ASIAN J. MATH.
Vol. 22, No. 6, pp. 1127–1156, December 2018

RIGIDITY THEOREM OF GRAPH-DIRECTED FRACTALS*

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Abstract. In this paper, we identify two fractals if and only if they are bilipschitz equivalent. Fix a ratio r, for dust-like graph-directed sets with ratio r and integer characteristic, we obtain a rigid theorem that these graph-directed sets are uniquely determined by their Hausdorff dimension (or integer characteristic) in the sense of bilipschitz equivalence. Using this rigidity theorem, we show that in a suitable class of self-similar sets, two totally disconnected self-similar sets without complete overlaps are bilipschitz equivalent. We also provide an algorithm to test complete overlaps in polynomial time.

Key words. Fractal, bilipschitz equivalence, graph-directed sets, self-similar set.

Mathematics Subject Classification. 28A80.

1. Introduction.

1.1. Rigidity of geometry. Roughly speaking, a rigidity theorem states that every element in a class of mathematical objects is uniquely determined by less information. For example, harmonic functions on the unit disk are rigid in the sense that they are uniquely determined by their boundary values.

In metric spaces, the uniformization theorem on quasisymmetric equivalence (Proposition 15.11 of [5]) states: a compact metric space is *quasisymmetrically equivalent* to the symbolic space $\Sigma_2 = \{0,1\}^\infty$ if and only if it is doubling, uniformly perfect and uniformly disconnected. In other words, the doubling, uniform perfectness and uniform disconnectedness are complete characteristics of symbolic space in the sense of quasisymmetric equivalence.

How about the rigidity theorem in geometry of fractals? What characteristics can be used to provide the main information in the rigidity theorem on fractals? It seems not a good choice to characterize fractals by quasisymmetric equivalence since quasisymmetric mappings do not preserve fractal dimensions. Falconer and Marsh [12] stated: "topology" may be regarded as the study of equivalence classes of sets under *homeomorphism*; in the same vein, "fractal geometry" is sometimes thought of as the study of equivalence classes of fractals under *bilipschitz mappings*. This leads to the following definition.

DEFINITION 1. We *identify* two metric spaces (X_1, ρ_1) and (X_2, ρ_2), denoted by $X_1 \simeq X_2$, if and only if they are bilipschitz equivalent, i.e., there is a bijection $f\colon (X_1,\rho_1) \to (X_2,\rho_2)$ such that for all $x, y \in X_1$, $C^{-1}\rho_1(x,y) \le \rho_2(f(x), f(y)) \le \rho_1(x,y)$, where $C > 1$ is a constant.

Bilipschitz mappings preserve many geometric properties, such as

- fractal dimensions: Hausdorff dimension, packing dimension etc;
- properties of measures: doubling, Ahlfors-David regularity etc;

*Received March 15, 2016; accepted for publication April 5, 2017. This work is supported by NSF of China (Nos. 11871227, 11831007, 11771226, 11471124, 11371329, 11071224, 11101159), K.C. Wong Magna Fund in Ningbo University, NCET, The Chinese University of Hong Kong and Morningside Center of Mathematics.

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- metric properties: uniform perfectness, uniform disconnectedness etc.

Consequently, bilipschitz equivalence provides much more information than fractal dimensions. Another motivation of studying bilipschitz equivalence of fractals comes from geometric group theory (see [3, 13]).

Falconer and Marsh [11] obtained a **rigidity theorem** on fractals in the sense of bilipschitz equivalence: the quasi-self-similar-circles are uniquely determined by their Hausdorff dimensions, i.e., two such circles are bilipschitz equivalent if and only if they have the same Hausdorff dimension. In this paper, we will show that this also holds for a suitable class of graph-directed fractals.

1.2. Graph-directed fractals with integer characteristic.

Definition 2 (graph-directed sets [10, 25]). Let (X,ρ) be a complete metric space. Let $G=(\mathcal{V},\mathcal{E})$ be a directed graph with vertex set $\mathcal{V}$ and directed-edge set $\mathcal{E}$. We write $\mathcal{E}_{i,j}$ for the set of edges from vertex i to vertex j, and $\mathcal{E}_{i,j}^k$ for the set of sequences of k edges $(e_1,\ldots,e_k)$ which form a directed path from vertex i to vertex j.

Suppose that for each edge $e\in\mathcal{E}$, there is a corresponding similarity $S_e\colon X\to X$ of ratio $r_e\in(0,1)$, i.e., S_e satisfies $\rho(S_e(x),S_e(y))=r_e\rho(x,y)$. The *graph-directed sets* on G with the similarities $\{S_e\}_{e\in\mathcal{E}}$ are defined to be the unique nonempty compact sets $\{K_i\}_{i\in\mathcal{V}}$ satisfying

$$K_i=\bigcup_{j\in\mathcal{V}}\bigcup_{e\in\mathcal{E}_{i,j}}S_e(K_j)\qquad\text{for } i\in\mathcal{V}. \tag{1.1}$$

In particular, if (1.1) is a disjoint union for each $i\in\mathcal{V}$, we call $\{K_i\}_{i\in\mathcal{V}}$ the *dust-like* graph-directed sets on $(\mathcal{V},\mathcal{E})$.

We say that graph-directed sets $\{K_i\}_i$ have ratio r if all the ratios r_e are equal to a common $r\in(0,1)$. If there is only one vertex in $\mathcal{V}$, the dust-like graph-directed set K is a self-similar set. Suppose that the number of edges is m. Let $\Sigma_m^r=\{1,2,\ldots,m\}^\infty$ be the symbolic space equipping with the metric

$$\rho(x_0x_1\ldots,y_0y_1\ldots)=r^{\min\{k\colon x_k\neq y_k\}}.$$

Then $K\simeq\Sigma_m^r$ if K has ratio r. To see this, notice that Σ_m^r can also be regarded as a dust-like self-similar set generated by m similitudes of ratio r.

Hence a natural question is: if there are two or more vertexes, what conditions ensure that the dust-like graph-directed sets are bilipschitz equivalent to the symbolic space Σ_m^r. To answer this question, we introduce the notion of integer characteristic.

Let $A=(a_{i,j})_{i,j\in\mathcal{V}}$ be the *adjacency matrix* of G defined by $a_{i,j}=\#\mathcal{E}_{i,j}$, the number of $\mathcal{E}_{i,j}$. We say that dust-like graph-directed sets $\{K_i\}_i$ with ratio r have integer characteristic $m\geq 2$ if there is a positive vector $v>0$ such that

$$Av=mv. \tag{1.2}$$

A directed graph is said to be *transitive* if for any vertices i,j, there exists a directed path starting at i and ending at j. In this case, the adjacency matrix A is *irreducible.* By the Perron-Frobenius Theorem for non-negative matrices, assumption (1.2) on a *transitive* graph is equivalent to that the Perron-Frobenius eigenvalue of A is an integer $m\geq 2$.

It will be proved in Section 3.3 that assumption (1.2) is equivalent to

$$0<\mathcal{H}^s(K_i)<\infty\quad\text{for } i\in\mathcal{V}\text{ with } s=-\log m/\log r. \tag{1.3}$$

Therefore, if $\{K_i\}_i$ are bilipschitz equivalent to the symbolic space Σ_m^r, they must have integer characteristic m. The following theorem states that the converse is also true and hence provides a natural and intrinsic characterization of the dust-like graph-directed sets which are bilipschitz equivalent to the symbolic space Σ_m^r.

THEOREM 1. *Let $\{K_i\}_{i=1}^N$ be dust-like graph-directed sets with ratio r. Then $K_1, \ldots, K_N$ are bilipschitz equivalent to Σ_m^r if and only if they have integer characteristic m.*

We remark that Theorem 1 doesn't need the transitive condition on the directed graph. In other words, the adjacency matrix A is not required to be irreducible.

Deng and He [6] first studied the Lipschitz equivalence of graph-directed fractals with integer characteristic. They proved the if part of Theorem 1 under some additional conditions which require the adjacency matrix to be "rearrangeable" or primitive (Theorem 1.3 and 1.4 in [6]). Such conditions are needed in their argument to avoid some complicated situations. We prove the general case by a careful analysis of the graph-directed structure (see the beginning of Section 4) and by making use of the technique of extension of bilipschitz mappings (see the proof of Proposition 4).

Let $\mathfrak{A}^r = \bigcup_{m=2}^{\infty} \mathfrak{A}_m^r$, where

$$\mathfrak{A}_m^r = \Big\{K\colon K = K_1 \text{ for some dust-like graph-directed sets } \{K_i\}_i \\ \text{with ratio } r \text{ and integer characteristic } m\Big\}.$$

As a corollary of Theorem 1, we have

THEOREM 2. *Two graph-directed fractals $K \in \mathfrak{A}_{m_1}^{r_1}$ and $K' \in \mathfrak{A}_{m_2}^{r_2}$ are bilipschitz equivalent if and only if $r_1^{k_1} = r_2^{k_2}$, $m_1^{k_1} = m_2^{k_2}$ for some integers $k_1, k_2 \geq 1$.*

The paper is organized as follows. In Section 2, we apply Theorem 1 to study the bilipschitz equivalence of a class of self-similar sets without complete overlaps and obtain Theorem 3. Section 2.3 provides an algorithm to test complete overlaps in polynomial time. We present some preliminaries in Section 3 including the techniques of number theory, non-negative matrix and bilipschitz equivalence. Section 4 is the proofs of Theorems 1 and 2. Section 5 is devoted to the proof of Theorem 3, in which total disconnectedness implies the graph-directed structure and the loss of complete overlaps insures the integer characteristic. The last section discusses some related open questions.

2. Bilipschitz equivalence of self-similar fractals.

2.1. Results on bilipschitz equivalence. The known results on the bilipschitz equivalence of self-similar sets can be divided into two main categories, according to whether the self-similar sets have overlaps or not. A self-similar set is said to be *without overlaps* if the strong separation condition (SSC) holds. Otherwise, it is said to be *with overlaps*.

Falconer and Marsh [12] gave two necessary conditions for self-similar sets satisfying the SSC (without overlaps) to be bilipschitz equivalent. Xi [37] further obtained a necessary and sufficient condition. For other characterizations of bilipschitz equivalence in this case, please also refer to Mattila and Saaranen [24], Llorente and Mattila [22], Deng, Wen, Xiong and Xi [8], Rao, Ruan and Wang [29] and Rao and Zhang [32].

Self-similar sets with overlaps have very *complicated* structures. Various conditions had been proposed to control the overlaps. For example, Moran [26], Hutchinson [16], Bandt and Graf [1] and Schief [34] studied the open set condition (OSC). Lau and Ngai [19] and Zerner [42] studied the weak separation condition. Ngai and Wang [27] and Lau and Ngai [20] studied the finite type condition.

There are also many efforts devoted to the study of bilipschitz equivalence of self-similar sets with overlaps. Wen and Xi [36] constructed two self-similar arcs which have the same dimension but are not Lipschitz equivalent. David and Semmes [5] asked whether two special self-similar sets, the {1,3,5}-set and {1,4,5}-set, are Lipschitz equivalent or not. Rao, Ruan and Xi [30], Xi and Xiong [40, 41] gave an affirmative answer to this problem in $\mathbb{R}^1$ and higher dimensional spaces respectively. In the case of different contraction ratios, Xi and Ruan [38], Ruan, Wang and Xi [33] studied the self-similar sets with touching structure on the line. Luo and Lau [23] and Deng, Lau and Luo [7] researched the Lipschitz equivalence of self-similar sets by using hyperbolic boundaries of trees.

Recently, Xi and Xiong [39] obtained a general result on the problem of the Lipschitz equivalence of self-similar sets in the OSC case. However, the argument in [39] is quite involved. As an application of Theorem 1, we can give a much simpler proof for self-similar sets without complete overlaps.

DEFINITION 3. Let $\{S_1,\dots,S_m\}$ be an IFS. We say that the corresponding self-similar set $E=\bigcup_{i=1}^m S_i(E)$ has complete overlaps, if there are two distinct sequences $i_1\dots i_t$, $j_1\dots j_{t'}$ such that

$$S_{i_1}\circ\cdots\circ S_{i_t}=S_{j_1}\circ\cdots\circ S_{j_{t'}}.$$

For surveys of the Hausdorff dimension of self-similar sets with complete overlaps, we refer to Kenyon [17] and Rao and Wen [31].

