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MANDELBROT CASCADES ON RANDOM WEIGHTED TREES AND NONLINEAR SMOOTHING TRANSFORMS*

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Abstract. We consider complex Mandelbrot multiplicative cascades on a random weighted tree. Under suitable assumptions, this yields a dynamics T on laws invariant by random weighted means (the so called fixed points of smoothing transformations) and which have a finite moment of order 2. We can exhibit two main behaviors: If the weights are conservative, i.e., sum up to 1 almost surely, we find a domain for the initial law μ such that a non-standard (functional) central limit theorem is valid for the orbit $(\mathsf{T}^n\mu)_{n\geq 0}$. The limit process possesses a structure combining multiplicative and additive cascade (this completes in a non trivial way our previous result in the case of nonnegative Mandelbrot cascades on a regular tree). If the weights are non conservative, we find a domain for the initial law μ over which $(\mathsf{T}^n\mu)_{n\geq 0}$ converges in law to a non trivial random variable whose law turns out to be a fixed point of a quadratic smoothing transformation, which naturally extends the usual notion of (linear) smoothing transformation; moreover, this limit law can be built as the limit of a nonnegative martingale. Also, the dynamics can be modified to build fixed points of higher degree smoothing transformations.

Key words. Multiplicative cascades, Mandelbrot martingales, smoothing transformations, dynamical systems, central limit theorem, Gaussian processes, Random fractals, Wasserstein distance, Galton-Watson tree.

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1. Introduction. We study dynamics on a class of probability distributions on $\mathbb{C}$, namely the fixed points of so-called smoothing transforms (in the framework of branching processes). This work is motivated by the special situation studied in [6].

Let N be a nonnegative integer valued random variable and $a = (a_j)_{j\geq 1}$ a sequence of nonnegative random variables such that

$$\mathbb{E}\sum_{j\geq 1} a_j = 1.$$

We assume that a and N are independent.

Let $\mathscr{L}W$ stand for the law of the random variable W. Consider the following map

$$\mathsf{S}_N \ : \ \mu \longmapsto \mathscr{L}\sum_{j\geq 1} a_j \prod_{1\leq k\leq N} W_k(j), \tag{1}$$

where μ is a probability measure on $\mathbb{C}$, all variables $W_k(j)$ are distributed according to μ, independent, and independent of a and N.

When N is a constant c, i.e., $\mathbb{P}(N = c) = 1$, denoted $N \equiv c$, we write S_c instead of S_N. In these conditions, S_1 is the well known linear smoothing transform associated with a [12, 17].

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Under suitable assumptions on a and N, we prove that this mapping S_N has a unique fixed point in a certain domain defined by inequalities on second moments. In other terms this fixed point is a solution to the equation

$$\mu = \mathscr{L} \sum_{j\geq 1} a_j \prod_{1\leq k\leq N} W_k(j), \tag{2}$$

where the unknown is μ (a and the other variables in the right hand side being independent and $\mathscr{L} W_k(j) = \mu$).

In the usual case, $N \equiv 1$, this equation is linear and its solutions are rather well understood [12, 17, 1, 2, 3, 14]. Otherwise, it is not so, and in case $N \equiv 2$, it reduces to a quadratic equation (see equation (26)). If $\sum a_j = 1$ with probability 1, the so called conservative case, this fixed point is the Dirac mass at 1, whereas in the non conservative case it is non trivial.

One way to construct these fixed points when $N \not\equiv 1$ is to iterate Mandelbrot cascades in a random scenery. Another one is to build them as limit of new kind of multiplicative process. The conservative case gives rise to a non standard central limit theorem, with a functional counterpart in the quadratic case. It is worth mentioning that in [6] only the quadratic case is considered, the a_j being equal and conservative, so that no non trivial solution to (2) appears; moreover, the weights W_k are non negative. Also, the central limit theorems established in the present paper (which are non standard in the sense that the limit laws are not necessarily Gaussian) cannot be guessed after the form they take in the case studied in [6].

Section 2 contains, for the reader's convenience, some known facts on multiplicative cascades; it also sets notation, defines what we mean by random scenery, and shows a few computations. Sections 3 to 6 deal with the case when N is the constant 2. Indeed we prefer to present calculations and ideas in this particular case. Section 3 defines the dynamics on Mandelbrot cascades in a random scenery and studies its fixed points, from the two viewpoints described above. The dynamical system acts on probability distributions with finite second moment. Section 4 studies the behavior of the third moment along the iteration. This prepares the proof of the central limit theorem associated with the attracting fixed point in the conservative case. Section 5 is dedicated to the nonnegative case, and Section 6 to the complex case. We treat the case of a general distribution for N in Section 7. In Section 8, we obtain a functional central limit theorem in case $N \equiv 2$. Section 9 contains some comments and questions.

2. Mandelbrot cascades.

2.1. Alphabets and trees. Let $\mathscr{A}$ be a countable set which we call alphabet.

Consider the tree $\mathscr{A}^* = \bigcup_{n\geq 0} \mathscr{A}^n$ whose root is the only element ϵ of $\mathscr{A}^0$. Endowed with concatenation, denoted by juxtaposition except in Section 8 where it will be occasionally denoted by a dot, $\mathscr{A}^*$ is also a monoid whose identity is ϵ. Its elements are written as words: If $w = x_1 x_2 \cdots x_n$, we set $|w| = n$, $w_j = x_j$, and $w|_k = x_1 \cdots x_k$ (with $w|_0 = \epsilon$).

We shall use the alphabet $\mathbb{N} = \{1, 2, \cdots\}$. Then the corresponding tree will be denoted by $\mathscr{T}$ and the set $\mathbb{N}^n$ will sometimes be denoted by $\mathscr{T}_n$.

2.2. Standard Mandelbrot cascades. A sequence of random variables $V = (V_j)_{j\in\mathscr{A}}$ such that $\sum_{j\in\mathscr{A}} \mathbb{E}\,|V_j| < \infty$ and $\sum_{j\in\mathscr{A}} \mathbb{E}\, V_j = 1$ defines a martingale $(Y_n)_{n\in\mathbb{N}}$ in the following way: consider a sequence $\big(V(w)\big)_{w\in\mathscr{A}^*}$ of independent random variables all equidistributed with V and set

$$Y_n = \sum_{w\in\mathscr{A}^n} \prod_{j=0}^{n-1} V_{w_{j+1}}(w|_j). \tag{3}$$

It is clear that the following recursion relation holds:

$$Y_{n+1} = \sum_{j\in\mathscr{A}} V_j(\epsilon) Y_n(j), \tag{4}$$

where all the variables $Y_n(j)$ are equidistributed with Y_n, independent, and independent of $(V_j(\epsilon))_{j\in\mathscr{A}}$ (indeed $Y_n(j)$ is constructed as Y_n on the subtree whose root is the word j).

If moments of order 2 exist, we have

$$\mathbb{E}\,|Y_{n+1}|^2 = \left(\sum_{j\in\mathscr{A}} \mathbb{E}\,|V_j|^2\right) \mathbb{E}\,|Y_n^2| + \mathbb{E}\sum_{j\neq k} V_j \overline{V}_k. \tag{5}$$

Therefore, if $\displaystyle\sum_{j\in\mathscr{A}} \mathbb{E}\,|V_j|^2 < 1$ this martingale converges to a limit Y.

Recall that, when the components of V are nonnegative variable, there is a necessary and sufficient condition insuring that this martingale converges in L^p. Namely, for $p > 1$ we have

$$\mathbb{E}\,Y^p < \infty \Longleftrightarrow \begin{cases} \mathbb{E}\left(\sum_{j\geq 1} V(j)\right)^p < \infty \\ \text{and} \\ \mathbb{E}\sum_{j\geq 1} V(j)^p < 1. \end{cases} \tag{6}$$

This equivalence due to Liu [18] is the generalization of the condition found in [15] for the finiteness of moments of order larger than 1 of non-degenerate Mandelbrot cascades on regular trees (in fact Liu assumes that there exists a random integer J such that $V_j = 0$ for $j \geq J+1$ almost surely, but this assumption can be removed).

2.3. Cascades in a random scenery. We are given once for all a sequence $a = (a_n)_{n\geq 1}$ of nonnegative random variables such that $\mathbb{E}\displaystyle\sum_{j\geq 1} a_j = 1$ and $\mathbb{E}\Big(\displaystyle\sum_{j\geq 1} a_j\Big)^2 < \infty$. We exclude that $a_j \in \{0,1\}$ for all $j \geq 1$ almost surely.

We set

$$b = \frac{\mathbb{E}\left(\sum a_j\right)^2}{\mathbb{E}\sum a_j^2} \quad \text{and} \quad q = \frac{1}{\mathbb{E}\left(\sum a_j\right)^2}.$$

Observe that $b \geq 1$, $0 < q \leq 1$, and that $q = 1$ if and only if $\sum_{j\geq 1} a_j = 1$ with probability 1. As already said, we refer to this last case as the *conservative case*.

