ASIAN J. MATH.
Vol. 22, No. 6, pp. 1117–1126, December 2018

007

NEW SURFACES WITH $K^2 = 7$ AND $p_g = q \leq 2$*

CARLOS RITO†

Abstract. We construct smooth minimal complex surfaces of general type with $K^2 = 7$ and: $p_g = q = 2$, Albanese map of degree 2 onto a $(1, 2)$-polarized abelian surface; $p_g = q = 1$ as a double cover of a quartic Kummer surface; $p_g = q = 0$ as a double cover of a numerical Campedelli surface with 5 nodes.

Key words. Surface of general type, Albanese map, double covering.

Mathematics Subject Classification. 14J29.

1. Introduction. Despite the efforts of several authors in past years, surfaces of general type with the lowest possible value of the holomorphic Euler characteristic $\chi = 1$ are still not classified. For these surfaces the geometric genus p_g, the irregularity q and the self-intersection of the canonical divisor K satisfy:

$$1 + p_g \leq K^2 \leq 9 \quad \text{if} \quad p_g \leq 1,$$

$$4 \leq K^2 \leq 9 \quad \text{if} \quad p_g = 2,$$

$$K^2 = 6 \text{ or } 8 \quad \text{if} \quad p_g = 3,$$

$$K^2 = 8 \quad \text{if} \quad p_g = 4,$$

from the Bogomolov-Miyaoka-Yau and Debarre inequalities, [HP] and the Beauville Appendix in [De] (cf. also [CCM], [Pi]).

According to Sai-Kee Yeung, the case with $p_g = q = 2$ and $K^2 = 9$ does not occur (see Section 6 in the revised version of the paper [Ye], available at `http://www.math.purdue.edu/~yeung/`).

So there are examples for all possible values of the invariants except for one mysterious case:

$$K^2 = 7, p_g = q = 2.$$

The cases $K^2 = 7$, $p_g = q = 1$ or 0 are also intriguing:

- $p_g = q = 1$. Lei Zhang [Zh] has shown that one of three cases occur:
  a) the bicanonical map is birational;
  b) the bicanonical map is of degree 2 onto a rational surface;
  c) the bicanonical map is of degree 2 onto a Kummer surface.
  The author has given examples for a) [Ri2] and b) [Ri3], but so far it is not known if c) can occur.

---

*Received January 20, 2016; accepted for publication March 23, 2017.

†Permanent address: Universidade de Trás-os-Montes e Alto Douro, UTAD, Quinta de Prados, 5000-801 Vila Real, Portugal (crito@utad.pt). Current address: Departamento de Matemática, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal (crito@fc.up.pt).

- $p_g = q = 0$. Yifan Chen [Ch2] considers the case when the automorphism group of the surface S contains a subgroup isomorphic to $\mathbb{Z}_2^2$. He shows that three different families of surfaces may exist:
  a) S is an Inoue surface [In];
  b) S belongs to the family constructed by him in [Ch1];
  c) a third case, in particular S is a double cover of a surface with $p_g = q = 0$ and $K^2 = 2$ with 5 nodes.
  The existence of this last case is an open problem.

In this paper we show the existence of the above three open cases. We give constructions for surfaces with $K^2 = 7$ and:

- $p_g = q = 2$, Albanese map of degree 2 onto a $(1, 2)$-polarized abelian surface;
- $p_g = q = 1$, bicanonical map of degree 2 onto a Kummer surface;
- $p_g = q = 0$ as a double cover of a numerical Campedelli surface with 5 nodes.

In all cases the surface can be seen as a double cover with branch locus as in the result below. In particular we show that a construction for the case $p_g = q = 2$ as suggested by Penegini and Polizzi [PP, Remark 2.2] does exist.

