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ON SINGULAR VARIETIES ASSOCIATED TO A POLYNOMIAL MAPPING FROM $\mathbb{C}^n$ TO $\mathbb{C}^{n-1}$*

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Abstract. We construct singular varieties $\mathcal{V}_G$ associated to a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ where $n \geqslant 2$. Let $G : \mathbb{C}^3 \to \mathbb{C}^2$ be a local submersion, we prove that if the homology or the intersection homology with total perversity (with compact supports or closed supports) in dimension two of any variety $\mathcal{V}_G$ is trivial then G is a fibration. In the case of a local submersion $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ where $n \geqslant 4$, the result is still true with an additional condition.

Key words. Complex polynomial mappings, intersection homology, singularities at infinity.

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1. Introduction.

Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a non-constant polynomial mapping ($n \geqslant 2$). It is well known [20] that G is a locally trivial fibration outside the bifurcation set $B(G)$ in $\mathbb{C}^{n-1}$. In a natural way appears a fundamental question: how to determine the set $B(G)$. In [12], Ha Huy Vui and Nguyen Tat Thang gave, for a generic class of $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ ($n \geqslant 2$), a necessary and sufficient condition for a point $z \in \mathbb{C}^{n-1}$ to be in the bifurcation set $B(G)$ in term of the Euler characteristic of the fibers at nearby points. The case n=2 was previously given in [11] .

In this paper, we want to construct singular varieties $\mathcal{V}_G$ associated to a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ ($n \geqslant 2$) such that the intersection homology of $\mathcal{V}_G$ can characterize the bifurcation set of G. The motivation for this paper comes from the paper [21], where Anna and Guillaume Valette constructed real pseudomanifolds, denoted V_F, associated to a given polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$, such that the singular part of the variety V_F is contained in $(S_F \times K_0(F)) \times \{0^p\}$ ($p > 0$), where $K_0(F)$ is the set of critical values and S_F is the set of non-proper points of F. In the case of dimension $n = 2$, the homology or intersection homology of V_F describes the geometry of the singularities at infinity of the mapping F. With Anna and Guillaume Valette, the first author generalized this result [18] for the general case $F : \mathbb{C}^n \to \mathbb{C}^n$ ($n \geqslant 2$). The idea to construct varieties V_F is the following: considering the polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ as a real one $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, then if we take a finite covering $\{V_i\}$ by smooth submanifolds of $\mathbb{R}^{2n} \backslash SingF$, the mapping F induces a diffeomorphism from V_i into its image $F(V_i)$. We use a technique in order to separate these $\{F(V_i)\}$ by embedding them in a higher dimensional space, then V_F is obtained by gluing $\{F(V_i)\}$ together along the set $S_F \cup K_0(F)$.

A natural question is that how can we apply this construction to the case of polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, or, $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$. The main difficulty of this case is that if we take an open submanifold V in $\mathbb{R}^{2n} \backslash SingF$, then locally we do not have a diffeomorphism from V into its image $G(V)$. So we consider a generic $(2n-2)$- real dimensional submanifold in the source space $\mathbb{R}^{2n}$, denoted $\mathcal{M}_G$, which

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is called *the Milnor set of* G. Then we can apply the construction of singular varieties V_F in [21] for $F := G|_{\mathcal{M}_G}$ the restriction of G to the Milnor set $\mathcal{M}_G$.

We obtain the following result: let $G : \mathbb{C}^3 \to \mathbb{C}^2$ be a local submersion, then if the homology or the intersection homology with total perversity (with compact supports or closed supports) in dimension two of any among of the constructed varieties $\mathcal{V}_G$ is trivial then G is a fibration (Theorem 5.1). In the case of a local submersion $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ where $n \geqslant 4$, the result is still true with an additional condition (Theorem 5.2). Comparing with the paper [12], we obtain the Corollary 5.9.

2. Preliminaries and basic definitions. In this section we set-up our framework. All the varieties we consider in this article are semi-algebraic.

2.1. Notations and conventions. Given a topological space X, singular simplices of X will be semi-algebraic continuous mappings $\sigma : T_i \to X$, where T_i is the standard i-simplex in $\mathbb{R}^{i+1}$. Given a subset X of $\mathbb{R}^n$ we denote by $C_i(X)$ the group of i-dimensional singular chains (linear combinations of singular simplices with coefficients in $\mathbb{R}$); if c is an element of $C_i(X)$, we denote by $|c|$ its support. By $Reg(X)$ and $Sing(X)$ we denote respectively the regular and singular locus of the set X. Given $X \subset \mathbb{R}^n$, $\overline{X}$ will stand for the topological closure of X. The smoothness to be considered as the differentiable smoothness.

Notice that, when we refer to the homology of a variety, the notation $H_*^c(X)$ refers to the homology with compact supports, the notation $H_*^{cl}(X)$ refers to the homology with closed supports (see [1]).

2.2. Intersection homology. We briefly recall the definition of intersection homology; for details, we refer to the fundamental work of M. Goresky and R. MacPherson [6] (see also [1]).

Definition 2.1. Let X be a m-dimensional semi-algebraic set. A *semi-algebraic stratification of* X is the data of a finite semi-algebraic filtration

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing,$$

such that for every i, the set $S_i = X_i \backslash X_{i-1}$ is either empty or a manifold of dimension i. A connected component of S_i is called *a stratum* of X.

We denote by cL the open cone on the space L, the cone on the empty set being a point. Observe that if L is a stratified set then cL is stratified by the cones over the strata of L and a 0-dimensional stratum (the vertex of the cone).

Definition 2.2. A stratification of X is said to be *locally topologically trivial* if for every $x \in X_i \backslash X_{i-1}$, $i \geqslant 0$, there is an open neighborhood U_x of x in X, a stratified set L and a semi-algebraic homeomorphism

$$h : U_x \to (0;1)^i \times cL,$$

such that h maps the strata of U_x (induced stratification) onto the strata of $(0;1)^i \times cL$ (product stratification).

The definition of perversities as originally given by Goresky and MacPherson:

Definition 2.3. A *perversity* is an $(m+1)$-uple of integers $\bar{p} = (p_0, p_1, p_2, p_3, \ldots, p_m)$ such that $p_0 = p_1 = p_2 = 0$ and $p_{k+1} \in \{p_k, p_k + 1\}$, for $k \geqslant 2$.

