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ON THE CONNECTEDNESS OF THE STANDARD WEB OF CALABI-YAU 3-FOLDS AND SMALL TRANSITIONS*

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Abstract. We supply a detailed proof of the result by P.S. Green and T. Hübsch that all complete intersection Calabi–Yau 3-folds in product of projective spaces are connected through projective conifold transitions (known as the standard web). We also introduce a subclass of small transitions which we call *primitive* small transitions and study such subclass. More precisely, given a small projective resolution $\pi : \widehat{X} \to X$ of a Calabi–Yau 3-fold X, we show that if the natural closed immersion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X)$ is an isomorphism then X has only ODPs as singularities.

Key words. Conifold transition, Calabi–Yau, Complete intersection, Deformation.

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1. Introduction. Calabi–Yau conifolds, i.e., Calabi–Yau 3-folds with only ordinary double points (ODPs), arise naturally in algebraic geometry and string theory, where a Calabi–Yau 3-fold X is a projective Gorenstein 3-fold with at worst terminal singularities such that $K_X \sim 0$ and $H^1(\mathscr{O}_X) = 0$. For example, every Calabi–Yau 3-fold can be deformed to a smooth one or to a conifold [7, 21]. M. Reid [27] had proposed to study the moduli spaces of simply connected smooth Calabi–Yau 3-folds through conifold transitions, by which we mean there is a small projective contraction from a smooth Calabi–Yau to a Calabi–Yau conifold so that the conifold is smoothable. This is usually referred as the *Reid's fantasy.* While non-projective conifold transitions are also considered in the literature, in this paper we stick on the projective ones.

In 1988, P.S. Green and T. Hübsch [10] discovered the remarkable connectedness phenomenon: *The moduli spaces of complete intersection Calabi–Yau 3-folds (CICYs) in product of projective spaces are connected with each other by a sequence of conifold transitions.* In [10, §3 p.435], the authors deferred the proof of the existence of conifold transitions to a forthcoming paper, which unfortunately has not yet been available. This result had since then been used again and again in the literature on Calabi–Yau geometry and string theory. While there is no doubt on its significance and correctness, a detailed complete proof to it is still long awaited. The first goal of this paper is to supply such a rigorous proof:

Theorem 1.1 (= Theorem 5.6). *Any two (parameter spaces of) complete intersection Calabi–Yau 3-folds in product of projective spaces are connected by a finite sequence of conifold transitions.*

In order to connect these parameter spaces of CICY 3-folds, the major idea is to use the determinantal contractions introduced in [1].

Let us recall a standard example to explain the process. Consider the smooth CICY 3-fold $\widehat{X}$ in $\mathbb{P}^1 \times \mathbb{P}^4$ defined by $p_j^0(z)t_0 + p_j^1(z)t_1 = 0$ for $j = 1, 2$, where t_0, t_1 are homogeneous coordinates on $\mathbb{P}^1$, $p_1^0(z), p_1^1(z)$ are two general quartic polynomials and $p_2^0(z), p_2^1(z)$ are two linear polynomials on $\mathbb{P}^4$. Since t_i's can not both vanish, it

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must be the case that the determinant

$$\Delta(z) := \det(p^i_j(z))$$

resulting from the projection along $\mathbb{P}^1$ vanishes (cf. §5). If we take $p^i_2(z) = z_i$ for $i = 0, 1$ and suitable quartic polynomials $p^0_1(z), p^1_1(z)$, then the quintic X defined by $\Delta(z)$ has 16 ODPs, where $p^i_j(z)$'s vanish simultaneously, along a projective plane in $\mathbb{P}^4$. Let $\widetilde{X}$ be a smooth quintic in $\mathbb{P}^4$. Note that all quintic hypersurfaces in $\mathbb{P}^4$ are deformation equivalent inside a flat family (cf. Proposition 3.1). Thus we get a conifold transition $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ which connects parameter spaces of $\widehat{X}$ and X.

In general, the task is to verify that the determinantal contraction is a small resolution of a Calabi–Yau conifold. The tool used in this paper is the following well known criterion (involved topological constraints) for ODPs.

Proposition 1.2 (= Proposition 2.3). *Let $\widehat{X} \to X$ be a small resolution of a Gorenstein terminal 3-fold X and $\widetilde{X}$ a smoothing of X. Then the difference of the topological Euler numbers $e(\widehat{X}) - e(\widetilde{X})$ equals the number $2\,|\mathrm{Sing}(X)|$ if and only if the singularities of X are ODPs.*

For 3-dimensional complete intersection varieties in a product of a projective space and a smooth projective variety, we give the formulas of the difference of the Euler numbers and the number of singularities involving Chern classes of vector bundles (see Proposition 3.4 and Corollary 5.4).

Another ingredient is a *Bertini-type theorem for vector bundles* (Theorem 2.1). The necessity for such a result with weaker positivity assumptions comes from the fact that the CICY 3-folds under consideration are not always cut out by *ample divisors*. Combining with the original ideas in [1, 10], we prove that the singular Calabi–Yau X defined by the determinantal equation (and other equations) has isolated singularities and the determinantal contraction is a small resolution of X (Theorem 5.2). According to Proposition 1.2, it follows that the determinantal contraction is a small resolution of a Calabi–Yau conifold as expected. We also give a formula of the second Betti number of CICYs (Proposition 4.7).

In the final section, we discuss the relationship between small transitions and conifold transitions. We introduce the primitive small transitions (Definition 6.3) and prove the following result:

Theorem 1.3 (= Theorem 6.8). *Let $\pi : \widehat{X} \to X$ be a small projective resolution of a Calabi–Yau 3-fold X. If the natural closed immersion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X)$ of Kuranishi spaces is an isomorphism then the singularities of X are ODPs. Moreover, the number of ODPs is equal to the relative Picard number $\rho(\widehat{X}/X)$.*

Theorem 1.3 is a generalization of the case of relative Picard number one which has been studied in [7, (5.1)]. Using the deformation properties of X and $\widehat{X}$ and the cone theorem, we prove it by induction on the relative Picard number.

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2. Preliminaries. Let $\sigma : \mathscr{E} \to \mathscr{F}$ be a morphism of bundles of ranks m and n on a variety M. Note that there is a natural bijection between morphisms $\mathscr{E} \to \mathscr{F}$ and global sections of $\mathscr{E}^\vee \otimes \mathscr{F}$.

For $k \leqslant \min(m, n)$, we define the k-th degeneracy locus of σ by

$$D_k(\sigma) = \{x \in M \mid \operatorname{rank}(\sigma(x)) \leqslant k\}.$$

Its ideal is locally generated by $(k+1)$-minors of a matrix for σ. We can show that the codimension of $D_k(\sigma)$ in M is less than or equal to $(m-k)(n-k)$ [5, Theorem 14.4 (b)], which is called its *expected codimension*. Notice that the 0-th degeneracy locus of σ is the zero scheme $Z(\sigma)$.

Now we state a Bertini-type theorem for vector bundles. The following statement is taken from [24, (2.8)].

THEOREM 2.1 ([24]). *Let $\mathscr{E}$ and $\mathscr{F}$ be vector bundles of ranks m and n on a smooth variety M and let $\mathscr{E}^\vee \otimes \mathscr{F}$ be generated by global sections. If $\sigma : \mathscr{E} \to \mathscr{F}$ is a general morphism, then one of the following holds:*

(1) *$D_k(\sigma)$ is empty;*
(2) *$D_k(\sigma)$ has expected codimension $(m-k)(n-k)$ and the singular locus of $D_k(\sigma)$ is $D_{k-1}(\sigma)$.*

Here the "general" means that there is a Zariski open set in the vector space $H^0(\mathscr{E}^\vee \otimes \mathscr{F})$ such that either (1) or (2) in Theorem 2.1 holds for each σ in the open set.

REMARK 2.2. Let D be a Cartier divisor on M. Assume that the linear system $\Lambda := |\mathscr{O}(D)|$ is base point free. Since the (-1)-th degeneracy locus is empty, the classical Bertini's second theorem follows from Theorem 2.1 by taking $k = 0$, $\mathscr{E} = \mathscr{O}$ and $\mathscr{F} = \mathscr{O}(D)$, i.e., a general member of Λ is smooth. We also know, by the Bertini's first theorem, that if Λ is not composed of a pencil then its general member is irreducible. However, the general degeneracy locus $D_k(\sigma)$ may not be connected.

For the case that $\mathscr{E}$ is a trivial line bundle, $\sigma : \mathscr{O}_M \to \mathscr{F}$ corresponds to a global section of $\mathscr{F}$. The wedge product by the section gives rise to a complex

$$\mathscr{O}_M \to \mathscr{F} \to \wedge^2 \mathscr{F} \to \cdots \to \wedge^{n-1}\mathscr{F} \to \wedge^n \mathscr{F}.$$

The dual complex

$$K^\bullet(\sigma) : \wedge^n \mathscr{F}^\vee \to \wedge^{n-1}\mathscr{F}^\vee \to \cdots \to \wedge^2 \mathscr{F}^\vee \to \mathscr{F}^\vee \xrightarrow{\sigma^\vee} \mathscr{O}_M \tag{2.1}$$

is called the *Koszul complex* of $Z(\sigma)$. Note that the image of $\sigma^\vee$ is the ideal sheaf of $Z(\sigma)$. We say that $Z(\sigma)$ is a *complete intersection* if the sequence

$$0 \to K^\bullet(\sigma) \to \mathscr{O}_{Z(\sigma)} \to 0.$$

is exact, i.e., the Koszul complex (2.1) is a resolution of $\mathscr{O}_{Z(\sigma)}$. In particular, the conormal bundle of $Z(\sigma)$ in M is isomorphic to $\mathscr{F}^\vee|_{Z(\sigma)}$. If M is Cohen-Macaulay, then $Z(\sigma)$ is complete intersection if and only if its codimension in M is equal to the rank n of $\mathscr{F}$ [5, p.431].

If M is a projective variety, endowed with an ample divisor $\mathscr{O}(1)$, then the Hilbert polynomial of a complete intersection $Z(\sigma)$ can be computed by the Koszul complex (2.1), that is, for $l \in \mathbb{Z}$

$$\chi(\mathscr{O}_{Z(\sigma)} \otimes \mathscr{O}(l)) = \sum_{i=0}^{n} (-1)^i \chi(\wedge^i \mathscr{F}^\vee \otimes \mathscr{O}(l)). \tag{2.2}$$

A birational morphism is called *small* if the exceptional set has codimension at least two. The following is a simple criterion of ODPs for a small resolution $\widehat{X} \to X$ if X admits a smoothing.

