ON EXTENDING SOULÉ'S VARIANT OF BLOCH-QUILLEN IDENTIFICATION*

SEN YANG†

Abstract. Based on Balmer's tensor triangular Chow group [2], we propose (Milnor) K-theoretic Chow groups of derived categories of schemes. These Milnor K-theoretic Chow groups recover the classical ones [6] for smooth projective varieties and can detect nilpotent, while the classical ones can't do.

As an application, we extend Soulé's variant of Bloch-Quillen identification from smooth projective varieties to their trivial infinitesimal thickenings. This answers affirmatively a question by Green-Griffiths for trivial deformations, see Question 1.1 below.

Key words. Chow groups, deformation, K-theory, Bloch formula, Chern character, negative cyclic homology, derived category.

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1. Introduction. Let $f : \mathcal{X} \to S$ be a smooth projective morphism, where $S = \mathrm{Spec}(k[[t]])$ and k is a field of characteristic 0. Let $X_j = \mathcal{X} \times_S S_j$, where $S_j = \mathrm{Spec}(k[t]/t^{j+1})$. We use X to denote X_0, and call the family $\{X_j\}_j$ a deformation of X, where X_j is called the j-th infinitesimal thickening of X. In particular, the family $\{X_j\}_j$ is a trivial deformation of X and X_j is called the j-th trivial infinitesimal thickening of X, if, for each j, $X_j = X \times_k S_j$.

In [8, 9], Green-Griffiths studies infinitesimal deformations of Chow groups. Fundamental to their work is the Soulé's variant of the Bloch-Quillen identification

$$CH^q(X)_\mathbb{Q} = H^q(X, K_q^M(O_X))_\mathbb{Q},$$

where $K_q^M(O_X)$ is the Milnor K-theory sheaf associated to the presheaf

$$U \to K_q^M(O_X(U)).$$

On page 471 of [8], Green-Griffiths suggested that it would be interesting to extend Bloch-Quillen identification to infinitesimal thickening X_j:

QUESTION 1.1 ([8]). Let X denote the closed fiber X_0 and X_j denote the j-th infinitesimal thickening of X (not necessarily trivial) as above, do we have the following identification

$$CH^q(X_j)_\mathbb{Q} = H^q(X_j, K_q^M(O_{X_j}))_\mathbb{Q} ?$$

where $K_q^M(O_{X_j})$ is the Milnor K-theory sheaf associated to the presheaf

$$U \to K_q^M(O_{X_j}(U)).$$

This question inspires us to propose a new definition of Chow groups capturing the nilpotent which is useful for studying deformation problems. Our starting point is

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†Current address: School of Mathematics, Southeast University, Nanjing 210096, Jiangsu, China (101012424@seu.edu.cn). Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China.

to look at the derived category $D^{\mathrm{perf}}(X)$ obtained from the exact category of perfect complexes of O_X-modules.

In [2], Balmer introduces tensor triangular Chow groups. The new insight is to allow the coefficients of q-cycles to lie in Grothendieck groups of suitable triangulated categories. Balmer's idea is followed by S. Klein [13] and the author [20].

In Section 2, we recall Balmer's tensor triangular Chow groups and propose K-theoretic Chow groups of derived categoies of schemes by slightly modifying Balmer's. In Section 3.1, we propose Milnor K-theoretic Chow groups with an additional assumption. We compute relative K-groups with support in Section 3.2 and prove the main result in Section 3.3, which answers Question 1.1 for trivial infinitesimal thickening, i.e., for each j, $X_j = X \times_k S_j$.

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Conventions. K-theory used in this note will be Thomason-Trobaugh K-theory [17], if not stated otherwise. For any abelian group M, $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$. "Dimension" always means Krull dimension, while Balmer uses general dimension functions in [2].

2. K-theoretic Chow groups of derived categories. In this section, X is an equidimensional noetherian scheme over a field of finite Krull dimension d. As explained in [1], one can filter the tensor triangulated category $\mathcal{L} = D^{\mathrm{perf}}(X)$ by dimension of support

$$\cdots \subset \mathcal{L}_{(p)}(X) \subset \mathcal{L}_{(p+1)}(X) \subset \cdots \subset \mathcal{L},$$

where $\mathcal{L}_{(p)}(X)$ is defined to be

$$\mathcal{L}_{(p)}(X) := \{E \in D^{\mathrm{perf}}(X) \mid \mathrm{codim}_{\mathrm{Krull}}(\mathrm{supph}(E)) \geq -p\},$$

where the closed subset $\mathrm{supph}(E) \subset X$ is the support of the total homology of the perfect complex E.

Let $(\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#}$ denote the idempotent completion of the Verdier quotient $\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X)$.

THEOREM 2.1 ([1]). *Localization induces an equivalence*

$$(\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#} \simeq \bigsqcup_{x \in X^{(-p)}} D_x^{\mathrm{perf}}(X)$$

between the idempotent completion of the quotient $\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X)$ and the coproduct over $x \in X^{(-p)}$ of the derived category of perfect complexes of $O_{X,x}$-modules with homology supported on the closed point $x \in \mathrm{Spec}(O_{X,x})$.

The short sequence

$$\mathcal{L}_{(p-1)}(X) \to \mathcal{L}_{(p)}(X) \to (\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#},$$

which is exact up to summand, induces the following homotopy fibration of K-theory spectrum:

$$\mathcal{K}(\mathcal{L}_{(p-1)}(X)) \to \mathcal{K}(\mathcal{L}_{(p)}(X)) \to \mathcal{K}((\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#}).$$

As pointed out in [1], this fibration gives rise to a long exact sequence:

$$\cdots \to K_n(\mathcal{L}_{(p-1)}(X)) \to K_n(\mathcal{L}_{(p)}(X)) \xrightarrow{i} K_n((\mathcal{L}_p(X)/\mathcal{L}_{p-1}(X))^{\#}) \xrightarrow{k} K_{n-1}(\mathcal{L}_{(p-1)}(X)) \to \cdots,$$

which produces an exact couple as usual and then gives rise to the associated coniveau spectral sequence with E_1-term:

$$E_1^{p,q} = K_{-p-q}((\mathcal{L}_{(-p)}(X)/\mathcal{L}_{(-p-1)}(X))^{\#}),$$

the differential $d_{1,X}^{p,q}$ is the composition $i \circ k$ as usual

$$d_{1,X}^{p,q} : K_{-p-q}((\mathcal{L}_{(-p)}(X)/\mathcal{L}_{(-p-1)}(X))^{\#}) \xrightarrow{k} K_{-p-q-1}(\mathcal{L}_{(-p-1)}(X))$$
$$\xrightarrow{i} K_{-p-q-1}((\mathcal{L}_{(-p-1)}(X)/\mathcal{L}_{(-p-2)}(X))^{\#}).$$

Balmer's definition of tensor triangular Chow groups applies to the tensor triangulated category $\mathcal{L} = D^{\mathrm{perf}}(X)$:

DEFINITION 2.2 ([2]). Let $q \in \mathbb{Z}$, one defines K-theoretic q-cycles associated to the tensor triangulated category $\mathcal{L} = D^{\mathrm{perf}}(X)$ to be

$$Z_q(\mathcal{L}) := K_0((\mathcal{L}_{(q)}/\mathcal{L}_{(q-1)})^{\#}) = \bigoplus_{x \in X^{(-q)}} K_0(O_{X,x} \text{ on } x),$$

where K_0 is the Grothendieck K-group (the quotient of the monoid of isomorphism class $[a]$ of objects under $\oplus$, by the submonoid of those $[a] + [\sum b] + [c]$ for which there exists a distinguish triangle $a \to b \to c \to \sum a$).

A K-theoretic q-cycles can be written as $\sum_{x \in X^{(-q)}} \lambda_x \cdot \overline{\{x\}}$, for $\sum_{x \in X^{(-q)}} \lambda_x \in \bigoplus_{x \in X^{(-q)}} K_0(O_{X,x} \text{ on } x)$. Balmer's new insight is to allow coefficients λ_x to live in the Grothendieck groups, not $\mathbb{Z}$.