Traditionally we denote the zero perversity by $\overline{0} = (0,0,0,\dots,0)$, the maximal perversity by $\overline{t} = (0,0,0,1,\dots,m-2)$, and the middle perversities by $\overline{m} = (0,0,0,0,1,1,\dots,[\frac{m-2}{2}])$ (lower middle) and $\overline{n} = (0,0,0,1,1,2,2,\dots,[\frac{m-1}{2}])$ (upper middle). We say that the perversities $\overline{p}$ and $\overline{q}$ are *complementary* if $\overline{p}+\overline{q}=\overline{t}$.

Let X be a semi-algebraic variety such that X admits a locally topologically trivial stratification. We say that a semi-algebraic subset $Y \subset X$ is $(\bar{p}, i)$-*allowable* if

$$\dim(Y \cap X_{m-k}) \leqslant i - k + p_k \text{ for all } k.$$

Define $IC_i^{\overline{p}}(X)$ to be the $\mathbb{R}$-vector subspace of $C_i(X)$ consisting in those chains ξ such that $|\xi|$ is $(\overline{p}, i)$-allowable and $|\partial\xi|$ is $(\overline{p}, i-1)$-allowable.

Definition 2.4. The i^{th} *intersection homology group with perversity* $\overline{p}$, denoted by $IH_i^{\overline{p}}(X)$, is the i^{th} homology group of the chain complex $IC_*^{\overline{p}}(X)$.

Notice that, the notation $IH_*^{\overline{p},c}(X)$ refer to the intersection homology with compact supports, the notation $IH_*^{\overline{p},cl}(X)$ refer to the intersection homology with closed supports.

Goresky and MacPherson proved that the intersection homology is independent of the choice of the stratification [6, 7].

The Poincaré duality holds for the intersection homology of a (singular) variety:

Theorem 2.5 (Goresky, MacPherson [6]). *For any orientable compact stratified semi-algebraic m-dimensional variety X, generalized Poincaré duality holds:*

$$IH_k^{\overline{p}}(X) \simeq IH_{m-k}^{\overline{q}}(X),$$

where $\overline{p}$ and $\overline{q}$ are complementary perversities.

For the non-compact case, we have:

$$IH_k^{\overline{p},c}(X) \simeq IH_{m-k}^{\overline{q},cl}(X).$$

A relative version is also true in the case where X has boundary.

2.3. The bifurcation set, the set of asymptotic critical values and the asymptotic set. Let $G : \mathbb{C}^n \to \mathbb{C}^m$ where $n \geqslant m$ be a polynomial mapping.

i) The bifurcation set of G, denoted by $B(G)$ is the smallest set in $\mathbb{C}^m$ such that G is not C^∞ - fibration on this set (see, for example, [20]).

ii) The set of asymptotic critical values, denoted by $K_\infty(G)$, is the set

$$K_\infty(G) = \{\alpha \in \mathbb{C}^m : \exists \{z_k\} \subset \mathbb{C}^n, \text{ such that } |z_k| \to \infty, G(z_k) \to \alpha \text{ and } |z_k||dG(z_k)| \to 0\}.$$

The set $K_\infty(G)$ is an approximation of the set $B(G)$. More precisely, we have $B(G) \subset K_\infty(G)$ (see, for example, [14] or [3]).

iii) When $n = m$, we denote by S_G the set of points at which the mapping G is not proper, *i.e.*

$$S_G = \{\alpha \in \mathbb{C}^m : \exists \{z_k\} \subset \mathbb{C}^n, |z_k| \to \infty \text{ such that } G(z_k) \to \alpha\},$$

and call it the *asymptotic variety.* In the case of polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$, the following holds: $B(G) = S_G$ ([9]).

3. The variety $\mathcal{M}_G$. We consider polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ as real ones $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$. By $Sing(G)$ we mean the singular locus of G, that is the set of points for which the (complex) rank of the Jacobian matrix is less than $n-1$. We denote by $K_0(G)$ the set of critical values. From here, we assume always $K_0(G) = \varnothing$.

Definition 3.1. Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial mapping. Consider G as a real polynomial mapping $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a real function such that $\rho(z) \geqslant 0$ for any $z \in \mathbb{C}^n$. Let

$$\varphi = \frac{1}{1+\rho}.$$

Consider (G, φ) as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n-1}$. Let us define

$$\mathcal{M}_G := Sing(G, \varphi) = \{x \in \mathbb{R}^{2n} \text{ such that } \mathrm{Rank} D_{\mathbb{R}}(G, \varphi)(x) \leqslant 2n-2\},$$

where $D_{\mathbb{R}}(G, \varphi)(x)$ is the Jacobian matrix of $(G, \varphi) : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ at x.

Remark 3.2. Since $K_0(G) = \varnothing$, then $\mathrm{Rank} D_{\mathbb{R}}(G) = 2n-2$, so we have

$$Sing(G, \varphi) = \{x \in \mathbb{R}^{2n} \text{ such that } \mathrm{Rank} D_{\mathbb{R}}(G, \varphi) = 2n-2\}.$$

Note that, from here, if we want to refer to the source space as a complex space, we will write $(G, \varphi) : \mathbb{C}^n \to \mathbb{R}^{2n-1}$, if we want to refer to the source space as a real space, we will write $(G, \varphi) : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$. Moreover, in general, we denote by z a complex element in $\mathbb{C}^n$ and by x a real element in $\mathbb{R}^{2n}$.

Lemma 3.3. *For any ρ, φ and (G, φ) as in the Definition 3.1 and for any $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, we have*

$$\mathrm{Rank} D_{\mathbb{R}}(G, \varphi)(x) = \mathrm{Rank} D_{\mathbb{R}}(G, \rho)(x),$$

so we have

$$\mathcal{M}_G = Sing(G, \varphi) = Sing(G, \rho).$$

Proof. For any $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, we have

$$D_{\mathbb{R}}(G, \rho)(x) = \begin{pmatrix} & D_{\mathbb{R}}(G) & \\ \rho_{x_1} & \cdots & \rho_{x_{2n}} \end{pmatrix},$$

$$D_{\mathbb{R}}(G, \varphi)(x) = \begin{pmatrix} & D_{\mathbb{R}}(G) & \\ \frac{-\rho_{x_1}}{(1+\rho)^2} & \cdots & \frac{-\rho_{x_{2n}}}{(1+\rho)^2} \end{pmatrix},$$

where ρ_{x_i} is the derivative of ρ with respect to x_i, for $i = 1, \ldots, 2n$. We have $\mathrm{Rank} D_{\mathbb{R}}(G, \varphi)(x) = \mathrm{Rank} D_{\mathbb{R}}(G, \rho)(x)$ for any $x \in \mathbb{R}^{2n}$ and $\mathcal{M}_G = Sing(G, \varphi) = Sing(G, \rho)$. □

Remark 3.4. From here, we consider the function ρ of the following form

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2,$$

where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$, and $a_i \in \mathbb{R}$ for $i = 1, \dots, n$.