2.2. A class of self-similar fractals in $\mathfrak{A}^{1/n}$. Let Γ be a discrete additive group in $\mathbb{R}^l$ and $\mathbb{G}$ a finite subgroup of the isometric group on Γ, i.e., for any $g\in\mathbb{G}$, we have $g\colon\Gamma\to\Gamma$, $g(0)=0$ and $|g(a_1)-g(a_2)|=|a_1-a_2|$ for all $a_1,a_2\in\Gamma$. This implies that each $g\in\mathbb{G}$ can be extended to a *linear* isometry of $\mathbb{R}^l$. In particular, $g(B(0,\delta))=B(0,\delta)$ for any closed ball with center 0 and radius δ. Fix an integer $n\ge 2$. Consider the similitude

$$S(x)=g(x)/n+b,\quad \text{where } g\in\mathbb{G} \text{ and } b\in\Gamma. \tag{2.1}$$

Let Λ be the collection of all the self-similar sets in $\mathbb{R}^l$ generated by contractive similitudes in the form of (2.1).

As an application of Theorem 1, we will prove the following theorem on the bilipschitz equivalence. For a one-dimensional analogue, we refer to [6, Theorem 3.7].

THEOREM 3. *Suppose $E=\bigcup_{i=1}^m(\frac{g_iE}{n}+b_i)\in\Lambda$ is a totally disconnected self-similar set without complete overlaps. Then $E\in\mathfrak{A}^{1/n}$ with characteristic m, and thus E is bilipschitz equivalent to $\Sigma_m^{1/n}$.*

We present some examples in the remainder of this subsection and give an algorithm to test complete overlaps in Section 2.3.

EXAMPLE 1. Let $n\ge 2$, $\Gamma_1=\mathbb{Z}^2$ and $\Gamma_2=\{(\frac{a}{2},\frac{\sqrt{3}}{2}b)\colon a,b\in\mathbb{Z}\text{ and }a\equiv b(\text{mod}2)\}$, then Γ_i are discrete additive groups as in Figure 1. The isometric group of Γ_2 has 12 elements, including rotations and reflections.

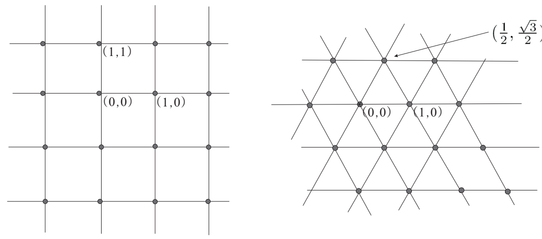

FIG. 1. *Discrete groups on the plane*

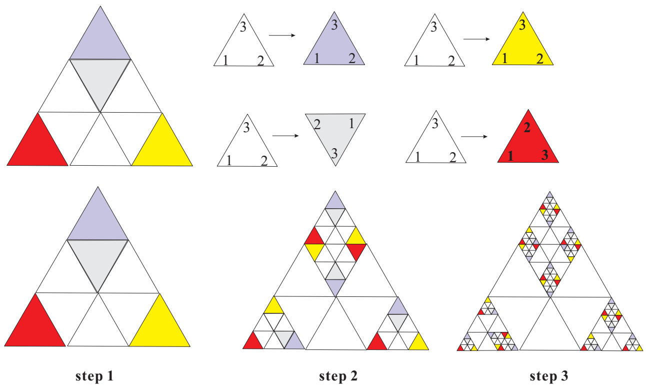

FIG. 2. *Initial pattern and first three steps in construction*

EXAMPLE 2. Let $n = 3$ and $\Gamma = \Gamma_2$ as in Example 1. As in Figure 2, we take 4 small colored triangles and select the corresponding isometries. We also show the first three steps in construction.

EXAMPLE 3. Let $\Gamma = \mathbb{Z}^l$ with $l \geq 2$; then its isometric group contains the element:

$$g(x_1, \ldots, x_l) = (s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, \ldots, s_l x_{\sigma(l)}),$$

where σ is a permutation on $\{1, \ldots, l\}$ and sign $s_i \in \{-1, 1\}$ for all i.

FIG. 3. *Different structures in F and F'*

EXAMPLE 4. For $n = 5$ and $l = 1$, let

$$F = \frac{F}{5} \cup \frac{-F+4}{5} \cup \frac{F+4}{5}, \quad F' = \frac{F'}{5} \cup \frac{F'+3}{5} \cup \frac{F'+4}{5}.$$

Due to the minus sign in the similarity $x \mapsto \frac{-x+4}{5}$, these two self-similar sets are quite different (Figure 3). It is easy to show that F and F' are totally disconnected self-similar sets without complete overlaps. Then it follows from Theorem 3 that F and F' are bilipschitz equivalent.

Finally, we consider the self-similar set E generated by

$$S_i(x) = x/n + b_i \text{ with } b_i \in \mathbb{Q}^l \text{ for } i = 1, \ldots, m.$$

Let Γ be the additive group generated by $\{b_i\}_i$. Suppose $n > m$; then $\dim_{\mathrm{H}} E \leq \log m/\log n < 1$ which implies E is totally disconnected. Theorem 3 implies that if E has no complete overlaps, then E is bilipschitz equivalent to $\Sigma_m^{1/n}$. Please see Example 6 for the self-similar set $E = \frac{E}{5} \cup (\frac{E}{5} + \frac{7}{10}) \cup (\frac{E}{5} + \frac{8}{10})$.

2.3. Algorithm in polynomial time to test complete overlaps in Λ. First we give the sketch of our algorithm. Considering the IFS in the form of (2.1), we can construct a directed graph (Steps 1-2) and calculate all vertexes named original vertexes and boundary vertexes (Step 3). Our algorithm is based on a criterion (Proposition 1) which states that the complete overlap exists if and only if there exists a directed path from an original vertex to a boundary vertex. By the criterion, we can use Dijkstra's algorithm [9] to test the existence of such a directed path (Step 4).

Now we describe our algorithm in detail. Given $E = \bigcup_{i=1}^m (\frac{g_i E}{n} + b_i) \in \Lambda$, let

$$M = 2 \max_{1 \leq i \leq m} |b_i|/(n-1).$$

Step 1. Set the vertex set

$$V = \{x \in \Gamma \colon |x| \leq M\} \times \mathbb{G} \times \mathbb{G}.$$

The elements of vertex set can be written to be (x, f, f').

Step 2. Set the edge set as follows.

We set an edge from vertex (x, f, f') to vertex (x_1, f_1, f_1'), if and only if there is a pair $(i, j) \in \{1, \ldots, m\}^2$ such that

$$(x_1, f_1, f_1') = (nx + f(b_i) - f'(b_j), fg_i, f'g_j),$$

where $f \circ g_i$ is written as fg_i to simplify the notation. We denote this edge by

$$(x, f, f') \xrightarrow{(i,j)} (x_1, f_1, f_1').$$

Step 3. Calculate all *original vertexes* and *boundary vertexes.*

The vertex (x, f, f') is called an *original vertex*, if $(x, f, f') = (b_i - b_j, g_i, g_j)$ for some $i \neq j$. We say that (x_1, f_1, f_1') is a *boundary vertex*, if $x_1 = 0$ and $f_1 = f_1'$. Using these definition, we can calculate all original vertexes and boundary vertexes.

Step 4. Test the existence of complete overlap by Dijkstra's algorithm.

Through steps 1–3, we obtain a finite graph since Γ is discrete and $\mathbb{G}$ is finite. We have the following criteria to test the existence of complete overlap.

Proposition 1. *E has complete overlaps if and only if there is a directed path starting at an original vertex and ending at a boundary vertex.*

Applying Dijkstra's algorithm, we can test whether there is such a directed path in polynomial time. Roughly speaking, Dijkstra's algorithm is an algorithm for finding the shortest paths between vertexes in a graph, runs in time $O((\#V)^2)$ where $\#V$ is the cardinality of the vertex set. When equipping each directed edge with weight 1, we find a directed path from one vertex to another if and only if their shortest distance between these two vertexes is finite. Suppose $E \subset \mathbb{R}^l$. Let $\delta = \min_{x \in \Gamma \setminus \{0\}} |x|$ be the least distance between any two distinct elements in Γ, then $\#V = O(M^l\delta^{-l}(\#\mathbb{G})^2)$. Thus the running time is at most $O\big(M^{2l}\delta^{-2l}(\#\mathbb{G})^4\big)$.

We provide two examples to illustrate the above algorithm.

Example 5. Let $n = 3$. Suppose $F \subset \mathbb{R}^2$ is generated by

$$S_1(x) = \frac{x}{3}, \quad S_2(x) = \frac{x}{3} + \alpha, \quad S_3(x) = \frac{x}{3} + 8\alpha, \quad S_4(x) = \frac{g(x)}{3} + 8\beta,$$

where $\alpha = (1, 0)$, $\beta = (1/2, \sqrt{3}/2)$, and g is an isometry such that

$$g^2 = \mathrm{id}, \quad g(\alpha) = \beta \quad \text{and} \quad g(\beta) = \alpha.$$

One can verify that the strong separation condition fails, i.e., there are overlaps. We will check the fact that there is no complete overlaps by the above algorithm. In fact, we have

$$\Gamma = \{k_1\alpha + k_2\beta \colon k_1, k_2 \in \mathbb{Z}\}$$

and $\mathbb{G} = \{\mathrm{id}, g\}$. Notice that $b_1 = 0$, $b_2 = \alpha$, $b_3 = 8\alpha$, $b_4 = 8\beta$, and $g_1 = g_2 = g_3 = \mathrm{id}$ and $g_4 = g$. We obtain that $M = 2\max_{1\le i\le 4}|b_i|/(n-1) = 8$. By programming, the set $\{x \in \Gamma \colon |x| \le M\}$ has 225 elements and the corresponding directed graph has 900 vertexes. Twelve original vertexes are

$$\begin{aligned}
&(\alpha, \mathrm{id}, \mathrm{id}), \quad (7\alpha, \mathrm{id}, \mathrm{id}), \quad (8\alpha, \mathrm{id}, \mathrm{id}), \quad (-\alpha, \mathrm{id}, \mathrm{id}),\\
&(-7\alpha, \mathrm{id}, \mathrm{id}), \quad (-8\alpha, \mathrm{id}, \mathrm{id}), \quad (-8\beta, \mathrm{id}, g), \quad (\alpha - 8\beta, \mathrm{id}, g),\\
&(8\alpha - 8\beta, \mathrm{id}, g), \quad (8\beta, g, \mathrm{id}), \quad (8\beta - \alpha, g, \mathrm{id}), \quad (8\beta - 8\alpha, g, \mathrm{id}),
\end{aligned}$$

and two boundary vertexes are $(0, \mathrm{id}, \mathrm{id})$ and $(0, g, g)$. Using Dijkstra's algorithm, we find that there does not exist any directed path from original vertexes to boundary vertexes. Therefore F is a self-similar set without complete overlaps.

If $\mathbb{G} = \{\mathrm{id}\}$, we need only consider the graph with the vertex set $\{x \in \Gamma \colon |x| \le M\}$, there is an edge from vertex x to vertex x_1, if and only if there is a pair $(i, j) \in \{1, \dots, m\}^2$ such that $x_1 = nx + b_i - b_j$. We denote the edge by $x \xrightarrow{(i,j)} x_1$.

Example 6. Let $5 = n > m = 3$ and $E = \frac{E}{5} \cup (\frac{E}{5} + \frac{7}{10}) \cup (\frac{E}{5} + \frac{8}{10})$. Here

$$n = 5, \quad \mathbb{G} = \{\mathrm{id}\}, \quad \Gamma = \mathbb{Z}/10, \quad M = 0.4$$

and $\{x \in \Gamma \colon |x| \le M\} = \{-0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4\}$. The original vertices are $\{-0.1, 0.1\}$ and the unique boundary vertex is $\{0\}$. If $x \xrightarrow{(i,j)} 0$ for some $x \in \{x \in \Gamma \colon |x| \le M\}$, then $0 = 5x + b_i - b_j$, that is

$$5x = b_j - b_i.$$

However, $b_j - b_i \in P = \{-0.8, -0.7, -0.1, 0, 0.1, 0.7, 0.8\}$ and

$$P \cap \{5x \colon x \in \Gamma, |x| \le M\} = \{0\}.$$

Therefore, $x = 0$. This means there is no directed path from original vertices to boundary vertices. Then E is totally disconnected self-similar set without complete overlaps and thus E is bilipschitz equivalent to $\Sigma_3^{1/5}$.