DEFINITION 2.3 ([2]). Let $q \in \mathbb{Z}$, we use $\mathrm{Ker}(j)$ to denote the Kernel of $K_0(\mathcal{L}_{(q)}) \xrightarrow{j} K_0(\mathcal{L}_{(q+1)})$. The K-theoretic q-boundaries $B_q(\mathcal{L})$ is defined as the image of $\mathrm{Ker}(j)$ in $Z_q(\mathcal{L})$

$$B_q(\mathcal{L}) := i \circ \mathrm{Ker}(j),$$

where $i : K_0(\mathcal{L}_{(q)}) \to K_0(\mathcal{L}_{(q)}/\mathcal{L}_{(q-1)})^{\#})(= Z_q(\mathcal{L}))$.

The K-theoretic Chow group of q-cycles in $\mathcal{L}$, denoted $CH_p(\mathcal{L})$, is defined to be the quotient of q-cycles by q-boundaries

$$CH_q(\mathcal{L}) := \frac{Z_q(\mathcal{L})}{B_q(\mathcal{L})}.$$

DEFINITION 2.4 ([1]). For X an equidimensional noetherian scheme over a field of finite Krull dimension d and for each integer q satisfying $1 \le q \le d+1$, the q-th

Gersten complex G_q is defined to be the $(-q)$-th line of E_1-page of the above coniveau spectral sequence

$$G_q : 0 \to \bigoplus_{x \in X^{(0)}} K_q(O_{X,x}) \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1(O_{X,x} \text{ on } x)$$

$$\xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}(O_{X,x} \text{ on } x) \to \cdots$$

$$\to \bigoplus_{x \in X^{(d)}} K_{q-d}(O_{X,x} \text{ on } x) \to 0.$$

The q-th augmented Gersten complex, still denoted G_q by abuse of notations, is defined to be the following complex,

$$G_q : 0 \to K_q(X) \to \bigoplus_{x \in X^{(0)}} K_q(O_{X,x}) \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1(O_{X,x} \text{ on } x)$$

$$\xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}(O_{X,x} \text{ on } x) \to \cdots$$

$$\to \bigoplus_{x \in X^{(d)}} K_{q-d}(O_{X,x} \text{ on } x) \to 0.$$

THEOREM 2.5. *Balmer's K-theoretic $(-q)$-boundaries $B_{-q}(\mathcal{L})$ for $\mathcal{L} = D^{\mathrm{perf}}(X)$ agrees with the image of the differential $d_{1,X}^{q-1,-q}$:*

$$B_{-q}(\mathcal{L}) = \mathrm{Im}(d_{1,X}^{q-1,-q}).$$

Proof. The long exact sequence

$$\cdots \to K_1((\mathcal{L}_{(-q+1)}/\mathcal{L}_{(-q)})^{\#}) \xrightarrow{k} K_0(\mathcal{L}_{(-q)}) \xrightarrow{j} K_0(\mathcal{L}_{(-q+1)}) \to \cdots$$

shows that $\mathrm{Ker}(j) = \mathrm{Im}(k)$. So $B_{-q}(\mathcal{L}) = i \circ \mathrm{Ker}(j) = i \circ \mathrm{Im}(k)$, where $i : K_0(\mathcal{L}_{(-q)}) \to K_0((\mathcal{L}_{(-q)}/\mathcal{L}_{(-q-1)})^{\#})$.

The conclusion follows, since the differential $d_{1,X}^{q-1,-q}$ is the composition $d = i \circ k$

$$d_{1,X}^{q-1,-q} : K_1((\mathcal{L}_{(-q+1)}/\mathcal{L}_{(-q)})^{\#}) \xrightarrow{k} K_0(\mathcal{L}_{(-q)}) \xrightarrow{i} K_0((\mathcal{L}_{(-q)}/\mathcal{L}_{(-q-1)})^{\#}).$$

□

As pointed out in [1], taking idempotent completion can result in the appearance of negative K-groups. In order to include this important information into our study, we propose our K-theoretic Chow groups of $D^{\mathrm{perf}}(X)$ by slightly modifying Balmer's as follows:

DEFINITION 2.6. The K-theoretic q-cycles and K-theoretic rational equivalence of (X, O_X), denoted $Z_q(D^{\mathrm{perf}}(X))$ and $Z_{q,\mathrm{rat}}(D^{\mathrm{perf}}(X))$ respectively, are defined to be

$$Z_q(D^{\mathrm{perf}}(X)) := \mathrm{Ker}(d_{1,X}^{q,-q}), \, Z_{q,\mathrm{rat}}(D^{\mathrm{perf}}(X)) := \mathrm{Im}(d_{1,X}^{q-1,-q}).$$

The q-th K-theoretic Chow group of (X, O_X), denoted by $CH_q(D^{\mathrm{perf}}(X))$, is defined to be

$$CH_q(D^{\mathrm{perf}}(X)) := \frac{Z_q(D^{\mathrm{perf}}(X))}{Z_{q,\mathrm{rat}}(D^{\mathrm{perf}}(X))}.$$

It is clear that our K-theoretic Chow groups are cohomology groups of the Gersten complex and that these K-theoretic Chow groups are subgroups of Balmer's:

COROLLARY 2.7. *Let $\mathcal{L} = D^{\mathrm{perf}}(X)$, we have the following:*

$$Z_q(D^{\mathrm{perf}}(X)) \subseteq Z_{-q}(\mathcal{L}),$$

$$Z_{q,\mathrm{rat}}(D^{\mathrm{perf}}(X)) = B_{-q}(\mathcal{L}),$$

$$CH_q(D^{\mathrm{perf}}(X)) \subseteq CH_{-q}(\mathcal{L}).$$

3. Milnor K-theoretic Chow groups of derived categories.

3.1. Definition. Let X be an equidimensional noetherian scheme of finite Krull dimension d over a field, and Y be a closed subset of X, by abuse of notations, we use $K_m^Q(X \text{ on } Y)$ to denote Quillen K-groups with supports. In [16], Soulé showed that there exists Adams operations ψ^k acting on Quillen K-groups with supports $K_m^Q(X \text{ on } Y)$, $m \geq 0$. According to Weibel [18], we can extend Adams operations ψ^k to negative range by descending induction.

For every integer $m \geq 0$, we have the following Bass fundamental exact sequence:

$$0 \to K_m^Q(X \text{ on } Y) \to K_m^Q(X[t] \text{ on } Y[t]) \oplus K_m^Q(X[t^{-1}] \text{ on } Y[t^{-1}])$$
$$\to K_m^Q(X[t, t^{-1}] \text{ on } Y[t, t^{-1}]) \to K_{m-1}^Q(X \text{ on } Y) \to 0.$$

For any $x \in K_{-1}^Q(X \text{ on } Y)$, we have $x \cdot t \in K_0^Q(X[t, t^{-1}] \text{ on } Y[t, t^{-1}])$, where $t \in K_1(k[t, t^{-1}])$. We have

$$\psi^k(x \cdot t) = \psi^k(x)\psi^k(t) = \psi^k(x)k \cdot t.$$

Tensoring with $\mathbb{Q}$, we have obtained Adams operations ψ^k on $K_{-1}^Q(X \text{ on } Y)$:

$$\psi^k(x) = \frac{\psi^k(x \cdot t)}{k \cdot t}.$$

Continuing this procedure, we obtain Adams operations on $K_m^Q(X \text{ on } Y)$, where $m \in \mathbb{Z}$.

We note that the Thomason-Trobaugh K-group with support $K_m(O_{X,x} \text{ on } x)$, appearing in Definition 2.4, is isomorphic to Quillen K-group with support $K_m^Q(O_{X,x} \text{ on } x)$. So Adams operations ψ^k exist on $K_m(O_{X,x} \text{ on } x)$. Keeping Soulé's variant of Bloch's formula in mind, we would like to define Milnor K-theoretic Chow groups of derived categories of schemes. However, we don't have Milnor K-group with support $K_m^M(O_{X,x} \text{ on } x)$ directly. The following theorem of Soulé [16] suggests that we can use suitable eigenspace of Adams operations to rationally replace Milnor K-group.