PROPOSITION 3.5. *Let $G = (G_1, \dots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geqslant 2)$ be a polynomial mapping such that $K_0(G) = \varnothing$ and $\rho : \mathbb{C}^n \to \mathbb{R}$ be such that $\rho = a_1|z_1|^2 + \dots + a_n|z_n|^2$, where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$ and $a_i \in \mathbb{R}$, for $i = 1, \dots, n$. Denote by $\mathbf{v}_i$ the determinant of the cofactor of $\frac{\partial}{\partial z_i}$ of the matrix*

$$\mathbf{V}(z) = \begin{pmatrix} \frac{\partial}{\partial z_1} & \dots & \frac{\partial}{\partial z_n} \\ \frac{\partial G_1}{\partial z_1} & \dots & \frac{\partial G_1}{\partial z_n} \\ & \dots & \\ \frac{\partial G_{n-1}}{\partial z_1} & \dots & \frac{\partial G_{n-1}}{\partial z_n} \end{pmatrix},$$

for $i = 1, \dots, n$. Then we have

$$\mathcal{M}_G = h^{-1}(0),$$

where

$$h : \mathbb{C}^n \to \mathbb{C}, \quad h(z) = 2\Sigma a_i \mathbf{v}_i(z)\overline{z_i}.$$

Proof. Let $G = (G_1, \dots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geqslant 2)$ and $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $\rho = a_1|z_1|^2 + \dots + a_n|z_n|^2$, where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$ and $a_i \in \mathbb{R}$. Let us consider the vector field

$$\mathbf{V}(z) = \begin{pmatrix} \frac{\partial}{\partial z_1} & \dots & \frac{\partial}{\partial z_n} \\ \frac{\partial G_1}{\partial z_1} & \dots & \frac{\partial G_1}{\partial z_n} \\ & \dots & \\ \frac{\partial G_{n-1}}{\partial z_1} & \dots & \frac{\partial G_{n-1}}{\partial z_n} \end{pmatrix}.$$

We have

$$\mathbf{V}(z) = \mathbf{v}_1 \frac{\partial}{\partial z_1} + \dots + \mathbf{v}_n \frac{\partial}{\partial z_n},$$

where $\mathbf{v}_i$ is the determinant of the cofactor of $\frac{\partial}{\partial z_i}$, for $i = 1, \dots, n$. The vector field $\mathbf{V}(z)$ is tangent to the curve $G = c$. Let $R(z) = a_1 z_1^2 + \dots + a_n z_n^2$, then we have $\mathcal{M}_G = h^{-1}(0)$, where

$$h : \mathbb{C}^n \to \mathbb{C}, \quad h(z) = < \mathbf{V}(z), \operatorname{Grad} R(z) > .$$

More precisely, we have $h(z) = 2\Sigma a_i \mathbf{v}_i(z)\overline{z_i}$. □

PROPOSITION 3.6. *For an open and dense set of polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ such that $K_0(G) = \varnothing$, the variety $\mathcal{M}_G$ is a smooth manifold of dimension $2n - 2$.*

Proof. The question is of local nature. In a neighbourhood of a point z_0 in $\mathbb{C}^n$, we can choose coordinates such that the level curve $G = c$, where $c = G(z_0) \in \mathbb{C}^{n-1}$ is parametrized

$$\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, z_0)$$

$$s \mapsto (\gamma_1(s), \ldots, \gamma_n(s)).$$

Since $\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2$, then $\rho \circ \gamma : (\mathbb{C}, 0) \to \mathbb{R}$ and

$$\rho \circ \gamma(s) = a_1|\gamma_1(s)|^2 + \cdots + a_n|\gamma_n(s)|^2.$$

If z_0 is a singular point of $\rho|_{G=c}$, then

$$\rho \circ \gamma(0) = \rho(\gamma(0)) = \rho(z_0),$$

$$(\rho \circ \gamma)'(0) = 0.$$

For an open and dense set of G, we have

$$(\rho \circ \gamma)''(0) \neq 0.$$

Hence, z_0 is a Morse singularity of $\rho|_{G=c}$. In particular, it is an isolated point of the level curve $G = c$. When c varies in $\mathbb{C}^{n-1}$, it follows that the set $\mathcal{M}_G$ has dimension $2n - 2$.

We prove now that $\mathcal{M}_G$ is smooth. By Proposition 3.5, the variety $\mathcal{M}_G$ is the set of solutions of $h = 0$, where

$$h(z) = 2\Sigma a_i \mathbf{v}_i(z)\overline{z_i},$$

and $\mathbf{v}_i$ is the determinant of the cofactor of $\frac{\partial}{\partial z_i}$ of $\mathbf{V}(z)$, for $i = 1, \ldots, n$. Since $K_0(G) = \varnothing$ then $\mathbf{V}(z) = (\mathbf{v}_1(z), \ldots, \mathbf{v}_n(z)) \neq 0$. We can assume that $\mathbf{V}(z_0) \neq 0$ for a fixed point z_0. For a generic polynomial mapping, we can solve the equation $h = 0$ in a neighbourhood of z_0. This shows that $h = 0$ is smooth in a neighbourhood of z_0. Then M_G is smooth. □

Remark 3.7. From here, we consider always generic polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ as in the Propostion 3.6.

4. The variety $\mathcal{V}_G$.

4.1. The construction of the variety $\mathcal{V}_G$. Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ and $\rho, \varphi : \mathbb{C}^n \to \mathbb{R}$ such that

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2, \quad \varphi = \frac{1}{1 + \rho},$$

where $\Sigma_{i=1,\ldots,n} a_i^2 \neq 0,\ a_i \geqslant 0$ and $a_i \in \mathbb{R}$. Let us consider:

a) $F := G_{|\mathcal{M}_G}$ the restriction of G on $\mathcal{M}_G$,

b) $\mathcal{N}_G = \mathcal{M}_G \backslash F^{-1}(K_0(F))$.

