ASIAN J. MATH.
Vol. 22, No. 5, pp. 919–940, October 2018

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ON THE CR ANALOGUE OF REILLY FORMULA AND YAU EIGENVALUE CONJECTURE*

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Abstract. In this paper, we derive the CR Reilly's formula and its applications to studying of the first eigenvalue estimate for CR Dirichlet eigenvalue problem and embedded p-minimal hypersurfaces. In particular, we obtain the first Dirichlet eigenvalue estimate in a compact pseudohermitian $(2n+1)$-manifold with boundary and the first eigenvalue estimate of the tangential sublaplacian on closed oriented embedded p-minimal hypersurfaces in a closed pseudohermitian $(2n+1)$-manifold of vanishing torsion.

Key words. Pseudohermitian minimal surface, CR Dirichlet eigenvalue, CR Reilly formula, Tangential sublaplacian, CR Yau eigenvalue conjecture.

Mathematics Subject Classification. Primary 32V05, 32V20; Secondary 53C56.

1. Introduction. In the paper of [Re], by integral version of Bochner-type formula, R. Reilly proved so-called Reilly formula which has numerous applications. For example, Reilly himself applied it to prove a Lichnerowicz type sharp lower bound for the first eigenvalue of Laplacian on compact Riemannian manifolds with boundary. In this paper, we will derive the CR version of Reilly's formula and give some applications. In particular, we obtain the first Dirichlet eigenvalue estimate in a compact pseudohermitian $(2n+1)$-manifold with boundary and the first eigenvalue estimate of the tangential sublaplacian on closed oriented embedded p-minimal hypersurfaces in a closed pseudohermitian $(2n+1)$-manifold of vanishing torsion. Finally, we will indicate the CR analogue of Yau conjecture ([Y]) and Lawson conjecture ([La]).

Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold (see next section for basic notions in pseudohermitian geometry). The CR Reilly's formula (1.3) involves terms which have no analogue in the Riemannian case. However, one can relate these extra terms to a third-order operator P which characterizes CR-pluriharmonic functions ([L1]) and the fourth-order CR Paneitz operator P_0 ([GL]).

DEFINITION 1.1 ([GL], [P]). Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold. We define

$$P\varphi = \textstyle\sum_{\gamma,\beta=1}^{n}(\varphi_{\overline{\gamma}}{}^{\overline{\gamma}}{}_{\beta} + inA_{\beta\gamma}\varphi^{\gamma})\theta^{\beta} = \sum_{\beta=1}^{n}(P_{\beta}\varphi)\theta^{\beta},$$

which is an operator that characterizes CR-pluriharmonic functions. Here

$$P_{\beta}\varphi = \textstyle\sum_{\gamma=1}^{n}(\varphi_{\overline{\gamma}}{}^{\overline{\gamma}}{}_{\beta} + inA_{\beta\gamma}\varphi^{\gamma}), \quad \beta = 1, \cdots, n,$$

and $\overline{P}\varphi = \sum_{\beta=1}^{n} \overline{P}_{\beta}\theta^{\overline{\beta}}$, the conjugate of P. The CR Paneitz operator P_0 is defined by

$$P_0\varphi = 4\delta_b(P\varphi) + 4\overline{\delta}_b(\overline{P}\varphi), \tag{1.1}$$

*Received September 5, 2015; accepted for publication November 25, 2016. Research supported in part by the MOST of Taiwan.

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where δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_\beta\theta^\beta) = \sigma_\beta{}^{\beta}$, and similarly, $\overline{\delta}_b(\sigma_{\overline{\beta}}\theta^{\overline{\beta}}) = \sigma_{\overline{\beta}}{}^{\overline{\beta}}$.

We observe that ([GL])

$$\begin{aligned} P_0\varphi &= 2\Box_b\overline{\Box}_b\varphi - 8in(A^{\beta\gamma}\varphi_\beta)_{,\gamma} \\ &= 2\overline{\Box}_b\Box_b\varphi + 8in(A^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}})_{,\overline{\gamma}} \\ &= 2(\Delta_b^2 + n^2T^2)\varphi - 8n\mathrm{Re}(\mathrm{i}\mathrm{A}^{\beta\gamma}\varphi_\beta)_{,\gamma} \end{aligned} \tag{1.2}$$

for $\Box_b\varphi = (\overline{\partial}_b\overline{\partial^*}_b + \overline{\partial^*}_b\overline{\partial}_b)\varphi = (-\Delta_b + inT)\varphi = -2\varphi_{\overline{\beta}}{}^{\overline{\beta}}$.

By using integrating by parts to the CR Bochner formula (3.1), we derive the following CR version of Reilly's formula.

THEOREM 1.1. *Let (M,J,θ) be a compact pseudohermitian $(2n+1)$-manifold with boundary Σ. Then for any real smooth function φ, we have*

$$\begin{aligned} &\frac{n+1}{n}\int_M[(\Delta_b\varphi)^2 - \frac{2n}{n+1}\sum_{\beta,\gamma}|\varphi_{\beta\gamma}|^2]d\mu \\ =&\frac{n+2}{4n}\int_M \varphi P_0\varphi d\mu + \int_M[2Ric - (n+1)Tor]((\nabla_b\varphi)_{\mathbb{C}},(\nabla_b\varphi)_{\mathbb{C}})d\mu \\ &-\frac{n+2}{2n}iC_n\int_\Sigma\varphi\left(P_n\varphi - P_{\overline{n}}\varphi\right)d\Sigma_p + \frac{i}{2}C_n\int_\Sigma(\varphi^{\overline{\beta}}B_{n\overline{\beta}}\varphi - \varphi^\beta B_{\overline{n}\beta}\varphi)d\Sigma_p \\ &+\frac{3}{4}C_n\int_\Sigma\varphi_0\varphi_{e_n}d\Sigma_p + \frac{3}{4n}C_n\int_\Sigma\varphi_{e_{2n}}\Delta_b\varphi d\Sigma_p + C_n\int_\Sigma\varphi_{e_{2n}}\Delta_b^t\varphi d\Sigma_p \\ &+\frac{1}{4}C_n\int_\Sigma H_{p.h}\varphi_{e_{2n}}^2 d\Sigma_p - \frac{1}{4}C_n\int_\Sigma\sum_{j=1}^{2n-1}\left\langle\nabla_{e_j}e_n, e_j\right\rangle\varphi_{e_n}\varphi_{e_{2n}}d\Sigma_p \\ &-\frac{1}{2}C_n\int_\Sigma\alpha\varphi_{e_n}\varphi_{e_{2n}}d\Sigma_p + \frac{1}{4}C_n\int_\Sigma\sum_{j,k=1}^{2n-1}\left\langle\nabla_{e_j}e_{2n}, e_k\right\rangle\varphi_{e_j}\varphi_{e_k}d\Sigma_p. \end{aligned} \tag{1.3}$$

Here P_0 is the CR Paneitz operator on M, $C_n := 2^n n!$; $B_{\beta\overline{\gamma}}\varphi := \varphi_{\beta\overline{\gamma}} - \frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}$, $\Delta_b^t := \frac{1}{2}\sum_{j=1}^{2n-1}[(e_j)^2 - (\nabla_{e_j}e_j)^t]$ is the tangential sublaplacian of Σ and $H_{p.h}$ is the p-mean curvature of Σ with respect to the Legendrian normal e_{2n}, $\alpha e_{2n} + T \in T\Sigma$ for some function α on $\Sigma\backslash S_\Sigma$, the singular set S_Σ consists of those points where the contact bundle $\xi = \ker\theta$ coincides with the tangent bundle $T\Sigma$ of Σ. $(\nabla_b\varphi)_{\mathbb{C}} = \varphi^\beta Z_\beta$ is the corresponding complex $(1,0)$-vector field of $\nabla_b\varphi$ and $d\Sigma_p = \theta\wedge e^1\wedge e^{n+1}\wedge\cdots\wedge e^{n-1}\wedge e^{2n-1}\wedge e^n$ is the p-area element on Σ.

If (M,J,θ) is a compact pseudohermitian $(2n+1)$-manifold without boundary, one can check easily that the fourth-order CR Paneitz P_0 is self-adjoint. That is

$$\int_M gP_0fd\mu = \int_M fP_0gd\mu \tag{1.4}$$

for all smooth functions f and g. However, if (M,J,θ) is a compact pseudohermitian $(2n+1)$-manifold with the smooth boundary Σ, it follows from (5.1) and (5.2) that (1.4) holds for all smooth functions with the Dirichlet condition or the Neumann condition as in (1.5) and (1.6) on Σ. In particular, it holds in the situation as in Theorem 1.2 and Theorem 1.3.

That is, one can have the following Dirichlet eigenvalue problem or Neumann eigenvalue problem, respectively:

$$\begin{cases} P_0\varphi &= \mu_{_D}\varphi \ \text{ on } M, \\ \varphi &= 0 \qquad \text{on } \Sigma, \\ \Delta_b\varphi &= 0 \qquad \text{on } \Sigma, \end{cases} \tag{1.5}$$

and

$$\begin{cases} P_0\phi &= \mu_{_N}\phi \ \text{ on } M, \\ \Delta_b\phi &= 0 \qquad \text{on } \Sigma, \\ (\Delta_b\phi)_{e_{2n}} &= 0 \qquad \text{on } \Sigma. \end{cases} \tag{1.6}$$

Hence

$$\int_M \varphi P_0\varphi d\mu \geq \mu_{_D}^1 \int_M \varphi^2 d\mu \tag{1.7}$$

for the first Dirichlet eigenvalue $\mu_{_D}^1$ and all smooth functions with $\varphi = 0 = \Delta_b\varphi$ on Σ. Similarly

$$\int_M \phi P_0\phi d\mu \geq \mu_{_N}^1 \int_M \phi^2 d\mu \tag{1.8}$$

for the first Neumann eigenvalue $\mu_{_N}^1$ and all smooth functions with $\Delta_b\phi = 0 = (\Delta_b\phi)_{e_{2n}}$ on Σ. In general, $\mu_{_D}^1$ and $\mu_{_N}^1$ are not always nonnegative.