PROPOSITION 2.3. *Let $\widehat{X} \to X$ be a small resolution of a Gorenstein terminal 3-fold X and $\widetilde{X}$ a smoothing of X. Then the difference of the topological Euler numbers $e(\widehat{X}) - e(\widetilde{X})$ equals the number $2\,|\mathrm{Sing}(X)|$ if and only if the singularities of X are ODPs.*

For the convenience of the reader, we supply a proof here.

Proof. Let C_i be the exceptional curve over an isolated hypersurface singularity p_i. We have the identity of the topological Euler numbers (for a proof see [28, Theorem 7])

$$e(\widehat{X}) - e(\widetilde{X}) = \sum m(p_i) + \sum (e(C_i) - 1)\,,$$

where $m(p_i)$ is the Milnor number of p_i. According to [25, Proposition 1], the exceptional curve C_i is a union of smooth rational curves which meet transversally and thus the number $e(C_i) - 1$ is equal to n_i the number of irreducible components of C_i. Observe that $m(p_i)$ and n_i are greater than or equal to one. Then $\sum m(p_i) + \sum n_i \geqslant 2\,|\mathrm{Sing}(X)|$, and the equality holds if and only if $n_i = m(p_i) = 1$ for all i. □

The following lemma is used in Remark 3.3 and the proof of Theorem 5.2.

LEMMA 2.4. *The product Zariski topology on $\mathbb{C}^n \times \mathbb{C}^m$ is strictly coarser than the Zariski topology on $\mathbb{C}^{n+m}$.*

Sketch of proof. Let I and J be ideals of $\mathbb{C}[x_1, \cdots, x_n]$ and $\mathbb{C}[y_1, \cdots, y_m]$, and let I^e and J^e be ideals generated by I and J in $\mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ respectively. Let $V(I)$ and $V(J)$ denote Zariski closed subsets defined by I and J respectively. Then the standard open subset $(\mathbb{C}^n \setminus V(I)) \times (\mathbb{C}^m \setminus V(J))$ of the product topology is the complement set of the Zariski closed subset $V(I^e) \cap V(J^e)$ of $\mathbb{C}^{n+m}$. Remark that not every open subset in the Zariski topology on $\mathbb{C}^{n+m}$ is open in the product Zariski topology. □

We conclude this section with an elementary lemma, which is used in the proof of Theorem 6.8.

LEMMA 2.5. *Let $C = \bigcup C_i$ be a curve in a smooth 3-fold Y such that the irreducible components C_i meet in a finite set of points. Then there exists an injection $\bigoplus_i H^2_{C_i}(Y, \Omega^2_Y) \hookrightarrow H^2_C(Y, \Omega^2_Y)$. Moreover, it is an isomorphism if C_i are mutually disjoint.*

Proof. By induction on the number of components of C, we may assume that $C = C_1 \cup C_2$. From the Mayer–Vietoris sequence, we get

$$H^2_{C_1 \cap C_2}(\Omega^2_Y) \to H^2_{C_1}(\Omega^2_Y) \oplus H^2_{C_2}(\Omega^2_Y) \to H^2_C(\Omega^2_Y).$$

Since Ω^2_Y is locally free and $\operatorname{depth}_{C_1\cap C_2}\mathscr{O}_Y = 3$, the local cohomology group $H^2_{C_1\cap C_2}(\Omega^2_Y)$ vanishes (cf. [12, III Ex.3.4]), which completes the proof. □

3. Configurations and Parameter Spaces. We start by introducing the configuration of *complete intersection* varieties of dimension d and constructing their parameter spaces.

A configuration of dimension d is a pair $[V\|\mathfrak{L}]$ of a smooth projective variety V with $\dim V = m+d$ and a sequence of line bundles $\mathfrak{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_m)$, where $\mathscr{L}_j$ is generated by global sections. Let X be a d-dimensional variety. The variety X is said to be a member of the configuration, denoted by $X \in [V\|\mathfrak{L}]$, if it is defined by global sections σ_j of $\mathscr{L}_j$ for $1 \leqslant j \leqslant m$.

If

$$V = \prod_{i=1}^k \mathbb{P}^{n_i} \text{ and } \mathscr{L}_j = \bigotimes_{i=1}^k pr_i^* \mathscr{O}_{\mathbb{P}^{n_i}}(q^i_j),$$

where $pr_i : V \to \mathbb{P}^{n_i}$ is the natural projection and $q^i_j \geqslant 0$ for all i, j, then we rewrite $[V\|\mathfrak{L}]$ as a configuration matrix

$$[\mathbf{n}\|\mathbf{q}] = \left[\begin{array}{c|c} n_1 & q^1_1 \ \cdots \ q^1_m \\ \vdots & \vdots \ \ \ddots \ \ \vdots \\ n_k & q^k_1 \ \cdots \ q^k_m \end{array}\right]. \tag{3.1}$$

The $(q^1_j, \cdots, q^k_j)$ and $\mathbf{q}$ will be called the multidegree of the line bundle $\mathscr{L}_j$ and a member $X \in [\mathbf{n}\|\mathbf{q}]$ respectively. We may assume that

$$\sum_{i=1}^k q^i_j \geqslant 2$$

for all $1 \leqslant j \leqslant m$ (otherwise a hyperplane section of only one factor $\mathbb{P}^n$ reduces the factor to $\mathbb{P}^{n-1}$). Note that the global sections of $\mathscr{L}_j$ are multi-homogeneous polynomials of multidegree $(q^1_j, \cdots, q^k_j)$.

Two configuration matrices are said to represent the same configuration if one can go from one to the other by a permutation of the rows or of the columns other than first. We say that $[\mathbf{n}_1\|\mathbf{q}_1]$ is a sub-configuration matrix of $[\mathbf{n}\|\mathbf{q}]$ if

$$\left[\begin{array}{c|cc} \mathbf{n}_1 & \mathbf{q}_1 & \mathbf{a} \\ \mathbf{m} & \mathbf{0} & \mathbf{b} \end{array}\right]$$

and $[\mathbf{n}\|\mathbf{q}]$ represent the same configuration.

In the case $V = \prod_{i=1}^k \mathbb{P}^{n_i}$, we can explain the meaning of a complete intersection $X \in [\mathbf{n}\|\mathbf{q}]$ precisely by defining a projective family for $[\mathbf{n}\|\mathbf{q}]$ whose fibers are complete intersections of multidegree $\mathbf{q}$ (cf. [31, §4.6.1]).

In the following we will write $\underline{T}_i$ and $\underline{u}$ as a short form for indeterminates $T_{i0}, \cdots, T_{in_i}$ and $u_1, \cdots, u_a$ respectively. Set $R = \mathbb{C}[\underline{T}_1; \cdots; \underline{T}_k]$.

Let X be a complete intersection variety defined by a sequence of multi-homogeneous polynomials $\sigma = (\sigma_j)$ of multidegree $\mathbf{q}$ and of dimension d. Let

$$\Phi^{(1)}, \cdots, \Phi^{(a)}$$

be a basis of $\bigoplus_{j=1}^m H^0(V, \mathscr{L}_j)$ and write $\Phi^{(h)} = (\phi^{(h)}_j)$ where $\phi^{(h)}_j \in R$ with multidegree $(q^1_j, \cdots, q^k_j)$.

Let $K_\bullet := K_\bullet(\sigma + \sum_{h=1}^a u_h \Phi^{(h)})$ be the Koszul complex (cf. (2.1)) and

$$D := \operatorname{Supp}(H_1(K_\bullet)) \subseteq \mathbb{A}^{a+N} = \operatorname{Spec}(\mathbb{C}[\underline{u}; \underline{T}_1; \cdots; \underline{T}_k])$$

where $N = \sum_i n_i + k$.

Set q be the projection from $\mathbb{A}^{a+N}$ onto $\mathbb{A}^a$. Then $U := q(\mathbb{A}^{a+N} \setminus D)$ is the open set of points $\underline{u} \in \mathbb{A}^a$ with $K_\bullet(\sigma + \sum_{h=1}^a u_h \Phi^{(h)})$ being exact. According to that $X = Z(\sigma)$ is a complete intersection, it follows that U contains the origin. Let

$$I = \left\langle \sigma_l + \sum_h u_h \phi_l^{(h)} \,\middle|\, 1 \leqslant l \leqslant m \right\rangle$$

and $\mathscr{X} = \operatorname{Proj}(R[\underline{u}]/I)$.

Consider the projection $P : \mathscr{X} \subseteq V \times \mathbb{A}^a \to \mathbb{A}^a$ and its restriction $P_U : \mathscr{X}_U \to U$. Since the the Koszul complex $K_\bullet$ is exact on U, all fibers of P_U are complete intersections of multidegree $\mathbf{q}$ and have the same Hilbert polynomial $P(t)$ which is computed by the Koszul resolution and depends on its multidegree (cf. (2.2)). Hence P_U is a flat family with the fiber $\mathscr{X}_0 = X$ ([12, III Theorem 9.9]).

To summarize what we have proved, we get the following proposition:

Proposition 3.1. *Let X be a variety. If $X \in [\mathbf{n}\|\mathbf{q}]$ then there is a Zariski open set U in $H^0(V, \bigoplus_{j=1}^m \mathscr{L}_j)$, a closed point $t_0 \in U$ and a flat projective morphism $P_U : \mathscr{X}_U \to U$ with the fiber $\mathscr{X}_{t_0} = X$ such that all complete intersections in V of multidegree $\mathbf{q}$ are parameterized by the pair (U, P_U).*

Hence we may use the configuration $[\mathbf{n}\|\mathbf{q}]$ to denote the parameter space of d-dimensional complete intersections in V of multidegree $\mathbf{q}$.

To point out what the fundamental cycle and the normal bundle of a smooth complete intersection is, we state the following result by using Theorem 2.1.

Proposition 3.2. *Let $\mathfrak{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_m)$ be a sequence of globally generated line bundles over a smooth projective variety V and $[V\|\mathfrak{L}]$ a configuration of dimension d. Then there is a Zariski open subset U in $H^0(V, \bigoplus_{j=1}^m \mathscr{L}_j)$ such that $Z(\sigma)$ is smooth and of dimension d for every element σ in U. Moreover, the normal bundle of $Z(\sigma)$ in V is $\bigoplus_{j=1}^m \mathscr{L}_j|_{Z(\sigma)}$ and the fundamental class $[Z(\sigma)]$ in $A_d(V)$ is the top Chern class of $\bigoplus_{j=1}^m \mathscr{L}_j$.*

Proof. Applying Theorem 2.1 to the case $k = 0$, $\mathscr{E} = \mathscr{O}_V$ and $\mathscr{F} = \bigoplus_{j=1}^m \mathscr{L}_j$, the zero locus $Z(\sigma)$ is smooth and has the expected codimension m for a general $\sigma : \mathscr{E} \to \mathscr{F}$. Namely, there is a Zariski open subset U in $H^0(V, \mathscr{F})$ such that every element σ in U defines a smooth complete intersection X in V of dimension $\dim V - m = d$. By [5, Example 3.2.16], the fundamental class of a general member in $A_d(V)$ is $c_m(\mathscr{F}) \cap [V]$. □

Remark 3.3. Since $\bigoplus_{j=1}^m H^0(V, \mathscr{L}_j)$ is naturally isomorphic to $H^0(V, \mathscr{F})$ (as vector spaces), any element σ in U corresponds (s_j) in $\bigoplus_{j=1}^m H^0(V, \mathscr{L}_j)$ and thus $X = Z(\sigma)$ is the complete intersection $\cap_{j=1}^m Z(s_j)$.