3. Preliminaries.

3.1. Combinatorial lemma from Frobenius coin problem. It is easy to check that $\{2k_1 + 5k_2 : k_1, k_2 \in \mathbb{N} \cup \{0\}\} \supset \{m : m > 3\}$. This is a special case of the so called Frobenius coin problem [28], which can be stated as follows. Given positive integers $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ with $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_\ell) = 1$, find the largest integer that *cannot* be expressed as a sum $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_\ell a_\ell$, where $k_1, k_2, \ldots, k_\ell$ are non-negative integers. We denote such integer by $\phi(\alpha_1, \ldots, \alpha_\ell)$. Hence $\phi(2, 5) = 3$.

Lemma 1. *Fix positive integers γ and $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ ($\ell \geq 2$) with $\gcd(\alpha_1, \ldots, \alpha_\ell) = 1$. Write $\phi^* = \phi(\alpha_1, \ldots, \alpha_\ell) + \max_{1 \leq j \leq \ell} \alpha_j$. Let $(\beta_\omega)_{\omega \in \Omega}$ be a sequence of positive integers, where Ω is a finite index set, such that*
(i) *$\beta_\omega \in \{\alpha_1, \ldots, \alpha_\ell\}$ for all $\omega \in \Omega$;*
(ii) *$\sum_{\omega \in \Omega} \beta_\omega = b \cdot \gamma$ for positive integers $b > 2\phi^* \sum_{j=1}^{\ell} \alpha_j$ and $\gamma \geq 1$;*
(iii) *for every $1 \leq j \leq \ell$, $\#\{\omega \in \Omega : \beta_\omega = \alpha_j\} \geq \gamma\phi^*$.*
Then there is a decomposition $\Omega = \bigcup_{1 \leq t \leq \gamma} \Omega_t$ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and

$$\sum_{\omega \in \Omega_t} \beta_\omega = b \quad \textit{for every } 1 \leq t \leq \gamma.$$

Proof. We will prove this by inductive on the integer $\gamma \geq 1$. It is trivial when $\gamma = 1$. Now suppose this is true for $\gamma - 1 \geq 1$, we shall prove that so does γ.

By (iii), there are disjoint $\Omega'_1, \ldots, \Omega'_\ell \subset \Omega$ such that, for $1 \leq j \leq \ell$,

$$\beta_\omega = \alpha_j \quad \text{for } \omega \in \Omega'_j \quad \text{and} \quad \#\Omega'_j = \gamma\phi^*. \tag{3.1}$$

Let $\Omega^* = \Omega \setminus \bigcup_{j=1}^{\ell} \Omega'_j$. By (ii), (3.1) and $\gamma \geq 2$, we have

$$\sum_{\omega \in \Omega^*} \beta_\omega = \sum_{\omega \in \Omega} \beta_\omega - \sum_{j=1}^{\ell} \sum_{\omega \in \Omega'_j} \beta_\omega = \gamma\left(b - \phi^* \sum_{j=1}^{\ell} \alpha_j\right) \geq \gamma b/2 \geq b.$$

Combining this with (i), we can select $\Omega^*_0 \subset \Omega^*$ such that

$$\phi(\alpha_1, \ldots, \alpha_\ell) < b - \sum_{\omega \in \Omega^*_0} \beta_\omega \leq \phi(\alpha_1, \ldots, \alpha_\ell) + \max_{1 \leq j \leq \ell} \alpha_j = \phi^*.$$

This together with the definition of the Frobenius number implies that we can find non-negative integers $k_1, \ldots, k_\ell$ such that

$$k_1\alpha_1 + \cdots + k_\ell\alpha_\ell = b - \sum_{\omega \in \Omega^*_0} \beta_\omega \leq \phi^*. \tag{3.2}$$

Clearly, $k_j \leq \phi^*$ for every $1 \leq j \leq \ell$.

By (3.1), we can take $\Omega^*_j \subset \Omega'_j$ with $\#\Omega^*_j = k_j \leq \phi^*$ for every $1 \leq j \leq \ell$. Now let $\Omega_1 = \Omega^*_0 \cup \bigcup_{j=1}^{\ell} \Omega^*_j$. Then by (3.1) and (3.2), we have

$$\sum_{\omega \in \Omega_1} \beta_\omega = \sum_{\omega \in \Omega^*_0} \beta_\omega + \sum_{j=1}^{\ell} \sum_{\omega \in \Omega^*_j} \beta_\omega = \sum_{\omega \in \Omega^*_0} \beta_\omega + \sum_{j=1}^{\ell} k_j\alpha_j = b.$$

Let $\Omega' = \Omega \setminus \Omega_1$, then

$$\sum_{\omega\in\Omega'} \beta_\omega = \sum_{\omega\in\Omega} \beta_\omega - \sum_{\omega\in\Omega_1} \beta_\omega = \gamma b - b = (\gamma-1)b.$$

And for every $1 \le t \le \ell$, we have

$$\#\{\omega \in \Omega' \colon \beta_\omega = \alpha_j\} \ge \#\Omega'_j - \#\Omega^*_j = \gamma\phi^* - k_j \ge \gamma\phi^* - \phi^* = (\gamma-1)\phi^*.$$

Hence the lemma follows by applying the inductive assumption to Ω'. □

3.2. Connected blocks of directed Graphs. Given a directed graph G, we give an equivalent relation on the vertex set as follows: let $i \sim i$; for distinct vertices i and j,

$$i \sim j \iff \text{there are directed paths from } i \text{ to } j \text{ and from } j \text{ to } i.$$

We call the equivalence class $[i] = \{j \colon j \sim i\}$ a *connected block*.

We also define a *partial order* among connected blocks as follows:

$$[j] \prec [j'] \iff \text{there is a directed path from a vertex in } [j] \text{ to a vertex in } [j'].$$

Under this partial order, we write

$$\mathfrak{B}_0 = \{[i] \colon [i] \text{ is maximal}\}.$$

Inductively, let

$$\mathfrak{B}_k = \left\{[i] \colon [i] \text{ is maximal in the complement of } \bigcup_{q\le k-1} \mathfrak{B}_q\right\}.$$

We say that a vertex has rank k if this vertex belongs to a connected block in $\mathfrak{B}_k$. By the definition of $\mathfrak{B}_k$, we have

Lemma 2. *Given a vertex i of rank k, then any directed path starting at i will end either at some vertex of rank $\le k-1$ or at some vertex j of rank k with $i \sim j$.*

3.3. Integer characteristic.

Lemma 3. *Assumptions* (1.2) *and* (1.3) *on integer characteristic are equivalent.*

Proof. Suppose the vertex set $\mathcal{V} = \{1, 2, \dots, N\}$. If assumption (1.3) holds, then by the definition of the adjacency matrix, we have

$$\mathcal{H}^s(K_i) = r^s \sum_{j=1}^{N} a_{i,j}\mathcal{H}^s(K_j) \quad \text{for } i \in \{1, 2, \dots, N\},$$

where $s = -\log m/\log r$. Since $r^s = 1/m$, we have

$$Av = mv$$

with $v = (\mathcal{H}^s(K_1), \dots, \mathcal{H}^s(K_p))^T > 0$.

If assumption (1.2) holds, we first show that $\mathcal{H}^s(K_i) < \infty$ for any vertex i. Take $v > 0$ such that $Av = mv$ and $v > (1, \dots, 1)^T$. Let $A^k = (a_{i,j}^{(k)})_{i,j}$. Then

$$\sum_{i,j} a_{i,j}^{(k)} = (1, \dots, 1)A^k(1, \dots, 1)^T \leq (1, \dots, 1)A^k v = m^k((1, \dots, 1) \cdot v). \tag{3.3}$$

For any k, we have $K_i = \bigcup_j \bigcup_{e^* \in \mathcal{E}_{i,j}^k} S_{e^*}(K_j)$, where $S_{e^*} = S_{e_1} \circ \cdots \circ S_{e_k}$ for $e^* = (e_1, \dots, e_k)$. By (3.3), for each i, we have

$$\begin{aligned}\mathcal{H}^s(K_i) &\leq \lim_{k\to\infty} \left(\sum_j a_{i,j}^{(k)}\right) \cdot (r^k)^s \max_j |K_j|^s \leq \lim_{k\to\infty} \left(\sum_{i,j} a_{i,j}^{(k)}\right) \cdot m^{-k} \max_j |K_j|^s \\ &\leq ((1, \dots, 1)v) \cdot \max_j |K_j|^s < \infty.\end{aligned}$$

where $|K_j|$ denotes the diameter of K_j.

Then we will show $\mathcal{H}^s(K_i) > 0$ for all i by induction on the rank of i.

Let i be a vertex of rank 0. By a permutation if necessary, we can assume that $[i] = \{1, 2, \dots, p\}$. According to the definition of rank 0, every edge starting at a vertex in $[i]$ is also ending at a vertex in $[i]$. Hence the adjacency matrix A must have the form

$$A = \begin{pmatrix} B_{p\times p} & 0 \\ C_{p\times q} & D_{q\times q} \end{pmatrix} \qquad \text{with } p + q = N.$$

Here $B_{p\times p}$ is a irreducible matrix since for any $j_1, j_2 \in [i] = \{1, 2, \dots, p\}$, there is a directed path starting at j_1 and ending at j_2.

Therefore $K_1, K_2, \dots, K_p$ are graph-directed sets with irreducible adjacency matrix $B = B_{p\times p}$. Since $Av = mv$ with $v = (v_1, \dots, v_N) > 0$, we have

$$B(v_1, \dots, v_p)^T = m(v_1, \dots, v_p)^T \quad \text{with } (v_1, \dots, v_p) > 0.$$

By the Perron-Frobenius Theorem for irreducible matrices, we have that m is the spectral radius of B. Using Corollary 3.5 of [10], which concerns dust-like graph-directed sets whose corresponding graphs are transitive, we have

$$0 < \mathcal{H}^s(K_i) < \infty \quad \text{for } i \text{ of rank } 0.$$

Inductively, we assume $\mathcal{H}^s(K_j) > 0$ for all j of rank $\leq k$. Now, let i be a vertex of rank $k + 1 > 0$. We have $[i] \prec [j]$ for some j of rank $\leq k$. For otherwise, the rank of the vertex i must be 0 by the definition of rank. The fact $[i] \prec [j]$ means that there are $i_0 \in [i]$ and $j_0 \in [j]$ such that there is a directed path from i_0 to j_0. Since $i \sim i_0$ and $j \sim j_0$, there are also directed paths from i to i_0 and from j_0 to j. Consequently, we can find a directed path e^* from i to j. It follows from the inductive assumption $\mathcal{H}^s(K_j) > 0$ that $\mathcal{H}^s(K_i) > 0$ since $K_i \supset S_{e^*}(K_j)$. □

3.4. Non-negative matrices. A non-negative matrix A is said to be primitive if $A^h > 0$ for some natural integer h. We say a matrix B is *irreducible* if B cannot be conjugated into block upper triangular form by a permutation matrix P:

$$PBP^{-1} \neq \begin{pmatrix} D_1 & D_2 \\ 0 & D_3 \end{pmatrix}.$$

The following lemma can be found in [2].

LEMMA 4. *Let $B_{n\times n}$ be a non-negative irreducible matrix, and $\rho(B) > 0$ its Perron-Frobenius eigenvalue. For any positive vector $x = (x_1, \dots, x_n)^T > 0$, we have $\rho(B) \le \max_i \frac{(Bx)_i}{x_i}$. And $\rho(B) = \max_i \frac{(Bx)_i}{x_i}$ if and only if $Bx = \rho(B)x$.*

For any irreducible matrix B, Corollary 3.2.3B of [21] indicates that: if a non-negative matrix B is irreducible, then there is a permutation matrix P and a natural integer d such that

$$PB^dP^{-1} = \mathrm{diag}(B_1, \dots, B_d),$$

where each B_i is primitive. Therefore, for each B_i there is a positive integer h_i such that $(B_i)^{h_i} > 0$. Take $u = d\prod_{i=1}^d h_i$, we have

LEMMA 5. *If a non-negative matrix B is irreducible, then there is a permutation matrix P and a natural integer u such that*

$$PB^uP^{-1} = \mathrm{diag}(D_1, \dots, D_d),$$

where $D_1, \dots, D_d > 0$ are positive square matrices.