Definition 1.2. Let (M, J, θ) be a compact pseudohermitian $(2n+1)$-manifold with smooth boundary Σ. We say that the CR Paneitz operator P_0 with respect to (J, θ) is nonnegative if

$$\int_M \varphi P_0\varphi d\mu \geq 0$$

for all smooth functions with suitable boundary conditions as in Dirichlet eigenvalue problem or Neumann eigenvalue problem of P_0.

Remark 1.1. Let (M, J, θ) be a compact pseudohermitian $(2n+1)$-manifold of vanishing torsion with smooth boundary Σ. It follows from (1.2) that the Kohn Laplacian $\Box_b$ and $\overline{\Box}$ commute and they are diagonalized simultaneously with

$$P_0\varphi = 2\Box_b\overline{\Box}_b\varphi = 2\overline{\Box}_b\Box_b\varphi.$$

Then the corresponding CR Paneitz operator P_0 is nonnegative ([CCC]). That is

$$\mu_{_D}^1 = 0 = \mu_{_N}^1.$$

For the first consequence of CR Reilly formula, we can consider the following Dirichlet eigenvalue problem:

$$\begin{cases} \Delta_b\varphi &= -\lambda_1\varphi \ \text{ on } M, \\ \varphi &= 0 \qquad \text{on } \Sigma. \end{cases} \tag{1.9}$$

Then we have the following first Dirichlet eigenvalue estimate:

THEOREM 1.2. *Let (M, J, θ) be a compact pseudohermitian $(2n+1)$-manifold with smooth boundary Σ. If the pseudohermitian mean curvature $H_{p.h}$ is nonnegative, $H_{p.h} + \tilde{\omega}_n^{\ n}(e_n)$ is also nonnegative and*

$$\left[Ric - \frac{n+1}{2}Tor\right](Z, Z) \geq k \langle Z, Z \rangle$$

for all $Z \in T_{1,0}$ and a positive constant k, then

(i) *For $n \geq 2$,*

$$\lambda_1 \geq \frac{nk}{n+1};$$

(ii) *For $n = 1$,*

$$\lambda_1 \geq \frac{k + \sqrt{k^2 + 6\mu_D^1}}{4}$$

with $\mu_D^1 \geq -\frac{k^2}{6}$. In addition if P_0 is nonnegative, in particular if the torsion is vanishing, then

$$\lambda_1 \geq \frac{k}{2}.$$

REMARK 1.2. It is known that the sharp first eigenvalue estimate is obtained as in [Gr], [LL], [Ch], [CC2] and [FK] in a closed pseudohermitian $(2n+1)$-manifold.

Next we can state the second consequence of the CR Reilly formula (1.3) which served as a CR analogue of Yau conjecture ([Y]) on the first eigenvalue estimate of embedded oriented minimal hypersurfaces. We refer to papers of Choi-Wang [CW] and Tang-Yan [TY] which are related to Yau conjecture.

As before, $\{e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{2n-1}, \alpha e_{2n} + T\}$ is the base of $T\Sigma$ for some function α on $\Sigma \backslash S_\Sigma$. It follows from (3.11) that $\Delta_b^t + \alpha e_n$ is a self-adjoint operator with respect to the p-area element $d\Sigma_p$ on Σ. Hence it is natural to consider the following CR analogue of eigenvalue problem on the embedded closed p-minimal ($H_{p.h} = 0$) hypersurface Σ in a closed pseudohermitian $(2n+1)$-manifold (M, J, θ):

$$L_\alpha u = -\lambda_1 u, \tag{1.10}$$

here

$$L_\alpha := \Delta_b^t + \alpha e_n. \tag{1.11}$$

In this paper, we consider the particular case that $\{e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{2n-1}, T\}$ are always tangent to Σ ($\alpha = 0$) as follows:

$$L_0 := \Delta_b^t. \tag{1.12}$$

That is, we have the first eigenvalue estimate of L_0 on embedded oriented hypersurfaces of nonnegative pseudohermitian mean curvature:

THEOREM 1.3. *Let Σ be a compact embedded oriented p-minimal hypersurface with $\alpha = 0$ in a closed pseudohermitian $(2n+1)$-manifold (M, J, θ) of vanishing* *torsion. Suppose that the pseudohermitian Ricci curvature of M is bounded from below by a positive constant k. Then*

(i) *The first non-zero eigenvalue λ_1 of L_0 on Σ has a lower bound given by*

$$\lambda_1 \geq \frac{k}{2}.$$

(ii) *In case of $n = 1$ if the equality holds, (M, J, θ) must be a closed spherical pseudohermitian 3-manifold and Σ be a compact embedded oriented p-minimal surface of genus ≤ 1. Moreover, (M, J, θ) is the the standard CR 3-sphere $(\mathbf{S}^3, \widehat{J}, \widehat{\theta})$ if it is simply connected.*

Let (M, J, θ) be a closed spherical pseudohermitian 3-manifold. Recall ([CC1]) that we call a CR structure J spherical if Cartan curvature tensor Q_{11} vanishes identically. Here

$$Q_{11} = \frac{1}{6}W_{11} + \frac{i}{2}WA_{11} - A_{11,0} - \frac{2i}{3}A_{11,\bar{1}1}.$$

Note that (M, J, θ) is called a spherical pseudohermitian 3-manifold if J is a spherical structure. We observe that the spherical structure is CR invariant and a closed spherical pseudohermitian 3-manifold (M, J, θ) is locally CR equivalent to the standard pseudohermitian 3-sphere $(\mathbf{S}^3, \widehat{J}, \widehat{\theta})$.

Note that for a p-minimal Clifford torus $\Sigma_0 = S^1(\frac{\sqrt{2}}{2}) \times S^1(\frac{\sqrt{2}}{2}) \subset \mathbb{R}^2 \times \mathbb{R}^2$ in the standard CR 3-sphere $\mathbf{S}^3$ (i.e. $k = 2$ and $A_{11} = 0$), T is always tangent to Σ_0 (i.e. $\alpha = 0$). Furthermore, the coordinate function x_i of Σ_0 is the eigenfunction of the tangential sublaplacian Δ_b^t with

$$\Delta_b^t x_i = -x_i, \quad i = 1, ...4.$$

Then in view of Theorem 1.3, we have the following CR analogue of Yau conjecture on the first eigenvalue estimate of embedded oriented p-minimal surfaces.

Conjecture 1.1. *The first eigenvalue of L_α on any closed embedded p-minimal surface of genus ≤ 1 in the standard CR 3-sphere $(\mathbf{S}^3, \widehat{J}, \widehat{\theta})$ is just 1.*

Finally, we propose a CR analogue of Lawson conjecture ([La]):

Conjecture 1.2. *Any closed embedded p-minimal torus in the standard CR 3-sphere $\mathbf{S}^3$ is the Clifford torus.*

If the Yau conjecture is true for the 2-torus, it was proved in [MR] that the Lawson conjecture holds which is to say that the only minimally embedded torus in $\mathbf{S}^3$ is the Clifford torus. However, Lawson conjecture was solved by S. Brendle [B] recently.

We briefly describe the methods used in our proofs. In section 3, by using integrating by parts to the CR Bochner formula (3.1), we can derive the CR version of Reilly's formula which involving a third order operator P which characterizes CR-pluriharmonic functions and the CR Paneitz operator P_0. By applying the CR Reilly's formula, we are able to obtain the first Dirichlet eigenvalue estimate as in section 4 and derive the first non-zero eigenvalue estimate of (1.10) on compact oriented embedded p-minimal hypersurfaces in a closed pseudohermitian $(2n + 1)$-manifold of vanishing torsion as in section 5.

2. Basic Notions in Pseudohermitian Geometry. We first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold. Let (M, J, θ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure $\xi = \ker\theta$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so is $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$. A CR structure J can extend to $\mathbb{C}\otimes\xi$ and decomposes $\mathbb{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to eigenvalues i and $-i$, respectively. A manifold M with a CR structure is called a CR manifold. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T transverse to ξ, which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$. Let $\left\{T, Z_\beta, Z_{\overline{\beta}}\right\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_β is any local frame of $T_{1,0}$, $Z_{\overline{\beta}} = \overline{Z_\beta} \in T_{0,1}$ and T is the characteristic vector field. Then $\left\{\theta, \theta^\beta, \theta^{\overline{\beta}}\right\}$, which is the coframe dual to $\left\{T, Z_\beta, Z_{\overline{\beta}}\right\}$, satisfies

$$d\theta = ih_{\beta\overline{\gamma}}\theta^\beta \wedge \theta^{\overline{\gamma}}, \tag{2.1}$$

for some positive definite Hermitian matrix of functions $(h_{\beta\overline{\gamma}})$. Actually we can always choose Z_β such that $h_{\beta\overline{\gamma}} = \delta_{\beta\gamma}$; hence, throughout this note, we assume $h_{\beta\overline{\gamma}} = \delta_{\beta\gamma}$.