Applying Theorem 2.1 repeatedly, we may assume that, for a general section (s_j) in $H^0(V, \mathscr{F})$, the divisor $Z(s_j)$ is smooth with all subsets of the $Z(s_j)$'s meeting transversely. For example, by Theorem 2.1, there is a Zariski open subset V_j of $H^0(V, \mathscr{L}_j)$ such that $Z(s_j)$ is smooth for $1 \leqslant j \leqslant m$. According to Lemma 2.4, it follows that $\prod_{j=1}^m V_j$ is Zariski open in $H^0(V, \mathscr{F})$. Replacing U by $U \cap (\prod_{j=1}^m V_j)$, which is Zariski open in $H^0(V, \mathscr{F})$, all $Z(s_j)$'s are smooth for $(s_j) \in U$.

In order to connect two configurations, we shall define a formal correspondence on configurations which is introduced in [1].

Let P be a smooth projective variety, and let p and π be the projections from $\mathbb{P}^n \times P$ onto $\mathbb{P}^n$ and P respectively. If $V = \mathbb{P}^n \times P$ and

$$\widehat{\mathscr{L}_j} = \begin{cases} p^*\mathscr{O}_{\mathbb{P}^n}(1) \otimes \pi^*\mathscr{L}_j & \text{if } 1 \leqslant j \leqslant n+1, \\ \pi^*\mathscr{L}_j & \text{if } n+2 \leqslant j \leqslant m, \end{cases}$$

where $\mathscr{L}_j$'s are globally generated line bundles on P, we rewrite the configuration $\widehat{\mathscr{C}} := [V \| \widehat{\mathscr{L}_1}, \cdots, \widehat{\mathscr{L}_m}]$ as

$$\left[\begin{array}{c||cccccc} n & 1 & \cdots & 1 & 0 & \cdots & 0 \\ P & \mathscr{L}_1 & \cdots & \mathscr{L}_{n+1} & \mathscr{L}_{n+2} & \cdots & \mathscr{L}_m \end{array} \right].$$

We introduce a new configuration

$$\mathscr{C} := \left[\begin{array}{c||cccc} P & \bigotimes_{i=1}^{n+1} \mathscr{L}_i & \mathscr{L}_{n+2} & \cdots & \mathscr{L}_m \end{array} \right]$$

so as to remove the $\mathbb{P}^n$ factor and denote the *formal correspondence* by

$$\widehat{\mathscr{C}} \twoheadleftarrow\!\!\!\longrightarrow \mathscr{C}. \tag{3.2}$$

Remark that the paper [1] refers to the correspondence of passing from the right hand side to the left as *splitting* and the reverse process as *contraction*.

We are going to compute the difference of topological Euler numbers of smooth members under the formal correspondence of 3-dimensional configurations.

Proposition 3.4. *Let S be a smooth variety of dimension 4,*

$$\mathscr{R} = \left[\begin{array}{c||ccc} n & 1 & \cdots & 1 \\ S & \mathscr{L}_1 & \cdots & \mathscr{L}_{n+1} \end{array} \right]$$

a configuration of dimension 3 and $\mathscr{E} = \bigoplus_{i=1}^{n+1} \mathscr{L}_i$ a vector bundle of rank $n+1$. Assume that $\widehat{X} \in \mathscr{R}$ and $\widetilde{X} \in \left[\ S \ \| \ \bigotimes_{i=1}^{n+1} \mathscr{L}_i \ \right]$ are smooth members. Then we have

$$e(\widehat{X}) - e(\widetilde{X}) = 2 \int_S \left(c_2(\mathscr{E})^2 - c_1(\mathscr{E}) c_3(\mathscr{E}) \right)$$

where $e(-)$ denotes the topological Euler number.

Proof. Let $\iota : \widetilde{X} \hookrightarrow S$ be the inclusion. Since S and $\widetilde{X}$ are smooth varieties, we have the normal bundle sequence

$$0 \to T_{\widetilde{X}} \to T_S|_{\widetilde{X}} \to N_{\widetilde{X}} \to 0.$$

By Proposition 3.2, the normal bundle of the hypersurface $\widetilde{X}$ in S is $\otimes_{i=1}^{n+1} \mathscr{L}_i|_S$. Let

$$p(t) := \iota_* c_t(T_{\widetilde{X}}) = c_t(T_S) s_t(\otimes_{i=1}^{n+1} \mathscr{L}_i), \tag{3.3}$$

where $c_t(\mathscr{V})$ is the Chern polynomial of a vector bundle $\mathscr{V}$ and $c_t(\mathscr{V}) s_t(\mathscr{V}) = 1$. Observe that $\otimes_{j=1}^{n+1} \mathscr{L}_j$ and $\mathscr{E}$ have the same first Chern class $\sum_{j=1}^{n+1} c_1(\mathscr{L}_j)$. Then the fundamental class of $\widetilde{X}$ in the Chow group $A_3(S)$ is

$$c_1(\otimes_{j=1}^{n+1} \mathscr{L}_j) \cap [S] = c_1(\mathscr{E}) \cap [S]. \tag{3.4}$$

We are going to calculate $e(\widetilde{X})$. According to that $s_t(\otimes_{j=1}^{n+1}\mathscr{L}_j)$ is the inverse of the Chern polynomial $c_t(\otimes_{j=1}^{n+1}\mathscr{L}_j) = 1 + c_1(\otimes_{j=1}^{n+1}\mathscr{L}_j)$, it follows that

$$s_t(\otimes_{j=1}^{n+1}\mathscr{L}_j) = \sum_{i=0}^{\infty}(-c_1(\mathscr{E}))^i t^i = \sum_{i=0}^{\infty} s_1(\mathscr{E})^i t^i. \tag{3.5}$$

Set $c_t(S) = c_t(T_S)$. Using (3.5) and collecting the coefficient of t^3 in (3.3), we obtain

$$\frac{1}{3!}p'''(0) = s_1(\mathscr{E})^3 + c_1(S)s_1(\mathscr{E})^2 + C_s, \tag{3.6}$$

where

$$C_s \coloneqq c_2(S)s_1(\mathscr{E}) + c_3(S). \tag{3.7}$$

By (3.4), (3.6), and the Gauss-Bonnet theorem, we get

$$e(\widetilde{X}) = \int_{\widetilde{X}} c_3(\widetilde{X}) = \int_S \frac{1}{3!}p'''(0)c_1(\mathscr{E}).$$

Let pr be the projection from $\mathbb{P}^n \times S$ onto $\mathbb{P}^n$ and $\widehat{\imath}$ the inclusion from $\widehat{X}$ into $\mathbb{P}^n \times S$. To compute $e(\widehat{X})$, we identify the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ with $pr^*\mathscr{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n \times S$ (similarly for vector bundles on S).

According to $N_{\widehat{X}} = (\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^n}(1))\,|_{\widehat{X}}$, it follows that

$$q(t) \coloneqq \widehat{\imath}_* c_t(T_{\widehat{X}}) = c_t(T_{\mathbb{P}^n} \oplus T_S)s_t(\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^n}(1)).$$

By Proposition 3.2, the fundamental class of $\widehat{X}$ in $A_3(\mathbb{P}^n \times S)$ is

$$c_{n+1}(\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^n}(1)) \cap [\mathbb{P}^n \times S].$$

Set $H = c_1(\mathscr{O}_{\mathbb{P}^n}(1))$ in $A_1(\mathbb{P}^n \times S)$. According to $c_t(T_{\mathbb{P}^n} \oplus T_S) = c_t(T_{\mathbb{P}^n})c_t(T_S)$, it follows that the coefficient of t^3 in $q(t)$ is

$$\frac{1}{3!}q'''(0) = \sum_{p=0}^{3}\binom{n+1}{p}\left[\sum_{i+j=3-p} c_i(S)s_j(\mathscr{E} \otimes \mathscr{O}(1))\right] H^p. \tag{3.8}$$

From [5, Example 3.1.1], we have

$$s_l(\mathscr{E} \otimes \mathscr{O}(1)) = \sum_{i=0}^{l}(-1)^{l-i}\binom{n+l}{n+i}s_i(\mathscr{E})H^{l-i}. \tag{3.9}$$

By substituting (3.9) into (3.8), we obtain

$$\frac{1}{3!}q'''(0) = s_1(\mathscr{E})H^2 - [2s_2(\mathscr{E}) + c_1(S)s_1(\mathscr{E})]\,H + [s_3(\mathscr{E}) + c_1(S)s_2(\mathscr{E}) + C_s]\,, \tag{3.10}$$

where C_s is the class as defined in (3.7). (For example, the coefficient of H^3 in (3.10) is $\binom{n+1}{3} - (n+1)\binom{n+1}{2} + (n+1)\binom{n+2}{n} - \binom{n+3}{n}$ which is equal to zero.)

We regard the class $\frac{1}{3!}q'''(0)c_{n+1}(\mathscr{E}\otimes\mathscr{O}_{\mathbb{P}^n}(1))$ as a polynomial in H, denoted it by $Q(H)$. Then

$$e(\widehat{X}) = \int_{\mathbb{P}^n\times S} Q(H) = \int_S \frac{1}{n!}Q^{(n)}(0).$$

If we can prove

$$\frac{1}{n!}Q^{(n)}(0) - \frac{1}{3!}p'''(0)c_1(\mathscr{E}) = 2\left(c_2(\mathscr{E})^2 - c_1(\mathscr{E})c_3(\mathscr{E})\right), \tag{3.11}$$

then the Proposition follows by integrating the equality (3.11) over S.