THEOREM 3.1 (Theorem 5 in page 526 of [16]). *Let X be a regular scheme of finite type over a field, with Krull dimension d, $X^{(p)}$ the set of points of X of codimension p. For $x \in X^{(p)}$, $k(x)$ is the residue field. There exists the following isomorphism*

$$K_m^M(k(x)) \xrightarrow{\cong} K_m^{(m)}(k(x)) \text{ modulo torsion,}$$

where $K_m^{(m)}$ is the eigenspace of $\psi^k = k^m$ and ψ^k is Adams operations.

Following Soulé's theorem, we define Milnor K-group with support $K_m^M(O_{X,x} \text{ on } x)$ to be suitable eigenspace of $K_m(O_{X,x} \text{ on } x)$:

DEFINITION 3.2. Let X be a d-equidimensional noetherian scheme and $x \in X^{(j)}$. For $m \in \mathbb{Z}$, Milnor K-group with support $K_m^M(O_{X,x} \text{ on } x)$ is rationally defined to be

$$K_m^M(O_{X,x} \text{ on } x) := K_m^{(m+j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}},$$

where $K_m^{(m+j)}$ is the eigenspace of $\psi^k = k^{m+j}$.

The reason why we choose $K_m^{(m+j)}$ to define K_m^M is inspired by another result of Soulé.

Let X be a regular scheme of finite type over a field, with Krull dimension d. For $x \in X^{(p)}$, $k(x)$ denotes the residue field. Let $f : \text{Spec}(k(x)) \to \text{Spec}(O_{X,x})$ denote the closed immersion. It is known that

$$f_* : K_m(k(x)) \xrightarrow{\cong} K_m(O_{X,x} \text{ on } x).$$

LEMMA 3.3 ([16]). *Let X be a regular scheme of finite type over a field, with Krull dimension d. For $x \in X^{(p)}$, let $f : \text{Spec}(k(x)) \to \text{Spec}(O_{X,x})$ be the closed immersion. We have (for any integer m and i)*

$$f_* : K_m^{(i)}(k(x)) \xrightarrow{\cong} K_m^{(p+i)}(O_{X,x} \text{ on } x), \text{ modulo torsion.}$$

Proof. We sketch the proof very briefly as follows. Given $a \in K_m^{(i)}(k(x))$, one has

$$\psi^k(f_*(a)) = f_*(\psi^k(p,a)) = f_*(k^p \psi^k(a)) = f_*(k^p k^i a) = k^{p+i} f_*(a),$$

The first identity is from Theorem 3 of [16](page 517), the second one uses the formula $\psi^k(p,a) = k^p \psi^k(a)$, see line 29 of page 522 of [16]. Hence, $f_*(a) \in K_m^{(p+i)}(O_{X,x} \text{ on } x)$.

See line 16- 30 of page 522 and line 8-11 of page 527 of [16] for related discussion if necessary. ❑

This lemma says that our definition of Milnor K-group with support is a honest generalization of the classical one, at least for regular case.

Next, we would like to define Milnor K-theoretic Chow groups of derived categories of schemes by mimicking Definition 2.6. In order to do that, we need to detect whether the differentials of the Gersten complex respect Adams operations.

If the differentials $d_{1,X}^{p,-q}$ of the Gersten complex in Definition 2.4 respect Adams' operations, for every $i \in \mathbb{Z}$, then there exists the following finer complex

$$0 \to \bigoplus_{x \in X^{(0)}} K_q^{(i)}(O_{X,x}) \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1^{(i)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(i)}} K_0^{(i)}(O_{X,x} \text{ on } x)$$

$$\xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}^{(i)}(O_{X,x} \text{ on } x) \to \cdots \to \bigoplus_{x \in X^{(d)}} K_{q-d}^{(i)}(O_{X,x} \text{ on } x) \to 0.$$

In particular, we obtain the following refiner complex by taking $i = q$:

$$0 \to \bigoplus_{x \in X^{(0)}} K_q^{(q)}(O_{X,x}) \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1^{(q)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^{(q)}(O_{X,x} \text{ on } x)$$

$$\xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}^{(q)}(O_{X,x} \text{ on } x) \to \cdots \to \bigoplus_{x \in X^{(d)}} K_{q-d}^{(q)}(O_{X,x} \text{ on } x) \to 0.$$

Tensoring each term with $\mathbb{Q}$, this complex can be written as

$$0 \to \bigoplus_{x \in X^{(0)}} K_q^M(O_{X,x}) \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1^M(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^M(O_{X,x} \text{ on } x)$$

$$\tag{3.1}$$

$$\xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}^M(O_{X,x} \text{ on } x) \to \ldots \cdots \to \bigoplus_{x \in X^{(d)}} K_{q-d}^M(O_{X,x} \text{ on } x) \to 0.$$

DEFINITION 3.4. For X an equidimensional noetherian scheme over a field of finite Krull dimension d, **if** the differentials $d_{1,X}^{p,-q}$ of the Gersten complex respect Adams operations, then the Milnor K-theoretic q-cycles and Milnor K-theoretic rational equivalence of (X, O_X), denoted $Z_q^M(D^{\mathrm{perf}}(X))$ and $Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X))$ respectively, are defined to be

$$Z_q^M(D^{\mathrm{perf}}(X)) := \mathrm{Ker}(d_{1,X}^{q,-q}), \, Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X)) := \mathrm{Im}(d_{1,X}^{q-1,-q}),$$

where $d_{1,X}^{q-1,-q}$ and $d_{1,X}^{q,-q}$ are the differentials of the complex (3.1).

The q-th Milnor K-theoretic Chow group of (X, O_X), denoted by $CH_q^M(D^{\mathrm{perf}}(X))$, is defined to be

$$CH_q^M(D^{\mathrm{perf}}(X)) := \frac{Z_q^M(D^{\mathrm{perf}}(X))}{Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X))}.$$

In the following, we show that this definition makes sense for smooth projective varieties and their trivial infinitesimal thickenings.

3.2. Goodwillie isomorphism. Let's recall that in [7] Goodwillie shows the relative Chern character is an isomorphism between the relative K-group $K_n(A, I)$ and negative cyclic homology $HN_n(A, I)$, where A is a commutative $\mathbb{Q}$-algebra and I is a nilpotent ideal in A. This is further studied by Cathelineau.

THEOREM 3.5 ([3, 7]). *Let I be a nilpotent ideal in a commutative $\mathbb{Q}$-algebra A, the relative Chern character induces an isomorphism between the relative K-group $K_n(A, I)$ and negative cyclic homology $HN_n(A, I)$:*

$$\mathrm{Ch} : K_n(A, I)_{\mathbb{Q}} \xrightarrow{\cong} HN_n(A, I).$$

Furthermore, the relative Chern character respects Adams operations. That is,

$$\mathrm{Ch} : K_n^{(i)}(A, I)_{\mathbb{Q}} \xrightarrow{\cong} HN_n^{(i)}(A, I),$$

here $K_n^{(i)}$ and $HN_n^{(i)}$ are respective eigenspaces of $\psi^k = k^i$ and $\psi^k = k^{i+1}$.

In the Appendix B of [4], Cortiñas-Haesemeyer-Weibel shows a space version of Goodwillie isomorphism. For every nilpotent sheaf of ideal I, they define $K(O, I)$ and $HN(O, I)$ as the following presheaves respectively:

$$U \to K(O(U), I(U)), U \to HN(O(U), I(U))$$

They write $\mathcal{K}(O, I)$ and $\mathcal{HN}(O, I)$ for the presheaves of spectrum whose initial spaces are $K(O, I)$ and $HN(O, I)$ respectively. Moreover, they define $\mathcal{K}^{(i)}(O, I)$ as the homotopy fiber of $\mathcal{K}(O, I)$ on which $\psi^k - k^i$ acts acyclicly. $\mathcal{HN}^{(i)}(O, I)$ is defined similarly. Goodwillie and Cathelineau's isomorphisms can be generalized in the following way.