Since the dimension of $\mathcal{M}_G$ is $2n-2$ (Proposition 3.6), then locally, in a neighbourhood of any point x_0 in $\mathcal{M}_G$, we get a mapping $F : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}$. Now, we can apply the construction of singular varieties V_F in [21] for $F := G_{|\mathcal{M}_G}$: there exists a covering $\{U_1, \ldots, U_p\}$ of $\mathcal{N}_G$ by open semi-algebraic subsets (in $\mathbb{R}^{2n}$) such that on every element of this covering, the mapping F induces a diffeomorphism onto its image (see Lemma 2.1 of [21], see also [16]). We can find semi-algebraic closed subsets $V_i \subset U_i$ (in $\mathcal{N}_G$) which cover $\mathcal{N}_G$ as well. Thanks to Mostowski's Separation Lemma (see Separation Lemma in [15], page 246), for each $i = 1, \ldots, p$, there exists a Nash function $\psi_i : \mathcal{N}_G \to \mathbb{R}$, such that ψ_i is positive on V_i and negative on $\mathcal{N}_G \setminus U_i$.

LEMMA 4.1. *We can choose the Nash functions ψ_i such that $\psi_i(x_k)$ tends to zero when $\{x_k\} \subset \mathcal{N}_G$ tends to infinity.*

Proof. If ψ_i is a Nash function, then with any $N_i \in (\mathbb{N}\setminus\{0\})$, the function

$$\psi_i'(x) = \frac{\psi_i(x)}{(1 + |x|^2)^{N_i}},$$

where $x \in \mathcal{N}_G$, is also a Nash function, for $i = 1, \ldots, p$. The Nash function ψ_i' satisfies the property: ψ_i' is positive on V_i and negative on $\mathcal{N}_G \setminus U_i$. With N_i large enough, $\psi_i'(x_k)$ tends to zero when $\{x_k\} \subset \mathcal{N}_G$ tends to infinity, for $i = 1, \ldots, p$. We replace the function ψ_i by ψ_i'. □

DEFINITION 4.2. Let the Nash functions ψ_i and ρ be such that $\psi_i(x_k)$ tends to zero and $\rho(x_k)$ tends to infinity when $x_k \subset \mathcal{N}_G$ tends to infinity. Define a variety $\mathcal{V}_G$ associated to (G, ρ) as

$$\mathcal{V}_G := \overline{(F, \psi_1, \ldots, \psi_p)(\mathcal{N}_G)}.$$

REMARK 4.3. For a given polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, the variety $\mathcal{V}_G$ is not unique. It depends on the choice of the function ρ and the Nash functions ψ_i.

PROPOSITION 4.4. *The real dimension of $\mathcal{V}_G$ is $2n - 2$.*

Proof. By Proposition 3.6, in the generic case, the real dimension of $\mathcal{M}_G$ is $2n-2$. Moreover, F is a local immersion in a neighbourhood of a point in $\mathcal{M}_G$. So, the real dimension of $F(\mathcal{M}_G)$ is also $2n - 2$. Since

$$F(\mathcal{N}_G) = F(\mathcal{M}_G) \setminus K_0(F),$$

so the real dimension of $F(\mathcal{N}_G)$ is $2n - 2$. By Definition 4.2, the real dimension of $\mathcal{V}_G$ is $2n - 2$. □

DEFINITION 4.5 (see, for example, [4]). Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial mapping and $\rho : \mathbb{C}^n \to \mathbb{R}$ a real function such that $\rho \geqslant 0$. Define

$$\mathcal{S}_G := \{\alpha \in \mathbb{C}^{n-1} : \exists \{z_k\} \subset Sing(G, \rho), \text{ such that } z_k \text{ tends to infinity, } G(z_k) \text{ tends to } \alpha\}.$$

REMARK 4.6. a) For any real function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $\rho \geqslant 0$, we have

$$B(G) \subset \mathcal{S}_G \subset K_\infty(G),$$

where $B(G)$ is the bifurcation set and $K_\infty(G)$ is the set of asymptotic critical values of G (see, for example, [4]).

b) By Lemma 3.3, we have $Sing(G, \rho) = \mathcal{M}_G$, so the set $\mathcal{S}_G$ can be written

$$\mathcal{S}_G := \{\alpha \in \mathbb{C}^{n-1} : \exists \{x_k\} \subset \mathcal{M}_G, \text{ such that } x_k \text{ tends to infinity}, G(x_k) \text{ tends to } \alpha\}.$$

Definition 4.7. The *singular set at infinity* of the variety $\mathcal{V}_G$ is the set

$$\{\beta \in \mathcal{V}_G : \exists \{x_k\} \subset \mathcal{N}_G, x_k \to \infty, (G, \psi_1, \dots, \psi_p)(x_k) \to \beta\}.$$

Proposition 4.8. *The singular set at infinity of the variety $\mathcal{V}_G$ is contained in the set $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$.*

Proof. At first, by Proposition 3.6, for the generic case, the real dimension of $\mathcal{V}_G$ associated to $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is $2n-2$. Moreover, we have the following facts:

a) $\mathcal{S}_G \subset K_\infty(G)$,

b) $\dim_{\mathbb{C}}(K_\infty(G)) \leqslant n-2$ (see [14]), so $\dim_{\mathbb{R}}(K_\infty(G)) \leqslant 2n-4$.

Hence, we have $\dim_{\mathbb{R}} \mathcal{S}_G \times \{0_{\mathbb{R}^p}\} \leqslant 2n-4$. Moreover, by Proposition 4.4, we have $\dim_{\mathbb{R}} \mathcal{V}_G = 2n-2$. Let β be a singular point at infinity of the variety $\mathcal{V}_G$, then there exists a sequence $\{x_k\}$ in $\mathcal{N}_G$ tending to infinity such that $(G, \psi_1, \dots, \psi_p)(x_k)$ tends to β. Assume that $G(x_k)$ tends to α, then α belongs to $\mathcal{S}_G$. Moreover, the Nash function $\psi_i(x_k)$ tends to 0, for any $i = 1, \dots, p$. So $\beta = (\alpha, 0_{\mathbb{R}^p})$ belongs to $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$. Notice that, by Definition of $\mathcal{V}_G$, the set $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$ is contained in $\mathcal{V}_G$. Then $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$ contains the singular set at infinity of the variety $\mathcal{V}_G$. □

Remark 4.9. The singular set at infinity of $\mathcal{V}_G$ depends on the choice of the function ρ, since when ρ changes, the set $\mathcal{S}_G$ also changes. But, the property $B(G) \subset \mathcal{S}_G$ does not depend on the choice of the function ρ (see, for example, [4]).