Note that the top Chern class of $\mathscr{E}\otimes\mathscr{O}_{\mathbb{P}^n}(1)$ is $\sum_{i=0}^{n+1} c_{n+1-i}(\mathscr{E})H^i$. According to (3.10), it follows that the coefficient $\frac{1}{n!}Q^{(n)}(0)$ of H^n in $Q(H)$ is equal to the coefficient of H^n in the class

$$\frac{1}{3!}q'''(0)\left(c_1(\mathscr{E})H^n + c_2(\mathscr{E})H^{n-1} + c_3(\mathscr{E})H^{n-2}\right).$$

From $H^{n+1}=0$ in $A_*(\mathbb{P}^n)$, it follows that

$$\begin{aligned}\frac{1}{n!}Q^{(n)}(0) =& s_1(\mathscr{E})c_3(\mathscr{E}) - [2s_2(\mathscr{E}) + c_1(S)s_1(\mathscr{E})]c_2(\mathscr{E}) \\ &+ [s_3(\mathscr{E}) + c_1(S)s_2(\mathscr{E}) + C_s]\, c_1(\mathscr{E}).\end{aligned} \tag{3.12}$$

Subtracting these two equations (3.12) and (3.6) from each other and rewriting Segre classes $s_i(\mathscr{E})$ in terms of Chern classes $c_i(\mathscr{E})$ by using the recurrence relations $s_l(\mathscr{E}) = -\sum_{i=1}^{l} c_i(\mathscr{E})s_{l-i}(\mathscr{E})$, we obtain the equation (3.11). □

Corollary 3.5. *Let $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}$ be as in (3.2), and let $\mathscr{E} = \bigoplus_{i=1}^{n+1}\mathscr{L}_i$ and $\mathscr{F} = \bigoplus_{i=n+2}^{m}\mathscr{L}_i$ be vector bundles of rank $n+1$ and $m-n-1$ respectively. Assume that $\widehat{\mathscr{C}}$ and $\mathscr{C}$ are configurations of dimension 3. Given smooth members $\widehat{X}\in\widehat{\mathscr{C}}$ and $\widetilde{X}\in\mathscr{C}$, we assume that there is a global section σ of $\mathscr{F}$ such that the zero locus $Z(\sigma)$ is smooth of dimension 4 and contains $\widetilde{X}$. Then*

$$e(\widehat{X}) - e(\widetilde{X}) = 2\int_P \left(c_2(\mathscr{E})^2 - c_1(\mathscr{E})c_3(\mathscr{E})\right)c_{m-n-1}(\mathscr{F}).$$

Proof. Let S be the zero locus $Z(\sigma)$. The corollary follows immediately from Proposition 3.4 and the fundamental class of S in $A_4(P)$ is $c_{m-n-1}(\mathscr{F})\cap[P]$. □

Example 3.6. Consider

$$\widehat{\mathscr{C}} := \left[\begin{array}{c\|ccc} 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 \end{array}\right] \twoheadleftarrow\!\!\longrightarrow \mathscr{C} := \left[\begin{array}{c\|c} 3 & 4 \\ 1 & 2 \end{array}\right].$$

For smooth member $\widehat{X}\in\widehat{\mathscr{C}}$ and $\widetilde{X}\in\mathscr{C}$, the Euler numbers $e(\widehat{X})$ and $e(\widetilde{X})$ are -112 and -168 respectively. Let s (resp. t) be the class of a hyperplane on $\mathbb{P}^3$ (resp. $\mathbb{P}^1$), and let $\mathscr{E}$ be the vector bundle $\mathscr{O}(1,0)\bigoplus\mathscr{O}(1,0)\bigoplus\mathscr{O}(2,2)$ of rank 3 on $\mathbb{P}^3\times\mathbb{P}^1$. Then the Chern classes of $\mathscr{E}$ are

$$c_1(\mathscr{E}) = 4s+2t,\ c_2(\mathscr{E}) = 5s^2+4st,\ c_3(\mathscr{E}) = 2s^3+2s^2t,$$

and the coefficient of s^3t in $c_2(\mathscr{E})^2 - c_1(\mathscr{E})c_3(\mathscr{E})$ is 28.

4. Calabi–Yau Configurations. From now on we assume that all configurations are of dimension 3.

Definition 4.1. A configuration matrix $[\mathbf{n}\|\mathbf{q}]$ is called a complete intersection Calabi–Yau (CICY) configuration if it satisfy the Calabi–Yau condition

$$\sum\nolimits_{j=1}^{m} q_j^i = n_i + 1$$

for all $1 \leqslant i \leqslant k$.

It is easy to see that CICY configuration matrices are preserved under formal correspondences (3.2). Note that the topological Euler number of a smooth member which belongs to a CICY configuration matrix is non-positive [1, (2.28)].

Remark 4.2. We do not allow that a Calabi–Yau 3-fold X is a product of three elliptic curves or of an elliptic curve and $K3$ surface since $H^1(\mathscr{O}_X) = 0$. Further, we are not interested in a configuration matrix which contains the sub-configuration $[1\|2]$ because the sub-configuration describes two points (counted with multiplicity) in $\mathbb{P}^1$. To exclude such cases, we only treat *non block-diagonal* CICY configuration matrices.

Let us consider the simple case for all $n_i = 1$ and $q_j^i = 0$ or 2.

Example 4.3 ([9]). Given a CICY configuration $k \times (m+1)$-matrix $[\mathbf{n}\|\mathbf{q}]$ with $n_i = 1$ and $q_j^i = 0$ or 2 for all i, j, we have $k = m + 3$. By Remark 4.2, we know that $[\mathbf{n}\|\mathbf{q}]$ is non block-diagonal and thus

$$\sum\nolimits_{i=1}^{k} q_j^i \geqslant 4$$

for each column of $\mathbf{q}$. According to the Calabi–Yau condition, it follows that

$$4(k-3) \leqslant \sum\nolimits_{i,j} q_j^i = 2k$$

and therefore $4 \leqslant k \leqslant 6$. When k equals 5 or 6, we get a product of an elliptic curve and $K3$ surface or of three elliptic curves respectively. By Remark 4.2, the CICY configuration matrix must be

$$\left[\begin{array}{c||c} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{array}\right]$$

in this simple case. We denote this configuration matrix by $\mathscr{C}_{1111}$.

We say that a configuration connects to another formally if, after finite formal correspondences (3.2), one represents the same configuration as the other one. The following proposition was proved in [10, Lemma 2], for the convenience of the readers we recall the proof here.

Proposition 4.4 ([10]). *Every CICY configuration matrices can be connected formally.*

Proof. Given a (non block-diagonal) CICY configuration matrix $[\mathbf{n}\|\mathbf{q}]$ as in (3.1). We perform formal correspondences iteratively until we arrive at a configuration matrix for which each row entries q_j^i with $n_i > 1$ are 0 or 1 (for example, introducing a sub-configuration matrix [1‖11] to split it). Perform next formal correspondences in a way that finally leaves each $n_i = 1$ and $q^i_j = 0$ or 2. Notice that non block-diagonal CICY configuration matrices are preserved under formal correspondences. According to Example 4.3, it follows that the configuration matrix is the simple configuration $\mathscr{C}_{1111}$. □

REMARK 4.5. To illustrate Proposition 4.4, we give formal correspondences connecting the configuration of quintic hypersurfaces in $\mathbb{P}^4$ to $\mathscr{C}_{1111}$:

$$[\ 4 \,\|\, 5\ ] \longleftrightarrow\!\!\!\!\rightarrow \left[\begin{array}{c||cc} 4 & 4 & 1 \\ 1 & 1 & 1 \end{array}\right] \longleftrightarrow\!\!\!\!\rightarrow \left[\begin{array}{c||ccc} 4 & 3 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \longleftrightarrow\!\!\!\!\rightarrow \left[\begin{array}{c||cccc} 4 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longleftrightarrow\!\!\!\!\rightarrow \left[\begin{array}{c||ccccc} 4 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \twoheadleftarrow\!\!\!\!\longrightarrow \left[\begin{array}{c||c} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{array}\right].$$

The following proposition is an application of Theorem 2.1, which is a well known result in [9]. The remaining task is to prove that a general CICY member is irreducible and $H^1(\mathscr{O}) = 0$ by using a suitable Lefschetz-type theorem for an ample reducible divisor.

PROPOSITION 4.6. *A general member of a CICY configuration matrix is a smooth Calabi–Yau 3-fold.*

Proof. Let $V = \prod_{i=1}^k \mathbb{P}^{n_i}$ and let $[\mathbf{n}\|\mathbf{q}]$ be a (non block-diagonal) CICY configuration matrix. Let $\mathscr{L}_j$ be the line bundle with the multidegree $(q^1_j, \cdots, q^k_j)$. Note that the canonical bundle of X is trivial by the adjunction formula. By Proposition 3.2, it suffices to prove that a general smooth member $X \in [\mathbf{n}\|\mathbf{q}]$ is connected and $H^1(\mathscr{O}_X) = 0$. Namely, we only need to prove that $H^0(X, \mathbb{C})$ and $H^1(X, \mathbb{C})$ have dimension one and zero respectively.

Pick a general section (s_j) in $H^0(V, \bigoplus_{j=1}^m \mathscr{L}_j)$ for which the divisor $D_j := Z(s_j)$ is a smooth with all subsets of the D_j's meeting transversely (cf. Remark 3.3). We notice that if all $q^{i_s}_j = 0$ for some i_s then D_J is of the form $D'_J \times \mathbb{P}^{n_{i_s}}$ where D'_J is a complete intersection in $\prod_{i \neq i_s} \mathbb{P}^{n_i}$. In particular, $H^0(D_j, \mathbb{C})$ and $H^1(D_j, \mathbb{C})$ are one and zero respectively by Lefschetz hyperplane theorem, and thus D_j is irreducible for all $1 \leqslant j \leqslant m$.

Using the mixed Hodge theory and Lefschetz hyperplane theorem on the ample divisor $\sum_{j=1}^m D_j$, we get exact sequences [2, (2.1)], for $i = 0, 1$,

$$0 \to H^i(V, \mathbb{C}) \cdots \to \bigoplus\nolimits_{|J|=r} H^i(D_J, \mathbb{C}) \to \cdots H^i(X, \mathbb{C}) \to 0 \tag{4.1}$$

where $D_J := D_{j_1} \bigcap \cdots \bigcap D_{j_r}$ for a multi-index $J = (j_1, \cdots, j_r)$ of length $|J| = r$ with $1 \leqslant j_1 < \cdots < j_r \leqslant m$ and $X = \bigcap_{|J|=m} D_J$. Note that $i + m < \dim V$ for $i = 0, 1$.

By induction, it follows that the dimension of $\bigoplus_{|J|=r} H^0(D_J, \mathbb{C})$ is $\binom{m}{r}$ and of $\bigoplus_{|J|=r} H^1(D_J, \mathbb{C})$ is zero for the length $r < m$. We remark that the induction process works because every D_J has the form $D'_J \times \prod \mathbb{P}^{n_l}$ with $D' = \sum D'_j$ is ample. Hence the connectedness and simple connectedness of D_J can be proved in the similar way as shown before. Using the sequence (4.1) and dimension counting, we get the dimension of $H^0(X,\mathbb{C})$ and $H^1(X,\mathbb{C})$ are one and zero respectively. □

As a byproduct of the proof of Proposition 4.6, we obtain the following second Betti number formula:

Proposition 4.7. *With the notation as in the proof of Proposition 4.6,*

$$b_2(X,\mathbb{C}) = (-1)^m \left(m + \sum_{r=1}^{m-1} (-1)^r \sum_{|J|=r} b_2(D_J,\mathbb{C}) \right).$$

Moreover, the second Betti number of X equals the second Betti number of the ambient space V if $b_2(D_J,\mathbb{C}) = b_2(V,\mathbb{C})$ for each $1 \leqslant |J| < m$.