Proposition 1. *Let X be an Abelian, $K3$ or Enriques surface containing n disjoint (-2)-curves $A_1, \ldots, A_n$, $n = 0, 16$ or 8, respectively. Assume that X contains a reduced curve B and a divisor L such that*

$$B + \sum_1^n A_i \equiv 2L,$$

B is disjoint from $\sum_1^n A_i$, $B^2 = 16$ and B contains a $(3, 3)$-point and no other singularity. Let S be the smooth minimal model of the double cover of X with branch locus $B + \sum_1^n A_i$. Then $\chi(\mathcal{O}_S) = 1$ and $K_S^2 = 7$.

Proof. This follows from the double cover formulas (see e.g. [BHPV, V.22]) and the fact that a $(3, 3)$-point decreases both χ and K^2 by 1 (see e.g. [Pe, p. 185]):

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_X) + \frac{1}{2}L(K_X + L) - 1 = 1,$$

$$K_S^2 = 2(K_X + L)^2 + n - 1 = 7.$$

□

Notation. We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. An (m_1, m_2)-point of a curve, or point of type (m_1, m_2), is a singular point of multiplicity m_1 which resolves to a point of multiplicity m_2 after one blow-up. Linear equivalence of divisors is denoted by $\equiv$. The rest of the notation is standard in Algebraic Geometry.

Acknowledgements. The author is a member of the Center for Mathematics of the University of Porto. This research was partially supported by FCT (Portugal) under the project PTDC/MAT-GEO/0675/2012 and by CMUP (UID/-MAT/00144/2013), which is funded by FCT with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

2. Bidouble covers. A bidouble cover is a finite flat Galois morphism with Galois group $\mathbb{Z}_2^2$. Following [Ca] or [Pa], to define a bidouble cover $\pi : V \to X$, with V, X smooth surfaces, it suffices to present:

- smooth effective divisors $D_1, D_2, D_3 \subset X$ with pairwise transverse intersections and no common intersection;
- line bundles L_1, L_2, L_3 such that $L_g + D_g \equiv L_j + L_k$ for each permutation (g, j, k) of $(1, 2, 3)$.

One has

$$\chi(\mathcal{O}_V) = 4\chi(\mathcal{O}_X) + \frac{1}{2}\sum_1^3 L_i(K_X + L_i),$$

$$p_g(V) = p_g(X) + \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i))$$

and

$$2K_V \equiv \pi^*\left(2K_X + \sum_1^3 L_i\right),$$

which implies

$$K_V^2 = \left(2K_X + \sum_1^3 L_i\right)^2.$$

3. Example with $p_g = q = 2$.

Step 1. Let $T_1, \ldots, T_4 \subset \mathbb{P}^2$ be distinct lines through a point p_0, let $p_1, p_2 \neq p_0$ be points in T_1, T_2, respectively, and C_1, C_2 be distinct smooth conics tangent to T_1, T_2 at p_1, p_2. Consider the map

$$\mu : X \longrightarrow \mathbb{P}^2$$

which resolves the singularities of the divisor $C_1 + C_2 + T_1 + \cdots + T_4$. Then μ is given by blow-ups at

$$p_0, p_1, p_1', p_2, p_2', p_3, \ldots, p_{10},$$

where p_i' is the point infinitely near to p_i corresponding to the line T_i, and $p_3, \ldots, p_{10}$ are nodes of $C_1 + C_2 + T_3 + T_4$. Let $E_0, E_1, E_1', E_2, E_2', E_3, \ldots, E_{10}$ be the corresponding exceptional divisors (with self-intersection -1) and let

$$\pi : V \longrightarrow X$$

be the bidouble cover defined by the divisors

$$\begin{aligned} D_1 &:= \left(\widetilde{T_1} + \widetilde{T_2} - 2E_0 - 2E_1' - 2E_2'\right) + \textstyle\sum_3^{10} E_i, \\ D_2 &:= \widetilde{T_3} + \widetilde{T_4} - 2E_0 - \textstyle\sum_3^{10} E_i, \\ D_3 &:= \widetilde{C_1} + \widetilde{C_2} - 2E_1 - 2E_1' - 2E_2 - 2E_2' - \textstyle\sum_3^{10} E_i, \end{aligned}$$

where the notation $\widetilde{\cdot}$ stands for the total transform $\mu^*(\cdot)$.