The Levi form $\langle\ ,\ \rangle$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle = -i\left\langle d\theta, Z \wedge \overline{W}\right\rangle.$$

We can extend $\langle\ ,\ \rangle$ to $T_{0,1}$ by defining $\left\langle \overline{Z}, \overline{W}\right\rangle = \overline{\langle Z, W\rangle}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, also denoted by $\langle\ ,\ \rangle$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta \wedge (d\theta)^n$, we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\beta \in T_{1,0}$ by

$$\nabla Z_\beta = \theta_\beta{}^\gamma \otimes Z_\gamma, \quad \nabla Z_{\overline{\beta}} = \theta_{\overline{\beta}}{}^{\overline{\gamma}} \otimes Z_{\overline{\gamma}}, \quad \nabla T = 0,$$

where $\theta_\beta{}^\gamma$ are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\gamma \wedge \theta_\gamma{}^\beta + \theta \wedge \tau^\beta, \\ 0 &= \tau_\beta \wedge \theta^\beta, \\ 0 &= \theta_\beta{}^\gamma + \theta_{\overline{\gamma}}{}^{\overline{\beta}}, \end{aligned} \tag{2.2}$$

We can write (by Cartan lemma) $\tau_\beta = A_{\beta\gamma}\theta^\gamma$ with $A_{\beta\gamma} = A_{\gamma\beta}$. The curvature of the Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\beta, \theta^{\overline{\beta}}\}$, is

$$\begin{aligned} &\Pi_\beta{}^\gamma = \overline{\Pi_{\overline{\beta}}{}^{\overline{\gamma}}} = d\theta_\beta{}^\gamma - \theta_\beta{}^\sigma \wedge \theta_\sigma{}^\gamma, \\ &\Pi_0{}^\beta = \Pi_\beta{}^0 = \Pi_0{}^{\bar{\beta}} = \Pi_{\bar{\beta}}{}^0 = \Pi_0{}^0 = 0. \end{aligned}$$

Webster showed that $\Pi_\beta{}^\gamma$ can be written

$$\Pi_\beta{}^\gamma = R_\beta{}^\gamma{}_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + W_\beta{}^\gamma{}_\rho\theta^\rho \wedge \theta - W^\gamma{}_{\beta\bar{\rho}}\theta^{\bar{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\gamma - i\tau_\beta \wedge \theta^\gamma$$

where the coefficients satisfy

$$R_{\beta\overline{\gamma}\rho\bar{\sigma}} = \overline{R_{\gamma\bar{\beta}\sigma\bar{\rho}}} = R_{\overline{\gamma}\beta\bar{\sigma}\rho} = R_{\rho\overline{\gamma}\beta\bar{\sigma}}, \quad W_{\beta\overline{\gamma}\rho} = W_{\rho\overline{\gamma}\beta}.$$

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\rho\beta,\gamma}$. The indices $\{0, \beta, \overline{\beta}\}$ indicate derivatives with respect to $\{T, Z_\beta, Z_{\overline{\beta}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_\beta = Z_\beta u,\ u_{\gamma\bar{\beta}} = Z_{\bar{\beta}} Z_\gamma u - \theta_\gamma{}^\rho(Z_{\bar{\beta}}) Z_\rho u,\ u_0 = Tu$ for a smooth function u .

For a real function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle = du(Z)$ for all vector fields Z tangent to the contact plane. Locally $\nabla_b u = u^\beta Z_\beta + u^{\overline{\beta}} Z_{\overline{\beta}}$. We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1} \;\text{ by }\; (\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$

In particular,

$$|\nabla_b u|^2 = 2\sum_\beta u_\beta u^\beta, \quad |\nabla_b^2 u|^2 = 2\sum_{\beta,\gamma}(u_{\beta\gamma}u^{\beta\gamma} + u_{\beta\overline{\gamma}}u^{\beta\overline{\gamma}}).$$

Also the sublaplacian is defined by

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \sum_\beta (u_\beta{}^\beta + u_{\overline{\beta}}{}^{\overline{\beta}}).$$

The pseudohermitian Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$\begin{aligned} Ric(X, Y) &= R_{\gamma\bar{\beta}} X^\gamma Y^{\bar{\beta}} \\ Tor(X, Y) &= i\sum_{\gamma,\beta}(A_{\overline{\gamma}\bar{\beta}} X^{\overline{\gamma}} Y^{\bar{\beta}} - A_{\gamma\beta} X^\gamma Y^\beta), \end{aligned}$$

where $X = X^\gamma Z_\gamma,\ Y = Y^\beta Z_\beta$.

3. The CR Reilly's Formula. Let M be a compact pseudohermitian $(2n+1)$-manifold with boundary Σ. We write $\theta_\gamma{}^\beta = \omega_\gamma{}^\beta + i\tilde{\omega}_\gamma{}^\beta$ with $\omega_\gamma{}^\beta = \mathrm{Re}(\theta_\gamma{}^\beta)$, $\tilde{\omega}_\gamma{}^\beta = \mathrm{Im}(\theta_\gamma{}^\beta)$ and $Z_\beta = \frac{1}{2}(e_\beta - ie_{n+\beta})$ for real vectors $e_\beta,\ e_{n+\beta},\ \beta = 1, \cdots, n$. It follows that $e_{n+\beta} = Je_\beta$. Let $e^\beta = \mathrm{Re}(\theta^\beta)$, $e^{n+\beta} = \mathrm{Im}(\theta^\beta)$, $\beta = 1, \cdots, n$. Then $\{\theta, e^\beta, e^{n+\beta}\}$ is dual to $\{T, e_\beta, e_{n+\beta}\}$. Now in view of (2.1) and (2.2), we have the following real version of structure equations:

$$\begin{cases} d\theta = 2\displaystyle\sum_\beta e^\beta \wedge e^{n+\beta}, \\ \nabla e_\gamma = \omega_\gamma{}^\beta \otimes e_\beta + \tilde{\omega}_\gamma{}^\beta \otimes e_{n+\beta}, \quad \nabla e_{n+\gamma} = \omega_\gamma{}^\beta \otimes e_{n+\beta} - \tilde{\omega}_\gamma{}^\beta \otimes e_\beta, \\ de^\gamma = e^\beta \wedge \omega_\beta{}^\gamma - e^{n+\beta} \wedge \tilde{\omega}_\beta{}^\gamma \bmod \theta;\ de^{n+\gamma} = e^\beta \wedge \tilde{\omega}_\beta{}^\gamma + e^{n+\beta} \wedge \omega_\beta{}^\gamma \bmod \theta. \end{cases}$$

Let Σ be a surface contained in M. The singular set S_Σ consists of those points where ξ coincides with the tangent bundle $T\Sigma$ of Σ. It is easy to see that S_Σ is a closed set. On ξ, we can associate a natural metric $\langle\ ,\ \rangle = \frac{1}{2}d\theta(\cdot, J\cdot)$ call the Levi metric. For a vector $v \in \xi$, we define the length of v by $|v|^2 = \langle v, v\rangle$. With respect to the Levi metric, we can take unit vector fields $e_1, \cdots, e_{2n-1} \in \xi \cap T\Sigma$ on $\Sigma \backslash S_\Sigma$, called the characteristic fields and $e_{2n} = Je_n$, called the Legendrian normal. The p(pseudohermitian)-mean curvature $H_{p.h}$ on $\Sigma\backslash S_\Sigma$ is defined by

$$H_{p.h} = \sum_{j=1}^{2n-1} \left\langle \nabla_{e_j} e_{2n}, e_j \right\rangle = -\sum_{j=1}^{2n-1} \left\langle \nabla_{e_j} e_j, e_{2n} \right\rangle .$$

For $e_1, \cdots, e_{2n-1}$ being characteristic fields, we have the p-area element

$$d\Sigma_p = \theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge e^{n-1} \wedge e^{2n-1} \wedge e^n$$

on Σ and all surface integrals over Σ are with respect to this $2n$-form $d\Sigma_p$. Note that $d\Sigma_p$ continuously extends over the singular set S_Σ and vanishes on S_Σ.

We also write $\varphi_{e_j} = e_j\varphi$ and $\nabla_b\varphi = \frac{1}{2}(\varphi_{e_\beta}e_\beta + \varphi_{e_{n+\beta}}e_{n+\beta})$. Moreover, $\varphi_{e_je_k} = e_ke_j\varphi - \nabla_{e_k}e_j\varphi$ and $\Delta_b\varphi = \frac{1}{2}\sum_\beta(\varphi_{e_\beta e_\beta} + \varphi_{e_{n+\beta}e_{n+\beta}})$. Next we define the subdivergence operator $div_b(\cdot)$ by $div_b(W) = W^\beta{}_{,\beta} + W^{\overline{\beta}}{}_{,\overline{\beta}}$ for all vector fields $W = W^\beta Z_\beta + W^{\overline{\beta}} Z_{\overline{\beta}}$ and its real version is $div_b(W) = \varphi_{\beta,e_\beta} + \psi_{n+\beta,e_{n+\beta}}$ for $W = \varphi_\beta e_\beta + \psi_{n+\beta}e_{n+\beta}$. We define the tangential subgradient ∇_b^t of a function φ by $\nabla_b^t\varphi = \nabla_b\varphi - \langle\nabla_b\varphi, e_{2n}\rangle e_{2n}$ and the tangent sublaplacian Δ_b^t of φ by $\Delta_b^t\varphi = \frac{1}{2}\sum_{j=1}^{2n-1}[(e_j)^2\varphi - (\nabla_{e_j}e_j)^t\varphi]$, where $(\nabla_{e_j}e_j)^t$ is the tangential part of $\nabla_{e_j}e_j$.

We first recall the following CR Bochner formula.