Proof. By $V = \prod_{i=1}^m \mathbb{P}^{n_i}$ and Künneth formula, the second Betti number of V equals m. Since $\dim V > m+2$, the exact sequence (4.1) holds for $i = 2$ and the proposition follows. □

Remark 4.8. For a smooth Calabi–Yau 3-fold X, the topological Euler number $e(X)$ is $2(h^{1,1}(X) - h^{2,1}(X))$. To know the Hodge number $h^{1,1}(X)$ and $h^{2,1}(X)$, it suffices to compute either one of these two Hodge numbers and $e(X)$. In [11], finding these Hodge numbers corresponding to a given CICY configuration matrix is in principle just a matter of looking up the *relevant* matrix in the list. Those calculated in [11] for the 7868 CICY matrices constructed in [1]. Proposition 4.7 gives a direct calculation of $h^{1,1}(X) = b_2(X,\mathbb{C})$ for X in *any* given CICY configuration matrix.

The remark is illustrated by the following example which was given in the appendix of [11].

Example 4.9 ([11]). Consider

$$X \in \left[\begin{array}{c||ccccc} 4 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 \end{array} \right].$$

Applying Lefschetz hyperplane theorem, Künneth formula and Proposition 4.7, we get

$$b_2(X,\mathbb{C}) = b_2(\mathbb{P}^4,\mathbb{C}) + b_2(D,\mathbb{C})$$

where $D \in \left[\begin{array}{c||cc} 2 & 1 & 1 \\ 2 & 1 & 1 \end{array} \right]$ is a smooth surface with Euler number 6. Therefore $b_2(D) = e(D) - 2 = 4$ and the second Betti number of X is 5.

5. Connecting the CICY Web via Determinantal Contractions. We first recall the definition of determinantal contractions, which is introduced in [1], between configurations of complete intersection varieties in a product of a projective space and a smooth projective variety.

Let P be a smooth projective variety, and let $\widehat{\mathscr{C}}$ be a configuration of dimension 3 of the type

$$\widehat{\mathscr{C}} = \left[\begin{array}{c||cccccc} n & 1 & \cdots & 1 & 0 & \cdots & 0 \\ P & \mathscr{L}_1 & \cdots & \mathscr{L}_{n+1} & \mathscr{L}_{n+2} & \cdots & \mathscr{L}_m \end{array} \right]$$

where $\mathscr{L}_j$'s are line bundles on P. Note that $\dim P = m - n + 3$ because $\widehat{\mathscr{C}}$ is of dimension 3. We have the formal correspondence

$$\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C} = \left[\begin{array}{c\|cccc} P & \bigotimes_{i=1}^{n+1} \mathscr{L}_i & \mathscr{L}_{n+2} & \cdots & \mathscr{L}_m \end{array}\right].$$

Let $\pi : \mathbb{P}^n \times P \to P$ be the projection and $\widehat{X}$, $X := \pi(\widehat{X})$ a member of the configuration $\widehat{\mathscr{C}}$, $\mathscr{C}$ respectively. We are going to define a determinantal contraction for the formal correspondence $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}$ and find a morphism $\pi : \widehat{X} \to X$ with each fiber is a point or a projective line in $\mathbb{P}^n$.

Writing $z = [z_0 : \cdots : z_n] \in \mathbb{P}^n$ and let $\widehat{X} \in \widehat{\mathscr{C}}$ be defined by global sections

$$\sum_{i=0}^{n} s_j^i(p) z_i = 0$$

and $t_l(p) = 0$, where $s_j^i \in H^0(P, \mathscr{L}_j)$ and $t_l \in H^0(P, \mathscr{L}_l)$ for $1 \leqslant j \leqslant n+1$, $n+2 \leqslant l \leqslant m$. Set

$$\Delta(p) := \det(s_j^i(p)),$$

which is a global section of the line bundle $\bigotimes_{j=1}^{n+1} \mathscr{L}_j$ on P. Since z_i cannot all vanish simultaneously, we have $\Delta(p) = 0$ for $(z, p) \in \mathbb{P}^n \times P$.

Obviously, the $X := \pi(\widehat{X})$ is defined by global sections

$$\Delta(p) = 0 \text{ and } t_l(p) = 0$$

for $n+2 \leqslant l \leqslant m$ and thus X belongs to the configuration $\mathscr{C}$.

DEFINITION 5.1. Assume that $\widehat{\mathscr{C}}$ and $\mathscr{C}$ are configurations of dimension 3. We say that a formal correspondence $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}$ gives a determinantal contraction if there is a smooth member $\widehat{X}$ in $\widehat{\mathscr{C}}$ such that the morphism $\pi : \widehat{X} \to X$ given in the above process is an isomorphism or a small resolution of a normal variety $X \in \mathscr{C}$ with only isolated singularities.

The proof of Theorem 5.2 follows the idea outlined in [1]. The main tool used in the proof is the Bertini-type theorem introduced in §2.

THEOREM 5.2. *Let $\widehat{\mathscr{C}}$ and $\mathscr{C}$ be 3-dimensional CICY configuration matrices as above. Then the formal correspondence $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\longrightarrow \mathscr{C}$ gives a determinantal contraction.*

Proof. Let

$$\widehat{\mathscr{L}_j} = \begin{cases} p^* \mathscr{O}_{\mathbb{P}^n}(1) \otimes \pi^* \mathscr{L}_j & \text{if } 1 \leqslant j \leqslant n+1, \\ \pi^* \mathscr{L}_j & \text{if } n+2 \leqslant j \leqslant m, \end{cases}$$

where p and π are the projections from $\mathbb{P}^n \times P$ onto $\mathbb{P}^n$ and P respectively. The basic idea of the proof is to find a suitable Zariski open subset in the space of global sections of $\bigoplus_{j=1}^m \widehat{\mathscr{L}_j}$ by repeatedly applying Theorem 2.1.

By Proposition 4.6, there is a Zariski open set $\widehat{U}$ in $H^0\left(\mathbb{P}^n \times P, \bigoplus_{j=1}^m \widehat{\mathscr{L}_j}\right)$ such that $\widehat{X} = Z(\sigma)$ is a smooth Calabi–Yau 3-fold for $\sigma \in \widehat{U}$. Under the isomorphism $H^0\left(\mathbb{P}^n \times P, \bigoplus_{j=1}^m \widehat{\mathscr{L}_j}\right) \simeq \bigoplus_{j=1}^m H^0\left(\mathbb{P}^n \times P, \widehat{\mathscr{L}_j}\right)$, we may assume that $\widehat{U} = \prod_{j=1}^m U_j$ where U_j is a Zariski open subset of $H^0\left(\mathbb{P}^n \times P, \widehat{\mathscr{L}_j}\right)$ (cf. Remark 3.3).

By Theorem 2.1, for a general morphism $\tau : \bigoplus_1^{n+1} \mathscr{O}_P \to \bigoplus_{j=1}^{n+1} \mathscr{L}_j$ the expected codimension of the degeneracy locus $D_k(\tau)$ is $(n+1-k)^2$. In particular, the expected codimension of the degeneracy loci $D_{n-2}(\tau)$ and $D_{n-1}(\tau)$ in P are nine and four. Using Lemma 2.4, we may assume that there are Zariski open subsets V_{ij} of $H^0(P, \mathscr{L}_j)$ for $1 \leqslant i, j \leqslant n+1$ such that sections $[s^i_j] \in \prod_{i,j=1}^{n+1} V_{ij}$ correspond to morphisms τ, by identifying τ with a global section of $(\bigoplus_{j=1}^{n+1} \mathscr{L}_j) \otimes (\bigoplus_{j=1}^{n+1} \mathscr{O}_j)^\vee$.

Applying Künneth formula, we have for $1 \leqslant j \leqslant n+1$

$$\begin{aligned} H^0(\mathbb{P}^n \times P, \widehat{\mathscr{L}_j}) &\simeq H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \otimes H^0(P, \mathscr{L}_j) \\ &\simeq \bigoplus_{i=1}^{n+1} \left(H^0(P, \mathscr{L}_j) \cdot z_{i-1}\right) \end{aligned} \tag{5.1}$$

and for $n+2 \leqslant j \leqslant m$

$$H^0(\mathbb{P}^n \times P, \widehat{\mathscr{L}_j}) \simeq H^0(P, \mathscr{L}_j),$$

where $\{z_0, \cdots, z_n\}$ is a basis of $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$. Therefore we can identify $\sigma \in \widehat{U}$ with global sections $\sum_{i=0}^n s^i_j(p) z_i = 0$ and $t_l(p) = 0$ where $s^i_j \in H^0(P, \mathscr{L}_j)$ and $t_l \in H^0(P, \mathscr{L}_l)$ for $1 \leqslant j \leqslant n+1$ and $n+2 \leqslant l \leqslant m$.

Using (5.1) and Lemma 2.4 again, $\prod_{i=1}^{n+1} V_{ij}$ can be thought of as a Zariski open subset of $H^0(\mathbb{P}^n \times P, \widehat{\mathscr{L}_j})$ for $1 \leqslant j \leqslant n+1$. Replacing $\widehat{U}$ with

$$\left(\prod_{j=1}^{n+1} (U_j \cap (\prod_{i=1}^{n+1} V_{ij}))\right) \times \prod_{j=n+2}^{m} U_j,$$

we get the desired Zariski open set.

We are now in a position to show the existence of determinantal contractions. Pick a section $\sigma = (\sum_i s^i_j z_i, t_j) \in \widehat{U}$, we notice that, for $p \in P$, the dimension of $\pi^{-1}(p)$ is less than two if and only if the corank of the matrix $[s^i_j(p)]$ is less than or equal to two, i.e., $\operatorname{rank}[s^i_j(p)] \geqslant n-1$. From $\dim P = m - n + 3$, the number of sections t_j's is equal to $\dim P - 4$. Set Y be the 4-dimensional smooth variety $Z(t_{n+2}, \cdots, t_m)$. Since the expected codimension $D_{n-2}([s^i_j])$ and $D_{n-1}([s^i_j])$ are nine and four, the intersection of Y with $D_{n-2}([s^i_j])$ and $D_{n-1}([s^i_j])$ are empty and isolated points respectively.