Notice that D_1 is the union of $\sum_3^{10} E_i$ with four (-2)-curves contained in the pullback of T_1+T_2, and D_2, D_3 are just the strict transforms of T_3+T_4, C_1+C_2, respectively.

One can easily see that the divisors D_1, D_2 and D_3 have pairwise transverse intersections and no common intersection.

Denote by T a general line of $\mathbb{P}^2$ and let

$$\begin{aligned} L_1 &:= 3\widetilde{T} - E_0 - E_1 - E_1' - E_2 - E_2' - \textstyle\sum_3^{10} E_i, \\ L_2 &:= 3\widetilde{T} - E_0 - E_1 - 2E_1' - E_2 - 2E_2', \\ L_3 &:= 2\widetilde{T} - 2E_0 - E_1' - E_2'. \end{aligned}$$

Then

$$\begin{aligned} K_X + L_1 &\equiv 0, \\ K_X + L_2 &\equiv -E_1' - E_2' + \textstyle\sum_3^{10} E_i, \\ K_X + L_3 &\equiv -\widetilde{T} - E_0 + E_1 + E_2 + \textstyle\sum_3^{10} E_i \end{aligned}$$

and

$$\chi(\mathcal{O}_V) = 4 + \frac{1}{2}(0-4-4) = 0,$$

$$p_g(V) = 0+1+0+0 = 1.$$

Let X_1 be the surface given by the double covering $\phi : X_1 \longrightarrow X$ with branch locus $D_2 + D_3$. The divisor $\phi^*\left(\widetilde{T_1} + \widetilde{T_2} - 2E_0 - 2E_1' - 2E_2'\right)$ is a disjoint union of 8 (-2)-curves, and the divisor $\phi^*\left(\sum_3^{10} E_i\right)$ is also a disjoint union of 8 (-2)-curves. Hence $\phi^*(D_1)$ is a disjoint union of 16 (-2)-curves. The canonical divisor K_V of V is the support of the pullback of D_1, a disjoint union of 16 (-1)-curves. So the minimal model V' of V is an abelian surface, with Kummer surface X_1. Notice that the lines $T_1, \ldots, T_4$ give rise to elliptic fibres of type I_0^* in X_1 (four disjoint (-2)-curves plus an elliptic curve with multiplicity 2).

Step 2. Now let R be the tangent line to C_1 at $p_3 \in C_1 \cap T_3$. We claim that the strict transform $\widehat{R} \subset V'$ of R is a curve with a tacnode (singularity of type $(2,2)$) at the pullback of p_3 and with self-intersection $\widehat{R}^2 = 8$. In fact, the covering π factors as

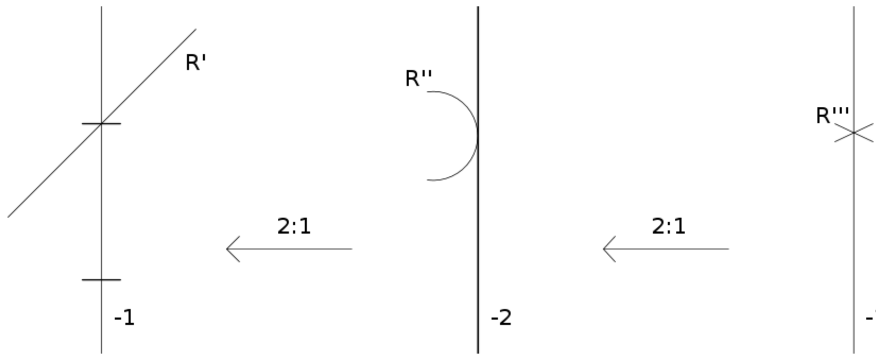
$$V \xrightarrow{\ \varphi\ } X_1 \xrightarrow{\ \phi\ } X.$$

The strict transforms $R', C_1' \subset X$ of R, C_1 meet at a point in the (-1)-curve E_3. Since C_1' is contained in the branch locus of the covering ϕ, then the curve $R'' := \phi^*(R')$ is tangent to the (-2)-curve $\phi^*(E_3)$. This curve is in the branch locus of φ, hence the curve $R''' := \varphi^*(R'')$ has a node at a point p in the (-1)-curve

$$\overline{E}_3 := \frac{1}{2}(\phi \circ \varphi)^*(E_3).$$

So the image of R''' in the minimal model V' of V is a curve $\widehat{R}$ with a tacnode.