Now, we define what we mean by Mandelbrot cascades on the tree $\mathscr{T} = \mathbb{N}^*$ endowed with a random scenery defined by a. More precisely, what we call scenery is a collection $\big(a(w)\big)_{w\in\mathscr{T}}$ of independent random variables equidistributed with a.

Also we are given a nonnegative integer valued random variable N whose probability generating function is

$$f_N(t) = \sum_{n\in\mathbb{N}} t^n \, \mathbb{P}(N = n).$$

If W is an integrable complex random variable of expectation 1, we define $W_{(N)}$ to be

$$W_{(N)} = \prod_{1\le j\le N} W_j,$$

where the variables W_j are independent and equidistributed with W.

We consider a sequence $\big(W_{(N)}(w)\big)_{w\in\mathscr{T}}$ of independent variables equidistributed with $W_{(N)}$ and independent of $\big(a(w)\big)_{w\in\mathscr{T}}$, and define

$$Y_n = \sum_{w=j_1j_2\ldots j_n\in\mathscr{T}_n} \prod_{k=0}^{n-1} a_{j_{k+1}}(w|_k) W_{(N)}(w|_{k+1}).$$

This is a Mandelbrot martingale as defined in the preceding section with $V = \big(a_j W_{(N)}(j)\big)_{j\in\mathbb{N}}$, the $W_{(N)}(j)$ being independent and independent of a. Accordingly, we have

$$Y_{n+1} = \sum_{j\ge 1} a_j W_{(N)}(j) Y_n(j), \tag{7}$$

where the variables $Y_{n-1}(j)$ are defined as Y_{n-1} but starting from j as a root, and all the variables $W_{(N)}(j)$ and $Y_n(j)$ in the sum are independent and independent of a. Notice that $Y_n(j)$ has the same distribution as Y_n for all $j \ge 1$. From this relation we obtain the following equality.

$$\mathbb{E}\,|Y_{n+1}|^2 = f_N(\mathbb{E}\,|W|^2)\,\mathbb{E}\,|Y_n|^2\,\mathbb{E}\sum_{j\ge 1} a_j^2 + \mathbb{E}\sum_{i\ne j} a_i a_j. \tag{8}$$

This means that this martingale is bounded in L^2 if and only if

$$f_N(\mathbb{E}\,|W|^2) < bq. \tag{9}$$

If it is so, which we assume from now on, let Y stand for the limit of this martingale. It results from (7) that Y fulfills the following equation.

$$Y = \sum_{j\ge 1} a_j W_{(N)}(j) Y(j), \tag{10}$$

where the variables $Y(i)$ are equidistributed with Y, and all the variables $W_{(N)}(j)$ and $Y(j)$ in the sum are independent and independent of a. Thus Y is an integrable fixed point of the following smoothing transformation $\mathsf{U}_{W_{(N)}}$ which to a given probability distribution μ on $\mathbb{C}$ associates

$$\mathsf{U}_{W_{(N)}}(\mu) = \mathscr{L}\sum_{j\ge 1} a_j W_{(N)}(j) Z(j),$$

where the $Z(j)$ are independent, distributed according to μ, and independent of a and the $W_{(N)}(j)$, themselves i.i.d. ($\mathscr{L}X$ stands for the probability distribution of X). The fixed points of such transformations have been studied for a long time, especially in the context of model for turbulence and branching processes, and much is known about their structure [19, 15, 7, 12, 13, 8, 17, 18, 9, 1, 2, 3].

We denote by T_N the operator which leads from the distribution of W to the one of Y. From time to time we shall make the following abuse of notation: $Y = \mathsf{T}_N W$.

A fixed point of T_N is a probability distribution μ on $\mathbb{C}$ satisfying the relation

$$\mu = \mathscr{L} \sum_{j \in \mathbb{N}} a_j \prod_{0 \le k \le N} W_k(j),$$

where the variables $W_k(j)$ are independent, distributed according to μ, and independent of a. In other terms, μ is also a fixed point of S_{N+1}. This is why we wish to iterate T_N and study its dynamics.

Also, Equation (8) can be rewritten as

$$\mathbb{E}\,|Y|^2 = \frac{b-1}{bq - f\big(\mathbb{E}\,|W|^2\big)}. \tag{11}$$

As said in the introduction, we prefer to expose the ideas and calculations in the case where $N \equiv 1$. The general case will be treated in Section 7. So except in this section $N \equiv 1$.

We adopt simplified notation: T instead of T_1. In the above formulae, when $N \equiv 1$, each occurrence of $W_{(N)}$ should be replaced by W. Also the function f is the identity, so that Equations (8) and (11) can be rewritten as

$$bq\,\mathbb{E}\,|Y_{n+1}|^2 = \mathbb{E}\,|W|^2\,\mathbb{E}\,|Y_n|^2 + b - 1, \tag{12}$$

and

$$\mathbb{E}\,|Y|^2 = \frac{b-1}{bq - \mathbb{E}\,|W|^2}, \tag{13}$$

and in this case this martingale is bounded in L^2 if and only if

$$\mathbb{E}\,|W|^2 < bq. \tag{14}$$

In this context, Liu's condition (6) is to be so stated: when W is a nonnegative variable, for $p > 1$ we have

$$\mathbb{E}\,Y^p < \infty \Longleftrightarrow \begin{cases} \mathbb{E}\left(\sum_{j \ge 1} a_j W(j)\right)^p < \infty \\ \text{and} \\ \mathbb{E}\sum_{j\ge 1} a_j^p W(j)^p < 1. \end{cases} \tag{15}$$

In [6], we considered the case where $b \ge 3$ is a fixed integer, $a_j = b^{-1}$ if $1 \le j \le b$ and 0 otherwise, and W is nonnegative. We proved that if one starts with W_0 such that $1 \le \mathbb{E}\,W_0^2 < b - 1$, one can indefinitely iterate T, and $\mathsf{T}^n W_0$ converges in law to δ_1, the unit Dirac mass at 1. Then, under additional assumptions on W_0, we proved that after centering $\mathsf{T}^n W_0$ and normalizing it by the resulting standard deviations one gets a sequence of probability distributions converging to the standard normal law; moreover, this result had a functional counterpart in which the limit process was obtained as the limit of a Gaussian additive cascade.

As we will see, in the present extended framework, the situation exhibits new features. First, when $q = 1$, i.e., $\sum_{j\geq 1} a_j = 1$ with probability 1, there is a more general non standard central limit theorem: the limit distribution is that of a complex centered Gaussian variable ξ multiplied by $\sqrt{U}$, where U is independent of ξ and is the limit of a non degenerate Mandelbrot martingale built on $\bigcup_{n\geq 1} \mathbb{N}^n \times \{0,1\}^n$ rather than on $\bigcup_{n\geq 0} \mathbb{N}^n$, which is an unexpected fact (Theorems 5.3 and 6.4). This result has a functional counterpart too (Theorem 8.4), the limit process being the limit of a mixture between additive and multiplicative cascades. Also, when $q < 1$, we find conditions under which there exists a non trivial fixed point of T (in the sense that it differs from δ_1) with a non trivial basin of attraction. As already said, this fixed point is also a fixed point of the quadratic smoothing transformation S_2. We will identify this fixed point as the probability distribution of the limit of a nonnegative martingale (Theorems 3.1 and 3.2).

Some useful preliminary facts about the mapping T are introduced in the next subsection.

2.4. Simultaneous cascades. This time we are given a random vector (W, W') such that $\mathbb{E}\, W = \mathbb{E}\, W' = 1$, $\mathbb{E}\,|W|^2 < bq$, and $\mathbb{E}\,|W'|^2 < bq$. We consider a family $\big\{\big(W(w), W'(w)\big)\big\}_{w\in\mathscr{T}}$ of independent copies of (W, W'), which are independent of all the $a(w)$, and perform the same construction as previously: one gets variables Y_n and Y'_n and their limits Y and Y'.

Thus T extends naturally to an operation $\mathsf{T}^{(2)}$ mapping the distribution of (W, W') to that of (Y, Y').

Let us perform a few computations.