THEOREM 3.6 ([4]). *The relative Chern character induces homotopy equivalence of spectra:*

$$\begin{cases} \text{Ch} : \mathcal{K}(O, I) \simeq \mathcal{HN}(O, I), \\ \text{Ch} : \mathcal{K}^{(i)}(O, I) \simeq \mathcal{HN}^{(i)}(O, I). \end{cases} \tag{3.2}$$

Now, let X be a noetherian scheme of finite type over a field k of characteristic 0. Let Y be a closed subset in X and $U = X - Y$ be the open complement. Let ε be a nilpotent of order n: $\varepsilon^n = 0$.

Let $\mathbb{H}(X, \bullet)$ denote Thomason's hypercohomology of spectra. We have the following commutative diagrams(each column and row is a homotopy fibration):

$$\begin{array}{ccccc}
\mathbb{H}_Y(X, \mathcal{K}(O, \varepsilon)) & \longrightarrow & \mathbb{H}(X, \mathcal{K}(O, \varepsilon)) & \longrightarrow & \mathbb{H}(U, \mathcal{K}(O, \varepsilon)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X, \mathcal{K}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(X, \mathcal{K}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(U, \mathcal{K}(O_U[\varepsilon])) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X, \mathcal{K}(O_X)) & \longrightarrow & \mathbb{H}(X, \mathcal{K}(O_X)) & \longrightarrow & \mathbb{H}(U, \mathcal{K}(O_U))
\end{array}$$

and

$$\begin{array}{ccccc}
\mathbb{H}_Y(X, \mathcal{HN}(O, \varepsilon)) & \longrightarrow & \mathbb{H}(X, \mathcal{HN}(O, \varepsilon)) & \longrightarrow & \mathbb{H}(U, \mathcal{HN}(O, \varepsilon)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X, \mathcal{HN}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(X, \mathcal{HN}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(U, \mathcal{HN}(O_U[\varepsilon])) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X, \mathcal{HN}(O_X)) & \longrightarrow & \mathbb{H}(X, \mathcal{HN}(O_X)) & \longrightarrow & \mathbb{H}(U, \mathcal{HN}(O_U)).
\end{array}$$

Since both $\mathcal{K}$ and $\mathcal{HN}$ satisfy Zariski descent, they satisfy

$$\begin{cases} \mathbb{H}_Y(X, \mathcal{K}(O_X)) = \mathcal{K}(X \text{ on } Y), \ \mathbb{H}_Y(X, \mathcal{HN}(O_X)) = \mathcal{HN}(X \text{ on } Y); \\ \mathbb{H}_Y(X, \mathcal{K}(O_X[\varepsilon])) = \mathcal{K}(X[\varepsilon] \text{ on } Y), \ \mathbb{H}_Y(X, \mathcal{HN}(O_X[\varepsilon])) = \mathcal{HN}(X[\varepsilon] \text{ on } Y). \end{cases}$$

The above diagrams show the following result

COROLLARY 3.7. $\mathbb{H}_Y(X, \mathcal{K}(O, \varepsilon))$ *and* $\mathbb{H}_Y(X, \mathcal{HN}(O, \varepsilon))$ *are the homotpy fibres of, respectively*

$$\mathbb{H}_Y(X, \mathcal{K}(O_X[\varepsilon])) \to \mathbb{H}_Y(X, \mathcal{K}(O_X)), \mathbb{H}_Y(X, \mathcal{HN}(O_X[\varepsilon])) \to \mathbb{H}_Y(X, \mathcal{HN}(O_X)).$$

Let $K_n(X[\varepsilon]$ on $Y[\varepsilon], \varepsilon)$ and $HN_n(X[\varepsilon]$ on $Y[\varepsilon], \varepsilon)$denote the relative groups with support, that is, the kernel of the natural projections:

$$K_n(X[\varepsilon] \text{ on } Y[\varepsilon]) \xrightarrow{\varepsilon=0} K_n(X \text{ on } Y), HN_n(X[\varepsilon] \text{ on } Y[\varepsilon]) \xrightarrow{\varepsilon=0} HN_n(X \text{ on } Y).$$

In other words, $K_n(X[\varepsilon]$ on $Y[\varepsilon], \varepsilon) = \mathbb{H}_Y^{-n}(X, \mathcal{K}(O, \varepsilon))$ and $HN_n(X[\varepsilon]$ on $Y[\varepsilon], \varepsilon) = \mathbb{H}_Y^{-n}(X, \mathcal{HN}(O, \varepsilon))$. So we have the following corollary of Theorem 3.6:

COROLLARY 3.8. *Let X be a noetherian scheme of finite type over a field k of characteristic 0. Let Y be a closed subset in X and let ε be a nilpotent of order n: $\varepsilon^n = 0$. We have*

$$K_n(X[\varepsilon] \text{ on } Y[\varepsilon], \varepsilon)_{\mathbb{Q}} = HN_n(X[\varepsilon] \text{ on } Y[\varepsilon], \varepsilon)_{\mathbb{Q}}.$$

According to the Appendix B of [4], there exists the following two splitting fibrations

$$\begin{cases} \mathcal{K}^{(i)}(O, \varepsilon) \to \mathcal{K}(O, \varepsilon) \to \prod_{j \neq i} \mathcal{K}^{(j)}(O, \varepsilon), \\ \mathcal{HN}^{(i)}(O, \varepsilon) \to \mathcal{HN}(O, \varepsilon) \to \prod_{j \neq i} \mathcal{HN}^{(j)}(O, \varepsilon). \end{cases} \tag{3.3}$$

Sine taking $\mathbb{H}_Y(X, -)$ preserves homotopy fibrations, there exists the following two splitting fibrations:

$$\begin{cases} \mathbb{H}_Y(X, \mathcal{K}^{(i)}(O, \varepsilon)) \to \mathbb{H}_Y(X, \mathcal{K}(O, \varepsilon)) \xrightarrow{\psi^k - k^i} \mathbb{H}_Y(X, \prod_{j \neq i} \mathcal{K}^{(j)}(O, \varepsilon)), \\ \mathbb{H}_Y(X, \mathcal{HN}^{(i)}(O, \varepsilon)) \to \mathbb{H}_Y(X, \mathcal{HN}(O, \varepsilon)) \xrightarrow{\psi^k - k^{i+1}} \mathbb{H}_Y(X, \prod_{j \neq i} \mathcal{HN}^{(j)}(O, \varepsilon)). \end{cases} \tag{3.4}$$

Passing to group level, we obtain the following results:

$$\begin{cases} \mathbb{H}_Y^{-n}(X, \mathcal{K}^{(i)}(O, \varepsilon))_{\mathbb{Q}} = \{x \in \mathbb{H}_Y^{-n}(X, \mathcal{K}(O, \varepsilon)) | \psi^k(x) - k^i x = 0\}, \\ \mathbb{H}_Y^{-n}(X, \mathcal{HN}^{(i)}(O, \varepsilon))_{\mathbb{Q}} = \{x \in \mathbb{H}_Y^{-n}(X, \mathcal{HN}(O, \varepsilon)) | \psi^k(x) - k^{i+1} x = 0\}. \end{cases} \tag{3.5}$$

Let $K_n^{(i)}(X[\varepsilon]$ on $Y[\varepsilon], \varepsilon)_{\mathbb{Q}}$ and $HN_n^{(i)}(X[\varepsilon]$ on $Y[\varepsilon], \varepsilon)_{\mathbb{Q}}$ denote the weight i eigenspaces of relative groups with support, that is, the kernel of the natural projections:

$$K_n^{(i)}(X[\varepsilon] \text{ on } Y[\varepsilon])_{\mathbb{Q}} \xrightarrow{\varepsilon=0} K_n^{(i)}(X \text{ on } Y)_{\mathbb{Q}}, HN_n^{(i)}(X[\varepsilon] \text{ on } Y[\varepsilon])_{\mathbb{Q}} \xrightarrow{\varepsilon=0} HN_n^{(i)}(X \text{ on } Y)_{\mathbb{Q}}.$$