The reduced strict transform of the conic C_1 passes through p, hence its image $\widehat{C}_1 \subset V'$ is tangent to $\widehat{R}$ at the tacnode. So the divisor $\widehat{R} + \widehat{C}_1$ is reduced and has



a singularity of type $(3, 3)$. We want to show that it is even, i.e. there is a divisor L such that

$$\widehat{R} + \widehat{C}_1 \equiv 2L,$$

and that

$$(\widehat{R} + \widehat{C}_1)^2 = 16.$$

Step 3. The pencil of lines through the point p_0 induces an elliptic fibration of the surface V. For $i = 1, 2$, the line T_i gives rise to a fibre (counted twice) which is the union of disjoint (-1)-curves $\xi_1^i, \ldots, \xi_4^i$ with an elliptic curve T_i' such that $\xi_j^i T_i' = 1$. These curves can be labeled such that ξ_1^i, ξ_2^i correspond to the strict transform of T_i and ξ_3^i, ξ_4^i correspond to the (-2)-curve $E_i - E_i'$. The curve R''' meets $\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2$, thus $\widehat{R}^2 = 4 + 4 = 8$ and then $(\widehat{R} + \widehat{C}_1)^2 = 8 + 0 + 2 \times 4 = 16$.

Let H be the line through the points p_1 and p_2. We have

$$\pi^*(R + H) = R''' + H' + 2\overline{E}_3 + \sum_1^2 (T_i' + 2\xi_3^i + 2\xi_4^i),$$

where $H' \subset V$ is the strict transform of H. Denote by $\widehat{R}$, $\widehat{H}$, $\widehat{T}_1$ and $\widehat{T}_2$ the projections of R''', H', T_1' and T_2' into the minimal model V' of V. Then there is a divisor L' such that

$$\widehat{R} + \widehat{H} + \widehat{T}_1 + \widehat{T}_2 \equiv 2L'.$$

The pencil of conics tangent to the lines T_1, T_2 at p_1, p_2 induces another elliptic fibration of the surface V'. The curves $\widehat{C}_1$ and $\widehat{H}$ are fibres of this fibration. We have

$$\widehat{R} + \widehat{C}_1 + \widehat{H} + \widehat{C}_1 + \widehat{T}_1 + \widehat{T}_2 \equiv 2(L' + \widehat{C}_1).$$

Since the above fibrations have elliptic bases, the sums $\widehat{H} + \widehat{C}_1$ and $\widehat{T}_1 + \widehat{T}_2$ are even, thus there exists a divisor L such that $\widehat{R} + \widehat{C}_1 \equiv 2L$.

Step 4. Finally, consider the double cover

$$\rho : S' \longrightarrow V'$$

with branch locus $\widehat{R}+\widehat{C}_1$, determined by L. It follows from Proposition 1 that the smooth minimal model S of S' is a surface of general type with $\chi=1$ and $K^2=7$. It is known that there is no smooth minimal surface of general type with $\chi=1$, $K^2=7$ and $q>2$ (see [HP] and the Beauville Appendix in [De]). Since $q(S)\geq q(V')=2$, we conclude that $p_g(S)=q(S)=2$.

Recall that $p_3\in C_1\cap T_3$ and assume that $p_4\in C_2\cap T_4$. The branch curve $C_1+C_2+T_1+\cdots+T_4$ is determined by the points $p_0,\ldots,p_4$. Since any two sequences of 4 points in $\mathbb{P}^2$, in general position, are projectively equivalent, we can fix $p_0,\ldots,p_3$. This implies that our family of examples is parametrized by a 2-dimensional open subset of $\mathbb{P}^2$.