Due to (10)

$$\begin{aligned}\mathbb{E}\,\overline{Y}Y' &= \sum_{i,j\geq 1} \mathbb{E}\Big(a_i a_j \overline{W(i)} W'(j) \overline{Y(i)} Y'(j)\Big)\\ &= \mathbb{E}(\overline{W}W')\,\mathbb{E}(\overline{Y}Y')\,\mathbb{E}\sum_{j\geq 1} a_j^2 + \sum_{i\neq j} a_i a_j,\end{aligned}$$

hence

$$\mathbb{E}\,\overline{Y}Y' = \frac{b-1}{bq - \mathbb{E}\,\overline{W}W'}. \tag{16}$$

Again, due to (10)

$$Y - Y' = \sum_{i\geq 1} a_i\big(W(i) - W'(i)\big)Y(i) + \sum_{i\geq 1} a_i\big(Y(i) - Y'(i)\big)W'(i),$$

so

$$\begin{aligned}bq\,\mathbb{E}\,|Y - Y'|^2 = {}& \mathbb{E}\,|W - W'|^2\,\mathbb{E}\,|Y|^2 + \mathbb{E}\,|Y - Y'|^2\,\mathbb{E}\,|W'|^2\\ &+2\Re\Big(\mathbb{E}\big((\overline{W} - \overline{W'})W'\big)\,\mathbb{E}\big(\overline{Y}(Y - Y')\big)\Big),\end{aligned}$$

and

$$\sqrt{bq}\ \|Y - Y'\|_2 \le \|W - W'\|_2\|Y\|_2 + \|Y - Y'\|_2\|W'\|_2. \tag{17}$$

2.5. Examples. The original Mandelbrot cascades correspond to the following choice of a:

$$a_j = \begin{cases} b^{-1}, & \text{if } 1 \le j \le b, \\ 0, & \text{if } j > b. \end{cases}$$

One also can associate a with a Galton-Watson process. More precisely, let J be a nonnegative integer valued random variable, not identically equal to 0, and define $q = \mathbb{P}\{J > 0\}$ and

$$a_j = \begin{cases} (qJ)^{-1}, & \text{if } 1 \le j \le J, \\ 0, & \text{if } j > J. \end{cases}$$

In this context, it is not difficult to see that

$$b^{-1} = \mathbb{E}(J^{-1} \mid J \neq 0).$$

We also use the following notation:

$$b_k = \frac{1}{\mathbb{E}\big(J^{-k} \mid J \neq 0\big)}. \tag{18}$$

Notice that

$$b_1 \le b_k \le b_1^k, \tag{19}$$

due to Hölder inequality. These inequalities are strict unless $\mathbb{P}(J > 1) = 0$.

This natural randomization of the scenery associated with original Mandelbrot cascades is enough to get, when $b > 2$, illustrations of the new phenomena exhibited in this paper, according to whether $q = 1$ (Theorems 3.2, 5.3, 6.4, 8.4 and 8.5) or not (Theorems 3.2 and 3.4, and Proposition 4.2).

3. A dynamical system on fixed points of smoothing transforms. Recall that from this section to Section 6 we assume that $N \equiv 1$. We wish to iterate $\mathsf{T} = \mathsf{T}_1$. So, we have to ensure that $\mathbb{E}\,|Y|^2 < bq$. In view of (13) this leads first to consider the iterates of the homography

$$\varphi(x) = \frac{b-1}{bq - x}. \tag{20}$$

3.1. Study of φ. There are two cases.

(1) The mapping φ has no fixed point. Then one of the iterates of any starting point $x_0 < bq$ is larger than bq.
(2) The mapping φ has two real fixed points $\alpha \le \beta$. Starting from $x_0 < \alpha$ the sequence of its iterates increases towards α. Starting from $x_0 \in (\alpha, \beta)$ the sequence decreases towards α. Starting from $x_0 > \beta$ leads, after some iterations, to values larger than bq.

This means that in case (1) there is no hope to indefinitely iterate T.

As we wish to start from $x_0 = \mathbb{E}\,|W_0|^2 \geq 1$, we must have $\beta \geq 1$. As a matter of fact the case $\beta = 1$ has not interest: we should have to start with $W_0 = 1$ and all the subsequent iterates of W_0 would be the constant 1.

In case $\beta > 1$, the following argument shows that $\alpha \geq 1$. By starting with $W = W_0$ such that $\mathbb{E}\,|W_0|^2 < \beta$, we get $W_1 = Y$ such that $\mathbb{E}\,|W_1|^2 < \beta$. So, by starting with W_1 instead of W_0 we get W_2 which still fulfills the non-degeneracy condition. And so on ... As previously said, $\lim_{n\to\infty} \mathbb{E}\,|W_n|^2 = \alpha$. But, as $\mathbb{E}\,W_n = 1$, one has $\alpha \geq 1$.

This homography, φ, has two real fixed points, $\alpha \leq \beta$, the roots of the polynomial $p(x) = x^2 - bqx + b - 1$, when $q \geq \dfrac{2\sqrt{b-1}}{b}$, what we assume from now on.

The case $b = 2$ presents no interest since in this case the discriminant of p is negative unless $q = 1$, and in this last case $\alpha = \beta = 1$.

Since $p(0) > 0$, $p(1) = b(1-q) \geq 0$, and $\alpha + \beta = bq > 0$, either $0 < \alpha \leq \beta \leq 1$ or $1 \leq \alpha \leq \beta$. In the first case, which as already said is of no interest to us, we have $2\sqrt{b-1} \leq bq = \alpha + \beta \leq 2$, which means $b \leq 2$. In the second case, we have $b \geq bq = \alpha + \beta \geq 2$.

So from now on, we assume that b $>$ 2.

Since $p(1) = p(bq-1) = b(1-q) \geq 0$ and $bq - 1 \geq 2\sqrt{b-1} - 1 > 1$, we have $1 \leq \alpha \leq \beta \leq bq - 1$. Observe that $\alpha = 1$ if and only if $q = 1$ and that $\beta = bq - 1$ if and only if $q = 1$.

Let us now examine the behavior of φ under iteration when there are real fixed points. Suppose first $\alpha < \beta$. By using the conservation of the cross-ratio we get

$$\frac{\varphi(x) - \alpha}{\varphi(x) - \beta} = \big(\varphi(x), \infty, \alpha, \beta\big) = \big(x, bq, \alpha, \beta\big) = \frac{\alpha}{\beta}\,\frac{x - \alpha}{x - \beta}, \tag{21}$$

which implies that, if $x_0 < \beta$ and $x_{n+1} = \varphi(x_n)$, one has

$$x_n - \alpha = \frac{(\alpha/\beta)^n}{1 - (\alpha/\beta)^n}\,\frac{\beta - \alpha}{\beta - x_0}\,(x_0 - \alpha). \tag{22}$$

Suppose now that $\alpha = \beta$, which means $b = \dfrac{2(1 + \sqrt{1 - q^2})}{q^2}$. Then

$$\frac{\alpha - \varphi(x)}{\alpha} = \big(\varphi(x), 0, \alpha, \infty\big) = \big(x, \infty, \alpha, bq\big) = \frac{\alpha - x}{2\alpha - x},$$

gives

$$\frac{1}{\alpha - \varphi(x)} = \frac{1}{\alpha} + \frac{1}{\alpha - x}.$$

It follows that if $x_0 < \alpha$,

$$\alpha - x_n = \frac{\alpha(\alpha - x_0)}{n(\alpha - x_0) + 1}. \tag{23}$$

3.2. The dynamical system. Let $\mathcal{P}$ be the set of Borel probability measures on $\mathbb{C}$, and $\mathcal{P}^{(2)}$ the set of Borel probability measures on $\mathbb{C}^2$.

If $\mu \in \mathcal{P}$ and $p > 0$, we denote by $\mathbf{m}_p(\mu)$ the moment of order p of μ, i.e.,

$$\mathbf{m}_p(\mu) = \int_{\mathbb{C}} |x|^p \, \mu(\mathrm{d}x).$$

Then let $\mathcal{P}_1$ be the set of elements of $\mathcal{P}$ with finite first moment and expectation 1:

$$\mathcal{P}_1 = \Big\{ \mu \in \mathcal{P} \ : \ \mathbf{m}_1(\mu) < \infty, \ \int_{\mathbb{C}} z \, \mu(\mathrm{d}z) = 1 \Big\}.$$

For $\gamma \geq 1$ we set

$$\mathscr{P}_\gamma = \big\{ \mu \in \mathcal{P}_1 \ : \ 1 \leq \mathbf{m}_2(\mu) \leq \gamma \big\}.$$

We also set

$$\mathscr{P}_\gamma^{(2)} = \big\{ \rho \in \mathcal{P}^{(2)} \ : \rho \circ \pi_1^{-1}, \rho \circ \pi_2^{-1} \in \mathscr{P}_\gamma \big\}.,$$

where π_1 and π_2 stand for the canonical projections on the first and second coordinates.

The set $\mathscr{P}_\gamma$ is endowed with the Wasserstein distance (see [21], p. 77 sqq)

$$\mathrm{d}_{W,2}(\mu, \mu')^2 = \inf \left\{ \int_{\mathbb{C}^2} |x - y|^2 \, \mathrm{d}\rho : \rho \in \mathcal{P}^{(2)}, \ \rho \circ \pi_1^{-1} = \mu, \ \rho \circ \pi_2^{-1} = \mu' \right\}.$$

The space $\big(\mathcal{P}_\gamma, \mathrm{d}_{W,2}(\mu, \mu')\big)$ is complete, and convergence in $(\mathcal{P}, \mathrm{d}_{W,2}(\mu, \mu'))$ implies convergence in distribution.