3.5. Bilipschitz equivalence. We say that a bijection f from a metric space (X_1, ρ_1) to another (X_2, ρ_2) is a bilipschitz mapping with bilipschitz constant $\mathrm{blip}(f) \ge 1$ if

$$\mathrm{blip}(f) = \inf\left\{c \ge 1 \colon c^{-1} \le \frac{\rho_2(f(x), f(y))}{\rho_1(x, y)} \le c \text{ for all } x \ne y\right\} \tag{3.4}$$

is finite. The next lemma follows from [4, 12] or Proposition 11.8 of [5].

LEMMA 6. *$\Sigma_{n_1}^{r_1}$ and $\Sigma_{n_2}^{r_2}$ are bilipschitz equivalent if and only if there are $k_1, k_2 \in \mathbb{N}$ such that $n_1^{k_1} = n_2^{k_2}$ and $r_1^{k_1} = r_2^{k_2}$.*

Theorem 2.1 of [30] yields the following lemma.

LEMMA 7. *Suppose $\{K_i\}_{i=1}^{\ell}$ and $\{K_i'\}_{i=1}^{\ell}$ are dust-like graph-directed sets on the same graph G satisfying*

$$K_i = \bigcup_j \bigcup_{e\in\mathcal{E}_{i,j}} S_e(K_j) \quad \text{and} \quad K_i' = \bigcup_j \bigcup_{e\in\mathcal{E}_{i,j}} S_e'(K_j'),$$

where S_e and S_e' are similitudes on complete metric spaces (X_1, ρ_1) and (X_2, ρ_2), respectively. If for each edge e the corresponding similarities S_e and S_e' have the same ratio r_e, then K_i and K_i' are bilipschitz equivalent for each i.

A special from of the following lemma is contained in [4, Proposition 2.2]. We omit its proof since the main idea is similar.

LEMMA 8. *Suppose E is a dust-like self-similar set in a complete metric space. Let $K = \bigcup_i f_i(E)$ be a disjoint union such that f_i is a bilipschitz mapping for each i. Then K and E are bilipschitz equivalent.*

3.6. Connectedness. Recall Lemma 2.3 and Lemma 2.4 of [40] as follows.

LEMMA 9. *Let Y be a compact subset of $\mathbb{R}^l$. Suppose $\{X_k\}_k$ are connected compact subsets of Y. Then there exist a subsequence $\{k_i\}_i$ and a connected compact set X such that $X_{k_i} \xrightarrow{\mathrm{d_H}} X$ as $i \to \infty$, where $\mathrm{d_H}$ is the Hausdorff metric.*

LEMMA 10. *Let $\{Y_1, Y_2, \dots, Y_k\}$ be totally disconnected compact subsets of $\mathbb{R}^l$. Then $Y = \bigcup_{i=1}^k Y_i$ is also totally disconnected.*

4. Graph-directed fractals with integer characteristic. Suppose $\{K_i\}_{i=1}^N$ are dust-like graph-directed sets with ratio r and integer characteristic m. For every vertex in the corresponding directed graph, we let the rank of that vertex be defined as in Section 3.2.

We notice that Theorem 2 follows from Theorem 1 and Lemma 6, where $r^{k_1} = (r')^{k_2}$ and $m^{k_1} = (m')^{k_2}$ with $k_1, k_2 \in \mathbb{N}$.

The necessary part of Theorem 1 follows from Lemma 3. We will prove the sufficient part of Theorem 1 by induction on the rank of the vertices. To this end, we first prove that K_i is bilipschitz equivalent to Σ_m^r if i is of rank 0. Since $\{K_j\}_{j\in[i]}$ are dust-like graph-directed sets with the adjacency matrix being irreducible, it suffices to show that

PROPOSITION 2. *Let $K_1, \dots, K_p$ be dust-like graph-directed sets of integer characteristic $m \geq 2$ and of ratio r such that the adjacency matrix is irreducible. Then $K_1, \dots, K_p$ are all bilipschitz equivalent to Σ_m^r.*

Then we assume inductively that K_j is bilipschitz equivalent to Σ_m^r if j is of rank $\leq k$. Let i be a vertex of rank $k+1$, we need to prove that K_i is bilipschitz equivalent to Σ_m^r.

Let V_i be the set of all vertices i_0 such that $[i] \prec [i_0]$ and $[i] \neq [i_0]$. Then i_0 is of rank $\leq k$ for every $i_0 \in V_i$ since i is of rank $k+1$. Hence K_{i_0} is bilipschitz equivalent to Σ_m^r. By a permutation if necessary, we can assume that $[i] = \{1, 2, \dots, p\}$ and that $V_i = \{p+1, \dots, p+q\}$. One can check that $K_1, \dots, K_{p+q}$ are also dust-like graph-directed sets and the adjacency matrix has the form

$$\begin{pmatrix} D_{p\times p} & C_{p\times q} \\ 0 & B_{q\times q} \end{pmatrix} \quad \text{where } D_{p\times p} \text{ is irreducible and } C_{p\times q} \neq 0.$$

Thus we complete the induction by proving

PROPOSITION 3. *Let $\{K_1, \dots, K_p, K_{p+1}, \dots K_{p+q}\}$ be dust-like graph-directed sets with ratio r, integer characteristic m and block upper triangular adjacency matrix*

$$\begin{pmatrix} D_{p\times p} & C_{p\times q} \\ 0 & B_{q\times q} \end{pmatrix},$$

where $D_{p\times p}$ is irreducible and $C_{p\times q} \neq 0$. If $K_{p+1}, \dots, K_{p+q}$ are all bilipschitz equivalent to Σ_m^r, then K_i is also bilipschitz equivalent to Σ_m^r for any $1 \leq i \leq p$.

The reminder of this section is devoted to prove Propositions 2 and 3.

4.1. Proof of Proposition 2. We assume that $K_1, \dots, K_p$ are dust-like graph-directed sets of ratio r with their adjacency matrix B irreducible such that

$$Bv = mv \quad \text{with } v > 0.$$

By Lemma 5, we can assume that $K_1, \dots, K_\ell$ $(\ell \le p)$ are dust-like graph-directed sets of ratio r^u with their adjacency matrix $D_1 > 0$ such that

$$D_1 v^* = m^u v^* \quad \text{with } v^* > 0. \tag{4.1}$$

If $\ell = 1$, we notice that K_1 is a disjoint union of m^u copies of itself with ratio r^u. We also notice that $\Sigma_{m^u}^{r^u}$ is a disjoint union of m^u copies of itself with ratio r^u, and $\Sigma_{m^u}^{r^u}$ and Σ_m^r are bilipschitz equivalent. Applying Lemma 7 to self-similar sets, we have K_1 and Σ_m^r are bilipschitz equivalent.

Now, suppose $\ell \ge 2$. The proof consists of two steps:

Step 1. prove that $\{K_1, \dots, K_\ell\}$ are dust-like graph-directed sets on a suitable graph G.

Step 2. construct a family of dust-like graph-directed sets on the same graph G such that those sets are all bilipschitz equivalent to the symbolic space.

Then we use Lemma 7 to complete the proof.

Step 1. We can assume that

$$u = 1. \tag{4.2}$$

Since $m \in \mathbb{N}$ and the entries of D_1 are natural integers, there is an integer eigenvector corresponding to the Perron-Frobenius eigenvalue m which is a simple eigenvalue, which means that v^* can be written as

$$v^* = (\alpha_1, \dots, \alpha_\ell)^T \text{ with } \alpha_1, \dots, \alpha_\ell \in \mathbb{N}.$$

Without loss of generality, we suppose that

$$\gcd(\alpha_1, \dots, \alpha_\ell) = 1.$$

We have

$$D_1(\alpha_1, \dots, \alpha_\ell)^T = m(\alpha_1, \dots, \alpha_\ell)^T. \tag{4.3}$$

Here m is the Perron-Frobenius eigenvalue of positive matrix D_1. Let $(D_1)^k = (d_{i,j}^{(k)})_{1 \le i,j \le \ell}$. Then for all k and all i, j, we have $d_{i,j}^{(k)} \ge C^{-1} m^k$ for some constant $C \ge 1$.

Now we use Lemma 1. Recall that $\phi^* = \phi(\alpha_1, \dots, \alpha_\ell) + \max_{1 \le j \le \ell} \alpha_j$. Take k^* such that

$$m^{k^*} > 2\phi^* \sum_{j=1}^{\ell} \alpha_j \quad \text{and} \quad d_{i,j}^{(k^*)} \ge C^{-1} m^{k^*} > \alpha_i \phi^* \quad \text{for each } i. \tag{4.4}$$

For each $i \in \{1, \dots, \ell\}$, let $\Omega_i = \bigcup_{j=1}^{\ell} \mathcal{E}_{i,j}^{k^*}$ be the set of all directed paths of length k^* and starting at i. For $e^* \in \Omega_i$, let $\beta_{e^*} = \alpha_j$ if e^* is ending at j. Then

$$\beta_{e^*} \in \{\alpha_1, \dots, \alpha_\ell\} \quad \text{for } e^* \in \Omega_i. \tag{4.5}$$

By the definition of the adjacency matrix, we have

$$\#\{e^* \in \Omega_i \colon \beta_{e^*} = \alpha_j\} = d_{i,j}^{(k^*)}. \tag{4.6}$$

It follows from (4.3) that

$$\sum_{e^* \in \Omega_i} \beta_{e^*} = \sum_{j=1}^{\ell} d_{i,j}^{(k^*)} \alpha_j = m^{k^*} \cdot \alpha_i. \tag{4.7}$$

Applying Lemma 1 by $b = m^{k^*}$ and $\gamma = \alpha_i$. It follows from (4.4)–(4.7) that the conditions of Lemma 1 are fulfilled. Thus we get the decomposition

$$\Omega_i = \bigcup_{t=1}^{\alpha_i} \Omega_{i,t}$$

such that for all $1 \le j \le \alpha_i$,

$$\sum_{e^* \in \Omega_{i,t}} \beta_{e^*} = m^{k^*}. \tag{4.8}$$

By the definition of the adjacency matrix, for each i, we have a disjoint union

$$K_i = \bigcup_{t=1}^{\alpha_i} \bigcup_{e^* \in \Omega_{i,t}} S_{e^*}(K_{j(e^*)}), \tag{4.9}$$

where $j(e^*)$ is the end point of the directed path e^*. Note that S_{e^*} is a similitude with ratio r^{k^*} for all e^*. Therefore, $\{K_1, \dots, K_\ell\}$ are dust-like graph-directed sets with ratio r^{k^*} satisfying (4.9).

Step 2. For $w = w_1 w_2 \dots w_k \in \{1, 2, \dots, m^{k^*}\}^k$, $x = x_1 x_2 \dots \in \Sigma_{m^{k^*}}^{r^{k^*}}$ and $A \subset \Sigma_{m^{k^*}}^{r^{k^*}}$, write

$$w * x = w_1 w_2 \dots w_k x_1 x_2 \dots \quad \text{and} \quad w * A = \{w * x \colon x \in A\}.$$

Let

$$[w] = w * \Sigma_{m^{k^*}}^{r^{k^*}} = \{x_1 x_2 \dots \in \Sigma_{m^{k^*}}^{r^{k^*}} \colon x_i = w_i \text{ for } 1 \le i \le k\}$$

be the cylinder set. For integers α, β with $1 \le \alpha \le \beta \le m^{k^*}$, write

$$\Pi_\alpha^\beta = [\alpha] \cup [\alpha + 1] \cup \dots \cup [\beta].$$

We claim that $\{\Pi_1^{\alpha_1}, \dots, \Pi_1^{\alpha_\ell}\}$ are dust-like graph-directed sets such that, for each i,

$$\Pi_1^{\alpha_i} = \bigcup_{t=1}^{\alpha_i} \bigcup_{e^* \in \Omega_{i,t}} T_{e^*}(\Pi_1^{\beta_{e^*}}), \tag{4.10}$$

where T_{e^*} is a similitude with ratio r^{k^*} for each e^*. Recall that $\beta_{e^*} = \alpha_j$ if e^* is ending at j.