4. Example with $p_g=q=1$. Let $T_1,T_2,T_3\subset\mathbb{P}^2$ be distinct lines through a point p_0 and $p_1,p_2,p_3\neq p_0$ be non-collinear points in T_1,T_2,T_3, respectively. For the construction of an example with $p_g=q=0$ and $K^2=7$, Y. Chen has shown that for a general point $p_4\neq p_0,\ldots,p_3$, there exist:

- an irreducible sextic curve C_6 with a node at p_0, a tacnode at p_i with tangent line T_i, $i=1,2,3$, and having a triple point at p_4;
- an irreducible quintic curve C_5 through p_0,p_4 and with a tacnode at p_i with tangent line T_i, $i=1,2,3$.

The curves C_5,C_6 correspond to the curves $\tilde{B}_2,\tilde{B}_3$ given in [Ch1, Proposition 2.5].

Let T be a general line through p_0. Keeping a notation analogous to the one in Section 3, consider the map

$$\mu: X\longrightarrow\mathbb{P}^2$$

which resolves the singularities of the curve C_6 and let

$$\pi: V\longrightarrow X$$

be the bidouble cover defined by the divisors

$$\begin{aligned}
&D_1:=\widetilde{T}-E_0+E_4,\\
&D_2:=\Big(\widetilde{T_1}+\widetilde{T_2}+\widetilde{T_3}-3E_0-\textstyle\sum_1^3 2E_i'\Big)+\Big(\widetilde{C_6}-2E_0-\sum_1^3(2E_i+2E_i')-3E_4\Big),\\
&D_3:=\widetilde{C_5}-E_0-\textstyle\sum_1^3(2E_i+2E_i')-E_4.
\end{aligned}$$

Notice that D_2 is the union of the strict transform of C_6 with six (-2)-curves contained in the pullback of $T_1+T_2+T_3$, and D_3 is the strict transform of C_5.

We verify that the divisors D_1, D_2 and D_3 have pairwise transverse intersections and no common intersection. Let $\widehat{C}_5,\widehat{C}_6$ be the strict transforms of C_5,C_6. These curves are disjoint from the (-2)-curves contained in the pullback of $T_1+T_2+T_3$. It is shown in [Ch1, Proposition 2.5] that the divisor $\widehat{C}_5+\widehat{C}_6+E_4$ has at most nodal singularities. Since the line T through p_0 is generic, the result follows.

We have

$$\begin{aligned}
&L_1:=7\widetilde{T}-3E_0-\textstyle\sum_1^3(2E_i+3E_i')-2E_4,\\
&L_2:=3\widetilde{T}-E_0-\textstyle\sum_1^3(E_i+E_i'),\\
&L_3:=5\widetilde{T}-3E_0-\textstyle\sum_1^3(E_i+2E_i')-E_4,
\end{aligned}$$

$$\begin{aligned}
&K_X+L_1\equiv 4\widetilde{T}-2E_0-\textstyle\sum_1^3(E_i+2E_i')-E_4,\\
&K_X+L_2\equiv E_4,\\
&K_X+L_3\equiv 2\widetilde{T}-2E_0-\textstyle\sum_1^3 E_i'
\end{aligned}$$

and

$$2K_X + \sum_1^3 L_i \equiv 9\widetilde{T} - 5E_0 - \sum_1^3 (2E_i + 4E_i') - E_4.$$

Thus

$$\chi(\mathcal{O}_V) = 4 + \frac{1}{2}(-4 + 0 - 2) = 1,$$

$$p_g(V) = 0 + 0 + 1 + 0 = 1$$

and

$$K_V^2 = -5.$$

Since the minimal model V' of V is obtained contracting the 12 (-1)-curves contained in $\pi^*\left(\widetilde{T_1} + \widetilde{T_2} + \widetilde{T_3}\right)$, then $K_{V'}^2 = 7$.

Notice that the minimal smooth resolution of the double plane $Q \to X$ with branch locus $D_1 + D_3$ is a K_3 surface with 16 disjoint (-2)-curves, and one can obtain the surface V as a double cover of Q with a branch curve B as in Proposition 1. It can be shown that the bicanonical map of V factors through this double covering. In fact, it follows from [Zh] that the bicanonical map is of degree 2 onto a Kummer surface.