When $\beta > \alpha \geq 1$, for any $\gamma \in [\alpha, \beta]$, the set $\mathscr{P}_\gamma$ is stable under operation T. This means that we can indefinitely iterate the process on $\mathscr{P}_\gamma$. Similarly, $\mathscr{P}_\gamma^{(2)}$ is stable under operation $\mathsf{T}^{(2)}$ defined in Section 2.4.

If $\mu \in \mathscr{P}_\gamma$, due to (10), we can associate with each $n \geq 0$ a random variable W_{n+1} as well as a copy of $(a_j)_{j \geq 1}$ and two sequences of random variables $(W_n(k))_{k \geq 1}$ and $(W_{n+1}(k))_{k \geq 1}$, such that the random variables a, $W_n(1), W_{n+1}(1), W_n(2), W_{n+1}(2), \dots$ are independent,

$$W_{n+1} = \sum_{j \geq 1} a_j W_n(j) W_{n+1}(j), \tag{24}$$

$\mathsf{T}^n \mu$ is the probability distribution of W_n and $W_n(k)$ for every $k \geq 1$, and $\mathsf{T}^{n+1}\mu$ is the probability distribution of W_{n+1} and $W_{n+1}(k)$ for every $k \geq 1$. One has to be aware that, if we also write Equation (24) for W_{n+2}, the variables $W_{n+1}(j)$ which appear in both formulae need not be the same.

3.3. Existence of fixed points for T (or S_2) as limit of a martingale. It turns out that, exactly like in the case of the classical linear smoothing transformation, it is possible to build special fixed points of S_2, and hence of T, as limit of a martingale whose successive terms are distributed according to $\mathsf{S}_2^n(\delta_1)$. However, while for the classical smoothing transform the convergence of the corresponding Mandelbrot martingale is a well understood problem, for the martingale we consider below we are only able to make it to converge in L^2, the study in L^p for $p \in [1,2)$ seeming out of reach at the moment.

Let $\{a(w,m)\}_{\substack{n\geq 0\\ (w,m)\in\mathscr{T}_n\times\{0,1\}^n}}$ be a sequence of independent copies of a. For all (w,m) we define $\Xi_1(w,m)=\sum_{j\geq 1} a_j(w,m)$, which is distributed according to $\mathsf{S}_2(\delta_1)$. Then we define recursively, for all $n\geq 2$ and (w,m),

$$\Xi_n(w,m)=\sum_{j\geq 1} a_j(w,m)\Xi_{n-1}(wj,m0)\Xi_{n-1}(wj,m1), \tag{25}$$

which, as easily seen by induction, is distributed according to $\mathsf{S}_2^n(\delta_1)$.

Set $\Xi_n=\Xi_n(\epsilon,\epsilon)$. The sequence $(\Xi_n)_{n\geq 1}$ is a nonnegative martingale with respect to the filtration $\mathcal{G}_n=\sigma\Big(a(w,m):(w,m)\in\bigcup_{k=0}^{n-1}\{0,1\}^k\times\mathscr{T}_k\Big)$, $n\geq 1$. To see this, define $\mathcal{G}_n(w,m)=\sigma\Big(a(ww',mm'):(w',m')\in\bigcup_{k=0}^{n-1}\{0,1\}^k\times\mathscr{T}_k\Big)$. We have

$$\begin{aligned}
&\mathbb{E}(\Xi_2(w,m)\mid\mathcal{G}_1(w,m))\\
&=\sum_{j_1\geq 1} a_{j_1}(w,m)\,\mathbb{E}(\Xi_1(wj_1,m0)\mid\mathcal{G}_1(w,m))\,\mathbb{E}(\Xi_1(wj_1,m1)\mid\mathcal{G}_1(w,m))\\
&=\sum_{j_1\geq 1} a_{j_1}(w,m)\,\mathbb{E}\left(\sum_{j_2\geq 1} a_{j_2}(wj_1,m0)\right)\mathbb{E}\left(\sum_{j_2\geq 1} a_{j_2}(wj_1,m1)\right)\\
&=\sum_{j_1\geq 1} a_{j_1}(w,m)\\
&=\Xi_1(w,m).
\end{aligned}$$

Then, suppose that for a given $n\geq 3$, for all (w,m) we have

$$\mathbb{E}(\Xi_{n-1}(w,m)\mid\mathcal{G}_{n-2}(w,m))=\Xi_{n-2}(w,m).$$

Using (25) and the independence between random variables, we get

$$\begin{aligned}
&\mathbb{E}\big(\Xi_n(m,w)\mid\mathcal{G}_{n-1}(w,m)\big)\\
&=\sum_{j\geq 1} a_j(w,m)\prod_{\varepsilon\in\{0,1\}}\mathbb{E}\big(\Xi_{n-1}(wj,m\varepsilon)\mid\mathcal{G}_{n-1}(w,m)\big)\\
&=\sum_{j\geq 1} a_j(w,m)\prod_{\varepsilon\in\{0,1\}}\mathbb{E}\big(\Xi_{n-1}(wj,m\varepsilon)|\mathcal{G}_{n-2}(wj,m\varepsilon)\big)\\
&=\sum_{j\geq 1} a_j(w,m)\Xi_{n-2}(wj,m0)\Xi_{n-2}(wj,m1)\\
&=\Xi_{n-1}(w,m).
\end{aligned}$$

Equation (25) also yields

$$\mathbb{E}\,\Xi_n^2=\left(\mathbb{E}\sum_{j\geq 1} a_j^2\right)\big(\mathbb{E}\,\Xi_{n-1}^2\big)^2+\mathbb{E}\sum_{i\neq j} a_i a_j=\frac{1}{bq}\big(\mathbb{E}\,\Xi_{n-1}^2\big)^2+\frac{b-1}{bq}$$

for all $n\geq 1$. Notice that the mapping $x\mapsto\frac{1}{bq}x^2+\frac{b-1}{bq}$ has exactly the same fixed points as φ, namely α and β. Since $\mathbb{E}\,\Xi_0^2=1$ and Ξ_n satisfies the recursion (25), we get the following result.

THEOREM 3.1. *Suppose that* $\alpha \geq 1$. *The martingale* $(\Xi_n)_{n\geq 1}$ *is bounded in* L^2, *and converges to a nonnegative limit* Ξ *such that* $\mathbb{E}(\Xi^2) = \alpha$. *Hence the probability distribution of* Ξ, *denoted by* M, *is a fixed point of* S_2 *and* T, *and* $\mathbf{m}_2(\mathsf{M}) = \alpha$.

3.4. Basin of attraction of M**, convergence speed, and explosion of moments when** $q < 1$**.**

THEOREM 3.2 (Basin of attraction of M and convergence speed). *Suppose that* $\beta \geq \alpha \geq 1$.

(1) *The probability* M *of Theorem 3.1 is the unique fixed point of* T *in* $\mathscr{P}_\alpha$.
(2) *If* $\alpha = \beta$, *then, for all* $\mu \in \mathscr{P}_\alpha$, $\mathrm{d}_{W,2}(\mathsf{T}^n\mu, \mathsf{M}) = \mathsf{O}(1/n)$.
(3) *If* $\beta > \alpha$, *then, for all* $\mu \in \bigcup_{\alpha \leq \gamma < \beta} \mathscr{P}_\gamma$, $\mathrm{d}_{W,2}(\mathsf{T}^n\mu, \mathsf{M}) = \mathsf{O}\big((\alpha/\beta)^n\big)$.
(4) *The fixed point* M *is a solution to the following equation:*

$$W = \sum_{j\geq 1} a_j W_1(j) W_2(j), \tag{26}$$

where W *is distributed according to* M, *the* $W_1(j)$, $W_2(j)$ *are independent copies of* W, *also independent of* a.

LEMMA 3.3.

(1) *Suppose* $\beta > \alpha \geq 1$, *fix* $\gamma \in [\alpha, \beta)$ *and* $\rho \in \mathscr{P}_\gamma^{(2)}$. *Let* (W_n, W_n') *be a sequence of variables distributed according to* $(\mathsf{T}^{(2)})^n(\rho)$. *Then* $\mathbb{E}\,|W_n - W_n'|^2 = \mathsf{O}\big((\alpha/\beta)^n\big)$.
(2) *If* $\alpha = \beta$, *fix* $\rho \in \mathscr{P}_\alpha^{(2)}$. *Let* (W_n, W_n') *be a sequence of variables distributed according to* $(\mathsf{T}^{(2)})^n(\rho)$. *Then* $\mathbb{E}\,|W_n - W_n'|^2 = \mathsf{O}(1/n)$.