The previous results show the following Proposition:

Proposition 4.10. *Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial mapping such that $K_0(G) = \varnothing$ and let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a real function such that*

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2,$$

where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$ and $a_i \in \mathbb{R}$ for $i = 1, \dots, n$. Then, there exists a real variety $\mathcal{V}_G$ in $\mathbb{R}^{2n-2+p}$, where $p > 0$, such that:

1) *The real dimension of $\mathcal{V}_G$ is $2n-2$,*

2) *The singular set at infinity of the variety $\mathcal{V}_G$ is contained in $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$.*

Remark 4.11. The variety $\mathcal{V}_G$ depends on the choice of the function ρ and the functions ψ_i. From now, we denote by $\mathcal{V}_G(\rho)$ any variety $\mathcal{V}_G$ associated to (G, ρ). If we refer to $\mathcal{V}_G$, that means a variety $\mathcal{V}_G$ associated to (G, ρ) for any ρ.

Remark 4.12. 1) In the construction of singular varieties $\mathcal{V}_G$, we can put $F := (G, \varphi)_{|\mathcal{M}_G}$, that means F is the restriction of (G, φ) on $\mathcal{M}_G$. In this case, since the dimension of $\mathcal{M}_G$ is $2n-2$ then locally, in a neighbourhood of any point x_0 in $\mathcal{M}_G$, we get a mapping $F : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-1}$. The construction of singular varieties $\mathcal{V}_G$ can be applied also in this case.

2) The construction of singular varieties $\mathcal{V}_G$ can be applied for polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^p$ where $p < n-1$ if the Milnor set $\mathcal{M}_G$ is smooth is this case.

4.2. A variety $\mathcal{V}_G$ in the case of the Broughton's Example.

EXAMPLE 4.13. We compute in this example a variety $\mathcal{V}_G$ in the case of the Broughton's example [2]:

$$G : \mathbb{C}^2 \to \mathbb{C}, \qquad G(z, w) = z + z^2 w.$$

We see that $K_0(G) = \varnothing$ since the system of equations $G_z = G_w = 0$ has no solutions. Let us denote

$$z = x_1 + ix_2, \qquad w = x_3 + ix_4,$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Consider G as a real polynomial mapping, we have

$$G(x_1, x_2, x_3, x_4) = (x_1 + x_1^2 x_3 - x_2^2 x_3 - 2x_1x_2x_4, x_2 + 2x_1x_2x_3 + x_1^2 x_4 - x_2^2 x_4).$$

Let $\rho = |w|^2$, then

$$\varphi = \frac{1}{1+\rho} = \frac{1}{1+|w|^2} = \frac{1}{1 + x_3^2 + x_4^2}.$$

The Jacobian matrix of (G, ρ) is

$$D_{\mathbb{R}}(G, \rho) = \begin{pmatrix} 1 + 2x_1x_3 - 2x_2x_4 & -2x_2x_3 - 2x_1x_4 & x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_2x_3 + 2x_1x_4 & 1 + 2x_1x_3 - 2x_2x_4 & 2x_1x_2 & x_1^2 - x_2^2 \\ 0 & 0 & 2x_3 & 2x_4 \end{pmatrix}.$$

By an easy computation, we have $\mathcal{M}_G = Sing(G, \rho) = M_1 \cup M_2$, where
$M_1 := \{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{R}\}$,
$M_2 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 1 + 2x_1x_3 - 2x_2x_4 = 2x_2x_3 + 2x_1x_4 = 0\right\}$.
Let us consider G as a real mapping from $\mathbb{R}^4_{(x_1,x_2,x_3,x_4)}$ to $\mathbb{R}^2_{(\alpha_1,\alpha_2)}$, then:

a) If $x = (x_1, x_2, 0, 0) \in M_1$, we have $G(x) = (x_1, x_2)$.

b) If $x = (x_1, x_2, x_3, x_4) \in M_2$, then we have $G(x) = (\alpha_1, \alpha_2)$, where

$$\alpha_1 = \frac{-x_3}{4(x_3^2 + x_4^2)}, \qquad \alpha_2 = \frac{x_4}{4(x_3^2 + x_4^2)}.$$

Let $F := G_{|\mathcal{M}_G}$. We can check easily that $K_0(F) = \varnothing$. Choosing $\mathcal{M}_G$ as a covering of $\mathcal{M}_G$ itself. We choose the Nash function $\psi = \varphi$, then ψ is positive on all $\mathcal{M}_G$. So, by Definition 4.2, we have

$$\mathcal{V}_G = \overline{(F, \varphi)(\mathcal{M}_G)} = \overline{(G, \varphi)(\mathcal{M}_G)} = (G, \varphi)(M_1) \cup (G, \varphi)(M_2) \cup (\mathcal{S}_G \times 0_{\mathbb{R}}),$$

where $(G, \varphi) : \mathbb{R}^4_{(x_1,x_2,x_3,x_4)} \to \mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3)}$. Then

a) $(G, \varphi)(M_1)$ is the plane $\{\alpha_3 = 1\} \subset \mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3,)}$.

b) Assume that $(\alpha_1, \alpha_2, \alpha_3) \in (G, \varphi)(M_2)$, and let

$$x_3 = rcos\theta, \qquad x_4 = rsin\theta,$$

where $r \in \mathbb{R}$, $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$, then

$$\alpha_1^2 + \alpha_2^2 = \frac{1}{16r^2}, \qquad \alpha_3 = \frac{1}{1 + r^2}.$$

So $(G, \varphi)(M_2)$ is a 2-dimensional open cone. In fact, when r tends to infinity, then α_1, α_2 and α_3 tend to 0, but the origin does not belong to this cone.

Moreover, by an easy computation, we can verify that the set $\mathcal{S}_G$ is $0 = (0,0) \in \mathbb{R}^2_{(\alpha_1,\alpha_2)}$. So the origin 0 of $\mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ belongs to $\mathcal{V}_G$. In conclusion, the variety $\mathcal{V}_G$ is the union of the plane $\alpha_3 = 1$ and a 2-dimensional cone $\mathcal{C}$ with vertex 0, where the cone $\mathcal{C}$ tends to infinity and asymptotic to the plane $\alpha_3 = 1$ in $\mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ (see Figure 1).