Then we have the following identifications:

$$\begin{cases} \mathbb{H}_Y^{-n}(X, \mathcal{K}^{(i)}(O, \varepsilon))_{\mathbb{Q}} = K_n^{(i)}(X[\varepsilon] \text{ on } Y[\varepsilon], \varepsilon)_{\mathbb{Q}}, \\ \mathbb{H}_Y^{-n}(X, \mathcal{HN}^{(i)}(O, \varepsilon))_{\mathbb{Q}} = HN_n^{(i)}(X[\varepsilon] \text{ on } Y[\varepsilon], \varepsilon)_{\mathbb{Q}}. \end{cases} \tag{3.6}$$

Therefore, the homotopy equivalences

$$\mathcal{K}(O, \varepsilon) \simeq \mathcal{HN}(O, \varepsilon), \mathcal{K}^{(i)}(O, \varepsilon) \simeq \mathcal{HN}^{(i)}(O, \varepsilon)$$

give us the following finer result:

THEOREM 3.9. *Let X be a noetherian scheme of finite type over a field k of characteristic 0. Let Y be a closed subset in X and let ε be a nilpotent of order n: $\varepsilon^n = 0$, we have the following identification:*

$$K_n^{(i)}(X[\varepsilon] \text{ on } Y[\varepsilon], \varepsilon)_{\mathbb{Q}} = HN_n^{(i)}(X[\varepsilon] \text{ on } Y[\varepsilon], \varepsilon)_{\mathbb{Q}}.$$

Let R be a regular noetherian domain, which is also a commutative $\mathbb{Q}$-algebra, and let ε be a nilpotent of order n: $\varepsilon^n = 0$. We consider $R[\varepsilon]$ as a graded $\mathbb{Q}$-algebra, it is known that the relative negative cyclic homology $HN_m^{(i)}(R[\varepsilon], \varepsilon)$ can be identified with absolute differentials as follows:

$$\begin{cases} HN_m^{(i)}(R[\varepsilon], \varepsilon) & = (\Omega_{R/\mathbb{Q}}^{2i-m-1})^{\oplus n-1}, \text{for } \frac{m}{2} < i \le m; \\ HN_m^{(i)}(R[\varepsilon], \varepsilon) & = 0, \text{else}. \end{cases} \tag{3.7}$$

$$HN_m(R[\varepsilon], \varepsilon) = (\Omega_{R/\mathbb{Q}}^{m-1} \oplus \Omega_{R/\mathbb{Q}}^{m-3} \oplus \dots)^{\oplus n-1}, \tag{3.8}$$

the last term in the direct sum is $\Omega_{R/\mathbb{Q}}^1$ or R, depending on m even or odd. E.g., Geller-Weibel [10] computed the eigenspace of K-groups of truncated polynomials, see Application 5.6 on page 29. See also the survey by Hesselholt [12].

Now, we explicitly compute the relative negative cyclic groups with support.

THEOREM 3.10. *Let X be a smooth projective variety over a field k of characteristic 0 and $y \in X^{(p)}$. Let ε be a nilpotent of order n: $\varepsilon^n = 0$. For any integer m, we have*

$$\begin{cases} HN_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \cong H_y^p(\Omega_{X/\mathbb{Q}}^{\bullet}), \\ HN_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \cong H_y^p(\Omega_{X/\mathbb{Q}}^{\bullet,(i)}), \end{cases} \tag{3.9}$$

where $(\Omega_{X/\mathbb{Q}}^{\bullet} = \Omega_{X/\mathbb{Q}}^{m+p-1} \oplus \Omega_{X/\mathbb{Q}}^{m+p-3} \oplus \dots)^{\oplus n-1}$ *and*

$$\begin{cases} \Omega_{X/\mathbb{Q}}^{\bullet,(i)} & = (\Omega_{X/\mathbb{Q}}^{2i-(m+p)-1})^{\oplus n-1}, \text{for } \frac{m+p}{2} < i \le m+p, \\ \Omega_{X/\mathbb{Q}}^{\bullet,(i)} & = 0, \text{else}. \end{cases} \tag{3.10}$$

Proof. $O_{X,y}$ is a regular local ring with dimension p, so the depth of $O_{X,y}$ is p. For each $n \in \mathbb{Z}$, $\Omega_{O_{X,y}/\mathbb{Q}}^n$ can be written as a direct limit of direct sum of $O_{X,y}$'s (as $O_{X,y}$-module), though $\Omega_{O_{X,y}/\mathbb{Q}}^n$ is not of finite type. Therefore, $\Omega_{O_{X,y}/\mathbb{Q}}^n$ has depth p.

$HN_m(O_{X,y}[\varepsilon]$ on $y[\varepsilon], \varepsilon)$ can be identified with the hypercohomology $\mathbb{H}_y^{-m}(O_{X,y}, HN(O_{X,y}[\varepsilon], \varepsilon))$, where $HN(O_{X,y}[\varepsilon], \varepsilon)$ is the relative negative cyclic complex, that is, the kernel of

$$HN(O_{X,y}[\varepsilon]) \xrightarrow{\varepsilon=0} HN(O_{X,y}).$$

There is a spectral sequence :

$$H_y^{-m-q}(O_{X,y}, H^q(HN(O_{X,y}[\varepsilon], \varepsilon))) \Longrightarrow \mathbb{H}_y^{-m}(HN(O_{X,y}[\varepsilon], \varepsilon)).$$

By Formula (3.8) above, we have

$$H^q(HN(O_{X,y}[\varepsilon], \varepsilon)) = HN_{-q}(O_{X,y}[\varepsilon], \varepsilon) = (\Omega_{O_{X,y}/\mathbb{Q}}^{-q-1} \oplus \Omega_{O_{X,y}/\mathbb{Q}}^{-q-3} \oplus \dots)^{\oplus n-1}.$$

As each $\Omega_{O_{X,y}/\mathbb{Q}}^n$ has depth p, only $H_y^p(X, H^q(HN(O_{X,y}[\varepsilon], \varepsilon)))$ can survive because of the depth condition. This means $q = -m - p$ and

$$H^{-m-p}(HN(O_{X,y}[\varepsilon], \varepsilon)) = HN_{m+p}(O_{X,y}[\varepsilon], \varepsilon) = (\Omega_{O_{X,y}/\mathbb{Q}}^{m+p-1} \oplus \Omega_{O_{X,y}/\mathbb{Q}}^{m+p-3} \oplus \dots)^{\oplus n-1}.$$

Let's write

$$\Omega^{\bullet}_{X/\mathbb{Q}} = (\Omega^{m+p-1}_{X/\mathbb{Q}} \oplus \Omega^{m+p-3}_{X/\mathbb{Q}} \oplus \dots)^{\oplus n-1},$$

thus

$$\mathbb{H}^{-m}_y(HN(O_{X,y}[\varepsilon], \varepsilon)) = H^p_y(\Omega^{\bullet}_{X/\mathbb{Q}}),$$

this means

$$HN_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) = H^p_y(\Omega^{\bullet}_{X/\mathbb{Q}}).$$

The proof for $HN^{(i)}_m$ works similarly. $\square$

COROLLARY 3.11. *Under the same assumption as above, relative Chern character induces the following isomorphisms between relative K-groups and local cohomology groups:*

$$\begin{cases} K_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}} \cong H^p_y(\Omega^{\bullet}_{X/\mathbb{Q}}), \\ K^{(i)}_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}} \cong H^p_y(\Omega^{\bullet,(i)}_{X/\mathbb{Q}}), \end{cases} \tag{3.11}$$

where $\Omega^{\bullet}_{X/\mathbb{Q}} = (\Omega^{m+p-1}_{X/\mathbb{Q}} \oplus \Omega^{m+p-3}_{X/\mathbb{Q}} \oplus \dots)^{\oplus n-1}$; and

$$\begin{cases} \Omega^{\bullet,(i)}_{X/\mathbb{Q}} &= (\Omega^{2i-(m+p)-1}_{X/\mathbb{Q}})^{\oplus n-1}, \text{for } \frac{m+p}{2} < i \leq m+p; \\ \Omega^{\bullet,(i)}_{X/\mathbb{Q}} &= 0, \text{else}. \end{cases} \tag{3.12}$$

3.3. Bloch's formula. In this subsection, X is a smooth projective variety over a field k of characteristic 0 and X_j is the j-th trivial infinitesimal thickening of X, that is, $X_j = X \times_k S_j$, where $S_j = \mathrm{Spec}(k[t]/(t^{j+1}))$. The aim of this subsection is to extend Bloch's formula from X to its trivial infinitesimal thickening X_j.