Proof. If $\alpha < \beta$, Equation (22) tells that $\mathbb{E}\,|W_n|^2 = \alpha + \mathsf{O}\big((\alpha/\beta)^n\big)$, $\mathbb{E}\,|W_n'|^2 = \alpha + \mathsf{O}\big((\alpha/\beta)^n\big)$, and $\mathbb{E}\,\overline{W}_n W_n' = \alpha + \mathsf{O}\big((\alpha/\beta)^n\big)$. So,

$$\mathbb{E}\,|W_n - W_n'|^2 = \mathbb{E}\,|W_n|^2 + \mathbb{E}\,|W_n'|^2 - 2\Re\,\mathbb{E}\,\overline{W}_n W_n' = \mathsf{O}\left((\alpha/\beta)^n\right).$$

For the second assertion, use Equation (23) instead of (22).

Proof of Theorem 3.2. Fix $\gamma \in [\alpha, \beta)$ if $\alpha < \beta$ or $\gamma = \alpha$ if $\alpha = \beta$.

Take $\mu \in \mathscr{P}_\gamma$. Let $\rho \in \mathscr{P}_\gamma^{(2)}$ such that $\pi_1(\rho) = \mu$ and $\pi_2(\rho) = \mathsf{M}$ (where M is defined in Theorem 3.1). Due to Lemma 3.3 and the fact that $\mathsf{TM} = \mathsf{M}$, we get that the Wasserstein distance between $\mathsf{T}^n\mu$ and M tends to 0, with the speed claimed in the statements.

When $\alpha < \beta$, one can give an alternate proof of the existence of a fixed point. Indeed, take $\mu \in \mathscr{P}_\gamma$ and set $\mu' = \mathsf{T}\mu$. It follows from Lemma 3.3 that $\mathbb{E}\,|W_{n+1} - W_{n+1}'|^2$ converges exponentially to 0. Consequently, so does the Wasserstein distance between $\mathsf{T}^{n+1}\mu$ and $\mathsf{T}^{n+1}\mu' = \mathsf{T}^{n+2}\mu$. It follows that $(\mathsf{T}^n\mu)_{n\geq 0}$ is a Cauchy sequence in $\mathscr{P}_\gamma$ endowed with $\mathrm{d}_{W,2}$, so $\mathsf{T}^n\mu$ converges in distribution as $n \to \infty$, to a limit law $\mathsf{M}(\mu)$, obviously in $\mathscr{P}_\alpha$. Equation (17) implies that T is continuous, so $\mathsf{M}(\mu)$ is a fixed point. Lemma 3.3 yields uniqueness.

When $q = 1$, we have $\alpha = 1$. So, in this case, the fixed point is the Dirac mass at 1.

The next theorem deals with the explosion of moments of M.

THEOREM 3.4 (Explosion of moments for M). *Suppose $\alpha > 1$ (which means $q < 1$). Then there exists $2 \le p_0 < \infty$ such that, if W is distributed according to M,*

$$\mathbb{E}\, W^p < \infty \Longleftrightarrow p \le p_0.$$

Proof. Since the variables Ξ_n are nonnegative the measure M is supported on $\mathbb{R}_+$. Then, due to (15) and (26), we have

$$\mathbb{E}\, W^p < \infty \Longleftrightarrow \begin{cases} \mathbb{E}\left(\sum_{j\ge 1} a_j W(j)\right)^p < \infty \\ \text{and} \\ \mathbb{E}\sum_{j\ge 1} a_j^p W(j)^p < 1. \end{cases}$$

But

$$\mathbb{E}\Big(\sum_{j\ge 1} a_j\Big)^p \le \mathbb{E}\left(\sum_{j\ge 1} a_j W(j)\right)^p$$

and

$$\sum_{j\ge 1}^{N} a_j^p W(j)^p = (\mathbb{E}\, W^p)\, \mathbb{E} \sum_{j\ge 1} a_j^p,$$

where we used conditional expectation with respect to $\sigma(a_j : j \ge 1)$ and Jensen's inequality to get the first inequality. So

$$\mathbb{E}\, W^p < \infty \Longleftrightarrow \left(\mathbb{E}\Big(\sum_{j\ge 1} a_j\Big)^p < \infty \quad \text{and} \quad \mathbb{E}\, W^p < \Big(\mathbb{E}\sum_{j\ge 1} a_j^p\Big)^{-1}\right).$$

Suppose all the moments of W are finite. We must have

$$\big(\mathbb{E}\, W^p\big)^{1/p} < \left(\mathbb{E}\sum_{j\ge 1} a_j^p\right)^{-1/p}$$

for all p. This imposes $\operatorname{ess\,sup} W \le \big(\sup_{j\ge 1} \operatorname{ess\,sup} a_j\big)^{-1}$. Let $m = \operatorname{ess\,sup} W$. Let $\varepsilon > 0$. By using (26) we see that, for all n, with positive probability, we have $W \ge (m - \varepsilon)^2 \sum_{j=1}^n a_j$. This means $m \ge (m-\varepsilon)^2 \operatorname{ess\,sup} \sum_{j\ge 1} a_j$, hence $m \le 1/\operatorname{ess\,sup} \sum_{j\ge 1} a_j$. But as $\sum_{j\ge 1} a_j$ is not constant and of expectation 1, we would have $m < 1$, which is impossible, since $\mathbb{E}\, W = 1$.

Define $p_0 = \sup\{p \ge 2 : \mathbb{E}\, W^p < \infty\}$ and $\Phi(p) = (\mathbb{E}\, W^p)\, \mathbb{E} \sum_{j\ge 1} a_j^p$. We necessarily have $\Phi(p_0) < 1$, if $\mathbb{E}(W^{p_0}) < \infty$ or $\Phi(p_0) = \infty$ if $\mathbb{E}(W^{p_0}) = \infty$. However, Φ is lower semi-continuous, so $\Phi(p_0) = \infty$ is impossible, for otherwise $\Phi(p)$ should tend to ∞ as p tends to p_0 from below, while it is bounded by 1.

Proposition 4.2, in the next section, gives examples for which $p_0 < 3$.

4. Moments of order 3. In this section we suppose that $\beta > 1$ and that W_0 is a *nonnegative* random variable. We study the moment of order 3 of the iterates of the probability distribution μ of W_0 under T. The discussion leads to Proposition 4.2, which gives sufficient conditions for the fixed point of T to have an infinite third moment, and it also provides a domain $\mathscr{D}_1$ such that $\mathsf{T}^n\mu$ and its third moment are defined for all $n \geq 0$ if $\mu \in \mathscr{D}_1$.

Define u, v and w as follows

$$\begin{aligned} \frac{1}{u} &= \mathbb{E}\sum a_j^3 \\ \frac{1}{v} &= \mathbb{E}\sum_{i\neq j} a_i^2 a_j \\ \frac{1}{w} &= \mathbb{E}\sum_{\operatorname{card}\{i,j,k\}=3} a_i a_j a_k, \end{aligned} \tag{27}$$

and suppose the right hand sides are finite.

Notice that we have $\frac{1}{u}+\frac{3}{v}+\frac{1}{w} = \mathbb{E}\left(\sum a_j\right)^3$, which in the conservative case implies $\frac{1}{u}+\frac{3}{v}+\frac{1}{w} = 1$.

Also, the Hölder inequality yields

$$\left(\mathbb{E}\sum a_j^2\right)^{1/2} \leq \left(\mathbb{E}\sum a_j\right)^{1/4}\left(\mathbb{E}\sum a_j^3\right)^{1/4}.$$

We also set, for $\kappa > 0$,

$$u_\kappa = \left(\mathbb{E}\sum_{j\geq 1} a_j^\kappa\right)^{-1}. \tag{28}$$

So we have $u = u_3$.

In the conservative case

$$b = u_2 \quad \text{and} \quad b \leq u \leq b^2. \tag{29}$$

It is easy to get the following formula from Equation (10)

$$\mathbb{E}\,Y^3 = \mathbb{E}\,W^3\,\mathbb{E}\,Y^3/u + 3\,\mathbb{E}\,W^2\,\mathbb{E}\,Y^2/v + 1/w \tag{30}$$

which can be written as

$$\mathbb{E}\,Y^3 = \frac{u(3w\,\mathbb{E}\,W^2\,\mathbb{E}\,Y^2 + v)}{vw(u - \mathbb{E}\,W^3)}. \tag{31}$$

Set

$$\psi_\theta(t) = \frac{u(3w\theta\varphi(\theta) + v)}{vw(u-t)}, \tag{32}$$

where $1 < \theta < \beta$, and

$$\Phi\;:\;(\theta,t) \longmapsto \big(\varphi(\theta),\psi_\theta(t)\big). \tag{33}$$

This means that, if $\theta = \mathbb{E}\, W^2$ and $t = \mathbb{E}\, W^3$, we have

$$\big(\mathbb{E}\, Y^2, \mathbb{E}\, Y^3\big) = \Phi(\theta, t).$$

This is why we wish to iterate Φ.