According to that $X = \pi(\widehat{X})$ is defined by $\Delta|_Y = \det(s^i_j)|_Y$ on the smooth variety Y and is irreducible, it follows that X is integral and satisfies Serre's S_2 condition [5, Theorem 14.4 (c)]. Since $\widehat{X} = Z(\sigma)$ is a smooth variety, we have now derived that, for all $\sigma = (\sum_i s^i_j z_i, t_j) \in \widehat{U}$, the morphism $\pi : \widehat{X} \to X$ is a small resolution of the normal variety X with only isolated singularities (which equals $Y \cap D_{n-1}([s^i_j])$). Hence the formal correspondence $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\!\longrightarrow \mathscr{C}$ gives a determinantal contraction. □

Remark 5.3. If corank of $[s^i_j(p)]$ is 1 or 2 then the solution space of the matrix defines a point or a projective line in $\mathbb{P}^n$ respectively. Namely, each fiber of π is a point or a projective line in $\mathbb{P}^n$.

Corollary 5.4. *With notation as in the proof of Theorem 5.2. Let* $\mathscr{E} = \bigoplus_{i=1}^{n+1} \mathscr{L}_i$ *and* $\mathscr{F} = \bigoplus_{i=n+2}^{m} \mathscr{L}_i$ *be vector bundles of rank* $n+1$ *and* $m-n-1$ *on* P *respectively. For the determinantal contraction* $\pi : \widehat{X} \to X$*, the number of singularities of* X *is equal to*

$$\int_P \left(c_2(\mathscr{E})^2 - c_1(\mathscr{E}) c_3(\mathscr{E})\right) c_{m-n-1}(\mathscr{F}).$$

Proof. As in the proof of Theorem 5.2, the number of singularities of X equals the intersection number $[D_{n-1}([s_j^i])] \cap [Z(t_{n+2}, \cdots, t_m)] \cap [P]$. By [5, Theorem 14.4, Example 14.4.1], for the smooth general member $\widehat{X}$ which is defined by a general section $\sigma = (\sum_i s_j^i z_i, t_j)$, the fundamental classes $[D_{n-1}([s_j^i])]$ and $[Z(t_{n+2}, \cdots, t_m)]$ are

$$\big(c_2(\mathscr{E})^2 - c_1(\mathscr{E})c_3(\mathscr{E})\big) \cap [P] \ \text{ and } \ c_{m-n-1}(\mathscr{F}) \cap [P]$$

respectively. This completes the proof. □

REMARK 5.5. If $\pi : \widehat{X} \to X$ is an isomorphism (that is, $\mathrm{Sing}(X) = \varnothing$), the $\widehat{\mathscr{C}} \twoheadleftarrow\!\!\!\longrightarrow \mathscr{C}$ is referred to as an ineffective splitting in [1, p.512]. It is easy to see that it is ineffective if and only if X and $\widehat{X}$ have the same Euler characteristic if and only if the intersection $D_{n-1}([s_j^i]) \cap Z(t_{n+2}, \cdots, t_m)$ is empty. In the case $n = 1$, the intersection is defined by $\dim P$ sections s_j^i and t_l. Therefore the splitting is ineffective if and only if the intersection number

$$c_2(\mathscr{E})^2 \cap [P] = D_1. \cdots .D_{\dim P} = 0$$

where $D_1, \cdots, D_{\dim P}$ are Cartier divisors defined by $s_1^0, s_1^1, s_2^0, s_2^1$, t_l's respectively.

We are now ready to prove the connectedness of parameter spaces of CICY configuration matrices.

THEOREM 5.6. *Any two (parameter spaces of) CICY 3-folds in product of projective spaces are connected by a finite sequence of conifold transitions.*

Proof. By Proposition 4.4 and Theorem 5.2, every CICY configuration matrices connect formally and each formal correspondence gives a determinantal contraction $\widehat{X} \to X$, which is an isomorphism or a small projective resolution, say $X \in \mathscr{C}$. According to Corollary 3.5 and Corollary 5.4, it follows that $e(\widehat{X}) - e(\widetilde{X}) = 2\,|\mathrm{Sing}(X)|$, where $\widetilde{X} \in \mathscr{C}$ is a general smooth member. By Proposition 2.3, the singularities of X are ODPs. Hence each parameter space $[\mathbf{n}\|\mathbf{q}]$ connects to $\mathscr{C}_{1111}$ by conifold transitions (cf. Remark 4.5). □

EXAMPLE 5.7 (Fiber products of elliptic surfaces). Consider

$$\widehat{\mathscr{C}} := \left[\begin{array}{c\|cc} 2 & 3 & 0 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{array}\right] \twoheadleftarrow\!\!\!\longrightarrow \mathscr{C} := \left[\begin{array}{c\|c} 2 & 3 \\ 2 & 3 \end{array}\right].$$

It shall be related to the fiber products of rational elliptic surfaces which was investigated in [30].

Let $f_i : S_i \to \mathbb{P}^1$ be a relatively minimal, rational, elliptic surface with section for $i = 1, 2$. Then S_i is the 9-fold blowing up of $\mathbb{P}^2$ at the base points of a cubic pencil which induces the fibration f_i [20, IV.1.2], that is, there are generic homogeneous cubic polynomials a_i and b_i such that $S_i \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ is a resolution of indeterminacy of the rational map $C_i : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by $C_i(x) = [a_i(x) : b_i(x)]$. Obviously, S_i is defined by

$$P_i(z, x) = z_1 a_i(x) - z_0 b_i(x) = 0$$

where $[z_0 : z_1] \in \mathbb{P}^1$ and $x \in \mathbb{P}^2$.

Let $W = S_1 \times_{\mathbb{P}^1} S_2$. It is well known that W is a Calabi–Yau 3-fold [30]. It is easy to see that W can be obtained as a CICY in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ defined by P_1 and P_2. Therefore $W \in \widehat{\mathscr{C}}$ and is birational to a member in $\mathscr{C}$ which is defined by the bicubic polynomial $a_0(x)b_1(x) - a_1(x)b_0(x) = 0$.

Example 5.8 (Double solids). Consider the CICY configuration matrix

$$\mathscr{C} := \left[\begin{array}{c\|c} 3 & 4 \\ 1 & 2 \end{array}\right].$$

Let x, y be a basis of $H^0(\mathscr{O}_{\mathbb{P}^1}(1))$. Let $\mathscr{L}$ be the line bundle of multidegree $(4, 2)$ on $P := \mathbb{P}^3 \times \mathbb{P}^1$ and $\Gamma = H^0(\mathscr{O}_{\mathbb{P}^3}(4))$. By Proposition 4.6, there is a Zariski open subset of

$$H^0(P, \mathscr{L}) \simeq (\Gamma \cdot x^2) \oplus (\Gamma \cdot xy) \oplus (\Gamma \cdot y^2)$$

such that each section in the open set defines a smooth Calabi–Yau 3-fold.

Choose general quartics A, B and C on $\mathbb{P}^3$ so that the octic hypersurface S in $\mathbb{P}^3$ defined by $\Delta := B^2 - 4AC \in H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(8))$ contains $4^3 = 64$ singular points (the three quartics vanish simultaneously at these 64 points) and the Calabi–Yau $\widehat{X} \in \mathscr{C}$ defined by

$$Ax^2 + Bxy + Cy^2 = 0$$

is smooth.

Let X be the double cover of $\mathbb{P}^3$ branched over S. We can show that the only singular points of X are ODPs, one for each singular point of S. These double covers X, called double solids, were firstly studied by Clemens [3].

Consider the Stein factorization of the natural projection ϕ of $\widehat{X}$ on $\mathbb{P}^3$:

$$\phi = \phi' \circ \pi$$

where ϕ' is finite and π has connected fibers. For each $p \in \mathbb{P}^3$, the fiber $\phi^{-1}(p)$ consists of (i) two points if $\Delta(p) \neq 0$, (ii) one point if $\Delta(p) = 0$ but at least one of $A(p)$, $B(p)$ and $C(p)$ does not vanish, (iii) a copy of $\mathbb{P}^1$ if $A(p) = B(p) = C(p) = 0$. Therefore the map ϕ' is a double cover of $\mathbb{P}^3$ (by (i)) branched over the octic surface S (by (ii)), and the map $\pi : \widehat{X} \to X$ is a small resolution (by (iii)).

For instance, we choose a open set U in $\mathbb{P}^3$ such that $\mathscr{O}_{\mathbb{P}^3}(4)|_U \simeq \mathscr{O}_U$ and $A|_U$ is nowhere zero. Let $V = \{[x : y] \in \mathbb{P}^1 \,|\, y \neq 0\}$. On $W := U \times V$, we rewrite the equation

$$Ax^2 + Bxy + Cy^2 = \frac{A}{4}\left[(2x + \frac{B}{A}y)^2 - \frac{\Delta}{A^2}y^2\right].$$

Then we get a commutative diagram

$$\begin{tikzcd} \widehat{X} & W \cap \widehat{X} \arrow[l, hook'] \arrow[r, equal, "\pi"] \arrow[dr, "\phi"'] & \operatorname{Spec}_U \frac{\mathscr{O}_{\mathbb{P}^3}(U)[T]}{(T^2-\Delta)} \arrow[r, hook] \arrow[d, "\phi'"] & X. \\ & & U & \end{tikzcd}$$

Let $\widetilde{X}$ be a smoothing of X, which is a double cover of $\mathbb{P}^3$ branched over a smooth octic surface $\widetilde{S} \in [3\|8]$. Then the topological Euler number $e(\widetilde{X}) = 2e(\mathbb{P}^3) - e(\widetilde{S}) = -296$. Hence, by Proposition 2.3, the difference of the Euler numbers $e(\widehat{X}) - e(\widetilde{X})$ is $128 = 2 \cdot 64$ as expected and X is a conifold.

6. Further discussions on small transitions. We review the definition of (projective) small transitions.

Definition 6.1. Let $\widehat{X} \to X$ be a small projective resolution of a Calabi–Yau 3-fold X, which has terminal singularities. If X can be smoothed to a Calabi–Yau manifold $\widetilde{X}$, then the process of going from $\widehat{X}$ to $\widetilde{X}$ is called a small transition and denoted by a diagram $\widehat{X} \to X \rightsquigarrow \widetilde{X}$. It is called a conifold transition if X has only ODPs.

Conifold transitions play a fundamental role in Reid's fantasy [27, Section 8] (cf. §1), which conjectures that all the moduli spaces of smooth Calabi–Yau 3-folds are connected through conifold transitions. As in the previous section, the moduli spaces of CICY 3-folds in product of projective spaces are connected to each other by conifold transitions (cf. [10] and Theorem 5.2). A special yet fundamental question arising from Reid's fantasy is the following:

Question 1. Let $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ be a small transition. Is it true $\widehat{X}$ can be connected to $\widetilde{X}$ by *a conifold transition* (through a different X of course)?

