorphic or anti-holomorphic.

THEOREM A.2 ([18]). *Suppose Φ is harmonic, $g_M = g_N, \deg(\Phi) = 1$ and $\kappa^N \leq 0$, then Φ is a diffeomorphism.*

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