LEMMA 3.1. *Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold. For a real function φ, we have*

$$\begin{aligned}\frac{1}{2}\Delta_b|\nabla_b\varphi|^2 = {} & |\nabla_b^2\varphi|^2 + \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle \\ & + [2Ric - (n-2)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) + 2\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle,\end{aligned} \tag{3.1}$$

where $(\nabla_b\varphi)_{\mathbb{C}} = \varphi^\beta Z_\beta$ is the corresponding complex $(1,0)$-vector field of $\nabla_b\varphi$.

The proof of the above formula follows from the Bochner formula (Lemma 3 in [Gr]) derived by A. Greenleaf and the commutation relation (see Lemma 2.2 in [CC1])

$$i\sum_\beta(\varphi_\beta\varphi_{\overline{\beta}0} - \varphi_{\overline{\beta}}\varphi_{\beta 0}) = i\sum_\beta(\varphi_\beta\varphi_{0\overline{\beta}} - \varphi_{\overline{\beta}}\varphi_{0\beta}) - Tor((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}).$$

From [CC1], we can relate $\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle$ with $\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle$ by

$$\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle = \frac{1}{n}\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle - 2Tor((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) - \frac{2}{n}\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle. \tag{3.2}$$

For the proof of Reilly's formula, we first need a series of formulae. In particular, one derives the following CR version of divergence theorem and Green's identity for a compact pseudohermitian $(2n+1)$-manifold M with boundary Σ. Note that $d\Sigma_p$ vanishes on S_Σ.

LEMMA 3.2 (Divergence Theorem). *Let (M, J, θ) be a compact pseudohermitian $(2n+1)$-manifold with boundary Σ. For a real function φ, we have*

$$\int_M \Delta_b\varphi d\mu = \int_M div_b(\nabla_b\varphi)d\mu = \frac{1}{2}C_n\int_\Sigma \varphi_{e_{2n}}d\Sigma_p = C_n\int_\Sigma\langle\nabla_b\varphi, e_{2n}\rangle d\Sigma_p, \tag{3.3}$$

$$\int_M \varphi\varphi_{00}d\mu + \int_M \varphi_0^2 d\mu = -C_n\int_\Sigma \alpha\varphi\varphi_0 d\Sigma_p. \tag{3.4}$$

Here $d\Sigma_p = \theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge e^{n-1} \wedge e^{2n-1} \wedge e^n$ *is the p-area element of* Σ *and* $C_n = 2^n n!$.

Proof. By Stoke's theorem, we have

$$\begin{aligned}\int_M \Delta_b \varphi d\mu &= \frac{1}{2}\int_M \sum_\beta (\varphi_{e_\beta e_\beta} + \varphi_{e_{n+\beta}e_{n+\beta}}) 2^n n! \theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge e^n \wedge e^{2n} \\ &= 2^{n-1} n! \int_M \sum_\beta d[-\varphi_{e_\beta}\theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge \widehat{e^\beta} \wedge e^{n+\beta} \wedge \cdots \wedge e^n \wedge e^{2n} \\ &\qquad + \varphi_{e_{n+\beta}}\theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge e^\beta \wedge \widehat{e^{n+\beta}} \wedge \cdots \wedge e^n \wedge e^{2n}] \\ &= 2^{n-1} n! \int_\Sigma \varphi_{e_{2n}} \theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge e^{n-1} \wedge e^{2n-1} \wedge e^n \\ &= C_n \int_\Sigma \langle \nabla_b \varphi, e_{2n} \rangle d\Sigma_p.\end{aligned}$$

Here we used $d\mu = \theta \wedge (d\theta)^n = C_n \theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge e^n \wedge e^{2n}$ and the fact that the $2n$-forms $\theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge \widehat{e^\beta} \wedge e^{n+\beta} \wedge \cdots \wedge e^n \wedge e^{2n}$ vanish on Σ for $\beta = 1, \cdots, n$ and so are $\theta \wedge e^1 \wedge e^{n+1} \wedge \cdots \wedge e^\beta \wedge \widehat{e^{n+\beta}} \wedge \cdots \wedge e^n \wedge e^{2n}$ for $\beta = 1, \cdots, n-1$, since e_j are tangent to Σ for $j = 1, \cdots, 2n-1$.

The second equation follows easily from Stoke's theorem as above

$$\begin{aligned}\int_M \varphi\varphi_{00} d\mu + \int_M \varphi_0^2 d\mu &= C_n \int_M d(\varphi\varphi_0 e^1 \wedge e^{n+1} \wedge \cdots \wedge e^n \wedge e^{2n}) \\ &= C_n \int_\Sigma \varphi\varphi_0 e^1 \wedge e^{n+1} \wedge \cdots \wedge e^n \wedge e^{2n}\end{aligned}$$

and the help of the identity $e^{2n} \wedge e^n = \alpha\theta \wedge e^n$ on $\Sigma \backslash S_\Sigma$. □

Corollary 3.1 (Green's identity). *Let* (M, J, θ) *be a compact pseudohermitian* $(2n+1)$*-manifold with boundary* Σ*. For real functions* φ *and* ψ*,*

$$\int_M \psi \Delta_b \varphi d\mu + \int_M \langle \nabla_b \varphi, \nabla_b \psi \rangle d\mu = \frac{1}{2} C_n \int_\Sigma \psi \varphi_{e_{2n}} d\Sigma_p. \tag{3.5}$$

Proof. It is easy to check that $div_b(\psi \nabla_b \varphi) = \psi \Delta_b \varphi + \langle \nabla_b \varphi, \nabla_b \psi \rangle$ and then the result follows from the CR version of divergence theorem. □

Lemma 3.3. *Let* (M, J, θ) *be a compact pseudohermitian* $(2n+1)$*-manifold with boundary* Σ*. For any real smooth function* φ*,*

$$\int_M \langle J\nabla_b \varphi, \nabla_b \varphi_0 \rangle d\mu + n \int_M \varphi_0^2 d\mu = \frac{1}{2} C_n \int_\Sigma \varphi_0 \varphi_{e_n} d\Sigma_p. \tag{3.6}$$

Proof. By using $div_b((J\nabla_b \varphi)\,\varphi_0) = \langle J\nabla_b \varphi, \nabla_b \varphi_0 \rangle + n\varphi_0^2$ and the divergence theorem (3.3), we have

$$\begin{aligned}&\int_M \langle J\nabla_b \varphi, \nabla_b \varphi_0 \rangle d\mu + n \int_M \varphi_0^2 d\mu \\ &= \int_M div_b((J\nabla_b \varphi)\,\varphi_0) d\mu = C_n \int_\Sigma \langle (J\nabla_b \varphi)\,\varphi_0, e_{2n} \rangle d\Sigma_p = \frac{1}{2} C_n \int_\Sigma \varphi_0 \varphi_{e_n} d\Sigma_p. \quad \square\end{aligned}$$

LEMMA 3.4. *Let (M, J, θ) be a compact pseudohermitian $(2n+1)$-manifold with boundary Σ. For any real smooth function φ,*

$$\int_M \langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle d\mu + \frac{1}{4}\int_M (P_0\varphi)\varphi d\mu = \frac{1}{2} iC_n \int_\Sigma \varphi\left(P_n\varphi - P_{\overline{n}}\varphi\right) d\Sigma_p. \tag{3.7}$$

Proof. It can be easily checked that

$$div_b\left((\varphi P^\beta\varphi)Z_\beta + (\varphi P^{\overline{\beta}}\varphi)Z_{\overline{\beta}}\right) = \langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle + \frac{1}{4}\varphi P_0\varphi.$$

We then have by the divergence theorem (3.3)

$$\begin{aligned}&\int_M \langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle d\mu + \frac{1}{4}\int_M (P_0\varphi)\varphi d\mu \\ &= C_n\int_\Sigma \langle(\varphi P^\beta\varphi)Z_\beta + (\varphi P^{\overline{\beta}}\varphi)Z_{\overline{\beta}}, e_{2n}\rangle d\Sigma_p = \frac{1}{2} iC_n \int_\Sigma \varphi\left(P_n\varphi - P_{\overline{n}}\varphi\right) d\Sigma_p.\end{aligned}$$

∎

LEMMA 3.5. *Let (M, J, θ) be a compact pseudohermitian $(2n+1)$-manifold with boundary Σ. For real-valued functions φ on Σ,*

$$\int_\Sigma \left(\varphi_{e_n} + 2\alpha\varphi\right) d\Sigma_p = 0; \tag{3.8}$$

$$\int_\Sigma \left[\varphi_{\overline{\beta}} + \left(\sum_{\gamma\neq n} \theta_{\overline{\beta}}{}^{\overline{\gamma}}(Z_{\overline{\gamma}}) + \frac{1}{2}\theta_{\overline{\beta}}{}^{\overline{n}}(e_n)\right)\varphi\right] d\Sigma_p = 0 \text{ for any } \beta \neq n; \tag{3.9}$$

$$\int_\Sigma [\varphi_0 + \alpha\varphi_{e_{2n}} - (\alpha\tilde{\omega}_n{}^n(e_n) - \mathrm{Re}A_{\overline{n}\overline{n}})\varphi] d\Sigma_p = 0. \tag{3.10}$$

Proof. By Stoke's theorem, we have

$$\begin{aligned}\frac{1}{2}C_n\int_\Sigma \varphi_{e_n} d\Sigma_p &= \int_\Sigma \varphi_{e_n}\theta\wedge(d\theta)^{n-1}\wedge e^n \\ &= -\int_\Sigma d\varphi\wedge\theta\wedge(d\theta)^{n-1} + \int_\Sigma \varphi_{e_{2n}} e^{2n}\wedge\theta\wedge(d\theta)^{n-1} \\ &= -\int_\Sigma d(\varphi\theta\wedge(d\theta)^{n-1}) + \int_\Sigma \varphi d\theta\wedge(d\theta)^{n-1} \\ &= \int_\Sigma 2\varphi e^n\wedge e^{2n}\wedge(d\theta)^{n-1} = -\int_\Sigma 2\alpha\varphi\theta\wedge e^n\wedge(d\theta)^{n-1} \\ &= -C_n\int_\Sigma \alpha\varphi d\Sigma_p,\end{aligned}$$

where we used the identities $\theta\wedge(d\theta)^{n-1}\wedge e^{2n} = 0$ on Σ since e_n is tangent to Σ, $d\theta = 2\sum_{\beta=1}^n e^\beta\wedge e^{n+\beta}$ and $e^{2n}\wedge e^n = \alpha\theta\wedge e^n$ on $\Sigma\backslash S_\Sigma$.