Finally we can see, as in [Ch1, Section 3], that this family of examples is parametrized by a 3-dimensional open subset of $\mathbb{P}^2 \times \mathbb{P}^1$: the point p_4 moves in an open subset of $\mathbb{P}^2$ and $\widetilde{T} - E_0$ moves in a pencil.

5. Example with $p_g = q = 0$. In [Ri2, §4.6], the author has computed points $p_0, \ldots, p_5 \in \mathbb{P}^2$ such that there exist:

- an irreducible curve C_7 of degree 7 with triple points at p_0, p_5 and tacnodes at $p_1, \ldots, p_4$ with tangent line the line T_i through p_0, p_i, $i = 1, \ldots, 4$;
- an irreducible curve C_6 of degree 6 with a node at p_0, tacnodes at $p_1, \ldots, p_4$ with tangent line the line T_i through p_0, p_i, $i = 1, \ldots, 4$, and passing through p_5 such that the singularity of $C_6 + C_7$ at p_5 is ordinary.

For the readers convenience, we give in the Appendix the equations of the curves C_6, C_7 computed in [Ri2, §4.6] (but with a different choice of $p_0, \ldots, p_5$ in order to get shorter equations) and we verify that the curves are exactly as stated above.

We note that for generic points $p_0, \ldots, p_5 \in \mathbb{P}^2$ there is no such curve C_7. This is because the dimension of the linear system of plane curves of degree 7 is 35, and the imposition of singularities as above puts 36 conditions. We don't know how to construct C_7 without using computer algebra. Thus here we compute just one surface, and we make no considerations about the dimension of the family of surfaces.

Keeping a notation as above, consider the map

$$\mu : X \longrightarrow \mathbb{P}^2$$

which resolves the singularities of the curve C_7 and let

$$\pi : V \longrightarrow X$$

be the bidouble cover defined by the divisors

$$\begin{aligned}
D_1 &:= \left(\widetilde{T_1} - E_0 - 2E_1'\right) + E_5,\\
D_2 &:= \left(\widetilde{T_4} - E_0 - 2E_4'\right) + \left(\widetilde{C_6} - 2E_0 - \textstyle\sum_1^4 (2E_i + 2E_i') - E_5\right),\\
D_3 &:= \left(\widetilde{T_2} + \widetilde{T_3} - 2E_0 - 2E_2' - 2E_3'\right) + \left(\widetilde{C_7} - 3E_0 - \textstyle\sum_1^4 (2E_i + 2E_i') - 3E_5\right).
\end{aligned}$$

Notice that D_1 is the union of E_5 with two (-2)-curves contained in the pullback of T_1, the divisor D_2 is the union of the strict transform of C_6 with two (-2)-curves contained in the pullback of T_4, and D_3 is the union of the strict transform of C_7 with four (-2)-curves contained in the pullback of $T_2 + T_3$.

To show that the divisors D_1, D_2 and D_3 have pairwise transverse intersections and no common intersection, notice that the strict transforms $\widehat{C}_6, \widehat{C}_7$ of C_6, C_7 meet at an unique point, because the intersection number of C_6 and C_7 at the points $p_0, \ldots, p_5$ is $6 + 4 \times 8 + 3 = 41$. It suffices to show that this point is not in E_5. In the Appendix we compute that in fact the singularities of $C_6 + C_7$ at $p_0, \ldots, p_5$ are no worse than stated; there is an ordinary double point not in $\{p_0, \ldots, p_5\}$.