If the claim is true, by (4.9) and (4.10), the two families $\{K_1, \ldots, K_\ell\}$ and $\{\Pi_1^{\alpha_1}, \ldots, \Pi_1^{\alpha_\ell}\}$ of dust-like graph-directed sets are on the same graph and with the same ratio r^{k^*}. By Lemma 7, $K_i \simeq \Pi_1^{\alpha_i}$; by Lemma 8, $\Pi_1^{\alpha_i} \simeq \Sigma_{m^{k^*}}^{r^{k^*}}$, since $\Pi_1^{\alpha_i} = [1] \cup \cdots \cup [\alpha_i]$ can be regard as α_i copies of $\Sigma_{m^{k^*}}^{r^{k^*}}$; by Lemma 6, $\Sigma_{m^{k^*}}^{r^{k^*}} \simeq \Sigma_m^r$. Consequently, K_i is bilipschitz equivalent to Σ_m^r for every i.

To prove the claim, we first define the similitude T_{e^*} for each $e^* \in \Omega_{i,t}$ and each i, t. Fix i, t, write $\#\Omega_{i,t} = N_{i,t}$ and $\Omega_{i,t} = \{e_1^*, \ldots, e_{N_{i,t}}^*\}$. For $e^* \in \Omega_{i,t}$, say, $e^* = e_j^*$. Write

$$\Pi_{e^*} = \Pi_{e_j^*} = t * \Pi_{\lambda(e_j^*)+1}^{\lambda(e_j^*)+\beta_{e_j^*}}$$

where $\lambda(e^*) = \lambda(e_j^*) = \beta_{e_1^*} + \beta_{e_2^*} + \cdots + \beta_{e_{j-1}^*}$. Then we define $T_{e^*} : \Pi_1^{\beta_{e^*}} \to \Pi_{e^*}$ by

$$T_{e^*} \colon x_1 x_2 x_3 \ldots \mapsto t * (x_1 + \lambda(e^*)) * x_2 x_3 \ldots$$

Clearly, T_{e^*} is a similitude with ratio r^{k^*}. Moreover, by (4.8), we have

$$\bigcup_{e^* \in \Omega_{i,t}} \Pi_{e^*} = \bigcup_{j=1}^{N_{i,t}} t * \Pi_{\lambda(e_j^*)+1}^{\lambda(e_j^*)+\beta_{e_j^*}} = t * \Sigma_{m^{k^*}}^{r^{k^*}} = [t].$$

So

$$\bigcup_{t=1}^{\alpha_i} \bigcup_{e^* \in \Omega_{i,t}} T_{e^*}(\Pi_1^{\beta_{e^*}}) = \bigcup_{t=1}^{\alpha_i} \bigcup_{e^* \in \Omega_{i,t}} \Pi_{e^*} = \bigcup_{t=1}^{\alpha_i} [t] = \Pi_1^{\alpha_i}.$$

Thus the claim is true and the proof of Proposition 2 is complete.

4.2. Proof of Proposition 3. Let $A = \begin{pmatrix} D_{p\times p} & C_{p\times q} \\ 0 & B_{q\times q} \end{pmatrix}$ and $A^k = (a_{i,j}^{(k)})_{i,j}$.

Suppose that $v = (v_1, \ldots, v_p, v_{p+1}, \ldots v_{p+q})^T$, with $v_i \in \mathbb{N}$ for all i, is the corresponding eigenvector of adjacency matrix, i.e.,

$$Av = mv.$$

Let $v^* = (v_1, \ldots, v_p)^T$ and $v_1 = (v_{p+1}, \ldots v_{p+q})^T > 0$. Since $C \geq 0, C \neq 0$ and $Dv^* + Cv_1 = mv^*$, we have

$$Dv^* \leq mv^* \text{ and } Dv^* \neq mv^*.$$

Lemma 11. *There exists k^* such that for any $k \geq k^*$ and any $1 \leq i \leq p$,*

$$\sum_{j=1}^{p} a_{i,j}^{(k)} v_j < m^k.$$

Proof. It suffices to verify that $\rho(D) < m$, where $\rho(D)$ is the spectral radius of D. Suppose otherwise that $\rho(D) \geq m$.

For the irreducible matrix D, by Lemma 4, we have

$$m \leq \rho(D) \leq \max_i \frac{(Dv^*)_i}{(v^*)_i} \leq m,$$

which implies $\rho(D) = \max_i \frac{(Dv^*)_i}{(v^*)_i} = m$. By using Lemma 4, we have

$$Dv^* = mv^*.$$

This contradicts that $Dv^* \neq mv^*$. □

Let k^* be as in Lemma 11. It is clear that

$$K_i = \bigcup_{j=1}^{p+q} \bigcup_{e^* \in \mathcal{E}_{i,j}^{k^*}} S_{e^*}(K_j) \quad \text{for } 1 \le i \le p+q.$$

In other words, $K_1, \dots, K_{p+q}$ can be regard as dust-like graph-directed sets with ratio r^{k^*} and the adjacency matrix

$$A^{k^*} = \begin{pmatrix} D_{p\times p}^{k^*} & \widetilde{C}_{p\times q} \\ 0 & B_{q\times q}^{k^*} \end{pmatrix},$$

where $D_{p\times p}^{k^*} = \big(a_{i,j}^{(k^*)}\big)_{i,j\in\{1,\dots,p\}}$ and $\widetilde{C}_{p\times q} \neq 0$ since $C_{p\times q} \neq 0$. By Lemma 11,

$$\sum_{j=1}^{p} a_{i,j}^{(k^*)} \le \sum_{j=1}^{p} a_{i,j}^{(k^*)} v_j < m^{k^*} \quad \text{for } 1 \le i \le p.$$

Since Σ_r^m and $\Sigma_{r^{k^*}}^{m^{k^*}}$ are bilipschitz equivalent, by rewriting A^{k^*} as A, m^{k^*} as m and r^{k^*} as r, Proposition 3 follows from

Proposition 4. *Let $K_1, \dots, K_p, K_{p+1}, \dots, K_{p+q}$ be a family of dust-like graph-directed sets with ratio r and the adjacency matrix*

$$A = (a_{i,j})_{i,j} = \begin{pmatrix} D_{p\times p} & C_{p\times q} \\ 0 & B_{q\times q} \end{pmatrix}$$

such that $Av = mv$ for $m \ge 2$, where $v = (v_1, \dots, v_{p+q})^T > 0$ with $v_i \in \mathbb{N}$ and

$$\sum_{j=1}^{p} a_{i,j} \le \sum_{j=1}^{p} a_{i,j} v_j < m \quad \text{for } 1 \le i \le p. \tag{4.11}$$

If $K_{p+1}, \dots, K_{p+q}$ are all bilipschitz to Σ_m^r, then K_i is also bilipschitz equivalent to Σ_m^r for any $1 \le i \le p$.

Proof. For $1 \le i \le p$, rewrite

$$K_i = \bigcup_{j=1}^{p+q} \bigcup_{e\in\mathcal{E}_{i,j}} S_e(K_j) = J_i \cup \bigcup_{j=1}^{p} \bigcup_{e\in\mathcal{E}_{i,j}} S_e(K_j), \tag{4.12}$$

where

$$J_i = \bigcup_{j=p+1}^{p+q} \bigcup_{e\in\mathcal{E}_{i,j}} S_e(K_j) \quad \text{for } 1 \le i \le p.$$

By (4.11), $J_i \neq \emptyset$. Hence by Lemma 8 and the fact that $K_{p+1}, \dots, K_{p+q}$ are all bilipschitz equivalent to Σ_m^r, we have

$$\text{all } J_i\text{'s are bilipschitz equivalent to } \Sigma_m^r. \tag{4.13}$$

It suffices to prove that K_1 and Σ_m^r are bilipschitz equivalent. Let

$$\widetilde{\mathcal{E}}_{i,j}^k = \left\{(e_1,\ldots,e_k) \in \mathcal{E}_{i,j}^k \colon (e_1,\ldots,e_l) \in \bigcup_{j=1}^p \mathcal{E}_{i,j}^l \text{ for all } 1 \le l \le k\right\}$$

be the set consisting of all directed paths of length k which don't pass though the vertexes $p+1,\ldots,p+q$. We deduce from (4.12) that

$$\begin{aligned} K_1 &= J_1 \cup \bigcup_{j=1}^p \bigcup_{e\in\mathcal{E}_{1,j}} S_e(K_j) = J_1 \cup \bigcup_{j=1}^p \bigcup_{e\in\mathcal{E}_{1,j}} S_e(J_j) \cup \bigcup_{j=1}^p \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^2} S_{e^*}(K_j) \\ &= J_1 \cup \bigcup_{k=1}^2 \bigcup_{j=1}^p \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^k} S_{e^*}(J_j) \cup \bigcup_{j=1}^p \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^3} S_{e^*}(K_j) = \cdots \\ &= \left(J_1 \cup \bigcup_{k\ge1} \bigcup_{j=1}^p \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^k} S_{e^*}(J_j)\right) \cup \left(\bigcap_{k\ge1} \bigcup_{j=1}^p \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^k} S_{e^*}(K_j)\right) =: J_\infty \cup K_\infty. \end{aligned} \tag{4.14}$$

Clearly, J_∞ is dense in K_1. We will find a dense subset $\widetilde{J}_\infty \subset \Sigma_m^r$ such that J_∞ and $\widetilde{J}_\infty$ are bilipschitz equivalent.

By the definition of adjacency matrix, we have $\#\mathcal{E}_{i,j} = a_{i,j}$ for $1 \le i,j \le p+q$. By (4.11), for $1 \le i \le p$,

$$\sum_{j=1}^p \#\mathcal{E}_{i,j} = \sum_{j=1}^p a_{i,j} < m. \tag{4.15}$$

Hence there are injections $\pi_i \colon \bigcup_{j=1}^p \mathcal{E}_{i,j} \to \{1,\ldots,m\}$ for $1 \le i \le p$. Fix a such π_i for each $i = 1,2,\ldots,p$, we define an injection $\pi \colon \bigcup_{k\ge1} \bigcup_{j=1}^p \widetilde{\mathcal{E}}_{1,j}^k \to \bigcup_{k\ge1} \{1,\ldots,m\}^k$ by

$$\pi \colon (e_1,e_2,\ldots,e_k) \mapsto w_1w_2\ldots w_k \in \{1,\ldots,m\}^k,$$

where $w_l = \pi_i(e_l)$ for $1 \le l \le k$ with i being the starting vertex of the edge e_l. For $e^* \in \bigcup_{k\ge1} \bigcup_{j=1}^p \widetilde{\mathcal{E}}_{1,j}^k$, define $T_{e^*} \colon \Sigma_m^r \to [\pi(e^*)]$ by

$$T_{e^*} \colon x_1x_2\ldots \mapsto \pi(e^*) * x_1x_2\ldots.$$

Let

$$\widetilde{J}_i = \Sigma_m^r \setminus \bigcup_{j=1}^p \bigcup_{e\in\mathcal{E}_{i,j}} [\pi_i(e)] \quad \text{for } 1 \le i \le p.$$

Notice that $\widetilde{J}_i \neq \emptyset$ since π_i is not a surjection by (4.15). Then

$$\Sigma_m^r = \widetilde{J}_i \cup \bigcup_{j=1}^p \bigcup_{e\in\mathcal{E}_{i,j}} [\pi_i(e)] = \widetilde{J}_i \cup \bigcup_{j=1}^p \bigcup_{e\in\mathcal{E}_{i,j}} T_e(\Sigma_m^r) \quad \text{for } 1 \le i \le p.$$

It follows that

$$\begin{aligned}\Sigma_m^r &= \widetilde{J}_1 \cup \bigcup_{j=1}^{p} \bigcup_{e\in\mathcal{E}_{1,j}} T_e(\Sigma_m^r) = \widetilde{J}_1 \cup \bigcup_{j=1}^{p} \bigcup_{e\in\mathcal{E}_{1,j}} T_e(\widetilde{J}_j) \cup \bigcup_{j=1}^{p} \bigcup_{e^*\in\widetilde{\mathcal{E}}_{i,j}^2} T_{e^*}(\Sigma_m^r)\\ &= \widetilde{J}_1 \cup \bigcup_{k=1}^{2} \bigcup_{j=1}^{p} \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^k} T_{e^*}(\widetilde{J}_j) \cup \bigcup_{j=1}^{p} \bigcup_{e^*\in\widetilde{\mathcal{E}}_{i,j}^3} T_{e^*}(\Sigma_m^r) = \cdots\\ &= \Bigg(\widetilde{J}_1 \cup \bigcup_{k\geq 1} \bigcup_{j=1}^{p} \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^k} T_{e^*}(\widetilde{J}_j)\Bigg) \cup \Bigg(\bigcap_{k\geq 1} \bigcup_{j=1}^{p} \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^k} T_{e^*}(\Sigma_m^r)\Bigg) =: \widetilde{J}_\infty \cup \widetilde{K}_\infty.\end{aligned} \tag{4.16}$$

Clearly, we also have $\widetilde{J}_\infty$ is dense in Σ_m^r.