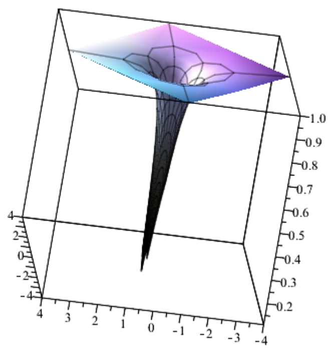

FIG. 1. *A variety $\mathcal{V}_G$ in the case of the Broughton's Example $G(z,w) = z + z^2 w$.*

REMARK 4.14. We can use the Proposition 3.5 with the view of mixed functions (see [19]) to determine the variety $\mathcal{M}_G$. Let us return to the Example 4.13. In this case $\rho = |w|^2$, then

$$\mathcal{M}_G = \left\{ (z,w) \in \mathbb{C}^2 : \frac{\partial G}{\partial z} \overline{w} = 0 \right\}.$$

Hence we have $(1 + 2zw)\overline{w} = 0$, that implies the following two cases:

i) $\overline{w} = 0$: We have $x_3 = x_4 = 0$, where $w = x_3 + ix_4$.

ii) $\overline{w} \neq 0$ and $z = -\frac{1}{2w} = -\frac{\overline{w}}{2|w|^2}$: We have

$$x_1 = \frac{-x_3}{2(x_3^2 + x_4^2)}, \qquad x_2 = \frac{x_4}{2(x_3^2 + x_4^2)},$$

where $z = x_1 + ix_2$.

So we get $\mathcal{M}_G = M_1 \cup M_2$ as the computations and notations in the Example 4.13.

5. Results.

THEOREM 5.1. *Let $G = (G_1, G_2) : \mathbb{C}^3 \to \mathbb{C}^2$ be a polynomial mapping such that $K_0(G) = \varnothing$. If one the groups $IH_2^{\overline{t},c}(\mathcal{V}_G, \mathbb{R})$, $IH_2^{\overline{t},cl}(\mathcal{V}_G, \mathbb{R})$, $H_2^c(\mathcal{V}_G, \mathbb{R})$ and $H_2^{cl}(\mathcal{V}_G, \mathbb{R})$ is trivial then the bifurcation set $B(G)$ is empty.*

THEOREM 5.2. *Let $G = (G_1, \ldots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geqslant 4)$ be a polynomial mapping such that $K_0(G) = \varnothing$ and $\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\ldots,n-1} > n - 3$, where $\hat{G}_i$*

is the leading form of G_i, that is the homogenous part of highest degree of G_i, for $i = 1, \ldots, n-1$. Then if one the groups $IH_2^{\bar{t},c}(\mathcal{V}_G, \mathbb{R})$, $IH_2^{\bar{t},cl}(\mathcal{V}_G, \mathbb{R})$, $H_2^c(\mathcal{V}_G, \mathbb{R})$, $H_2^{cl}(\mathcal{V}_G, \mathbb{R})$, $H_{2n-4}^c(\mathcal{V}_G, \mathbb{R})$ and $H_{2n-4}^{cl}(\mathcal{V}_G, \mathbb{R})$ is trivial then the bifurcation set $B(G)$ is empty.

Before proving these Theorems, we recall some necessary Definitions and Lemmas.

Definition 5.3. A *semi-algebraic family of sets (parametrized by $\mathbb{R}$)* is a semi-algebraic set $A \subset \mathbb{R}^n \times \mathbb{R}$, the last variable being considered as parameter.

Remark 5.4. A semi-algebraic set $A \subset \mathbb{R}^n \times \mathbb{R}$ will be considered as a family parametrized by $t \in \mathbb{R}$. We write A_t, for "the fiber of A at t", *i.e.*:

$$A_t := \{x \in \mathbb{R}^n : (x, t) \in A\}.$$

Lemma 5.5 ([21]). *Let β be a j-cycle and let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a compact semi-algebraic family of sets with $|\beta| \subset A_t$ for any t. Assume that $|\beta|$ bounds a $(j+1)$-chain in each A_t, $t > 0$ small enough. Then β bounds a chain in A_0.*

Definition 5.6 ([21]). Given a subset $X \subset \mathbb{R}^n$, we define the *"tangent cone at infinity"*, called "*contour apparent à l'infini*" in [16] by:

$$C_\infty(X) := \{\lambda \in \mathbb{S}^{n-1}(0,1) \text{ such that } \exists \eta : (t_0, t_0 + \varepsilon] \to X \text{ semi-algebraic},$$
$$\lim_{t \to t_0} \eta(t) = \infty, \lim_{t \to t_0} \frac{\eta(t)}{|\eta(t)|} = \lambda\}.$$

Lemma 5.7 ([18]). *Let $G = (G_1, \ldots, G_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial mapping and V the zero locus of $\hat{G} := (\hat{G}_1, \ldots, \hat{G}_m)$, where $\hat{G}_i$ is the leading form of G_i. If X is a subset of $\mathbb{R}^n$ such that $G(X)$ is bounded, then $C_\infty(X)$ is a subset of $\mathbb{S}^{n-1}(0,1) \cap V$, where $V = \hat{G}^{-1}(0)$.*

Proof of the Theorem 5.1. Recall that in this case, $\dim_{\mathbb{R}} \mathcal{V}_G = 4$ (Proposition 4.4) and $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$ is not smooth in general. Consider a stratification of $\mathcal{V}_G$, the strata of which are the strata of $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$ union the strata of the stratification of $K_0(F)$ defined by the rank, according to Thom [20]. Assume that $B(G) \neq \emptyset$, then by Remark 4.6, the set $\mathcal{S}_G$ is not empty. This means that there exists a complex Puiseux arc in $\mathcal{M}_G$

$$\gamma : D(0, \eta) \to \mathbb{R}^6, \quad \gamma = uz^\alpha + \ldots,$$

(with α negative integer and u is an unit vector of $\mathbb{R}^6$) tending to infinity such a way that $G(\gamma)$ converges to a generic point $x_0 \in \mathcal{S}_G$. Then, the mapping $h_F \circ \gamma$, where $h_F = (F, \varphi_1, \ldots, \varphi_p)$ and F is the restriction of G on $\mathcal{M}_G$ provides a singular 2-simplex in $\mathcal{V}_G$ that we will denote by c. We prove now the simplex c is $(\bar{t}, 2)$-allowable for total perversity $\bar{t}$. In fact, by [14], in this case we have $\dim_{\mathbb{C}} \mathcal{S}_G \leqslant 1$, then $\alpha = \mathrm{codim}_{\mathbb{R}} \mathcal{S}_G \geqslant 2$. The condition