For the second equation, we compute

$$\int_\Sigma \varphi_{\overline{\beta}}\theta\wedge(d\theta)^{n-1}\wedge e^n = \int_\Sigma \varphi_{\overline{\beta}}\theta\wedge\theta^\beta\wedge\theta^{\overline{\beta}}\wedge\left(\sum_{j=1}^{n-1}\underset{j\neq\beta}{\wedge}\theta^j\wedge\theta^{\overline{j}}\right)\wedge e^n$$

$$= \int_\Sigma d\varphi\wedge\theta\wedge\theta^\beta\wedge\left(\sum_{j=1}^{n-1}\underset{j\neq\beta}{\wedge}\theta^j\wedge\theta^{\overline{j}}\right)\wedge e^n = -\int_\Sigma \varphi d\left[\theta\wedge\theta^\beta\wedge((d\theta)^{n-2})\wedge e^n\right]$$

$$= \int_\Sigma \varphi\left[\theta\wedge d\theta^\beta\wedge((d\theta)^{n-2})\wedge e^n\right] - \int_\Sigma \varphi\left[\theta\wedge\theta^\beta\wedge((d\theta)^{n-2})\wedge de^n\right]$$

$$= \int_\Sigma \varphi\left[\theta\wedge(\theta^\gamma\wedge{\theta_\gamma}^\beta+\theta\wedge\tau^\beta)\wedge\left(\sum_{j=1}^{n-1}\underset{j\neq\gamma}{\wedge}\theta^j\wedge\theta^{\overline{j}}\right)\wedge e^n\right]$$

$$-\int_\Sigma \frac{1}{2}\varphi\left[\theta\wedge\theta^\beta\wedge\left(\sum_{j=1}^{n-1}\underset{j\neq\beta}{\wedge}\theta^j\wedge\theta^{\overline{j}}\right)\wedge\left(\sum_{\gamma\neq n}{\theta_{\overline{\gamma}}}^{\overline{n}}(e_n)\theta^{\overline{\gamma}}\right)\wedge e^n\right]$$

$$= \int_\Sigma\left(\sum_{\gamma\neq n}{\theta_\gamma}^\beta(Z_{\overline{\gamma}})-\frac{1}{2}{\theta_{\overline{\beta}}}^{\overline{n}}(e_n)\right)\varphi\theta\wedge\theta^\beta\wedge\theta^{\overline{\beta}}\wedge\left(\sum_{j=1}^{n-1}\underset{j\neq\beta}{\wedge}\theta^j\wedge\theta^{\overline{j}}\right)\wedge e^n$$

$$= -\int_\Sigma\left(\sum_{\gamma\neq n}{\theta_{\overline{\beta}}}^{\overline{\gamma}}(Z_{\overline{\gamma}})+\frac{1}{2}{\theta_{\overline{\beta}}}^{\overline{n}}(e_n)\right)\varphi\theta\wedge(d\theta)^{n-1}\wedge e^n,$$

where we used $de^n = \frac{1}{2}(\theta^\gamma\wedge{\theta_\gamma}^n+\theta^{\overline{\gamma}}\wedge{\theta_{\overline{\gamma}}}^{\overline{n}}) = \frac{1}{2}\sum_{\gamma\neq n}{\theta_{\overline{\gamma}}}^{\overline{n}}(e_n)\theta^{\overline{\gamma}}\wedge e^n$ mod θ, e^{2n} on Σ.

The same computation for the third equation yields

$$\int_\Sigma \varphi_0\theta\wedge(d\theta)^{n-1}\wedge e^n$$

$$= \int_\Sigma d\varphi\wedge(d\theta)^{n-1}\wedge e^n - \int_\Sigma \varphi_{e_{2n}}e^{2n}\wedge e^n\wedge(d\theta)^{n-1}$$

$$= \int_\Sigma d(\varphi\,(d\theta)^{n-1}\wedge e^n) - \int_\Sigma \varphi\,(d\theta)^{n-1}\wedge de^n - \int_\Sigma \alpha\varphi_{e_{2n}}\theta\wedge(d\theta)^{n-1}\wedge e^n$$

$$= \int_\Sigma \varphi\,(d\theta)^{n-1}\wedge\left[{\tilde{\omega}_n}^{\;n}(e_n)e^{2n}\wedge e^n - \mathrm{Re}A_{\overline{n}\overline{n}}\theta\wedge e^n\right] - \int_\Sigma \alpha\varphi_{e_{2n}}\theta\wedge(d\theta)^{n-1}\wedge e^n$$

$$= \int_\Sigma \left[(\alpha{\tilde{\omega}_n}^{\;n}(e_n)-\mathrm{Re}A_{\overline{n}\overline{n}})\varphi-\alpha\varphi_{e_{2n}}\right]\theta\wedge(d\theta)^{n-1}\wedge e^n.$$

∎

LEMMA 3.6. *Let (M, J, θ) be a compact pseudohermitian $(2n+1)$-manifold with boundary Σ. For real-valued functions φ and ψ on Σ, we have*

$$\int_\Sigma \psi(\Delta_b^t+\alpha e_n)\varphi d\Sigma_p = \int_\Sigma \varphi(\Delta_b^t+\alpha e_n)\psi d\Sigma_p. \tag{3.11}$$

This lemma implies that $\Delta_b^t+\alpha e_n$ is a self-adjoint operator with respect to the p-area element $d\Sigma_p$ on Σ.

Proof of Theorem 1.1. By integrating the CR version of Bochner formula (3.1), we have

$$\begin{aligned}\frac{1}{2}\int_M \Delta_b|\nabla_b\varphi|^2 d\mu = & \int_M |\nabla_b^2\varphi|^2 d\mu + \int_M \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle d\mu \\ & + \int_M [2Ric - (n-2)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) d\mu \\ & +2\int_M \langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle d\mu.\end{aligned}$$

Note that

$$\sum_{\beta,\gamma}|\varphi_{\beta\overline{\gamma}}|^2 = \sum_{\beta,\gamma}|\varphi_{\beta\overline{\gamma}} - \frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}|^2 + \frac{1}{4n}(\Delta_b\varphi)^2 + \frac{n}{4}\varphi_0^2.$$

It follows from the CR Green's identity (3.5) with $\psi = \Delta_b\varphi$ and (3.6), that

$$\begin{aligned} & \frac{1}{2}\int_M \Delta_b|\nabla_b\varphi|^2 d\mu \\ = & 2\int_M \sum_{\beta,\gamma}|\varphi_{\beta\gamma}|^2 d\mu + 2\int_M \sum_{\gamma,\beta}|\varphi_{\beta\overline{\gamma}} - \frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}|^2 d\mu \\ & - \frac{3n}{2}\int_M \varphi_0^2 d\mu + C_n\int_\Sigma \varphi_0\varphi_{e_n} d\Sigma_p + \frac{1}{2}C_n\int_\Sigma (\Delta_b\varphi)\varphi_{e_{2n}} d\Sigma_p \\ & - \frac{2n-1}{2n}\int_M (\Delta_b\varphi)^2 d\mu + \int_M [2Ric - (n-2)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}).\end{aligned} \tag{3.12}$$

By combining (3.6), (3.2), (3.5) and (3.7), we have

$$\begin{aligned} n\int_M \varphi_0^2 d\mu = & \frac{1}{n}\int_M (\Delta_b\varphi)^2 d\mu - \frac{1}{2n}C_n\int_\Sigma (\Delta_b\varphi)\varphi_{e_{2n}} d\Sigma_p \\ & - \frac{1}{2n}\int_M \varphi P_0\varphi d\mu + \frac{1}{n}iC_n\int_\Sigma \varphi\left(P_n\varphi - P_{\overline{n}}\varphi\right) d\Sigma_p \\ & + \frac{1}{2}C_n\int_\Sigma \varphi_0\varphi_{e_n} d\Sigma_p + 2\int_M Tor\left((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\right) d\mu.\end{aligned} \tag{3.13}$$

Also applying the divergence theorem to the equation

$$(B^{\beta\overline{\gamma}}\varphi)(B_{\beta\overline{\gamma}}\varphi) = (\varphi^\beta B_{\beta\overline{\gamma}}\varphi),^{\overline{\gamma}} - \frac{n-1}{n}(\varphi P_\beta\varphi),^\beta + \frac{n-1}{8n}\varphi P_0\varphi$$

with $B_{\beta\overline{\gamma}}\varphi = \varphi_{\beta\overline{\gamma}} - \frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}$, we obtain

$$\begin{aligned} & \int_M \sum_{\beta,\gamma}|\varphi_{\beta\overline{\gamma}} - \frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}|^2 d\mu \\ = & \frac{n-1}{8n}\int_M \varphi P_0\varphi d\mu - \frac{n-1}{4n}iC_n\int_\Sigma \varphi\left(P_n\varphi - P_{\overline{n}}\varphi\right) d\Sigma_p \\ & + \frac{1}{4}iC_n\int_\Sigma (\varphi^{\overline{\beta}}B_{n\overline{\beta}}\varphi - \varphi^\beta B_{\overline{n}\beta}\varphi) d\Sigma_p.\end{aligned} \tag{3.14}$$