Let $i_j : X \to X_{j+1}$ denote the closed immersion and $p_j : X_j \to X$ denote the natural projection. We have $p_j \circ i_j = \mathrm{Identity}$. The morphism i^*_j induces pull-back between K-theory spectra: $i^*_j : \mathcal{K}(X_j) \to \mathcal{K}(X)$. Furthermore it induces maps between coniveau spectral sequences, recalled in Section 2:

$$i^*_j : E^{p,q}_1(X_j) \to E^{p,q}_1(X).$$

Since the pull-back of p_j:

$$p^*_j : E^{p,q}_1(X) \to E^{p,q}_1(X_j),$$

splits

$$i^*_j : E^{p,q}_1(X_j) \to E^{p,q}_1(X),$$

this gives us the following split commutative diagram:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\overline{K}_q(k(X)[t]/(t^{j+1})) & \longleftarrow & K_q(k(X)[t]/(t^{j+1})) & \xrightarrow{\;i_j^*\;} & K_q(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\displaystyle\bigoplus_{x \in X^{(1)}} \overline{K}_{q-1}(O_{X_j,x_j}\text{ on }x_j) & \longleftarrow & \displaystyle\bigoplus_{x_j \in X_j^{(1)}} K_{q-1}(O_{X_j,x_j}\text{ on }x_j) & \longrightarrow & \displaystyle\bigoplus_{x \in X^{(1)}} K_{q-1}(O_{X,x}\text{ on }x) \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \longleftarrow & \cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\displaystyle\bigoplus_{x \in X^{(q-1)}} \overline{K}_1(O_{X_j,x_j}\text{ on }x_j) & \longleftarrow & \displaystyle\bigoplus_{x_j \in X_j^{(q-1)}} K_1(O_{X_j,x_j}\text{ on }x_j) & \longrightarrow & \displaystyle\bigoplus_{x \in X^{(q-1)}} K_1(O_{X,x}\text{ on }x) \\
\overline{d}_{1,X_j}^{q-1,-q}\downarrow & & d_{1,X_j}^{q-1,-q}\downarrow & & d_{1,X}^{q-1,-q}\downarrow \\
\displaystyle\bigoplus_{x \in X^{(q)}} \overline{K}_0(O_{X_j,x_j}\text{ on }x_j) & \longleftarrow & \displaystyle\bigoplus_{x_j \in X_j^{(q)}} K_0(O_{X_j,x_j}\text{ on }x_j) & \longrightarrow & \displaystyle\bigoplus_{x \in X^{(q)}} K_0(O_{X,x}\text{ on }x) \\
\overline{d}_{1,X_j}^{q,-q}\downarrow & & d_{1,X_j}^{q,-q}\downarrow & & d_{1,X}^{q,-q}\downarrow \\
\displaystyle\bigoplus_{x \in X^{(q+1)}} \overline{K}_{-1}(O_{X_j,x_j}\text{ on }x_j) & \longleftarrow & \displaystyle\bigoplus_{x_j \in X_j^{(q+1)}} K_{-1}(O_{X_j,x_j}\text{ on }x_j) & \longrightarrow & \displaystyle\bigoplus_{x \in X^{(q+1)}} K_{-1}(O_{X,x}\text{ on }x) \\
\overline{d}_{1,X_j}^{q+1,-q}\downarrow & & d_{1,X_j}^{q+1,-q}\downarrow & & d_{1,X}^{q+1,-q}\downarrow \\
\cdots & \longleftarrow & \cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\displaystyle\bigoplus_{x \in X^{(d)}} \overline{K}_{q-d}(O_{X_j,x_j}\text{ on }x_j) & \longleftarrow & \displaystyle\bigoplus_{x_j \in X_j^{(d)}} K_{q-d}(O_{X_j,x_j}\text{ on }x_j) & \longrightarrow & \displaystyle\bigoplus_{x \in X^{(d)}} K_{q-d}(O_{X,x}\text{ on }x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0,
\end{array}
$$

where $\overline{K}_*(O_{X_j,x_j}\text{ on }x_j)$ denotes the kernel of the natural projection

$$
K_*(O_{X_j,x_j}\text{ on }x_j) \to K_*(O_{X,x}\text{ on }x).
$$

And the differentials $\overline{d}_{1,X_j}^{p,q}$, which is induced from $d_{1,X_j}^{p,q}$, satisfies $d_{1,X_j}^{p,q} = d_{1,X}^{p,q} \oplus \overline{d}_{1,X_j}^{p,q}$.

Each term of the left column in the above diagram can be identified with local cohomology group, using Corollary 3.11. In fact, the relative Chern character induces

the following commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^\bullet_{k(X)/\mathbb{Q}} & \xleftarrow{\;\cong\;} & \overline{K}_q(k(X)[t]/(t^{j+1}))_\mathbb{Q} \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{x\in X^{(1)}} H^1_x(\Omega^\bullet_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(1)}} \overline{K}_{q-1}(O_{X_j,x_j}\text{ on } x_j)_\mathbb{Q} \\
\downarrow & & \downarrow \\
\cdots & \longleftarrow & \cdots \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{x\in X^{(q-1)}} H^{q-1}_x(\Omega^\bullet_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(q-1)}} \overline{K}_1(O_{X_j,x_j}\text{ on } x_j)_\mathbb{Q} \\
\partial^{q-1,-q}_1 \downarrow & & \overline{d}^{q-1,-q}_{1,X_j} \downarrow \\
\displaystyle\bigoplus_{x\in X^{(q)}} H^q_x(\Omega^\bullet_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(q)}} \overline{K}_0(O_{X_j,x_j}\text{ on } x_j)_\mathbb{Q} \\
\partial^{q,-q}_1 \downarrow & & \overline{d}^{q,-q}_{1,X_j} \downarrow \\
\displaystyle\bigoplus_{x\in X^{(q+1)}} H^{q+1}_x(\Omega^\bullet_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(q+1)}} \overline{K}_{-1}(O_{X_j,x_j}\text{ on } x_j)_\mathbb{Q} \\
\partial^{q+1,-q}_1 \downarrow & & \overline{d}^{q+1,-q}_{1,X_j} \downarrow \\
\cdots & \longleftarrow & \cdots \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{x\in X^{(d)}} H^d_x(\Omega^\bullet_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(d)}} \overline{K}_{q-d}(O_{X_j,x_j}\text{ on } x_j)_\mathbb{Q} \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array}
$$

where

$$
\begin{cases}
\Omega^\bullet_{X/\mathbb{Q}} & = (\Omega^{q-1}_{X/\mathbb{Q}} \oplus \Omega^{q-3}_{X/\mathbb{Q}} \oplus \dots)^{\oplus j}, \\
\Omega^\bullet_{k(X)/\mathbb{Q}} & = (\Omega^{q-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{q-3}_{k(X)/\mathbb{Q}} \oplus \dots)^{\oplus j}.
\end{cases}
$$

The left column is the classical Cousin complex [11]. Combining this diagram with the last one, one has:

THEOREM 3.12. *Let X be a smooth projective variety over a field k of characteristic 0 and X_j be the j-th trivial infinitesimal thickening of X. For each integer $q \geq 1$, there exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $\Omega^\bullet_{X/\mathbb{Q}}$, $K_q(O_{X_j})_\mathbb{Q}$ and $K_q(O_X)_\mathbb{Q}$ respectively. The left arrows are induced by Chern characters from K-theory to negative cyclic homology.*