$$0 = \dim_{\mathbb{R}}\{x_0\} = \dim_{\mathbb{R}}((\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}) \cap |c|) \leqslant 2 - \alpha + t_\alpha,$$

implies $t_\alpha \geqslant \alpha - 2$, with $\alpha \geqslant 2$, which is true for total perversity $\bar{t}$. Take now a stratum V_i of $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$. Denote by $\beta = \mathrm{codim}_{\mathbb{R}} V_i$. If $\beta \geqslant 2$, we can choose the Puiseux arc γ such that c lies in the regular part of $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$. In fact, this comes from the generic position of transversality. So c is $(\bar{t}, 2)$-allowable in this case. We need to consider only the cases $\beta = 0$ and $\beta = 1$. We have the following two cases:

1) If V_i intersects c, again by the generic position of transversality, we can choose the Puiseux arc γ such that $0 \leqslant \dim_{\mathbb{R}}(V_i \cap |c|) \leqslant 1$. The condition

$$\dim_{\mathbb{R}}(V_i \cap |c|) \leqslant 2 - \beta + t_\beta$$

holds since $2 - \beta + t_\beta \geqslant 1$, for $\beta = 0$ and $\beta = 1$.

2) If V_i does not meet c, then the condition

$$-\infty = \dim_{\mathbb{R}} \varnothing = \dim_{\mathbb{R}}(V_i \cap |c|) \leqslant 2 - \beta + t_\beta$$

holds always.

So the simplex c is $(\bar{t}, 2)$-allowable for total perversity $\bar{t}$.

From here, the proof of the Theorem follows the ideas of [21]: We can always choose the Puiseux arc such that the support of ∂c lies in the regular part of $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$. We have

$$H_1(\mathrm{Reg}(\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}))) = 0,$$

then the chain ∂c bounds a singular chain $e \in C^2(\mathrm{Reg}(\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})))$, where e is a chain with compact supports or closed supports. So $\sigma = c - e$ is a $(\bar{t}, 2)$-allowable cycle of $\mathcal{V}_G$, with compact supports or closed supports.

We claim that σ may not bound a 3-chain in $\mathcal{V}_G$. Assume otherwise, *i.e.* assume that there is a chain $\tau \in C_3(\mathcal{V}_G)$, satisfying $\partial \tau = \sigma$. Let

$$A := h_F^{-1}(|\sigma| \cap (\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}))),$$

$$B := h_F^{-1}(|\tau| \cap (\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}))).$$

By definition, $C_\infty(A)$ and $C_\infty(B)$ are subsets of $\mathbb{S}^5(0,1)$. Observe that, in a neighborhood of infinity, A coincides with the support of the Puiseux arc γ. The set $C_\infty(A)$ is equal to $\mathbb{S}^1.a$ (denoting the orbit of $a \in \mathbb{C}^3$ under the action of $\mathbb{S}^1$ on $\mathbb{C}^3$, $(e^{i\eta}, z) \mapsto e^{i\eta} z$). Let V be the zero locus of the leading forms $\hat{G} := (\hat{G}_1, \hat{G}_2)$. Since $G(A)$ and $G(B)$ are bounded, by Lemma 5.7, $C_\infty(A)$ and $C_\infty(B)$ are subsets of $V \cap \mathbb{S}^5(0,1)$.

For R large enough, the sphere $\mathbb{S}^5(0,R)$ with center 0 and radius R in $\mathbb{R}^6$ is transverse to A and B (at regular points). Let

$$\sigma_R := \mathbb{S}^5(0,R) \cap A, \qquad \tau_R := \mathbb{S}^5(0,R) \cap B.$$

Then σ_R is a chain bounding the chain τ_R. Consider a semi-algebraic strong deformation retraction $\Phi : W \times [0;1] \to \mathbb{S}^1.a$, where W is a neighborhood of $\mathbb{S}^1.a$ in $\mathbb{S}^5(0,1)$ onto $\mathbb{S}^1.a$. Considering R as a parameter, we have the following semi-algebraic families of chains:

1) $\tilde{\sigma}_R := \frac{\sigma_R}{R}$, for R large enough, then $\tilde{\sigma}_R$ is contained in W,

2) ${\sigma'}_R = \Phi_1(\tilde{\sigma}_R)$, where $\Phi_1(x) := \Phi(x,1), \qquad x \in W$,

3) $\theta_R = \Phi(\tilde{\sigma}_R)$, we have $\partial \theta_R = \sigma'_R - \tilde{\sigma}_R$,

4) ${\theta'}_R = \tau_R + \theta_R$, we have $\partial \theta'_R = \sigma'_R$.

As, near infinity, σ_R coincides with the intersection of the support of the arc γ with $\mathbb{S}^5(0,R)$, for R large enough the class of σ'_R in $\mathbb{S}^1.a$ is nonzero.

Let $r = 1/R$, consider r as a parameter, and let $\{\tilde{\sigma}_r\}$, $\{\sigma'_r\}$, $\{\theta_r\}$ as well as $\{\theta'_r\}$ the corresponding semi-algebraic families of chains.

Denote by $E_r \subset \mathbb{R}^6 \times \mathbb{R}$ the closure of $|\theta_r|$, and set $E_0 := (\mathbb{R}^6 \times \{0\}) \cap E$. Since the strong deformation retraction Φ is the identity on $C_\infty(A) \times [0,1]$, we see that

$$E_0 \subset \Phi(C_\infty(A) \times [0,1]) = \mathbb{S}^1.a \subset V \cap \mathbb{S}^5(0,1).$$

Denote by $E'_r \subset \mathbb{R}^6 \times \mathbb{R}$ the closure of $|\theta'_r|$, and set $E'_0 := (\mathbb{R}^6 \times \{0\}) \cap E'$. Since A bounds B, then $C_\infty(A)$ is contained in $C_\infty(B)$. We have

$$E'_0 \subset E_0 \cup C_\infty(B) \subset V \cap \mathbb{S}^5(0,1).$$

The class of σ'_r in $\mathbb{S}^1.a$ is, up to a product with a nonzero constant, equal to the generator of $\mathbb{S}^1.a$. Therefore, since σ'_r bounds the chain θ'_r, the cycle $\mathbb{S}^1.a$ must bound a chain in $|\theta'_r|$ as well. By Lemma 5.5, this implies that $\mathbb{S}^1.a$ bounds a chain in E'_0 which is included in $V \cap \mathbb{S}^5(0,1)$.