Here

$$
\begin{aligned}
& i(\varphi^{\overline{\beta}} B_{n\overline{\beta}}\varphi - \varphi^{\beta} B_{\overline{n}\beta}\varphi) \\
= & \frac{1}{4}\sum_{\beta\neq n}\left[\varphi_{e_{n+\beta}}(\varphi_{e_\beta e_n} + \varphi_{e_{n+\beta}e_{2n}}) + \varphi_{e_\beta}(\varphi_{e_\beta e_{2n}} - \varphi_{e_{n+\beta}e_n})\right] \\
& + \frac{1}{4}\varphi_{e_{2n}}\left[(\varphi_{e_n e_n} + \varphi_{e_{2n}e_{2n}}) - \frac{2}{n}\Delta_b\varphi\right].
\end{aligned}
$$

Substituting these into the right hand side of (3.12), we get

$$
\begin{aligned}
& \frac{1}{2}\int_M \Delta_b|\nabla_b\varphi|^2 d\mu \\
= & 2\int_M \sum_{\beta,\gamma}|\varphi_{\beta\gamma}|^2 d\mu - \frac{n+1}{n}\int_M (\Delta_b\varphi)^2 d\mu \\
& + \frac{n+2}{4n}\int_M \varphi P_0\varphi d\mu - \frac{n+2}{2n} iC_n\int_\Sigma \varphi\left(P_n\varphi - P_{\overline{n}}\varphi\right) d\Sigma_p \\
& + \int_M [2Ric - (n+1)Tor]((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) d\mu + \frac{1}{4}C_n\int_\Sigma \varphi_0\varphi_{e_n} d\Sigma_p \\
& + \frac{1}{2} iC_n\int_\Sigma (\varphi^{\overline{\beta}} B_{n\overline{\beta}}\varphi - \varphi^{\beta} B_{\overline{n}\beta}\varphi) d\Sigma_p + \frac{2n+3}{4n}C_n\int_\Sigma (\Delta_b\varphi)\varphi_{e_{2n}} d\Sigma_p.
\end{aligned}
\tag{3.15}
$$

On the other hand, the divergence theorem (3.3) implies that

$$
\begin{aligned}
\int_M \Delta_b|\nabla_b\varphi|^2 d\mu &= \frac{1}{2}C_n\int_\Sigma \left(|\nabla_b\varphi|^2\right)_{e_{2n}} d\Sigma_p \\
&= \frac{1}{2}C_n\int_\Sigma \sum_{\beta\neq n}\left(\varphi_{e_\beta}\varphi_{e_\beta e_{2n}} + \varphi_{e_{n+\beta}}\varphi_{e_{n+\beta}e_{2n}}\right) d\Sigma_p \\
&\quad + \frac{1}{2}C_n\int_\Sigma (\varphi_{e_n}\varphi_{e_n e_{2n}} + \varphi_{e_{2n}}\varphi_{e_{2n}e_{2n}}) d\Sigma_p.
\end{aligned}
$$

Substituting the commutation relations

$$
\begin{aligned}
\varphi_{e_\beta e_{n+\gamma}} &= \varphi_{e_{n+\gamma}e_\beta},\ \varphi_{e_{n+\beta}e_{n+\gamma}} = \varphi_{e_{n+\gamma}e_{n+\beta}} \text{ for all } \beta\neq\gamma, \\
\varphi_{e_n e_{2n}} &= \varphi_{e_{2n}e_n} + 2\varphi_0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{\beta\neq n} 2(\varphi_{\beta\overline{\beta}} + \varphi_{\overline{\beta}\beta}) + \varphi_{e_n e_n} \\
= & \sum_{j=1}^{2n-1}\varphi_{e_j e_j} = 2\Delta_b^t\varphi + H_{p.h}\varphi_{e_{2n}}\varphi_{e_{2n}e_{2n}} = 2\Delta_b\varphi - \sum_{j=1}^{2n-1}\varphi_{e_j e_j}
\end{aligned}
\tag{3.16}
$$

into the above equation, also integrating by parts from (3.8) and (3.9) yields

$$\begin{aligned}
&\int_M \Delta_b|\nabla_b\varphi|^2 d\mu \\
=&\frac{1}{2}C_n\int_\Sigma\sum_{\beta\neq n}\left(\varphi_{e_\beta}\varphi_{e_{2n}e_\beta}+\varphi_{e_{n+\beta}}\varphi_{e_{2n}e_{n+\beta}}\right)d\Sigma_p \\
&+\frac{1}{2}C_n\int_\Sigma\varphi_{e_n}\left(\varphi_{e_{2n}e_n}+2\varphi_0\right)d\Sigma_p+\frac{1}{2}C_n\int_\Sigma\varphi_{e_{2n}}\varphi_{e_{2n}e_{2n}}d\Sigma_p \\
=&\frac{1}{2}C_n\int_\Sigma\left[\sum_{\beta\neq n}2\left(\varphi_{\overline{\beta}}\varphi_{e_{2n}Z_\beta}+\varphi_\beta\varphi_{e_{2n}Z_{\overline{\beta}}}\right)+\varphi_{e_n}\varphi_{e_{2n}e_n}\right]d\Sigma_p \\
&+C_n\int_\Sigma\varphi_{e_n}\varphi_0 d\Sigma_p+\frac{1}{2}C_n\int_\Sigma\varphi_{e_{2n}}\varphi_{e_{2n}e_{2n}}d\Sigma_p \\
=&-\frac{1}{2}C_n\int_\Sigma\varphi_{e_{2n}}\left[\sum_{\beta\neq n}2\left(\varphi_{\beta\overline{\beta}}+\varphi_{\overline{\beta}\beta}\right)+\varphi_{e_ne_n}\right]d\Sigma_p \\
&-\frac{1}{2}C_n\int_\Sigma\left[\varphi_{e_n}(\nabla_{e_n}e_{2n})\varphi+\varphi_{e_{2n}}(\nabla_{e_n}e_n)\varphi\right]d\Sigma_p \\
&+C_n\int_\Sigma\varphi_{e_n}[\varphi_0-\alpha\varphi_{e_{2n}}]d\Sigma_p+\frac{1}{2}C_n\int_\Sigma\varphi_{e_{2n}}\varphi_{e_{2n}e_{2n}}d\Sigma_p \\
&+C_n\int_\Sigma\sum_{\beta\neq n}\left[\theta_{\overline{n}}{}^{\overline{\beta}}(Z_{\overline{\beta}})\varphi_n-\frac{1}{2}\theta_{\overline{\beta}}{}^{\overline{n}}(e_n)\varphi_\beta\right]\varphi_{e_{2n}}d\Sigma_p \\
&+C_n\int_\Sigma\sum_{\beta\neq n}\left[\theta_n{}^\beta(Z_\beta)\varphi_{\overline{n}}-\frac{1}{2}\theta_\beta{}^n(e_n)\varphi_{\overline{\beta}}\right]\varphi_{e_{2n}}d\Sigma_p \\
&-C_n\int_\Sigma\sum_{\beta\neq n}\left[\varphi_\beta(\nabla_{Z_{\overline{\beta}}}e_{2n})\varphi+\varphi_{\overline{\beta}}(\nabla_{Z_\beta}e_{2n})\varphi\right]d\Sigma_p \\
=&C_n\int_\Sigma\varphi_{e_{2n}}\left(\Delta_b\varphi-2\Delta_b^t\varphi\right)d\Sigma_p-\frac{1}{2}C_n\int_\Sigma H_{p.h}\varphi_{e_{2n}}^2 d\Sigma_p \\
&-C_n\int_\Sigma\varphi_0\varphi_{e_n}d\Sigma_p+\frac{1}{2}C_n\int_\Sigma\sum_{j=1}^{2n-1}\left\langle\nabla_{e_j}e_n,e_j\right\rangle\varphi_{e_n}\varphi_{e_{2n}}d\Sigma_p \\
&+C_n\int_\Sigma\alpha\varphi_{e_n}\varphi_{e_{2n}}d\Sigma_p-\frac{1}{2}C_n\int_\Sigma\sum_{j,k=1}^{2n-1}\left\langle\nabla_{e_j}e_{2n},e_k\right\rangle\varphi_{e_j}\varphi_{e_k}d\Sigma_p.
\end{aligned}\tag{3.17}$$

Here we use

$$\begin{aligned}
&2\sum_{\beta\neq n}\left[\theta_{\overline{n}}{}^{\overline{\beta}}(Z_{\overline{\beta}})\varphi_n-\frac{1}{2}\theta_{\overline{\beta}}{}^{\overline{n}}(e_n)\varphi_\beta+\theta_n{}^\beta(Z_\beta)\varphi_{\overline{n}}-\frac{1}{2}\theta_\beta{}^n(e_n)\varphi_{\overline{\beta}}\right] \\
=&\sum_{j=1}^{2n-1}\left\langle\nabla_{e_j}e_n,e_j\right\rangle\varphi_{e_n}+(\nabla_{e_n}e_n)\varphi+H_{p.h}\varphi_{e_{2n}}
\end{aligned}$$

and

$$\sum_{\beta\neq n}2\left[\varphi_\beta(\nabla_{Z_{\overline{\beta}}}e_{2n})\varphi+\varphi_{\overline{\beta}}(\nabla_{Z_\beta}e_{2n})\varphi\right]+\varphi_{e_n}(\nabla_{e_n}e_{2n})\varphi=\sum_{j,k=1}^{2n-1}\left\langle\nabla_{e_j}e_{2n},e_k\right\rangle\varphi_{e_j}\varphi_{e_k},$$

the fact that (3.16) holds only on $\Sigma\backslash S_\Sigma$. However, $d\Sigma_p$ can be continuously extends over the singular set S_Σ and vanishes on S_Σ. Finally, by combining the equations (3.15) and (3.17), we can then obtain (1.3). This completes the proof of the theorem. □

4. The CR First Non-Zero Dirichlet Eigenvalue Estimate. In this section, we derive the first Dirichlet eigenvalue estimate in a compact pseudohermitian $(2n+1)$-manifold (M,J,θ) with boundary Σ.