Since $\widetilde{J}_i$ is a disjoint finite union of cylinders, we have $\widetilde{J}_i \simeq \Sigma_m^r$ by Lemma 8. So $J_i \simeq \widetilde{J}_i$ for $1 \leq i \leq p$ by (4.13). Fix a bilipschitz bijections $f_i\colon J_i \mapsto \widetilde{J}_i$ for each $1 \leq i \leq p$. In view of (4.14) and (4.16), define $f\colon J_\infty \mapsto \widetilde{J}_\infty$ by

$$f|_{J_1} = f_1 \quad \text{and} \quad f|_{S_{e^*}(J_j)} = T_{e^*} \circ f_j \circ S_{e^*}^{-1} \quad \text{for all } e^* \in \bigcup_{k\geq 1} \bigcup_{j=1}^{p} \widetilde{\mathcal{E}}_{1,j}^k.$$

Notice that S_{e^*} and T_{e^*} are similarities of same ratio r^k, where k is the length of e^*. We claim that f is a bilipschitz bijection. If the claim is true, we can extend f to be a bilipschitz bijection from K_1 onto Σ_m^r since J_∞ are dense in K_1 and $\widetilde{J}_\infty$ is dense in Σ_m^r. Hence the proof is complete.

It remains to prove the claim. Let

$$x, y \in J_\infty = J_1 \cup \bigcup_{k\geq 1} \bigcup_{j=1}^{p} \bigcup_{e^*\in\widetilde{\mathcal{E}}_{1,j}^k} S_{e^*}(J_j).$$

There are three cases to consider.

Case 1. $x, y \in J_1$ or $x, y \in S_{e^*}(J_j)$ for some e^*. Let $c_1 = \max_{1\leq i\leq p} \operatorname{blip}(f_i)$, where $\operatorname{blip}(f_i)$ is the bilipschitz constant of f_i defined by (3.4). Then by the definition of f, if $x, y \in J_1$, then

$$\frac{\rho(f(x), f(y))}{|x-y|} = \frac{\rho(f_1(x), f_1(y))}{|x-y|} \in [c_1^{-1}, c_1].$$

If $x, y \in S_{e^*}(J_j)$, recall that S_{e^*} and T_{e^*} are similarities of same ratio r^k, where k is the length of e^*. Hence

$$\begin{aligned}\frac{\rho(f(x), f(y))}{|x-y|} &= \frac{\rho(T_{e^*} \circ f_j \circ S_{e^*}^{-1}(x), T_{e^*} \circ f_j \circ S_{e^*}^{-1}(y))}{|x-y|}\\ &= \frac{\rho(f_j \circ S_{e^*}^{-1}(x), f_j \circ S_{e^*}^{-1}(y))}{r^{-k}|x-y|}\\ &= \frac{\rho(f_j \circ S_{e^*}^{-1}(x), f_j \circ S_{e^*}^{-1}(y))}{|S_{e^*}^{-1}(x) - S_{e^*}^{-1}(y)|} \in [c_1^{-1}, c_1].\end{aligned}$$

Case 2. $x \in J_1$ and $y \in S_{e^*}(J_j)$ for some e^*. Then $y \notin J_1$, and so

$$\operatorname{dist}(J_1, K_1 \setminus J_1) \le |x-y| \le |K_1|,$$

where $|K_1|$ is the diameter of K_1. By the definition of f, we also have $f(x) \in \widetilde{J}_1$ and $f(y) \notin \widetilde{J}_1$, and so

$$r^{-1} = \operatorname{dist}(\widetilde{J}_1, \Sigma_m^r \setminus \widetilde{J}_1) \le \rho(f(x), f(y)) \le |\Sigma_m^r| = 1.$$

Consequently, there exists $c_2 > 1$ such that

$$\frac{\rho(f(x), f(y))}{|x-y|} \in [c_2^{-1}, c_2].$$

Case 3. $x \in S_{e_1^*}(J_{j_1})$ and $y \in S_{e_2^*}(J_{j_2})$ for distinct e_1^* and e_2^*. Suppose that

$$\begin{aligned} e_1^* &= (e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_l), \\ e_2^* &= (e_1, e_2, \ldots, e_k, e'_{k+1}, \ldots, e'_{l'}), \end{aligned} \tag{4.17}$$

where $e_{k+1} \neq e'_{k+1}$. So $e^* = (e_1, \ldots, e_k)$ be the common directed path prefixed to e_1^* and e_2^*. There are two subcases to consider.

Case 3.1. $e^* = e_1^*$ or $e^* = e_2$. If $e^* = e_1^*$, then $x \in S_{e^*}(J_{j_1})$ and $y \in S_{e_*}(K_{j_1} \setminus J_{j_1})$, and so $f(x) \in T_{e^*}(\widetilde{J}_{j_1})$ and $f(y) \in T_{e^*}(\Sigma_m^r \setminus \widetilde{J}_{j_1})$. Recall that S_{e^*} and T_{e^*} are similarities of the same ratio r^k. Therefore,

$$\begin{aligned} r^k \min_{1\le j\le p} \operatorname{dist}(J_j, K_j \setminus J_j) \le |x-y| \le r^k \max_{1\le j\le p} |K_j|, \\ r^{k+1} = r^k \min_{1\le j\le p} \operatorname{dist}(\widetilde{J}_j, \Sigma_m^r \setminus \widetilde{J}_j) \le \rho(f(x), f(y)) \le r^k |\Sigma_m^r| = r^k. \end{aligned}$$

The same conclusion holds in the case $e^* = e_2^*$.

Case 3.2. $e^* \neq e_1^*$ and $e^* \neq e_2^*$. By (4.17), let

$$e_3^* = (e_1, \ldots, e_k, e_{k+1}) \quad \text{and} \quad e_4^* = (e_1, \ldots, e_k, e'_{k+1}).$$

Let j, j_3 and j_4 be the end points of e^*, e_3^* and e_4^*, respectively. Then we have

$$x, y \in S_{e^*}(K_j), \quad f(x), f(y) \in T_{e^*}(\Sigma_m^r)$$

and

$$\begin{aligned} x &\in S_{e_3^*}(K_{j_3}) = S_{e^*} \circ S_{e_{k+1}}(K_{j_3}), & y &\in S_{e_4^*}(K_{j_4}) = S_{e^*} \circ S_{e'_{k+1}}(K_{j_4}), \\ f(x) &\in T_{e_3^*}(\Sigma_m^r) = T_{e^*} \circ T_{e_{k+1}}(\Sigma_m^r), & f(y) &\in T_{e_4^*}(\Sigma_m^r) = T_{e^*} \circ T_{e'_{k+1}}(\Sigma_m^r). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & r^k \delta_* \le |x-y| \le r^k \max_{1\le j\le p} |K_j|, \\ & r^{k+1} = r^k \min_{e\neq e'} \operatorname{dist}(T_e(\Sigma_m^r), T_{e'}(\Sigma_m^r)) \le \rho(f(x), f(y)) \le r^k |\Sigma_m^r| = r^k, \end{aligned}$$

where

$$\delta_* = \min_{1\le j, j_3, j_4 \le p} \Big\{ \operatorname{dist}(S_e(K_{j_3}), S_{e'}(K_{j_4})) \colon e \in \mathcal{E}_{j,j_3}, e' \in \mathcal{E}_{j,j_4}, e \neq e' \Big\}.$$

Combining Cases 3.1 and 3.2, we conclude that there exists $c_3 > 1$ with

$$\frac{\rho(f(x), f(y))}{|x-y|} \in [c_3^{-1}, c_3].$$

Combining all the cases, we have f is a bilipschitz bijection from J_∞ onto $\widetilde{J}_\infty$. $\square$

5. Self-similar sets. In this section, under the assumptions of total disconnectedness and non-existence of complete overlaps, we will show that the self-similar sets in Theorem 3 can be regarded as the graph-directed fractals with integer characteristic. Thus Theorem 3 follows from Theorem 1. Finally we prove Proposition 1, which is the basis of efficient algorithm to test the existence of complete overlaps.

5.1. Total disconnectedness and finite type. Let E be a totally disconnected self-similar set generated by $\{S_i\}_{i=1}^m$, where

$$S_i(x) = g_i(x)/n + b_i$$

with the isometry $g_i \in \mathbb{G}$ and $b_i \in \Gamma$.

Fix a positive number r_0 such that

$$B(0, r_0) \supset \bigcup_{i=1}^{m} \left(\frac{B(0, r_0)}{n} + b_i \right) \tag{5.1}$$

and that

$$\bigcup_{i=1}^{m} \left(\frac{B(0, r_0)}{n} + b_i \right) \quad \text{is connected,} \tag{5.2}$$

where $B(0, r_0)$ is the closed ball with center 0 and radius r_0.

We introduce types to describe the structure of the self-similar set E. For $(g, b) \in \mathcal{G} \times \Gamma$, denote by $S(g, b)$ the similarity $x \mapsto gx/n + b$ and let

$$\mathfrak{S} = \{S(g, b)\colon (g, b) \in \mathbb{G} \times \Gamma\}. \tag{5.3}$$

Definition 4. A finite set $V \subset \mathfrak{S}$ is called a type if

$$\chi_V := \bigcup_{S \in V} S\big(B(0, r_0)\big) \quad \text{is connected.}$$

We say two types V_1, V_2 are equivalent, denoted by $V_1 \sim V_2$, if there exists an isometry I such that

$$V_1 = \big\{I \circ S\colon S \in V_2\big\}. \tag{5.4}$$

Denote by $[V]$ the equivalence class containing V.

If V is a type, we consider the compact set

$$K_V = \bigcup_{S \in V} S(E). \tag{5.5}$$

In particular, by (5.2), $U_1 = \{S_1, \ldots, S_m\} = \{S(g_1, b_1), \ldots, S(g_m, b_m)\}$ is a type and

$$E = K_{U_1}.$$

Lemma 12. *If $V_1 \sim V_2$, then $K_{V_1} = I(K_{V_2})$ and $\chi_{V_1} = I(\chi_{V_2})$ for some isometry I.*

Proof. Since $V_1 \sim V_2$, there is an isometry I satisfying (5.4). So

$$K_{V_1} = \bigcup_{S\in V_1} S(E) = \bigcup_{S\in V_2} I\circ S(E) = I\Big(\bigcup_{S\in V_2} S(E)\Big) = I(K_{V_2})$$

and

$$\chi_{V_1} = \bigcup_{S\in V_1} S(B(0,r_0)) = \bigcup_{S\in V_2} I\circ S(B(0,r_0)) = I\Big(\bigcup_{S\in V_2} S(B(0,r_0))\Big) = I(\chi_{V_2}).$$

$\square$

Now substituting $E = \bigcup_{i=1}^m S_i(E)$ into (5.5), we get

$$\begin{aligned} K_V &= \bigcup_{S\in V} S\Big(\bigcup_{i=1}^m S_i(E)\Big) = \frac{1}{n}\bigcup_{S\in V}\bigcup_{i=1}^m n\cdot S\circ S_i(E) \\ &= \frac{1}{n}\bigcup_{S(g,b)\in V}\bigcup_{i=1}^m\Big(\frac{gg_iE}{n} + gb_i + nb\Big) = \frac{1}{n}\bigcup_{S\in\mathfrak{S}(V)} S(E). \end{aligned}$$

Here the set

$$\begin{aligned} \mathfrak{S}(V) &:= \big\{n\cdot S\circ S_i\colon S\in V, 1\le i\le m\big\} \\ &= \big\{S(gg_i, gb_i+nb)\colon S(g,b)\in V, 1\le i\le m\big\}\subset\mathfrak{S}, \end{aligned} \tag{5.6}$$

since $gg_i \in \mathbb{G}$ and $gb_i + nb \in \Gamma$. We divide the set

$$\chi_{\mathfrak{S}(V)} = \bigcup_{S\in\mathfrak{S}(V)} S\big(B(0,r_0)\big) = \bigcup_{S(g,b)\in V}\bigcup_{i=1}^m\Big(\frac{B(0,r_0)}{n} + gb_i + nb\Big)$$

into connected components. Since $B(0,r_0)/n$ are connected, every connected component must have the form χ_U with U being a type. Let $\mathcal{T}_V$ be the family of all such types. Clearly, $\mathfrak{S}(V) = \bigcup_{U\in\mathcal{T}_V} U$.