Let T be a general line through p_0. We have

$$\begin{aligned}
L_1 &:= 8\widetilde{T} - 4E_0 - (2E_1 + 2E_1') - \textstyle\sum_2^4 (2E_i + 3E_i') - 2E_5,\\
L_2 &:= 5\widetilde{T} - 3E_0 - \textstyle\sum_1^3 (E_i + 2E_i') - (E_4 + E_4') - E_5,\\
L_3 &:= 4\widetilde{T} - 2E_0 - (E_1 + 2E_1') - \textstyle\sum_2^3 (E_i + E_i') - (E_4 + 2E_4'),
\end{aligned}$$

$$\begin{aligned}
K_X + L_1 &\equiv \left(\widetilde{T_2} + \widetilde{T_3} + \widetilde{T_4} - 3E_0 - \textstyle\sum_2^4 2E_i'\right) + \left(2\widetilde{T} - (E_1 + E_1') - \textstyle\sum_2^5 E_i\right),\\
K_X + L_2 &\equiv 2\widetilde{T} - 2E_0 - \textstyle\sum_1^3 E_i',\\
K_X + L_3 &\equiv \widetilde{T} - E_0 - E_1' - E_4' + E_5
\end{aligned}$$

and

$$2K_X + \sum_1^3 L_i \equiv 11\widetilde{T} - 7E_0 - \sum_1^4 (2E_i + 4E_i') - E_5.$$

The divisor

$$\widetilde{T_2} + \widetilde{T_3} + \widetilde{T_4} - 3E_0 - \sum_2^4 2E_i'$$

is a disjoint union of 6 (-2)-curves, each meeting $K_X + L_1$ with intersection number -1. Hence $K_X + L_1$ is effective only if

$$2\widetilde{T} - (E_1 + E_1') - \sum_2^5 E_i$$

is effective. This is not the case, we can verify that the conic through the points $p_1, \ldots, p_5$ is not tangent to the line T_1. Therefore $h^0(X, \mathcal{O}_X(K_X + L_1)) = 0$ and then

$$p_g(V) = 0 + 0 + 0 + 0 = 0.$$

Also

$$\chi(\mathcal{O}_V) = 4 + \frac{1}{2}(-2 - 2 - 2) = 1$$

and

$$K_V^2 = -9.$$

Since the minimal model V' of V is obtained contracting the 16 (-1)-curves contained in $\pi^*\left(\widetilde{T_1} + \cdots + \widetilde{T_4}\right)$, then $K_{V'}^2 = 7$.

The covering π factors as

$$V \longrightarrow Y \longrightarrow X,$$

where $Y \to X$ is the double cover with branch locus $D_2 + D_3$. Using the double cover formulas, one can verify that the smooth minimal model of Y is a numerical Campedelli surface ($p_g = q = 0$, $K^2 = 2$). The double cover $V \to Y$ is ramified over the pullback of D_1 (which contains four (-2)-curves) and over the node corresponding to the transverse intersection of D_2 and D_3.

Appendix A. Here we use the computer algebra system Magma [BCP] to show that the curves C_6 and C_7 referred in Section 5 are exactly as stated there. This code can be tested on the online Magma calculator [MC].

```
R<i>:=PolynomialRing(Rationals());
K<i>:=ext<Rationals()|i^2+1>;
P<x,y,z>:=ProjectiveSpace(K,2);

F6:=4*x^6-273*x^4*y^2-258*x^2*y^4-481*y^6+720*x^4*y*z+1740*x^2*y^3*z+
4020*y^5*z-520*x^4*z^2-3190*x^2*y^2*z^2-12670*y^4*z^2+1200*x^2*y*z^3+
17700*y^3*z^3+900*x^2*z^4-9225*y^2*z^4;

F7:=12*x^7+(8*i+420)*x^6*y+1611*x^5*y^2+(174*i+3060)*x^4*y^3+
4086*x^3*y^4+(924*i+3360)*x^2*y^5+987*x*y^6+(-242*i+720)*y^7-560*x^6*z-
4320*x^5*y*z+(-480*i-13580)*x^4*y^2*z-23940*x^3*y^3*z+
(-5160*i-24980)*x^2*y^4*z-10620*x*y^5*z+(1320*i-6960)*y^6*z+
2760*x^5*z^2+(240*i+16200)*x^4*y*z^2+44970*x^3*y^2*z^2+
(9780*i+63900)*x^2*y^3*z^2+39210*x*y^4*z^2+(-2460*i+25200)*y^5*z^2-
4400*x^4*z^3-28800*x^3*y*z^3+(-7200*i-62300)*x^2*y^2*z^3-
60300*x*y^3*z^3+(1800*i-40400)*y^4*z^3+2700*x^3*z^4+
(1800*i+16500)*x^2*y*z^4+33075*x*y^2*z^4+(-450*i+24000)*y^3*z^4;

C6:=Curve(P,F6); C7:=Curve(P,F7);
IsAbsolutelyIrreducible(C6);
IsAbsolutelyIrreducible(C7);

p:=[P![0,0,1],P![-2,1,1],P![2,1,1],P![-1,2,1],P![1,2,1],P![3,2*i,1]];

[ResolutionGraph(C6,p[i]):i in [1..5]];
[ResolutionGraph(C7,p[i]):i in [1..6]];
[ResolutionGraph(C6 join C7,p[i]):i in [1..6]];
SingularPoints(C6 join C7);
```