For a Calabi–Yau 3-fold X, Namikawa and Steenbrink proved that X can be deformed to a Calabi–Yau 3-fold with at worst ODPs [21]. In view of this result, it seems that one may possibly answer Question 1 affirmatively by finding a deformation direction of $\widehat{X}$ which deforms $\widehat{X} \to X$ into $\widehat{X}_1 \to X_1$ with X_1 being a Calabi–Yau conifold. Unfortunately, Namikawa produced a counterexample to this in [23, Remark 2.8]. We recall it briefly as follows:

Choose a suitable rational elliptic surface S with six singular fibers of type II (i.e., cuspidal rational curves). Let $X = S \times_{\mathbb{P}^1} S$. Then X is a Calabi–Yau 3-fold with six singular points of cA_2 type:

$$x^2 - y^3 = u^2 - v^3,$$

which admits smoothings to $\widetilde{X} = S_1 \times_{\mathbb{P}^1} S_2$ with $S_i \to \mathbb{P}^1$ having disjoint discriminant loci. A small resolution $\pi : \widehat{X} \to X$ can also be constructed (see below). Namikawa observed that the exceptional loci cannot be deformed to a disjoint union of $(-1,-1)$-curves. The reason is that a singular fiber of type II splits up into at most two singular fibers of type I, and a general fiber of small deformation of a singularity of X which preserves the small resolution has three ODPs.

To search for a modification of Question 1, we need to study Namikawa's construction of the small resolution π carefully. Notice that the diagonal $D \cong S$ in X is a smooth Weil divisor which contains the six singular points and is thus not $\mathbb{Q}$-Cartier[1]. On the other hand, by [23, Example, p.1220], there is a nontrivial automorphism $\tau \in \mathrm{Aut}(X)$ such that $D_\tau := \tau(D)$ has the same properties as D. Then $X' := \mathrm{Bl}_D X$ has six ODPs and the exceptional locus of $X' \to X$ consists of six mutually disjoint $\mathbb{P}^1$s, with each of them passing through one of the six ODPs. Now the small resolution can be constructed as the blowing up of X' along the proper transform $\widetilde{D}_\tau$ of D_τ, with π being composed of morphisms $\widehat{X} \to X' \to X$. It admits exceptional trees, composed of couples of rational curves intersecting at one point.

[1]In fact the divisor class group of a terminal Gorenstein 3-fold is torsion-free [13, (5.1)]. Hence it suffices to show that D is not Cartier. It follows from commutative algebra: If $(A, \mathfrak{m})$ is a Noetherian local ring, $f \in \mathfrak{m}$ and $A/(f)$ is a regular local ring of dimension $\dim A - 1$, then A is regular. Then we are done since the smooth Weil divisor D contains $\mathrm{Sing}(X)$.

Now comes the key point. Using Friedman's criterion, $X' \to X$ can be deformed to a small resolution $Y' \to Y$ where Y' is smooth and Y has only ODPs. Thus we have decomposed the small transition $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ into two conifold transitions $\widehat{X} \to X' \rightsquigarrow Y'$ and $Y' \to Y \rightsquigarrow \widetilde{X}$:

$$\begin{tikzcd}
\widehat{X} = \mathrm{Bl}_{\widetilde{D}_\tau} X' \arrow[d] & Y' \arrow[d] & \\
X' = \mathrm{Bl}_D X \arrow[d] \arrow[ur, rightsquigarrow] & Y \arrow[dr, rightsquigarrow] & \\
X \arrow[ur, rightsquigarrow] \arrow[rr, rightsquigarrow] & & \widetilde{X}.
\end{tikzcd}$$

Combining the above discussions, we modify Question 1 as follows:

Question 2. Let $\widehat{X} \to X \rightsquigarrow \widetilde{X}$ be a small transition. Up to deformations of contractions and flops, is it true $\widehat{X}$ can be connected to $\widetilde{X}$ through *a sequence of conifold transitions*?

Remark 6.2. We prefer to identify Calabi–Yau 3-folds which can be connected by a sequence of flops. The reason is that many invariants are preserved under flops, e.g. quantum invariance [19, 18] (see also [33] for a survey on recent development), the Kuranishi (miniversal deformation) spaces [16, (12.6)], analytic type of singularities [14, (4.11)], integral cohomology groups, etc. (see [15, (3.2.2)]).

Before proceeding further, we review here the deformation theory of Calabi–Yau 3-folds. Fix a Calabi–Yau 3-fold X. Let $\mathrm{Def}(X)$ be the Kuranishi space for flat deformations of X (see [16, (11.3)] and the references therein). By [22, Theorem A], the Kuranishi space $\mathrm{Def}(X)$ is smooth[2].

Fix a small projective *partial* resolution $\pi : X' \to X$, where the Calabi–Yau 3-fold X' has at worst terminal singularities. Since X has only rational singularities, there is a natural map of germs of smooth complex spaces $\pi_* : \mathrm{Def}(X') \to \mathrm{Def}(X)$ (cf. [16, (11.4)] or [32, (1.4)] on the level of deformation functors). According to that π is small, it follows that the natural map π_* is a closed immersion (cf. [32, (1.12)], [23, (2.3)]).

Let $\widehat{X} \to X$ be a small projective resolution. We can show that there is a closed immersion $\mathrm{Def}(\widehat{X})$ into $\mathrm{Def}(X')$. Indeed, let $\widehat{X}'$ be a $\mathbb{Q}$-factorialization of X' [13, (4.5)]. Since Calabi–Yau 3-folds $\widehat{X}$ and $\widehat{X}'$ are connected by a sequence of flops [13, 14] and the Kuranishi spaces are unchanged under flops [16, (12.6)], we get

$$\mathrm{Def}(\widehat{X}) \simeq \mathrm{Def}(\widehat{X}') \hookrightarrow \mathrm{Def}(X').$$

Remark that $\widehat{X}'$ is also smooth (see Remark 6.2 or [14, (4.11)]).

According to that the Gorenstein terminal 3-fold singularities $p \in X$ are precisely the isolated compound Du Val (cDV for short) hypersurface singularities [26, (1.1)], it follows that the miniversal deformation space $\mathrm{Def}\,(p \in X)$ is smooth (see

[2]In this paper, we stick to Calabi–Yau 3-folds X with at worst terminal singularities. If we relax the class of singularities, then $\mathrm{Def}(X)$ might be singular. Indeed, there is a Calabi–Yau 3-fold with canonical singularities whose Kuranishi space is singular [8].

[17, (4.61)] and the references therein). There is a natural restriction morphism $\mathrm{Def}(X) \to \mathrm{Def}\,(p \in X)$ for every singular point $p \in X$.

To attack Question 2, we introduce *primitive* small transitions:

Definition 6.3. A small transition $\widehat{X} \xrightarrow{\pi} X \rightsquigarrow \widetilde{X}$ is said to be primitive if it satisfies the following two conditions:

(1) For any small projective partial resolution $X'(\neq X)$ of X, the closed immersion $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X')$ of Kuranishi spaces is an isomorphism.

(2) The composition

$$\mathrm{Def}(\widehat{X}) \xhookrightarrow{\pi_*} \mathrm{Def}(X) \to \prod_{p \in \mathrm{Sing}(X)} \mathrm{Def}\,(p \in X) \tag{6.1}$$

is trivial.

Remark 6.4. In [22], Namikawa discusses a natural stratification on the Kuranishi space $\mathrm{Def}(X)$ by means of small projective partial resolutions of X. Let $Y_1 := \mathrm{Def}(\widehat{X})$ and let Y_0 be the complement of Y_1 in $\mathrm{Def}(X)$. The condition (1) in Definition 6.3 means that strata of $\mathrm{Def}(X)$ are only Y_0 and Y_1. Note that if the relative Picard number of π is one, then the condition (1) is automatically satisfied.

To explain the condition (2) in Definition 6.3, let $\Pi : \widehat{\mathcal{X}} \to \mathcal{X}$ be a flat family over the unit disk Δ in $\mathbb{C}$ with $\Pi|_{t=0} = \pi$. By restricting the deformation $\mathcal{X} \to \Delta$ of X to a sufficiently small open neighborhood V_i of a singular point $p_i \in X$, we get a deformation $\mathcal{V}_i \to \Delta$ of the singular point p_i. Then the condition (2) implies that $\Pi^{-1}(\mathcal{V}_i) \to \mathcal{V}_i$ is isomorphic to the trivial family $\pi^{-1}(V_i) \times \Delta \to V_i \times \Delta$ over the unit disk Δ.

We recall the Namikawa's criterion for the smoothability [23, (2.5)].

Theorem 6.5 ([23]). *Let X be a Calabi–Yau 3-fold. The following two conditions are equivalent:*

(1) *X is smoothable by a flat deformation;*

(2) *for any small projective partial resolution $X'(\neq X)$ of X, $\mathrm{Def}(X') \hookrightarrow \mathrm{Def}(X)$ is not a surjection.*

As an immediate consequence of Theorem 6.5, we give an equivalent formulation of the condition (1) in Definition 6.3.

Proposition 6.6. *Let $\pi : \widehat{X} \to X$ be a small projective resolution of a Calabi–Yau 3-fold X. Then X satisfies the condition (1) in Definition 6.3 if and only if for any small projective partial resolution $X' \to X$ with $X' \neq X$ the Calabi–Yau 3-fold X' is not smoothable.*

Proof. First, we observe that the "only if" implication follows immediately from Theorem 6.5. To proof the "if" implication, we recall the defect $\sigma(Y)$ of a variety Y. It is the rank of $\mathrm{WDiv}(Y)/\mathrm{CDiv}(Y)$, where $\mathrm{WDiv}(Y)$ (resp. $\mathrm{CDiv}(Y)$) is the Abelian group of Weil (resp. Cartier) divisors of Y. Then $\sigma(Y) < \infty$ (resp. $= 0$) if Y has at most rational singularities [13, (1.1)] (resp. if Y is $\mathbb{Q}$-factorial). Remark that if Y admits a nontrivial small birational morphism then $\sigma(Y) > 0$.

Fix a small projective partial resolution $X' \to X$ with $X' \neq X$. By Theorem 6.5 and X' is not smoothable, there is a small projective partial resolution $X''(\neq X')$ of X' such that $\mathrm{Def}(X'') \hookrightarrow \mathrm{Def}(X')$ is an isomorphism. Clearly $\sigma(X'') < \sigma(X') < \infty$. Since X'' is also a small projective partial resolution of X, it is not smoothable.

Hence we can repeat this process until we reach a $\mathbb{Q}$-factorial variety $\widehat{X}''$ with the isomorphism $\mathrm{Def}(\widehat{X}'') \hookrightarrow \mathrm{Def}(X'')$. According to that $\widehat{X}''$ is also a $\mathbb{Q}$-factorialization of X, it follows that the composition

$$\mathrm{Def}(\widehat{X}) \simeq \mathrm{Def}(\widehat{X}'') \hookrightarrow \mathrm{Def}(X'') \hookrightarrow \mathrm{Def}(X')$$

is an isomorphism, which completes the proof of the "if" implication. □

The following proposition explains why we use primitive small transitions as building blocks of general small transitions.