Remarks 4.1. Let us first make some simple observations on homographies. Consider $\chi(x) = c/(d - x)$, where c and d are positive parameters. Then

(1) $x\chi(x)$ increases with x for $x \in (-\infty, d)$,
(2) when $d^2 > 4c$, χ has two real fixed points; when d is fixed, the smaller fixed point w_- is an increasing function of c and the larger one w_+ is decreasing.
(3) If $x < w_-$ then $x < \chi(x)$, and if $w_- < x < w_+$ then $w_- < \chi(x) < x$.

First, one can check that, when $v(uw - 4) + 12w(b - 1) > 0$, ψ_θ has real fixed points if and only if

$$\theta \le \frac{vbq(uw - 4)}{v(uw - 4) + 12w(b - 1)}.$$

If it is so, let $\gamma_-(\theta) \le \gamma_+(\theta)$ stand for the fixed points.

Define

$$\theta_a = \frac{vbq(uw - 4)}{v(uw - 4) + 12w(b - 1)} \quad \text{and} \quad \vartheta = \min\{\beta, \theta_a\}. \tag{34}$$

In the Galton-Watson case (see (18)), $v(uw - 4) + 12w(b - 1)$ has the same sign as $b_2^2 q^4 + 8b_2 - 12b_1 + 4$. But, since $2 < b_1 < b_2$ (see (19)) and $q \le 1$, we have

$$b_2^2 + 8b_2 - 12b_1 + 4 > b_2^2 - 4b_2 + 4 > 0.$$

So, if q is large enough $b_2^2 q^4 + 8b_2 - 12b_1 + 4$ is positive.

In this setting we have

$$\theta_a = \frac{b_1 b_2^2 q^5 + 4(3 - b_1) b_2 q - 8b_1 q}{b_2^2 q^4 + 8b_2 - 12b_1 + 4}.$$

Proposition 4.2. *If*

(1) $v(uw - 4) + 12w(b - 1) > 0$,
(2) $12w(b - 1) > v(uw - 4) > 0$,
(3) $\big(12w(b - 1) + v(uw - 4)\big)^2 > 12vw(uw - 4)b^2q^2$,

then the third moment of M *is infinite.*

Proof. The above conditions mean $2\theta_a < bq$ and $\theta_a^2 - bq\theta_a + b - 1 > 0$, which implies that $\theta_a < \alpha$ and ψ_α has no fixed point. Suppose that M has a finite third moment t. Then $t = \mathbf{m}_3\mathsf{M} = \mathbf{m}_3\mathsf{TM} = \psi_\alpha(t)$, which is not possible since ψ_α has no fixed point.

To be complete, one should prove that these conditions can be fulfilled. Indeed, in the case of Galton-Watson (see Section 2.5), with $b_2 = b_1^2$ the three requirements are

$$\begin{aligned} &b^4q^4 + 8b^2 - 12b + 4 > 0 \\ &16b^2 - 36b + 20 - b^4q^4 > 0 \\ &b^8q^8 - 12b^6(b - 1)q^6 + 8b^4(b - 1)(2b - 1)q^4 \\ &\qquad\qquad + 48b^2(b - 1)^2(b - 2)q^2 + 16(b - 1)^2(2b - 1)^2 > 0. \end{aligned}$$

The first inequality always holds (do not forget that $b > 2$). If

$$q^4 < \min\left\{\frac{16b^2 - 36b + 20}{b^4}, \frac{48b^2(b-1)^2(b-2)}{12b^6(b-1)}\right\} = \frac{4(b-1)(b-2)}{b^4}$$

the remaining inequalities are fulfilled.

The following facts, when $v(uw - 4) + 12w(b-1) > 0$, easily result from Remarks 4.1:

(1) If $\alpha \leq \theta < \vartheta$ and $\gamma_-(\theta) \leq t \leq \gamma_+(\theta)$, then

$$\gamma_-\big(\varphi(\theta)\big) \leq \gamma_-(\theta) \leq \psi_\theta(t) \leq t \leq \gamma_+(\theta) \leq \gamma_+\big(\varphi(\theta)\big). \tag{35}$$

(2) If $\theta \leq \alpha \leq \theta_a$ and $t \leq \gamma_-(\theta)$, then

$$t \leq \psi_\theta(t) \leq \gamma_-(\theta) \leq \gamma_-\big(\varphi(\theta)\big). \tag{36}$$

Let us consider the following subsets of $\mathbb{R}^2$:

$$\Omega_1 = \big\{(\theta, t) \ : \ \alpha \leq \theta < \vartheta, \ \gamma_-(\theta) \leq t \leq \gamma_+(\theta)\big\},$$
$$\Omega_2 = \big\{(\theta, t) \ : \ \theta \leq \alpha, \ t \leq \gamma_-(\theta)\big\} \quad \text{if} \quad \alpha \leq \theta_a.$$

The set Ω_1 is invariant under Φ, and, if $\alpha \leq \theta_a$, so is Ω_2 (notice that if $\alpha = 1$ then $\theta_a > 1$ and Ω_2 reduces to δ_1).

Set, for $j = 1, 2$,

$$\mathscr{D}_j = \big\{\mu \in \mathscr{P} \ : \ \mathbf{m}_1(\mu) = 1, \, (\mathbf{m}_2(\mu), \mathbf{m}_3(\mu)) \in \Omega_j)\big\}.$$

Then it follows from the above analysis that both these sets are invariant under the transformation T.

So, if $\mu \in \mathscr{D}_1 \cup \mathscr{D}_2$, one has

$$\big(\mathbf{m}_2(\mathsf{T}^n\mu), \mathbf{m}_3(\mathsf{T}^n\mu) = \Phi^n(\mathbf{m}_2(\mu), \mathbf{m}_3(\mu))\big)$$

and

$$\lim \mathbf{m}_2(\mathsf{T}^n\mu) = \alpha, \quad \text{and} \quad \lim \mathbf{m}_3(\mathsf{T}^n\mu) = \gamma_-(\alpha).$$

Of course this is of interest only if $\mathscr{D}_1 \cup \mathscr{D}_2$ is non-empty. In particular, one has to take into account the inequalities $1 \leq \mathbf{m}_2(\mu)^{1/2} \leq \mathbf{m}_3(\mu)^{1/3}$.

Let us show that there are parameters such that $\mathscr{D}_1$ is nonempty. Consider the Galton-Watson case with $q = 1$. Then one has $\alpha = 1 < \beta = b_1 - 1$, $\gamma_-(\alpha) = 1$, $\gamma_+(\alpha) = b_2 - 1 > 1$, and $\theta_a - 1 = \dfrac{(b_2-2)^2(b_1-1)}{b_2^2 + 8b_2 - 12b_1 + 4} > 0$. It results that for a Galton-Watson process with (q, b_1, b_2) in a neighborhood of $(1, b, b^2)$, where b in an integer larger than or equal to 3, the set $\mathscr{D}_1$ is nonempty.

5. Central limit theorem in the non negative case when $q = 1$. We assume that $q = 1$ and we still suppose that $\beta > 1$ and W is a nonnegative random variable. In this case, we know that $\mathsf{T}^n\mu$ weakly converges towards the Dirac mass at 1.

For $\mu \in \bigcup_{1<\gamma<\beta} \mathscr{P}_\gamma$ and $n \geq 1$, we define $\sigma_n = \left(\int (x-1)^2\, \mathsf{T}^n\mu(\mathrm{d}x)\right)^{1/2}$.

Then Equations (13) and (21) give

$$\sigma_{n+1}^2 = \frac{\sigma_n^2}{b-1-\sigma_n^2} \quad \text{and} \quad \frac{\sigma_n^2}{b-2-\sigma_n^2} = (b-1)^{-n}\,\frac{\sigma_0^2}{b-2-\sigma_0^2}. \tag{37}$$

Our goal is Theorem 5.3 below which establishes the convergence in law of $\frac{W_n-1}{\sigma_n}$ under suitable conditions. This statement is a non trivial extension of the special case considered in [6] where b is an integer ≥ 3 and $a_j = b^{-1}$ if $1 \leq j \leq b$ and $a_j = 0$ otherwise. It requires some preparation achieved in the next subsection.