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{\ \mathrm{Chern}\ } & K_q(k(X)[t]/(t^{j+1}))_{\mathbb{Q}} & \xrightarrow{\ i_j^*\ } & K_q(k(X))_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus\limits_{x\in X^{(1)}} H^1_x(\Omega^{\bullet}_{X/\mathbb{Q}}) & \longleftarrow & \bigoplus\limits_{x_j\in X_j^{(1)}} K_{q-1}(O_{X_j,x_j}\text{ on } x_j)_{\mathbb{Q}} & \longrightarrow & \bigoplus\limits_{x\in X^{(1)}} K_{q-1}(O_{X,x}\text{ on } x)_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \longleftarrow & \cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus\limits_{x\in X^{(q-1)}} H^{q-1}_x(\Omega^{\bullet}_{X/\mathbb{Q}}) & \longleftarrow & \bigoplus\limits_{x_j\in X_j^{(q-1)}} K_1(O_{X_j,x_j}\text{ on } x_j)_{\mathbb{Q}} & \longrightarrow & \bigoplus\limits_{x\in X^{(q-1)}} K_1(O_{X,x}\text{ on } x)_{\mathbb{Q}} \\
\partial_1^{q-1,-q}\downarrow & & d_{1,X_j}^{q-1,-q}\downarrow & & d_{1,X}^{q-1,-q}\downarrow \\
\bigoplus\limits_{x\in X^{(q)}} H^{q}_x(\Omega^{\bullet}_{X/\mathbb{Q}}) & \longleftarrow & \bigoplus\limits_{x_j\in X_j^{(q)}} K_0(O_{X_j,x_j}\text{ on } x_j)_{\mathbb{Q}} & \longrightarrow & \bigoplus\limits_{x\in X^{(q)}} K_0(O_{X,x}\text{ on } x)_{\mathbb{Q}} \\
\partial_1^{q,-q}\downarrow & & d_{1,X_j}^{q,-q}\downarrow & & d_{1,X}^{q,-q}\downarrow \\
\bigoplus\limits_{x\in X^{(q+1)}} H^{q+1}_x(\Omega^{\bullet}_{X/\mathbb{Q}}) & \longleftarrow & \bigoplus\limits_{x_j\in X_j^{(q+1)}} K_{-1}(O_{X_j,x_j}\text{ on } x_j)_{\mathbb{Q}} & \longrightarrow & \bigoplus\limits_{x\in X^{(q+1)}} K_{-1}(O_{X,x}\text{ on } x)_{\mathbb{Q}} \\
\partial_1^{q+1,-q}\downarrow & & d_{1,X_j}^{q+1,-q}\downarrow & & d_{1,X}^{q+1,-q}\downarrow \\
\cdots & \longleftarrow & \cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus\limits_{x\in X^{(d)}} H^{d}_x(\Omega^{\bullet}_{X/\mathbb{Q}}) & \longleftarrow & \bigoplus\limits_{x_j\in X_j^{(d)}} K_{q-d}(O_{X_j,x_j}\text{ on } x_j)_{\mathbb{Q}} & \longrightarrow & \bigoplus\limits_{x\in X^{(d)}} K_{q-d}(O_{X,x}\text{ on } x)_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0,
\end{array}
$$

where

$$
\begin{cases}
\Omega^{\bullet}_{X/\mathbb{Q}} & = \ (\Omega^{q-1}_{X/\mathbb{Q}} \oplus \Omega^{q-3}_{X/\mathbb{Q}} \oplus \dots)^{\oplus j}, \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & = \ (\Omega^{q-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{q-3}_{k(X)/\mathbb{Q}} \oplus \dots)^{\oplus j}.
\end{cases}
$$

Proof. Since X is smooth, the sheafifications of the left and right column are flasque resolutions of $\Omega^{\bullet}_{X/\mathbb{Q}}$ and $K_q(O_X)_{\mathbb{Q}}$ respectively. So the sheafification of the middle column is a flasque resolution of $K_q(O_{X_j})_{\mathbb{Q}}$. $\square$

Next, we want to use Adams operations ψ^k to refine the diagram in Theorem 3.12. Since q can be any integer (≥ 1) there, negative K-groups might appear. One can use Weibel's method to extend Adams operations to negative K-groups, as recalled in the beginning of Section 3.1. In the following, $K_*^{(i)}$ denotes the eigenspace of $\psi^k = k^i$.

Since X is smooth, we note that negative K-groups of the right column Theorem 3.12 are 0 and this Gersten complex agrees with the one in [15],

$$
0 \to \bigoplus_{x\in X^{(0)}} K_q(k(X)) \to \cdots \to \bigoplus_{x\in X^{(q-1)}} K_1(k(x)) \xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x\in X^{(q)}} K_0(k(x)) \xrightarrow{d_{1,X}^{q,-q}} 0.
$$

Thus, we can use Adams operations, defined at space level [16], to decompose this complex directly. And for every $i \in \mathbb{Z}$, then there exists the following finer complex

$$0 \to K_q^{(i)}(k(X))_{\mathbb{Q}} \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1^{(i)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} \xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^{(i)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} \xrightarrow{d_{1,X}^{q,-q}} 0,$$

which agrees with the following complex in [16],

$$0 \to K_q^{(i)}(k(X))_{\mathbb{Q}} \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1^{(i-(q-1))}(k(x))_{\mathbb{Q}} \xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^{(i-q)}(k(x))_{\mathbb{Q}} \xrightarrow{d_{1,X}^{q,-q}} 0.$$

One notes the differentials $\partial_1^{p,q}$ of the Cousin complex (the left column in Theorem 3.12) respects Adams operations (Adams operations on $H_x^*(\Omega_{X/\mathbb{Q}}^\bullet)$ is induced from the isomorphism in Theorem 3.10):

$$\partial_1^{p,q} : \bigoplus_{x \in X^{(p)}} H_x^p(\Omega_{X/\mathbb{Q}}^{\bullet,(i)}) \to \bigoplus_{x \in X^{(p+1)}} H_x^{p+1}(\Omega_{X/\mathbb{Q}}^{\bullet,(i)}),$$

where

$$\begin{cases} \Omega_{X/\mathbb{Q}}^{\bullet,(i)} & = (\Omega_{X/\mathbb{Q}}^{2i-q-1})^{\oplus j}, \text{for } \frac{q}{2} < i \leq q; \\ \Omega_{X/\mathbb{Q}}^{\bullet,(i)} & = 0, \text{else.} \end{cases} \tag{3.13}$$

Since the relative Chern character respects Adams operations, see Theorem 3.6

and Corollary 3.11, there exists the following commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^{\bullet,(i)}_{k(X)/\mathbb{Q}} & \xleftarrow{\;\cong\;} & \overline{K}^{(i)}_q(k(X)[t]/(t^{j+1}))_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{x\in X^{(1)}} H^1_x(\Omega^{\bullet,(i)}_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(1)}} \overline{K}^{(i)}_{q-1}(O_{X_j,x_j}\ \text{on}\ x_j)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\cdots & \longleftarrow & \cdots \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{x\in X^{(q-1)}} H^{q-1}_x(\Omega^{\bullet,(i)}_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(q-1)}} \overline{K}^{(i)}_1(O_{X_j,x_j}\ \text{on}\ x_j)_{\mathbb{Q}} \\
\partial^{q-1,-q}_1\downarrow & & \overline{d}^{q-1,-q}_{1,X_j}\downarrow \\
\displaystyle\bigoplus_{x\in X^{(q)}} H^q_x(\Omega^{\bullet,(i)}_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(q)}} \overline{K}^{(i)}_0(O_{X_j,x_j}\ \text{on}\ x_j)_{\mathbb{Q}} \\
\partial^{q,-q}_1\downarrow & & \overline{d}^{q,-q}_{1,X_j}\downarrow \\
\displaystyle\bigoplus_{x\in X^{(q+1)}} H^{q+1}_x(\Omega^{\bullet,(i)}_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(q+1)}} \overline{K}^{(i)}_{-1}(O_{X_j,x_j}\ \text{on}\ x_j)_{\mathbb{Q}} \\
\partial^{q+1,-q}_1\downarrow & & \overline{d}^{q+1,-q}_{1,X_j}\downarrow \\
\cdots & \longleftarrow & \cdots \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{x\in X^{(d)}} H^d_x(\Omega^{\bullet,(i)}_{X/\mathbb{Q}}) & \longleftarrow & \displaystyle\bigoplus_{x\in X^{(d)}} \overline{K}^{(i)}_{q-d}(O_{X_j,x_j}\ \text{on}\ x_j)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array}
$$