LEMMA 4.1. *Let (M,J,θ) be a compact pseudohermitian $(2n+1)$-manifold with the smooth boundary Σ of pseudohermitian mean curvature $H_{p.h}$ for $n\geq 2$. For the first eigenfunction φ of Dirichlet eigenvalue problem (1.9), we have*

$$\frac{n-1}{8n}\int_M \varphi P_0\varphi d\mu = \int_M \sum_{\beta,\gamma}|\varphi_{\beta\overline{\gamma}} - \frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}|^2 d\mu + \frac{1}{16}C_n\int_\Sigma (H_{p.h}+\tilde{\omega}_n{}^n(e_n))\varphi_{e_{2n}}^2 d\Sigma_p$$

which implies

$$\int_M \varphi P_0\varphi d\mu \geq 0 \tag{4.1}$$

if $H_{p.h}+\tilde{\omega}_n{}^n(e_n)$ is nonnegative.

Proof. Since $\varphi=0$ on Σ and e_j is tangent along Σ for $1\leq j\leq 2n-1$, then $\varphi_{e_j}=0$ for $1\leq j\leq 2n-1$ and $\Delta_b^t\varphi=\frac{1}{2}\sum_{j=1}^{2n-1}[(e_j)^2\varphi-(\nabla_{e_j}e_j)^t\varphi]=0$ on Σ. Furthermore, since $\Delta_b\varphi=\lambda_1\varphi$ on M and $\varphi=0$ on Σ, then $\Delta_b\varphi=0$ on Σ. It follows from (3.16) that

$$\begin{aligned}
&4iC_n\int_\Sigma(\varphi^{\overline{\beta}}B_{n\overline{\beta}}\varphi-\varphi^\beta B_{\overline{n}\beta}\varphi)d\Sigma_p\\
&=C_n\int_\Sigma\sum_{\beta\neq n}\left[\varphi_{e_{n+\beta}}(\varphi_{e_\beta e_n}+\varphi_{e_{n+\beta}e_{2n}})+\varphi_{e_\beta}(\varphi_{e_\beta e_{2n}}-\varphi_{e_{n+\beta}e_n})\right]d\Sigma_p\\
&\quad+C_n\int_\Sigma\varphi_{e_{2n}}\left[(\varphi_{e_ne_n}+\varphi_{e_{2n}e_{2n}})-\frac{2}{n}\Delta_b\varphi\right]d\Sigma_p\\
&=C_n\int_\Sigma\varphi_{e_{2n}}\left\{\left[(e_n)^2-(\nabla_{e_n}{}^{e_n})\right]\varphi+(2\Delta_b\varphi-2\Delta_b^t\varphi-H_{p.h}\varphi_{e_{2n}})\right\}d\Sigma_p\\
&=-C_n\int_\Sigma(H_{p.h}+\tilde{\omega}_n{}^n(e_n))\varphi_{e_{2n}}^2 d\Sigma_p.
\end{aligned}$$

Substituting the above equation into (3.14), we get

$$\begin{aligned}
&\frac{n-1}{8n}\int_M\varphi P_0\varphi d\mu\\
&=\int_M\sum_{\beta,\gamma}|\varphi_{\beta\overline{\gamma}}-\frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}|^2d\mu+\frac{n-1}{4n}iC_n\int_\Sigma\varphi\left(P_n\varphi-P_{\overline{n}}\varphi\right)d\Sigma_p\\
&\quad-\frac{1}{4}iC_n\int_\Sigma\left(\varphi^{\overline{\beta}}B_{n\overline{\beta}}\varphi-\varphi^\beta B_{\overline{n}\beta}\varphi\right)d\Sigma_p\\
&=\int_M\sum_{\beta,\gamma}|\varphi_{\beta\overline{\gamma}}-\frac{1}{n}\varphi_\sigma{}^\sigma h_{\beta\overline{\gamma}}|^2d\mu+\frac{1}{16}C_n\int_\Sigma(H_{p.h}+\tilde{\omega}_n{}^n(e_n))\varphi_{e_{2n}}^2d\Sigma_p.
\end{aligned}$$

□

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. It follows the CR Reilly formula (1.3) that

$$\begin{aligned} &\frac{n+1}{n}\int_M (\Delta_b\varphi)^2 d\mu \\ &\geq \frac{n+2}{4n}\int_M \varphi P_0\varphi d\mu + \int_M [2Ric-(n+1)Tor]((\nabla_b\varphi)_{\mathbb{C}},(\nabla_b\varphi)_{\mathbb{C}})d\mu. \end{aligned} \tag{4.2}$$

Since

$$\varphi = 0 \text{ and } \Delta_b\varphi = 0 \text{ on } \Sigma,$$

(1.7) and (4.2) imply

$$\frac{n+1}{n}\int_M (\Delta_b\varphi)^2 d\mu \geq \frac{n+2}{4n}\mu_{_D}^1\int_M \varphi^2 d\mu + \int_M [2Ric-(n+1)Tor]((\nabla_b\varphi)_{\mathbb{C}},(\nabla_b\varphi)_{\mathbb{C}})d\mu.$$

Moreover, by using

$$[2Ric-(n+1)Tor]((\nabla_b\varphi)_{\mathbb{C}},(\nabla_b\varphi)_{\mathbb{C}}) \geq k|\nabla_b\varphi|^2$$

and

$$\int_M |\nabla_b\varphi|^2 d\mu = \lambda_1\int_M \varphi^2 d\mu,$$

we obtain

$$\frac{n+1}{n}\lambda_1^2\int_M \varphi^2 d\mu \geq \left(k\lambda_1 + \frac{n+2}{4n}\mu_{_D}^1\right)\int_M \varphi^2 d\mu.$$

Hence

$$\frac{n+1}{n}\lambda_1^2 - k\lambda_1 - \frac{n+2}{4n}\mu_{_D}^1 \geq 0$$

and thus

$$\lambda_1 \geq \frac{nk+\sqrt{n^2k^2+(n+1)(n+2)\mu_{_D}^1}}{2(n+1)}.$$

(i) In case of $n=1$, we have

$$\lambda_1 \geq \frac{k+\sqrt{k^2+6\mu_{_D}^1}}{4},$$

for $\mu_{_D}^1 \geq -\frac{k^2}{6}$. In addition if P_0 is nonnegative, we have

$$\lambda_1 \geq \frac{k}{2}.$$

(ii) In case of $n \geq 2$, it follows from (4.1) and (4.2) that

$$\frac{n+1}{n}\lambda_1^2 - k\lambda_1 \geq 0$$

and then

$$\lambda_1 \geq \frac{nk}{n+1}.$$

∎

5. The First Eigenvalue Estimate of Embedded P-minimal hypersurfaces. In this section, we study a CR analogue of Yau conjecture [Y] on the first eigenvalue estimate of embedded p-minimal hypersurfaces.

Proof of Theorem 1.3. Since M has vanishing torsion and positive pseudohermitian Ricci curvature, it follows from [CC1] that M has positive Ricci curvature with respect to the Webster metric. Hence its first homology group $H^1(M,\mathbb{R})$ is trivial. By an exact sequence argument, we conclude that Σ divides M into two connected components M_1 and M_2 with $\partial M_1 = \Sigma = \partial M_2$. Let us denote D to be one of two components to be chosen later. If u is the first nonconstant eigenfunction on Σ, satisfying

$$L_\alpha u = -\lambda_1 u.$$

We first let φ be the solution of

$$\Delta_b \varphi = 0 \text{ on } D$$

with the boundary condition

$$\varphi = u \text{ on } \Sigma.$$

If D is a compact pseudohermitian $(2n+1)$-manifold with the smooth boundary Σ, then P_0 is self-adjoint on the space of all smooth functions with $\Delta_b\varphi = 0$ and $(\Delta_b\varphi)_{e_{2n}} = 0$ on Σ. In fact, it suffices to check that

$$\begin{aligned}\int_D g\Delta_b^2 f d\mu &= -\int_D \langle \nabla_b g, \nabla_b \Delta_b f\rangle d\mu + C_n\int_\Sigma g(\Delta_b f)_{e_{2n}} d\Sigma_p \\ &= \int_D \Delta_b f\Delta_b g d\mu - C_n\int_\Sigma g_{e_{2n}}\Delta_b f d\Sigma_p + C_n\int_\Sigma g(\Delta_b f)_{e_{2n}} d\Sigma_p \\ &= \int_D \Delta_b f\Delta_b g d\mu = \int_D f\Delta_b^2 g d\mu\end{aligned} \tag{5.1}$$

and for $\alpha = 0$

$$\begin{aligned}\int_D g f_{00} d\mu &= -\int_D g_0 f_0 d\mu + 2C_n\int_\Sigma \alpha g f_0 d\Sigma_p \\ &= \int_D f g_{00} d\mu - 2C_n\int_\Sigma \alpha f g_0 d\Sigma_p + 2C_n\int_\Sigma \alpha g f_0 d\Sigma_p \\ &= \int_D f g_{00} d\mu.\end{aligned} \tag{5.2}$$