5.1. Recursive decomposition of $\frac{W_n-1}{\sigma_n}$. We set $Z_n = \dfrac{W_n-1}{\sigma_n}$. Equation (24) yields

$$Z_{n+1} = \sum_{j\geq 1} a_j \left[\sigma_n\, Z_n(j)\, Z_{n+1}(j) + \frac{\sigma_n}{\sigma_{n+1}}\, Z_n(j) + Z_{n+1}(j)\right]. \tag{38}$$

If we set

$$R_n = \sum_{j\geq 1} a_j Z_n(j) Z_{n-1}(j)\sigma_{n-1} + \left(\frac{\sigma_{n-1}}{\sigma_n} - \sqrt{b-1}\right)\sum_{j\geq 1} a_j Z_{n-1}(j), \tag{39}$$

then Equation (38) rewrites as

$$Z_{n+1} = R_{n+1} + \sum_{j\geq 1} a_j Z_{n+1}(j) + \sqrt{b-1}\sum_{j\geq 1} a_j Z_n(j). \tag{40}$$

We are going to use repeatedly Formula (40). Let ϵ stand for empty word on any alphabet. For this purpose, fix $n > 1$, define $R_n(\epsilon,\epsilon) = R_n$ as well as $Z_n(\epsilon,\epsilon) = Z_n$, and write (40) in the following way

$$Z_n = Z_n(\epsilon,\epsilon) = R_n(\epsilon,\epsilon) + \sum_{j\geq 1} a_j(\varepsilon,\varepsilon) Z_n(j,0) + \sqrt{b-1}\sum_{j\geq 1} a_j(\varepsilon,\varepsilon) Z_{n-1}(j,1). \tag{41}$$

Since we are interested in distributions only, we can take copies of these variables so that we can write

$$Z_n(j,0) = R_n(j,0) + \sum_{k\geq 1} a_k(j,0) Z_n(jk,00) + \sqrt{b-1}\sum_{k\geq 1} a_k(j,0) Z_{n-1}(jk,01)$$
$$Z_{n-1}(j,1) = R_{n-1}(j,1) + \sum_{k\geq 1} a_k(j,1) Z_{n-1}(jk,10) + \sqrt{b-1}\sum_{k\geq 1} a_k(j,1) Z_{n-2}(jk,11).$$

Notice that by definition in Formula (41) the random variables of the form $Z_{n-1}(j,w)$ and $Z_n(j,w)$ are mutually independent and independent of a, and the same holds for the random variables $R_n(j,w)$ and $R_{n-1}(j,w)$, as well as for the random variables $Z_{n-2}(jk,w)$, $Z_{n-1}(jk,w)$ and $Z_n(jk,w)$.

Then Formula (41) rewrites as

$$\begin{aligned} Z_n(\epsilon,\epsilon) = R_n(\epsilon,\epsilon) + \sum_{j\geq 1} a_j(\epsilon,\epsilon)\left(\sqrt{b-1}\, R_{n-1}(j,1) + R_n(j,0)\right) \\ + \sum_{j,k\geq 1}(b-1)a_j(\epsilon,\epsilon)a_k(j,1)Z_{n-2}(jk,11) \\ + \sum_{j,k\geq 1}\sqrt{b-1}\, a_j(\epsilon,\epsilon)a_k(j,1)Z_{n-1}(jk,10) \\ + \sum_{j,k\geq 1}\sqrt{b-1}a_j(\epsilon,\epsilon)a_k(j,0)Z_{n-1}(jk,01) \\ + \sum_{j,k\geq 1} a_j(\epsilon,\epsilon)a_k(j,0)Z_n(jk,00), \end{aligned}$$

and so on. At last we get $Z_n = T_{1,n} + T_{2,n}$, with

$$T_{1,n} = \sum_{k=0}^{n-1}\sum_{\substack{m\in\{0,1\}^k\\ w\in\mathscr{T}_k}} (b-1)^{\frac{k-\varsigma(m)}{2}} R_{n-k+\varsigma(m)}(w,m)\prod_{j=0}^{k-1} a_{w_{j+1}}(w|_j, m|_j) \tag{42}$$

$$T_{2,n} = \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} (b-1)^{\frac{n-\varsigma(m)}{2}} Z_{\varsigma(m)}(w,m)\prod_{j=0}^{n-1} a_{w_{j+1}}(w|_j, m|_j), \tag{43}$$

where $\varsigma(m)$ stands for the number of zeroes in m. Moreover, all variables in Equation (43) are independent, and in Equation (42), the variables corresponding to the same k are independent.

We need to make precise the above construction of this decomposition of Z_n. At first, we notice that the meaning of Equation (40) is the following: given independent variables $Z_n(j)$ and $Z_{n+1}(j)$ (for $j \geq 1$) equidistributed with Z_n and Z_{n+1}, and independent of a, if we define R_n by Equation (39), then the left hand side of Equation (40) is equidistributed with Z_{n+1}.

Let $\{a(w,m)\}_{\ell\geq 0,\, w\in\mathcal{T}_\ell,\, m\in\{0,1\}^\ell}$ be a collection of independent random variables equidistributed with a.

For each n larger than 2 one starts with a collection

$$\{Z_\ell(w,m)\}_{0\leq\ell\leq n,\, w\in\mathcal{T}_n,\, m\in\{0,1\}^n}$$

such that all these variables are independent and independent of the $a(w,m)$, and the $Z_\ell(\cdot,\cdot)$ have the same distribution as Z_ℓ.

One defines by descending recursion on the length of m

$$\begin{aligned} R_\ell(w,m) = \sum_{j\geq 1} a_j(w,m)Z_{\ell-1}(wj,m0)Z_\ell(wj,m1)\sigma_{\ell-1} \\ + \left(\frac{\sigma_{\ell-1}}{\sigma_\ell} - \sqrt{b-1}\right)\sum_{j\geq 1} a_j(w,m)Z_{\ell-1}(wj,m0) \end{aligned}$$

and

$$Z_\ell(w,m) = R_\ell(w,m) + \sqrt{b-1}\sum_{j\geq 1} a_j(w,m)Z_{\ell-1}(wj,m0) + \sum_{j\geq 1} a_j(w,m)Z_\ell(wj,m1),$$

for $0 \le \ell \le n$, $(w,m) \in \mathbb{Z}_+^j \times \{0,1\}^j$ with $j \ge n-\ell$, with the convention $R_0(\cdot,\cdot) = 0 = Z_{-1}(\cdot,\cdot)$.

Due to (40), all these new variables $Z_\ell(\cdot,\cdot)$ are equidistributed with Z_ℓ, and we get $Z_n(\epsilon,\epsilon) = T_{1,n} + T_{2,n}$.

It will be convenient to denote by $\mathcal{A}_n$ the σ-field generated by the variables $a(w,m)$ with $|w| = |m| < n$, and by $\mathcal{A}$ the σ-field generated by all the variables $a(w,m)$.

Now we study the respective behaviors of $T_{1,n}$ and $T_{2,n}$.

PROPOSITION 5.1. *We have* $\lim_{n\to\infty} \mathbb{E}\, T_{1,n}^2 = 0$, *so* $T_{1,n}$ *converges in distribution to 0.*

Proof. Set $r_n^2 = \mathbb{E}\, R_n^2$. We have

$$b\, r_n^2 = \sigma_{n-1}^2 + \left(\frac{\sigma_{n-1}}{\sigma_n} - \sqrt{b-1}\right)^2,$$

which together with Formulae (37) implies that there exists $C > 0$ such that $r_n^2 \le C^2(b-1)^{-n}$ for all $n \ge 1$. By using the independence properties of random variables in (42) as well as the triangle inequality, we obtain

$$\begin{aligned}\left(\mathbb{E}\, T_{1,n}^2\right)^{1/2} &\le \sum_{0\le k<n} \left(\sum_{|w|=|m|=k} (b-1)^{k-\varsigma(m)} r_{n-k+\varsigma(m)}^2\, \mathbb{E} \prod_{j=0}^{k-1} a_{w_{j+1}}^2 \right)^{1/2} \\ &= \sum_{0\le k<n} \left(\sum_{j=0}^{k} \binom{k}{j} \mathbb{E}\left(\sum_{x\ge 1} a_x^2\right)^k (b-1)^{k-j} r_{n-k+j}^2 \right)^{1/2} \\ &= \sum_{0\le k<n} \left(\sum_{j=0}^{k} \binom{k}{j} b^{-k} (b-1)^{k-j} r_{n-k+j}^2 \right)^{1/2}.\end{aligned}$$

Thus, due to our estimate on $(r_j)_{j\ge1}$,

$$\begin{aligned}\left(\mathbb{E}\, T_{1,n}^2\right)^{1/2} &\le C \sum_{0\le k<n} \left(\sum_{j=0}^{k} \binom{k}{j} b^{-k} (b-1)^{k-j} (b-1)^{k-j-n} \right)^{1/2} \\ &= C \sum_{0\le k<n} b^{-k/2} (b-1)^{-n/2} \left((b-1)^2+1\right)^{k/2} \\ &= C\,(b-1)^{-n/2} \sum_{0\le k<n} \left(\frac{(b-1)^2+1}{b}\right)^{k/2} \\ &= \mathrm{O}\left(\left(1 - \frac{b-2}{b(b-1)}\right)^{n/2}\right),\end{aligned}$$

and $\frac{b-2}{b(b-1)} < 1$ since $b > 2$. ☐

Now we study the main term $T_{2,n}$.