The set V is a projective variety which is an union of cones in $\mathbb{R}^6$. Since $\dim_{\mathbb{C}} V \leqslant 1$, so $\dim_{\mathbb{R}} V \leqslant 2$ and $\dim_{\mathbb{R}} V \cap \mathbb{S}^5(0,1) \leqslant 1$. The cycle $\mathbb{S}^1.a$ thus bounds a chain in $E'_0 \subseteq V \cap \mathbb{S}^5(0,1)$, which is a finite union of circles, that provides a contradiction. □

Now we provides the proof of the Theorem 5.2.

Proof. [Proof of the Theorem 5.2]

The proof of this Theorem follows the idea of [18] and the proof of Theorem 5.1.

Assume that $B(G) \neq \emptyset$. Similarly to the proof of the Theorem 5.1 and with the same notations in this proof but for the general case, we have: since

$$\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\dots,n-1} > n-3$$

then

$$\mathrm{corank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\dots,n-1} = \dim_{\mathbb{C}} V \leqslant 1,$$

so $\dim_{\mathbb{R}} V \leqslant 2$ and $\dim_{\mathbb{R}} V \cap \mathbb{S}^{2n-1}(0,1) \leqslant 1$. The cycle $\mathbb{S}^1.a$ bounds a chain in $E'_0 \subseteq V \cap \mathbb{S}^{2n-1}(0,1)$, which is a finite union of circles, that provides a contradiction. So we have

$$IH_2^{\bar{t},c}(\mathcal{V}_G, \mathbb{R}) \neq 0, \quad IH_2^{\bar{t},cl}(\mathcal{V}_G, \mathbb{R}) \neq 0, \quad H_2^{c}(\mathcal{V}_G, \mathbb{R}) \neq 0 \text{ and } H_2^{cl}(\mathcal{V}_G, \mathbb{R}) \neq 0.$$

From the Goresky-MacPherson Poincaré duality Theorem, we have

$$IH_2^{\bar{t},c}(\mathcal{V}_G, \mathbb{R}) = IH_{2n-4}^{\bar{0},cl}(\mathcal{V}_G, \mathbb{R}) \text{ and } IH_2^{\bar{t},cl}(\mathcal{V}_G, \mathbb{R}) = IH_{2n-4}^{\bar{0},c}(\mathcal{V}_G, \mathbb{R}),$$

that implies $H_{2n-4}^{c}(\mathcal{V}_G, \mathbb{R}) \neq 0$ and $H_{2n-4}^{cl}(\mathcal{V}_G, \mathbb{R}) \neq 0$. □

REMARK 5.8. The variety $\mathcal{V}_G$ associated to a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is not unique, but the result of the theorems 5.1 and 5.2 hold for any variety $\mathcal{V}_G$ among the constructed varieties $\mathcal{V}_G$ associated to G.

With the conditions of Theorem 5.2, the result of [12] also holds, hence as a consequence of Theorem 5.2 in this paper and Theorems 2.1 and 2.6 in [12], we obtain the following corollary.

COROLLARY 5.9. *Let $G = (G_1, \dots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geqslant 4$, be a polynomial mapping such that $K_0(G) = \emptyset$. Assume that the zero set $\{z \in \mathbb{C}^n :$*

$\hat{G}_i(z) = 0, i = 1, \ldots, n-1\}$ has complex dimension one, where $\hat{G}_i$ is the leading form of G_i. If the Euler characteristic of $G^{-1}(z^0)$ is bigger than that of the generic fiber, where $z^0 \in \mathbb{C}^{n-1}$, then

1) *$H_2(\mathcal{V}_G(\rho), \mathbb{R}) \neq 0$, for any ρ,*
2) *$H_{2n-4}(\mathcal{V}_G(\rho), \mathbb{R}) \neq 0$, for any ρ,*
3) *$IH_2^{\bar{t}}(\mathcal{V}_G(\rho), \mathbb{R}) \neq 0$, for any ρ, where $\bar{t}$ is the total perversity.*

Proof. At first, since the zero set $\{z \in \mathbb{C}^n : \hat{G}_i(z) = 0, i = 1, \ldots, n-1\}$ has complex dimension one, then by the Theorem 2.6 in [12], any generic linear mapping L is a very good projection with respect to any regular value z^0 of G. Then if the Euler characteristic of $G^{-1}(z^0)$ is bigger than that of the generic fiber, where $z^0 \in \mathbb{C}^{n-1}$, then by the Theorem 2.1 of [12], the set $B(G) \neq \emptyset$. Moreover, the complex dimension of the set $\{z \in \mathbb{C}^n : \hat{G}_i(z) = 0, i = 1, \ldots, n-1\}$ is the complex *corank* of $(D\hat{G}_i)_{i=1,\ldots,n-1}$. Hence $\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\ldots,n-1} = n-2$, and by the Theorem 5.2, we finish the proof. □

EXAMPLE 5.10. Consider the suspension of the Broughton's example:

$$G : \mathbb{C}^3 \to \mathbb{C}^2, \quad G(z, w, \zeta) = (z + z^2 w, \zeta),$$

or, more general $G(z, w, \zeta) = (z + z^2 w, g(\zeta))$ where $g(\zeta)$ is any polynomial of variable ζ and $g'(\zeta) \neq 0$. We can check that, for any function ρ, we have always $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) \neq 0$.

REMARK 5.11. The condition $B(G) = \emptyset$ does not imply $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) = 0$, since in this case $\mathcal{S}_G$ maybe not empty.

EXAMPLE 5.12. Let

$$G : \mathbb{C}^3 \to \mathbb{C}^2, \quad G(z, w, \zeta) = (z, z\zeta^2 + w).$$

1) If we choose the function $\rho = |\zeta|^2$, then $\mathcal{S}_G = \emptyset$ and $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) = 0$.
2) If we choose the function $\rho = |w|^2$, then $\mathcal{S}_G \neq \emptyset$ and $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) \neq 0$.

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