It follows that if the pseudohermitian torsion is vanishing

$$\int_D \varphi P_0 \varphi d\mu \geq 0. \tag{5.3}$$

By applying the CR Reilly formula (1.1), we have

$$\begin{aligned} 0 \geq k \int_D |\nabla_b \varphi|^2 d\mu + \frac{3}{4} C_n \int_\Sigma \varphi_0 \varphi_{e_n} d\Sigma_p - \frac{n+2}{2n} i C_n \int_\Sigma \varphi \left(P_n \varphi - P_{\overline{n}} \varphi \right) d\Sigma_p \\ + \frac{i}{2} C_n \int_\Sigma (\varphi^{\overline{\beta}} B_{n\overline{\beta}} \varphi - \varphi^\beta B_{\overline{n}\beta} \varphi) d\Sigma_p + \frac{3}{4n} C_n \int_\Sigma \varphi_{e_{2n}} \Delta_b \varphi d\Sigma_p \\ + C_n \int_\Sigma \varphi_{e_{2n}} \Delta_b^t \varphi d\Sigma_p - \frac{1}{4} C_n \int_\Sigma \sum_{j=1}^{2n-1} \left\langle \nabla_{e_j} e_n, e_j \right\rangle \varphi_{e_n} \varphi_{e_{2n}} d\Sigma_p \\ - \frac{1}{2} C_n \int_\Sigma \alpha \varphi_{e_n} \varphi_{e_{2n}} d\Sigma_p + \frac{1}{4} C_n \int_\Sigma \sum_{j,k=1}^{2n-1} \left\langle \nabla_{e_j} e_{2n}, e_k \right\rangle \varphi_{e_j} \varphi_{e_k} d\Sigma_p. \end{aligned} \tag{5.4}$$

Now we are going to estimate all terms in RHS of (5.4):

(i) By the CR divergence theorem and $\Delta_b \varphi^2 = 2\varphi \Delta_b \varphi + 2|\nabla_b \varphi|^2 = 2|\nabla_b \varphi|^2$, we have

$$\begin{aligned} & C_n \int_\Sigma \varphi_{e_{2n}} \Delta_b^t \varphi d\Sigma_p \\ = & - C_n \int_\Sigma \alpha \varphi_{e_n} \varphi_{e_{2n}} d\Sigma_p - \lambda_1 C_n \int_\Sigma \varphi \varphi_{e_{2n}} d\Sigma_p \\ = & - C_n \int_\Sigma \alpha \varphi_{e_n} \varphi_{e_{2n}} d\Sigma_p - \frac{1}{2} \lambda_1 C_n \int_\Sigma (\varphi^2)_{e_{2n}} d\Sigma_p \\ = & - C_n \int_\Sigma \alpha \varphi_{e_n} \varphi_{e_{2n}} d\Sigma_p - \lambda_1 \int_D \Delta_b (\varphi^2) d\mu \\ = & - C_n \int_\Sigma \alpha \varphi_{e_n} \varphi_{e_{2n}} d\Sigma_p - 2\lambda_1 \int_D |\nabla_b \varphi|^2 d\mu. \end{aligned} \tag{5.5}$$

(ii) By the CR Green's theorem

$$C_n \int_\Sigma \varphi_{e_{2n}} \Delta_b \varphi d\Sigma_p = \int_D (\Delta_b \varphi)^2 d\mu + \int_D \left\langle \nabla_b \Delta_b \varphi, \nabla_b \varphi \right\rangle d\mu = 0. \tag{5.6}$$

(iii) The computation for $\alpha = 0$, the p-area element $d\Sigma_p$ is the area form $d\Sigma$ on Σ,

$$\begin{aligned} & 2iC_n \int_\Sigma (\varphi^{\overline{\beta}} B_{n\overline{\beta}} \varphi - \varphi^\beta B_{\overline{n}\beta} \varphi) d\Sigma_p \\ = & \frac{1}{2} C_n \int_\Sigma \varphi_{e_{2n}} \left[(\varphi_{e_n e_n} + \varphi_{e_{2n} e_{2n}}) - \frac{2}{n} \Delta_b \varphi - \sum_{j \neq n, 2n} \varphi_{e_j e_j} \right] d\Sigma_p \\ & + \frac{1}{2} C_n \int_\Sigma \sum_{\beta \neq n} (\varphi_{e_\beta e_{n+\beta}} - \varphi_{e_{n+\beta} e_\beta}) \varphi_{e_n} d\Sigma_p \\ = & C_n \int_\Sigma \varphi_{e_{2n}} \left[\frac{n-1}{n} \Delta_b \varphi - \sum_{j \neq n, 2n} \varphi_{e_j e_j} \right] d\Sigma_p + (n-1) C_n \int_\Sigma \varphi_0 \varphi_{e_n} d\Sigma_p. \end{aligned} \tag{5.7}$$

(iv) By straightforward computation, since $A_{\beta\gamma} = 0$,

$$i \left(P_n \varphi - P_{\overline{n}} \varphi \right) = i \left(\varphi_{\overline{\beta}}{}^{\overline{\beta}}{}_n - \varphi_\beta{}^\beta{}_{\overline{n}} \right) = \frac{1}{2} [n \varphi_{0 e_n} + (\Delta_b \varphi)_{e_{2n}}].$$

From (3.9), (5.7) and $\int_\Sigma \varphi(\Delta_b\varphi)_{e_{2n}} d\Sigma_p = 0$ that

$$\begin{aligned} -2iC_n\int_\Sigma \varphi(P_n\varphi - P_{\overline{n}}\varphi)d\Sigma_p &= -C_n\int_\Sigma \varphi[n\varphi_{0e_n} + (\Delta_b\varphi)_{e_{2n}}]d\Sigma_p \\ &= nC_n\int_\Sigma \varphi_0\varphi_{e_n}d\Sigma_p + 2nC_n\int_\Sigma \alpha\varphi_0\varphi d\Sigma_p. \end{aligned} \tag{5.8}$$

By combining (5.4), (5.5), (5.6), (5.7) and (5.8) for $\alpha = 0$,

$$\begin{aligned} 0 \geq (k - 2\lambda_1)\int_D |\nabla_b\varphi|^2 d\mu &- \frac{1}{4}C_n\int_\Sigma \sum_{j=1}^{2n-1}\left\langle \nabla_{e_j}e_{2n}, e_j\right\rangle \varphi_{e_n}\varphi_{e_{2n}}d\Sigma \\ &+ \frac{n}{2}C_n\int_\Sigma \varphi_0\varphi_{e_n}d\Sigma - \frac{1}{4}C_n\int_\Sigma \sum_{j\neq n,2n}\varphi_{e_je_j}\varphi_{e_{2n}}d\Sigma \\ &+ \frac{1}{4}C_n\int_\Sigma \sum_{j,k=1}^{2n-1}\left\langle \nabla_{e_j}e_{2n}, e_k\right\rangle \varphi_{e_j}\varphi_{e_k}d\Sigma. \end{aligned} \tag{5.9}$$

Next we observe that T is always tangent to Σ due to $\alpha = 0$. Then $\int_\Sigma \varphi_0\varphi_{e_n}d\Sigma$ is independent of the extended function φ. If we choose a different component of $M\backslash\Sigma$ to perform this computation, $u_{e_n}u_0$, $\sum_{j\neq n,2n} u_{e_je_j}u_{e_{2n}}$, $\sum_{j=1}^{2n-1}\left\langle \nabla_{e_j}e_n, e_j\right\rangle u_{e_n}u_{e_{2n}}$ and $\sum_{j,k=1}^{2n-1}\left\langle \nabla_{e_j}e_{2n}, e_k\right\rangle u_{e_j}u_{e_k}$ will differ by a sign, hence we may choose a component, say M_1, so that

$$\begin{aligned} &2n\int_\Sigma \varphi_0\varphi_{e_n}d\Sigma - \int_\Sigma \sum_{j=1}^{2n-1}\left\langle \nabla_{e_j}e_{2n}, e_j\right\rangle \varphi_{e_n}\varphi_{e_{2n}}d\Sigma \\ &- \int_\Sigma \sum_{j\neq n,2n}\varphi_{e_je_j}\varphi_{e_{2n}}d\Sigma + \int_\Sigma \sum_{j,k=1}^{2n-1}\left\langle \nabla_{e_j}e_{2n}, e_k\right\rangle \varphi_{e_j}\varphi_{e_k}d\Sigma \geq 0. \end{aligned} \tag{5.10}$$

By combining (5.9) and (5.10) that we have

$$0 \geq (k - 2\lambda_1)\int_D |\nabla_b\varphi|^2 d\mu$$

with $D = M_1$. This implies

$$0 \geq k - 2\lambda_1$$

and thus

$$\lambda_1 \geq \frac{k}{2}$$

because φ has boundary value u which is nonconstant.

Now if the equality holds for $n = 1$, then

$$W = k.$$

Since $A_{11} = 0$,

$$Q_{11} = 0$$

and then (M, J, θ) is a closed spherical pseudohermitian 3-manifold. On the other hand, it follows from ([CHMY]) that any embedded p-minimal surface in a closed spherical pseudohermitian 3-manifold must have genus less than two. In addition, if M is simply connected, then (M, J, θ) is the standard pseudohermitian 3-sphere. This completes the proof. $\square$

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