Therefore, for each type V, we have a family $\mathfrak{S}(V)$ of similarities and a family $\mathcal{T}_V$ of types. Using $\mathcal{T}_V$, we get the decomposition

$$K_V = \frac{1}{n}\bigcup_{S\in\mathfrak{S}(V)} S(E) = \frac{1}{n}\bigcup_{U\in\mathcal{T}_V} K_U. \tag{5.7}$$

$\{K_U\}_{U\in\mathcal{T}_V}$ are *disjoint* since $\{\chi_U\}_{U\in\mathcal{T}_V}$ are disjoint and

$$K_U = \bigcup_{S\in U} S(E) \subset \bigcup_{S\in U} S\big(B(0,r_0)\big) = \chi_U.$$

Here we use the fact that $E\subset B(0,r_0)$.

We say that U is generated by V *directly* (or after 1 step), denoted by $V\to U$ if $U\in\mathcal{T}_V$. We also say that V_k is generated by V after k steps if

$$V\to V_1\to\cdots\to V_k.$$

We say U is generated by V if U is generated by V after k steps for some $k\ge 1$.

We define the generation of equivalent classes similarly. $[U]$ is said to be generated by $[V]$ *directly* (or after 1 step), denoted by $[V] \to [U]$ if $V' \to U'$ for some $V' \in [V]$ and some $U' \in [U]$. $[V_k]$ is said to be generated by $[V]$ after k steps if

$$[V] \to [V_1] \to \cdots \to [V_k].$$

$[U]$ is said to be generated by $[V]$ if $[U]$ is generated by $[V]$ after k steps for some $k \geq 1$.

We present two lemmas on the types generated by $U_1 = \{S_1, \dots, S_m\}$, which is the IFS of the self-similar set E.

LEMMA 13. *If U is generated by U_1 after k steps, then*

$$\{S/n^k \colon S \in U\} \subset \{S_{i_1 \dots i_{k+1}} \colon i_1 \dots i_{k+1} \in \{1, \dots, m\}^{k+1}\}.$$

Proof. We prove this by induction on k. If $k = 1$, then U is generated by U_1 directly. For every $S \in U$, we have $S \in \mathfrak{S}(U_1)$ since $U \subset \mathfrak{S}(U_1)$. By (5.6), every $S \in U$ has the form $S = n \cdot S_{i_1} \circ S_i = nS_{i_1 i}$ with $i_1 i \in \{1, \dots, m\}^2$ since $U_1 = \{S_1, \dots, S_m\}$. So the lemma holds for $k = 1$.

Now suppose the lemma holds for all U^* generated by U_1 after $k-1$ steps. If U is generated by U_1 after k steps, then U is generated by U^* directly for some U^* generated by U_1 after $k-1$ steps. For every $S \in U$, we have $S \in \mathfrak{S}(U^*)$ since $U \subset \mathfrak{S}(U^*)$. By (5.6), there are an $S^* \in U^*$ and $1 \leq i \leq m$ such that $S = n \cdot S^* \circ S_i$. By induction assumption, we have $S^*/n^{k-1} = S_{i_1 \dots i_{k-1}}$ for some $i_1 \dots i_{k-1} \in \{1, \dots, m\}^{k-1}$. Hence $S/n^k = n^{k-1} S^* \circ S_i = S_{i_1 \dots i_{k-1} i}$ with $i_1 \dots i_{k-1} i \in \{1, \dots, m\}^k$. This completes the proof. ∎

The following lemma is the key point to describe the structure of the self-similar set E.

LEMMA 14. *There are only finitely many equivalent classes of types generated by*

$$[U_1] = [\{S_1, \dots, S_m\}].$$

To prove Lemma 14, we need a lemma related to the total disconnectedness of E. Let $\mathbb{H}_k = \bigcup_{S \in \mathfrak{S}} S(E_k)$ and $\mathbb{H} = \bigcup_{S \in \mathfrak{S}} S(E)$, where

$$E_k = \bigcup_{i_1 \dots i_k \in \{1, \dots, m\}^k} S_{i_1, \dots i_k}(B(0, r_0)).$$

We have $E_k \xrightarrow{\mathrm{d_H}} E$ and thus $\mathbb{H}_k \cap B(0, r) \xrightarrow{\mathrm{d_H}} \mathbb{H} \cap B(0, r)$ for all $r > 0$. Note that (5.1) implies that

$$E \subset E_k \subset B(0, r_0) \quad \text{for all } k \geq 1. \tag{5.8}$$

LEMMA 15. *There exists an integer k_0 such that every connected component in $\mathbb{H}_{k_0}$ touching $B(0, r_0)$ cannot touch $\{x \colon |x| \geq r_0 + 1\}$.*

Proof. Suppose on the contrary that for each k there is a connected components X_k in $\mathbb{H}_k$ touching $B(0, r_0)$ and $\{x \colon |x| \geq r_0 + 1\}$. Write

$$\widetilde{X}_k = X_k \cap B(0, r_0 + 1) \quad \text{and} \quad \widehat{X}_k = X_k \cap \{x \colon |x| \geq r_0 + 1\}.$$

Let A_k be a connected component of $\widetilde{X}_k$ such that $A_k \cap B(0, r_0) \neq \emptyset$. We claim that A_k touches $\{x\colon |x| = r_0 + 1\}$. If otherwise, $A_k \cap \{x\colon |x| = r_0 + 1\} = \emptyset$, since A_k is a component of $\widetilde{X}_k$, then for each $x \in \widetilde{X}_k$ with $|x| = r_0 + 1$, there are two compact sets A_x and A_x^* such that $\widetilde{X}_k = A_x \cup A_x^*$, $A_x \cap A_x^* = \emptyset$, $x \in A_x$ and $A_k \subset A_x^*$. Note that A_x is relative open in $\widetilde{X}_k$ and $\widetilde{X}_k \cap \{x\colon |x| = r_0 + 1\}$ is compact. So there is a finite cover $A_{x_1}, \ldots, A_{x_t}$ of $\widetilde{X}_k \cap \{x\colon |x| = r_0 + 1\}$. Let $A = \bigcup_{i=1}^t A_{x_t}$ and $A^* = \bigcap_{i=1}^t A_{x_t}^*$. Then $\widetilde{X}_k = A \cup A^*$, $A \cap A^* = \emptyset$ and $\widetilde{X}_k \cap \{x\colon |x| = r_0 + 1\} \subset A$. It follows that $A^* \cap \widehat{X}_k = \emptyset$ since $A^* \subset \widetilde{X}_k \setminus \{x\colon |x| = r_0 + 1\}$. Hence the two disjoint nonempty compact sets $\widehat{X}_k \cup A$ and A^* form a decomposition of X_k. This contradicts the connectedness of X_k, and so the claim follows.

By Lemma 9, for some subsequence $\{k_i\}_i$, there is a connected compact set A_∞ such that $A_{k_i} \xrightarrow{\mathrm{d_H}} A_\infty$. Since $A_k \subset \mathbb{H}_k \cap B(0, r_0 + 1) \xrightarrow{\mathrm{d_H}} \mathbb{H} \cap B(0, r_0)$ and A_k touches $B(0, r_0)$ and $\{x\colon |x| = r_0 + 1\}$, $A_\infty \subset \mathbb{H} \cap B(0, r_0 + 1)$ is a connected set touching $B(0, r_0)$ and $\{x\colon |x| = r_0 + 1\}$. Hence $\mathbb{H} \cap B(0, r_0 + 1)$ is not totally disconnected.

Recall that $\mathbb{H} \cap B(0, r_0 + 1) = B(0, r_0 + 1) \cap \bigcup_{S \in \mathfrak{S}} S(E)$. By (5.3), there are only finitely many $S \in \mathfrak{S}$ such that $S(E) \cap B(0, r_0 + 1) \neq \emptyset$ since $\mathbb{G}$ is finite and Γ is a discrete additive group. Thus, by Lemma 10 and the fact that the E is totally disconnected, $\mathbb{H} \cap B(0, r_0 + 1)$ is also totally disconnected. This contradiction completes the proof. ∎

Proof of Lemma 14. Let k_0 be as in Lemma 15. Note that there are only finitely many types generated by U_1 after k steps with $k \leq k_0$.

Let U be a type generated by U_1 after k steps with $k > k_0$. We claim that

$$\chi_U / n^{k_0} \subset \mathbb{H}_{k_0}. \tag{5.9}$$

To prove the claim, we use Lemma 13 to see that

$$\bigl\{S/n^{k_0}\colon S \in U\bigr\} \subset \bigl\{n^{k-k_0} \cdot S_{i_1 \ldots i_{k+1}}\colon i_1 \ldots i_{k+1} \in \{1, \ldots, m\}^{k+1}\bigr\}.$$

Routine computations show that

$$n^{k-k_0} S_{i_1 \ldots i_{k-k_0+1}} = \frac{g_{i_1} \cdots g_{i_{k-k_0+1}}}{n} + (g_{i_1} \cdots g_{i_{k-k_0}} b_{i_{k-k_0+1}} + \cdots + n^{k-k_0} b_{i_1}).$$

By (5.3), we have $n^{k-k_0} S_{i_1 \ldots i_{k-k_0+1}} \in \mathfrak{S}$ since Γ is a discrete additive group and $\mathbb{G}$ is a finite subgroup of the isometric group on Γ. Combining this with Lemma 13, we have

$$\begin{aligned}
\chi_U / n^{k_0} &= \bigcup_{S \in U} S\bigl(B(0, r_0)\bigr)/n^{k_0} \subset \bigcup_{i_1 \ldots i_{k+1} \in \{1, \ldots, m\}^{k+1}} n^{k-k_0} S_{i_1 \ldots i_{k+1}}\bigl(B(0, r_0)\bigr) \\
&= \bigcup_{i_1 \ldots i_{k+1} \in \{1, \ldots, m\}^{k+1}} n^{k-k_0} S_{i_1 \ldots i_{k-k_0+1}} \circ S_{i_{k-k_0+2} \ldots i_{k+1}}\bigl(B(0, r_0)\bigr) \\
&\subset \bigcup_{S \in \mathfrak{S}} \bigcup_{j_1 \ldots j_{k_0}} S \circ S_{j_1 \ldots j_{k_0}}\bigl(B(0, r_0)\bigr) = \mathbb{H}_{k_0}.
\end{aligned}$$

Now pick $S^* \in \mathfrak{S}$ such that $\chi_U / n^{k_0} \cap S^*\bigl(B(0, r_0)\bigr) \neq \emptyset$. Since $S^*\bigl(B(0, r_0)\bigr) = S^*(0) + B(0, r_0)/n$, we have

$$\bigl(\chi_U / n^{k_0} - S^*(0)\bigr) \cap B(0, r_0) \neq \emptyset. \tag{5.10}$$

Let $U^* = \{S - n^{k_0}S^*(0)\colon S \in U\}$. Notice that $S^*(0) \in \Gamma$, hence U^* is also a type, $U^* \sim U$ and $\chi_{U^*} = \chi_U - n^{k_0}S^*(0)$. By (5.9) and (5.10),

$$\chi_{U^*}/n^{k_0} \subset \mathbb{H}_{k_0} - n^{k_0}S^*(0) = \mathbb{H}_{k_0} \quad \text{and} \quad \chi_{U^*}/n^{k_0} \cap B(0, r_0) \neq \emptyset.$$

By Lemma 15,

$$\chi_{U^*}/n^{k_0} \subset B(0, r_0 + 1).$$

We conclude that, if U is generated by U_1, then either U is generated by U_1 after at most k_0 steps or there is an U^* with $U^* \sim U$ such that $\chi_{U^*}/n^{k_0} \subset B(0, r_0 + 1)$. Since Γ is discrete and $\mathbb{G}$ is finite, we get the finiteness of equivalent classes of types generated by $[U_1]$. □

5.2. Proof of Theorem 3. Lemmas 12 and 14 ensures the *dust-like* graph-directed structure of fractals related to E. In fact, let

$$U_1 = \{S_1, \ldots, S_m\}.$$

By Lemma 14, there are only finitely many equivalent classes generated by $[U_1]$. Suppose all such classes are $[U_2], \ldots, [U_p]$ except for $[U_1]$. Note that the equivalent classes of types generated by $[U_i]$, for $2 \leq i \leq p$, must also be generated by $[U_1]$. So by (5.7) and Lemma 12, $\{K_{U_1}, K_{U_2}, \ldots, K_{U_p}\}$ forms a family of graph-directed sets with ratio $1/n$.