REFERENCES

[BHPV] W. Barth, K. Hulek, C Peters, and A. Van de Ven, *Compact complex surfaces*, Berlin: Springer, 2nd enlarged edition, 2004.

[BCP] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24 (1997), no. 3-4, pp. 235–265.

[CCM] F. Catanese, C. Ciliberto, and M. Mendes Lopes, *On the classification of irregular surfaces of general type with nonbirational bicanonical map*, Trans. Amer. Math. Soc., 350:1 (1998), pp. 275–308.

[Ca] F. Catanese, *Singular bidouble covers and the construction of interesting algebraic surfaces*, in "Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)", volume 241 of Contemp. Math., pp. 97–120. Amer. Math. Soc., Providence, RI, 1999.

[Ch1] Y. Chen, *A new family of surfaces of general type with $K^2 = 7$ and $p_g = 0$*, Math. Z., 275 (2013), no. 3-4, pp. 1275–1286.

[Ch2] Y. Chen, *Commuting involutions on surfaces of general type with $p_g = 0$ and $k^2 = 7$*, Manuscripta Math., 147 (2015), no. 3-4, pp. 547–575.

[De] O. Debarre, *Inégalités numériques pour les surfaces de type général*, Bull. Soc. Math. France, 110:3 (1982), pp. 319–346. With an appendix by A. Beauville.

[HP] C. D. Hacon and R. Pardini, *Surfaces with $p_g = q = 3$*, Trans. Amer. Math. Soc., 354:7 (2002), pp. 2631–2638.

[In] M. Inoue, *Some new surfaces of general type*, Tokyo J. Math., 17:2 (1994), pp. 295–319.

[MC] Magma online calculator. `http://magma.maths.usyd.edu.au/calc/`.

[Pa] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math., 417 (1991), pp. 191–213.

[PP] M. Penegini and F. Polizzi, *Surfaces with $p_g = q = 2, K^2 = 6$, and Albanese map of degree* 2, Can. J. Math., 65:1 (2013), pp. 195–221.

[Pe] U. Persson, *Double coverings and surfaces of general type*, in "Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977), volume 687 of Lecture Notes in Math., pp. 168–195. Springer, Berlin, 1978.

[Pi] G. P. Pirola, *Surfaces with $p_g = q = 3$*, Manuscripta Math., 108:2 (2002), pp. 163–170.

[Ri2] C. Rito, *Involutions on surfaces with $p_g = q = 1$*, Collect. Math., 61:1 (2010), pp. 81–106.

[Ri3] C. Rito, *On equations of double planes with $p_g = q = 1$*, Math. Comp., 79:270 (2010), pp. 1091–1108.

[Ye] S.-K. Yeung, *Classification of surfaces of general type with Euler number* 3, J. Reine Angew. Math., 679 (2013), pp. 1–22.

[Zh] L. Zhang, *Surfaces with $p_g = q = 1$, $K^2 = 7$ and non-birational bicanonical maps*, Geom. Dedicata, 177 (2015), pp. 293–306.