Proposition 6.7. *Every small transition of Calabi–Yau 3-folds can be decomposed into primitive small transitions up to deformations and flops.*

Proof. Let $\widehat{X} \xrightarrow{\pi} X \rightsquigarrow \widetilde{X}$ be a small transition of Calabi–Yau 3-folds. We use induction on the relative Picard number $\rho := \rho(\widehat{X}/X)$ to prove the proposition.

Observe that the condition (1) in Definition 6.3 is automatically satisfied in the case $\rho = 1$. Suppose that the composition map (6.1) in Definition 6.3 is not trivial. The key point is just that the Du Val surface singularities have no moduli. In fact, given such a nontrivial deformation $\Pi : \widehat{\mathcal{X}} \to \mathcal{X}$ of π over the unit disk Δ in $\mathbb{C}$ with $\Pi|_{t=0} = \pi$, there is a nontrivial holomorphic map $\Delta \to \mathrm{Def}(p_i \in X)$. Since $(p_i \in X)$ is an isolated cDV singularity, it is a 1-parameter family f_i over a 1-dimensional disk Δ_i of Du Val surface singularities (cf. [26] or [22, §1]). Let $S_i := f_i^{-1}(0)$. By the versality of $\mathrm{Def}(S_i)$ there is a nontrivial homomorphic map $\psi_i : \Delta \times \Delta_i \to \mathrm{Def}(S_i)$.

Let F_i be the miniversal family for the deformation of S_i. Since the Milnor number of hypersurface singularities is upper semicontinuous under deformations, the Milnor number of the isolated Du Val surface singularity S_i is greater than or equal to the sum of the Milnor numbers at all singularities of the Du Val surface $F_i^{-1}(\psi_i(t,w))$ for $(t,w) \in \Delta \times \Delta_i$. Recall that the isolated Du Val surface singularity S_i is simple (and therefore of type A_n, D_n, E_6, E_7, E_8). Hence, for sufficiently small $t \in \Delta \setminus \{0\}$, the Milnor number at a singularity of $F_i^{-1}(\psi_i(t,0))$ is less than the Milnor number of the isolated Du Val surface singularity $S_i = F_i^{-1}(\psi_i(0,0))$ (cf. [6, Theorem 5] or [17, Remark 4.42]) and then we replace the original small transition $\widehat{X} \xrightarrow{\pi} X \rightsquigarrow \widetilde{X}$ with $\widehat{\mathcal{X}}_t \xrightarrow{\Pi|_t} \mathcal{X}_t \rightsquigarrow \widetilde{X}$. Repeating this process finitely many times leads to a primitive small transition.

Now assume that $\rho \geqslant 2$ and that there is a small projective partial resolution $X' \to X$ with $X' \neq X$ such that the Calabi–Yau 3-fold X' is smoothable. Take a $\mathbb{Q}$-factorialization $\widehat{X}'$ of X'. Since $\widehat{X}$ and $\widehat{X}'$ are both $\mathbb{Q}$-factorialization of X and $\widehat{X}$ is smooth, they are connected by flops

$$\begin{tikzcd} \widehat{X} \arrow[r, dashed, "\text{flops}"] \arrow[rrr, bend right, "\pi"'] & \widehat{X}' \arrow[r] & X' \arrow[r] & X \end{tikzcd}$$

and $\widehat{X}'$ is also smooth. Then we replace $\widehat{X} \xrightarrow{\pi} X \rightsquigarrow \widetilde{X}$ with the new small transition $\widehat{X}' \to X \rightsquigarrow \widetilde{X}$. By assumption, $X' \to X$ can be deformed to a small projective resolution $Y' \to Y$ where Y' is smooth. Thus we have decomposed $\widehat{X}' \to X \rightsquigarrow \widetilde{X}$ into two small transitions $\widehat{X}' \to X' \rightsquigarrow Y'$ and $Y' \to Y \rightsquigarrow \widetilde{X}$.

By induction, we may assume that, for any small projective partial resolution $X'(\neq X)$ of X, the Calabi–Yau 3-fold X' is not smoothable. By Proposition 6.6, the small transition $\widehat{X} \xrightarrow{\pi} X \rightsquigarrow \widetilde{X}$ satisfies the condition (1) in Definition 6.3. Using the same argument as in the case $\rho = 1$ yields a primitive small transition, and the proof is completed. □

If we want to approach Question 2, understanding primitive small transitions becomes essential. The following theorem provides the first step towards this problem:

THEOREM 6.8. *Let* $\pi : \widehat{X} \to X$ *be a small projective resolution of a Calabi–Yau 3-fold* X. *If the natural closed immersion* $\mathrm{Def}(\widehat{X}) \hookrightarrow \mathrm{Def}(X)$ *of Kuranishi spaces is an isomorphism then the singularities of* X *are ODPs. Moreover, the number of ODPs is equal to the relative Picard number* $\rho(\widehat{X}/X)$.

We note that Theorem 6.8 is a generalization of [7, (5.1)].

Proof. The proof is by induction on the relative Picard number $\rho \coloneqq \rho(\widehat{X}/X)$. Observe that X is not smoothable by Theorem 6.5. For the case $\rho = 1$, the result follows from the above observation and [7, (5.1)].

To prove the case $\rho \geqslant 2$, we recall some facts about extremal rays. Let D be the pullback of an ample divisor under the morphism π. By Kodaira's Lemma, a linear system $|mD - A|$ is nonempty for any ample divisor A on $\widehat{X}$ and $m \gg 0$. Pick a divisor $E \in |mD - A|$, which is relatively antiample by the relative Kleiman's criterion for ampleness. Let $\overline{NE}(\widehat{X}/X)$ be the relative Mori cone. It is a convex (polyhedral) cone generated by (finitely many) exceptional curves of π. Using the Cone Theorem [17, (3.25)], we have a klt pair $(\widehat{X}, \varepsilon E)$ for $0 < \varepsilon \ll 1$ with $\mathscr{O}_{\widehat{X}}(-E)$ being π-ample such that

$$\overline{NE}(\widehat{X}/X) = \sum_{i=1}^{k} \mathbb{R}_{\geqslant 0}[C_i],$$

where $\mathbb{R}_{\geqslant 0}[C_i]$ are different extremal rays and $k \geqslant \rho$. Notice that every face of $\overline{NE}(\widehat{X}/X)$ is a $(K_{\widehat{X}} + \varepsilon E)$-negative extremal face. It is also evident that the number of irreducible components of $\mathrm{Exc}(\pi)$ is at least ρ.

Suppose that our assertion is valid for small resolutions with the relative Picard number less than ρ, and let $\pi : \widehat{X} \to X$ be a small projective resolution with $\rho(\widehat{X}/X) = \rho$. We first claim that the number of irreducible components of $\mathrm{Exc}(\pi)$ is the relative Picard number ρ.

Let $U = \widehat{X} \setminus \pi^{-1}(\mathrm{Sing}(X))$. Consider the following long exact sequence

$$0 \to H^1(\Omega^2_{\widehat{X}}) \to H^1(U, \Omega^2_U) \to \bigoplus_{p \in \mathrm{Sing}(X)} H^2_{\pi^{-1}(p)}(\Omega^2_{\widehat{X}}) \xrightarrow{\alpha} H^2(\Omega^2_{\widehat{X}}) \tag{6.2}$$

where $H^1_{\pi^{-1}(p)}(\Omega^2_{\widehat{X}})$ is vanishing for all $p \in \mathrm{Sing}(X)$ by the depth argument (cf. Lemma 2.5). Since X is Calabi-Yau, $\mathrm{Def}(X)$ is smooth [22] and the tangent space of $\mathrm{Def}(X)$ is isomorphic to $H^1(U, \Omega^2_U)$, by Schlessinger's result [4, 29]. According to the assumption of the theorem, the dimension of $\mathrm{Def}(\widehat{X})$ and $\mathrm{Def}(X)$ are the same. Then we get $h^1(\Omega^2_{\widehat{X}}) = h^1(U, \Omega^2_U)$ and thus α is injective. Since the image of α is just the vector space generated by the fundamental classes of irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$, we get $\mathrm{rank}(\alpha) = \rho$. According to Lemma 2.5, it follows that the dimension of $\bigoplus_p H^2_{\pi^{-1}(p)}(\Omega^2_{\widehat{X}})$ is greater than or equal to the number of irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$ which is at least ρ. Hence we conclude that the number of irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$ is exactly ρ.

Notice that now we have

$$\overline{NE}(\widehat{X}/X) = \bigoplus_{i=1}^{\rho} \mathbb{R}_{\geqslant 0}[C_i].$$

If any two curves have non-empty intersection, say C_1 and C_2, we let F be the cone generated by $[C_1]$ and $[C_2]$. It is indeed a face since there are precisely ρ generators of the ρ-dimensional cone $\overline{NE}(\widehat{X}/X)$. Let $\pi' : \widehat{X} \to X'$ be the contraction of the $(K_{\widehat{X}} + \varepsilon E)$-negative extremal face F. By the induction hypothesis, the singularities of X' consist of exactly two ODPs and $\mathrm{Exc}(\pi') = C_1 \coprod C_2$. This contradicts to that $C_1 \cap C_2 \neq \varnothing$, and thus $\mathrm{Exc}(\pi)$ is a disjoint union of irreducible rational curves. By the above argument using the induction hypothesis and the Cone theorem, we infer that the singularities of X are ODPs (the normal bundle of an irreducible exceptional curve in $\widehat{X}$ is $\mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$). □

We can use Theorem 6.8 to give a necessary condition for primitive small transitions with the relative Picard number $\geqslant 2$.

COROLLARY 6.9. *Let $\widehat{X} \xrightarrow{\pi} X \rightsquigarrow \widetilde{X}$ be a primitive small transition. If the relative Picard number of π is great than or equal to two, then for any nontrivial factorization $\widehat{X} \to X' \to X$ with $X' \neq X$ the singularities of X' are ODPs. Moreover, the number of ODPs of X' equals $\rho(\widehat{X}) - \rho(X')$.*

QUESTION 3. Can one classify primitive small transitions? Or more ambitiously, is it true a primitive small transition is necessarily a conifold transition?

It amounts to studying the *global* deformation theory of the small contraction $\widehat{X} \to X$. Notice that in the case of standard web (CICY inside product of projective spaces, Theorem 5.6), we have used a Bertini-type theorem for degeneracy loci to play the role of the required deformation theory. For a general small transition, a deeper analysis of globalizing the local deformations is needed.

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