To show that these sets have integer characteristic m, letting $A = (a_{i,j})_{1 \leq i,j \leq p}$ be the corresponding adjacency matrix, it suffices to prove that, for $1 \leq i \leq p$,

$$m \cdot \#U_i = \sum_{j=1}^{p} a_{i,j} \#U_j. \tag{5.11}$$

By (5.7), this is equivalent to that, for each $V \in \{U_1, \ldots, U_p\}$,

$$m \cdot \#V = \sum_{U \in \mathcal{T}_V} \#U.$$

Note that $\bigcup_{U \in \mathcal{T}_V} U = \mathfrak{S}(V)$ by the definition of $\mathcal{T}_V$. Combining this and (5.6), we have

$$\sum_{U \in \mathcal{T}_V} \#U = \#\mathfrak{S}(V) = \#\big\{S \circ S_i\colon S \in V, 1 \leq i \leq m\big\} = m \cdot \#V,$$

if we can show that $S \circ S_i \neq S^* \circ S_j$ if $(S, i) \neq (S^*, j)$.

To see this, suppose that $[V]$ is generated by $[U_1]$ after k steps and that $(S, i), (S^*, j) \in V \times \{1, \ldots, m\}$ are distinct. By the definition of generation of the equivalent classes of types and Lemma 13, there is an isometry I such that

$$S/n^k, S^*/n^k \in \big\{I \circ S_{i_1 \ldots i_{k+1}}\colon i_1 \ldots i_{k+1} \in \{1, \ldots, m\}^{k+1}\big\}.$$

Therefore,

$$S \circ S_i = n^k I \circ S_{i_1 \ldots i_{k+1} i} \quad \text{and} \quad S^* \circ S_j = n^k I \circ S_{j_1 \ldots j_{k+1} j},$$

where $i_1 \ldots i_{k+1} i \neq j_1 \ldots j_{k+1} j$ since $(S, i) \neq (S^*, j)$. We have $S \circ S_i \neq S^* \circ S_j$ since E has no complete overlaps.

We conclude that $\{K_{U_1}, K_{U_2}, \ldots, K_{U_p}\}$ are dust-like graph-directed sets with ratio $1/n$ and integer characteristic m. Therefore, it follows from Theorem 1 that $E = K_{U_1}$ is bilipschitz equivalent to $\Sigma_m^{1/n}$.

5.3. Proof of Proposition 1. Recall that E is generated by $\{S_i\}_{i=1}^m$, where

$$S_i(x) = g_i(x)/n + b_i.$$

The self-similar set E has complete overlaps if and only if there are two sequences $i_1 \cdots i_t$ and $j_1 \cdots j_t$ with $i_1 \neq j_1$ such that

$$S_{i_1} \circ \cdots \circ S_{i_t} = S_{j_1} \circ \cdots \circ S_{j_t}.$$

This is equivalent to

$$g_{i_1} \cdots g_{i_t} = g_{j_1} \cdots g_{j_t} \tag{5.12}$$

and $S_{i_1} \circ \cdots \circ S_{i_t}(0) = S_{j_1} \circ \cdots \circ S_{j_t}(0)$ which means

$$b_{i_1} + \frac{g_{i_1} b_{i_2}}{n} + \cdots + \frac{g_{i_1} \cdots g_{i_{t-1}} b_{i_t}}{n^{t-1}} = b_{j_1} + \frac{g_{j_1} b_{j_2}}{n} + \cdots + \frac{g_{j_1} \cdots g_{j_{t-1}} b_{j_t}}{n^{t-1}}. \tag{5.13}$$

We will show that (5.12) and (5.13) are equivalent to that there is an edge chain starting at an original vertex $(b_{i_1} - b_{j_1}, g_{i_1}, g_{j_1})$ and ending at a boundary vertex $(0, g_{i_1} \cdots g_{i_t}, g_{j_1} \cdots g_{j_t})$ with $g_{i_1} \cdots g_{i_t} = g_{j_1} \cdots g_{j_t}$, i.e.,

$$(b_{i_1} - b_{j_1}, g_{i_1}, g_{j_1}) \xrightarrow{(i_2,j_2)} \cdots \xrightarrow{(i_t,j_t)} (0, f_t, f_t'), \quad \text{where } f_t = f_t' = g_{i_1} \cdots g_{i_t}. \tag{5.14}$$

Thus Proposition 1 follows.

By the definition of edges (see Step 2 of the algorithm),

$$\begin{aligned}(b_{i_1} - b_{j_1}, g_{i_1}, g_{j_1}) = (x_1, f_1, f_1') &\xrightarrow{(i_2,j_2)} (x_2, f_2, f_2') \xrightarrow{(i_3,j_3)} \cdots \xrightarrow{(i_k,j_k)} (x_k, f_k, f_k') \\ &\xrightarrow{(i_{k+1},j_{k+1})} \cdots \xrightarrow{(i_t,j_t)} (x_t, f_t, f_t') = (x_t, g_{i_1} \cdots g_{i_t}, g_{j_1} \cdots g_{j_t}),\end{aligned}$$

where $f_k = g_{i_1} \cdots g_{i_k}$, $f_k' = g_{j_1} \cdots g_{j_k}$ and

$$x_{k+1} = n x_k + f_k b_{i_{k+1}} - f_k' b_{j_{k+1}} \in \Gamma.$$

Therefore,

$$n^{-t+1} x_t = (b_{j_1} - b_{i_1}) + \left(\frac{g_{j_1} b_{j_2}}{n} - \frac{g_{i_1} b_{i_2}}{n} \right) + \cdots + \left(\frac{g_{j_1} \cdots g_{j_{t-1}} b_{j_t}}{n^{t-1}} - \frac{g_{i_1} \cdots g_{i_{t-1}} b_{i_t}}{n^{t-1}} \right).$$

Consequently, if (5.12) and (5.13) hold, then $x_t = 0$ and (5.14) follows. Conversely, if (5.14) holds, then $f_t = f_t'$ implies (5.12) and $x_t = 0$ implies (5.13). This completes the proof of Proposition 1.

6. Open questions.

QUESTION 1. How to generalize the result to the case of *irrational* characteristic?

An interesting class of self-similar sets with overlaps is $\{E_\lambda\}_\lambda$, where E_λ is generated by

$$S_1(x) = x/3, \quad S_\lambda = x/3 + \lambda/3 \quad \text{and} \quad S_3(x) = x/3 + 2/3.$$

This class has been studied by Kenyon [17], Rao and Wen [31], Świątek and Veerman [35] and Hochman [15].

It is proved in [14] that $E_{2/3^n}$ and $E_{2/3^{n'}}$ are bilipschitz equivalent for any $n, n' \geq 1$, and that $\dim_{\mathrm{H}} E_{2/3^n} = \dim_{\mathrm{H}} E_{2/3^{n'}} = \left(\log \frac{3+\sqrt{5}}{2}\right)/\log 3$.

In fact, $E_{2/3^n}$ can generate graph-directed sets with adjacency matrix

$$M_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 2 \end{pmatrix}_{2^n \times 2^n}.$$

The matrix M_n has Perron-Frobenius eigenvalue $(3+\sqrt{5})/2$ and corresponding positive eigenvector.

Let λ_1 =6/7 and λ_2 =8/9, then their adjacency matrices of corresponding graph-directed sets are

$$B_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Using Mathematica 7.0, we find these two matrices have the same Perron-Frobenius eigenvalue $2.879\ldots$, we ask

whether E_{λ_1} and E_{λ_2} are bilipschitz equivalent or not?

Looking at Theorem 3, we find there is an algorithm to test the existence of complete overlaps, so the following question arises naturally.

Question 2. How to test in polynomial time the total disconnectedness of self-similar sets in Λ, especially for fractal cubes?

A self-similar set E is called a fractal cube if its IFS $\{S_i\}_i$ has the form $S_i(x) = (x + b_i)/n$ with $b_i \in \{0, 1, \ldots, n-1\}^l$ with $l \geq 2$. Lau, Luo and Rao [18] gave an extensive study of the topological structure of fractal squares (the case $l = 2$). Among other things, they obtained an algorithm to test the total disconnectedness. However, their argument depends heavily on the topological property of $\mathbb{R}^2$. It seems difficult to find an algorithm for $l \geq 3$. Even for fractal squares, there seems no obvious manner to see whether they are totally disconnected. Please see the following interesting example.

Example 7. Consider the initial self-similar pattern in Figure 4. At first sight, one may guess that the self-similar set is totally disconnected. However, this self-similar set includes infinitely many lines.

In the unit square, we have γ_1 from placement 2 to placement 3, γ_2 from placement 1 to placement 2 and γ_3 from placement 3 to placement 1. In the small squares with

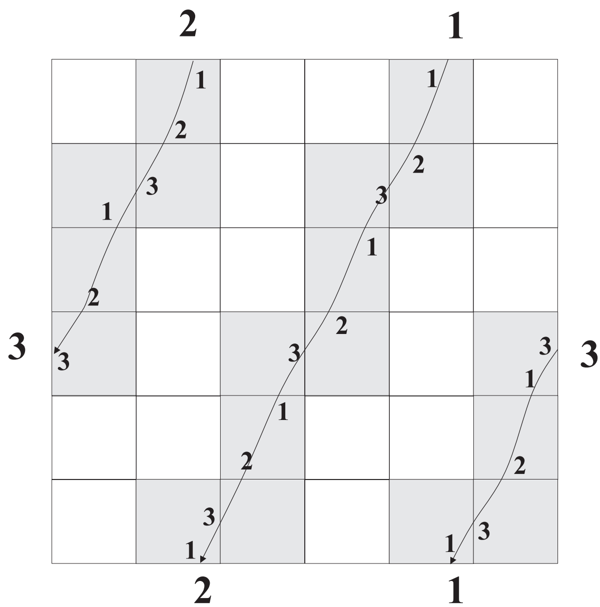

FIG. 4. *A self-similar set which is not totally disconnected*

side length 1/6, we also have small curves which are similar to $\gamma_1, \gamma_2, \gamma_3$ respectively. Therefore, this self-similar set includes $\gamma_1, \gamma_2, \gamma_3$. In fact, the three curves $\{\gamma_1, \gamma_2, \gamma_3\}$ are the straight lines of slope 2 connecting the six points $(0, 2/5)$, $(3/10, 1)$, $(3/10, 0)$, $(4/5, 1)$, $(4/5, 0)$, $(1, 2/5)$, respectively. They are also graph-directed sets (satisfying the open set condition) with ratio 1/6 and adjacency matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 3 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

Acknowledgement. The authors are very grateful to the anonymous referee for his/her patience and carefulness to improve the quality of the manuscript.

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