where $\overline{K}^{(i)}_*(O_{X_j,x_j}\ \text{on}\ x_j)$ denotes the eigenspace of $\psi^k = k^i$. Consequently, $\overline{d}^{p,q}_{1,X_j}$ respects Adams operations. The differential $d^{p,q}_{1,X_j}$ splits as $d^{p,q}_{1,X} \oplus \overline{d}^{p,q}_{1,X_j}$, so $d^{p,q}_{1,X_j}$ respects Adams operations:

$$
d^{p,q}_{1,X_j} : E^{p,q}_1(X_j)^{(i)} \to E^{p,q}_1(X_j)^{(i)}.
$$

So we have the following complex:

$$0 \to K_q^{(i)}(X_j)_{\mathbb{Q}} \to K_q^{(i)}(k(X_j))_{\mathbb{Q}} \to \cdots \to \bigoplus_{x_j \in X_j^{(q-1)}} K_1^{(i)}(O_{X_j, x_j} \text{ on } x_j)_{\mathbb{Q}}$$

$$\xrightarrow{d_{1, X_j}^{q-1, -q}} \bigoplus_{x_j \in X_j^{(q)}} K_0^{(i)}(O_{X_j, x_j} \text{ on } x_j)_{\mathbb{Q}}$$

$$\xrightarrow{d_{1, X_j}^{q, -q}} \bigoplus_{x_j \in X_j^{(q-1)}} K_{-1}^{(i)}(O_{X_j, x_j} \text{ on } x_j)_{\mathbb{Q}} \to \cdots \to \bigoplus_{x_j \in X_j^{(d)}} K_{q-d}^{(i)}(O_{X_j, x_j} \text{ on } x_j)_{\mathbb{Q}} \to 0.$$

THEOREM 3.13. *Let X be a smooth projective variety over a field k of characteristic 0 and X_j be the j-th trivial infinitesimal thickening of X. For each integer $q \geq 1$, there exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $\Omega_{X/\mathbb{Q}}^{\bullet, (i)}$, $K_q^{(i)}(O_{X_j})_{\mathbb{Q}}$ and $K_q^{(i)}(O_X)_{\mathbb{Q}}$ respectively. The left arrows are induced by Chern characters from K-theory to negative cyclic homology.*

where

$$\begin{cases} \Omega_{X/\mathbb{Q}}^{\bullet,(i)} &= (\Omega_{X/\mathbb{Q}}^{2i-q-1})^{\oplus j}, \text{for } \frac{q}{2} < \ i \leq q; \\ \Omega_{X/\mathbb{Q}}^{\bullet,(i)} &= 0, \text{else.} \end{cases} \qquad (3.14)$$

Proof. The sheafications of the left and right columns are flasque resolutions of $\Omega_{X/\mathbb{Q}}^{\bullet,(i)}$, and $K_q^{(i)}(O_X)_{\mathbb{Q}}$ respectively. So the sheafification of the middle column is a flasque resolution of $K_q^{(i)}(O_{X_j})_{\mathbb{Q}}$. $\square$

In particular, we are interested in the "Milnor K-theory". Letting $i = q$ in Theorem 3.13, one have the following theorem.

THEOREM 3.14. *Let X be a smooth projective variety over a field k of characteristic 0 and X_j be the j-th trivial infinitesimal thickening of X. For each integer $q \geq 1$, there exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $(\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}$, $K_q^M(O_{X_j})_{\mathbb{Q}}$ and $K_q^M(O_X)_{\mathbb{Q}}$ respectively. The left arrows are induced by Chern characters from K-theory to negative cyclic homology.*

The middle and right columns are complexes, so Definition 3.4 applies:

DEFINITION 3.15. The Milnor K-theoretic q-cycles and rational equivalence of (X, O_X) are defined to be

$$Z_q^M(D^{\mathrm{perf}}(X)) = \mathrm{Ker}(d_{1,X}^{q,-q}), \ \ Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X)) = \mathrm{Im}(d_{1,X}^{q-1,-q}).$$

The q-th Milnor K-theoretic Chow group of (X, O_X) is defined to be

$$CH_q^M(D^{\mathrm{perf}}(X)) = \frac{Z_q^M(D^{\mathrm{perf}}(X))}{Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X))}.$$

The Milnor K-theoretic q-cycles and rational equivalence of (X, O_{X_j}) are defined to be

$$Z_q^M(D^{\mathrm{perf}}(X_j)) = \mathrm{Ker}(d_{1,X_j}^{q,-q}), \ \ Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X_j)) = \mathrm{Im}(d_{1,X_j}^{q-1,-q}).$$

The q-th Milnor K-theoretic Chow group of (X, O_{X_j}) is defined to be

$$CH_q^M(D^{\mathrm{perf}}(X_j)) = \frac{Z_q^M(D^{\mathrm{perf}}(X_j))}{Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X_j))}.$$

Agreement. We now prove that our Milnor K-theoretic Chow group rationally agrees with the classical ones for smooth projective varieties.

THEOREM 3.16. *For X a smooth projective variety over a field k of characteristic 0, let $Z^q(X)$, $Z_{\mathrm{rat}}^q(X)$ and $CH^q(X)$ denote the classical q-cycles, rational equivalence and Chow groups respectively, then we have the following identifications*

$$Z_q^M(D^{\mathrm{perf}}(X)) = Z^q(X)_{\mathbb{Q}},$$

$$Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X)) = Z_{\mathrm{rat}}^q(X)_{\mathbb{Q}},$$

$$CH_q^M(D^{\mathrm{perf}}(X)) = CH^q(X)_{\mathbb{Q}}.$$

Proof. Since X is smooth, by Theorem 3.1 and Theorem 3.3, the right column in Theorem 3.14 agrees with the following classical sequence,

$$0 \to K_q^M(k(X))_{\mathbb{Q}} \to \bigoplus_{x \in X^{(1)}} K_{q-1}^M(k(x))_{\mathbb{Q}} \to \cdots \to \bigoplus_{x \in X^{(q-1)}} K_1^M(k(x))_{\mathbb{Q}}$$

$$\xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^M(k(x))_{\mathbb{Q}} \xrightarrow{d_{1,X}^{q,-q}} 0.$$

Noting $K_0^M(k(x))_{\mathbb{Q}} = \mathbb{Q}$, one has

$$Z_q^M(D^{\mathrm{perf}}(X)) = \mathrm{Ker}(d_{1,X}^{q,-q}) = \bigoplus_{x \in X^{(q)}} K_0^M(k(x))_{\mathbb{Q}} = Z^q(X)_{\mathbb{Q}}.$$

Since $K_1^M(k(x))_{\mathbb{Q}} = K_1(k(x))_{\mathbb{Q}}$, as explained in Quillen's proof of Bloch's formula, the image of $d_{1,X}^{q-1,-q}$ gives the rational equivalence. Hence,

$$Z_{q,\mathrm{rat}}^M(D^{\mathrm{perf}}(X)) = \mathrm{Im}(d_{1,X}^{q-1,-q}) = Z_{\mathrm{rat}}^q(X)_{\mathbb{Q}}.$$

Therefore, we have the following identification

$$CH_q^M(D^{\mathrm{perf}}(X)) = CH^q(X)_{\mathbb{Q}}.$$

$\square$

We obtain the following **Bloch's formulas**, which gives a positive answer to Green-Griffiths' **Question 1.1** for trivial deformations.

THEOREM 3.17. *Let X be a smooth projective variety over a field k of characteristic 0 and X_j be the j-th trivial infinitesimal deformation of X. For q a non negative integer, we have the following identifications:*

$$CH_q^M(D^{\mathrm{perf}}(X)) = H^q(X, K_q^M(O_X))_{\mathbb{Q}}.$$

$$CH_q^M(D^{\mathrm{perf}}(X_j)) = H^q(X, K_q^M(O_{X_j}))_{\mathbb{Q}}.$$

Proof. Immediately from Theorem 3.14. $\square$

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