LEMMA 5.2. *$U_n = \mathbb{E}(T_{2,n}^2 \mid \mathcal{A})$ is a nonnegative martingale. Denote its almost sure limit by U.*

Proof. We have

$$U_n = \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} (b-1)^{n-\varsigma(m)} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j).$$

It is to be noticed that, although the variables $T_{2,n}^2$ are not defined on the same probability space, the variables U_n live in the same space.

We have

$$\begin{aligned}
&\mathbb{E}(U_{n+1} \mid \mathcal{A}_n)\\
&= \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j)\, \mathbb{E} \sum_{\substack{k\in\{0,1\}\\ x\geq 1}} (b-1)^{(n+1-\varsigma(mk))} a_x^2(w,m)\\
&= \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) \sum_{k\in\{0,1\}} b^{-1}(b-1)^{(n+1-\varsigma(mk))}\\
&= U_n.
\end{aligned}$$

In fact, U_n is a standard Mandelbrot multiplicative martingale built on the tree $\bigcup_{n\geq 1} \mathscr{T}_n \times \{0,1\}^n$. Indeed, for each $(w,m) \in \bigcup_{n\geq 1} \mathscr{T}_n \times \{0,1\}^n$ define the vector $A(w,m) = (A_{j,\varepsilon}(w,m))_{(j,\varepsilon)\in\mathscr{T}_1\times\{0,1\}}$, where $A_{j,0}(w,m) = a_j^2(w,m)$ and $A_{j,1}(m,w) = (b-1)a_j^2(w,m)$.

By construction we have $\mathbb{E}\sum_{(j,\varepsilon)\in\mathscr{T}_1\times\{0,1\}} A_{j,\varepsilon}(w,m) = 1$, and

$$U_n = \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} \prod_{j=0}^{n-1} A_{w_{j+1},m_{j+1}}(w|_j, m|_j).$$

5.2. The result. We can now state and prove the main result of this section.

THEOREM 5.3. *Suppose that $(b-1)^3 < (u-1)^2$ and $\mu \in \mathscr{D}_1 \setminus \{\delta_1\}$. The limit U of $(U_n)_{n\geq 1}$ is non degenerate and the sequence $(\sigma_n^{-1}(W_n - 1))_{n\geq 1}$ converges in distribution to $\sqrt{U}\xi$, where ξ is a standard normal law independent of U.*

The proof requires several preliminary facts.

LEMMA 5.4. *Let $\kappa > 0$. Suppose that $u_\kappa = \left(\mathbb{E}\sum_{j\geq 1} a_j^\kappa\right)^{-1} > 0$. Then*

$$V_{\kappa,n} = \left(\frac{(b-1)^{\kappa/2}+1}{u_\kappa}\right)^{-n} \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} (b-1)^{\frac{\kappa\left(n-\varsigma(m)\right)}{2}} \prod_{j=0}^{n-1} a_{w_{j+1}}^\kappa(w|_j, m|_j),$$

is a martingale.

Proof. This results from a computation similar to that used in the proof of Lemma 5.2, or the observation that $V_{\kappa,n}$ is the Mandelbrot martingale associated with the random vectors

$$A_\kappa(m,w) = \left(\frac{(b-1)^{\kappa/2}+1}{u_\kappa}\right)^{-1} (A_{j,\varepsilon}^{\kappa/2}(w,m))_{(j,\varepsilon)\in\mathscr{T}_1\times\{0,1\}}.$$

Corollary 5.5.
(1) *For $\kappa > 0$, if $u_\kappa > 0$,*

$$\sup_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} (b-1)^{(n-\varsigma(m))} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) \le \left(\frac{(b-1)^{\kappa/2}+1}{u_\kappa}\right)^{2n/\kappa} V_{\kappa,n}^{2/\kappa}.$$

(2) *If $(b-1)^{\kappa/2}+1 < u_\kappa$ for some $\kappa > 0$, with probability 1*

$$\lim_{n\to\infty} \sup_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} (b-1)^{(n-\varsigma(m))} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) = 0.$$

(3) *If $\kappa > 2$, $(b-1)^{\kappa/2}+1 < u_\kappa$ and $\mathbb{E}(\sum_{j\ge 1} a_j^2)^{\kappa/2} < \infty$, then $\mathbb{P}(U > 0) > 0$, and $\mathbb{P}(U > 0) = 1$ if and only if $\mathbb{P}(\#\{j \ge 1 : a_j > 0\} \ge 1) = 1$.*

Proof. Let V_κ be the a.s. limit of the nonnegative martingale $V_{\kappa,n}$ of Lemma 5.4. Since V_κ is integrable, it is a.s. finite. So

$$\sup_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} (b-1)^{\kappa(n-\varsigma(m))/2} \prod_{j=0}^{n-1} a_{w_{j+1}}^\kappa(w|_j, m|_j) \le \left(\frac{(b-1)^{\kappa/2}+1}{u_\kappa}\right)^{n} V_{\kappa,n}.$$

This accounts for the first and second assertions.

For the third assertion, we notice that our assumptions are exactly those required for the Mandelbrot martingale U_n to be bounded in $L^{\kappa/2}$, hence have a non degenerate limit:
$\mathbb{E}\sum_{(j,\varepsilon)\in\mathscr{T}_1\times\{0,1\}} A_{j,\varepsilon}^{\kappa/2}(w,m) < 1$ and $\mathbb{E}(\sum_{(j,\varepsilon)\in\mathscr{T}_1\times\{0,1\}} A_{j,\varepsilon}(w,m))^{\kappa/2} < \infty$. The assertion on the possibility that U_n vanishes is then standard.

Now we state conditions under which if $\mu \ne \delta_1$ and μ belongs to the domain $\mathscr{D}_1$ pointed out in the Section 4, then $\big(Z_n\big)_{n\ge 0}$ is bounded in L^3:

Lemma 5.6. *There exists C such that for all nonnegative W whose distribution is in $\mathscr{D}_1 \setminus \{\delta_1\}$ one has*

$$(u - \mathbb{E}\,W^3)\,\mathbb{E}\,Z_Y^3 \le (b-1)^{3/2}\,\mathbb{E}\,Z_W^3 + C\big((\mathbb{E}\,Z_W^3)^{2/3} + (\mathbb{E}\,Z_W^3)^{1/3} + 1\big),$$

where $Y = \mathsf{T}W$, $Z_W = |W-1|/\sigma_W$, and $Z_Y = |Y-1|/\sigma_Y$.

The proof, as well as that of the following corollary, follows the same lines as in the special case studied in [6].

Corollary 5.7. *If $(b-1)^3 < (u-1)^2$ and $\mu \in \mathscr{D}_1 \setminus \{\delta_1\}$, then*

$$\sup_n \int \sigma_n^{-3}|x-1|^3 \mathsf{T}^n\mu(\mathrm{d}x) < \infty.$$

Proof of Theorem 5.3. At first we notice that the assumptions of Corollary 5.5 are satisfied with $\kappa = 3$. Indeed $(b-1)^3 < (u-1)^2$ is just $(b-1)^{3/2}+1 < u_3$, and

$\mathbb{E}(\sum_{j\geq 1} a_j^2)^{3/2} \leq \mathbb{E}(\sum_{j\geq 1} a_j)^3 < \infty$ due to our assumption on u, v, and w. Then, due to Proposition 5.1 if suffices to prove the same convergence for the sequence $(T_{2,n})_{n\geq 0}$.

We adapt a proof given by Breiman [11] for Lindeberg's theorem. First we remark that, if X is a centered random variable with standard deviation σ and $t \in \mathbb{R}$, one has $|\mathbb{E}\, \mathrm{e}^{\mathrm{i}tX} - 1| \leq \frac{t^2\sigma^2}{2}$, and, if $\mathbb{E}\,|X|^3$ is finite, $|\mathbb{E}\, \mathrm{e}^{\mathrm{i}tX} - 1 + \frac{\sigma^2 t^2}{2}| \leq \frac{|t^3|\,\mathbb{E}\,|X|^3}{6}$. Also, if $|z| \leq 1/2$, $|\log(1+z) - z| \leq |z|^2$.

For $n \geq 1$, let $A_n = \{V_{3,n} \leq n\}$. Since $V_{3,n}$ has a finite limit with probability 1, the variable $\mathbf{1}_{A_n}$ converges towards 1 with probability 1.

For $w \in \mathscr{T}_n$ and $m \in \{0,1\}^n$, set

$$f_{n,w,m}(t) = \mathbb{E}\left(\mathrm{e}^{\mathrm{i}t(b-1)^{(n-\varsigma(m))/2} Z_{\varsigma(m)}(w,m) \prod_{j=0}^{n-1} a_{w_{j+1}}(w|_j, m|_j)} \middle| \mathscr{A}_n\right).$$

According to Corollary 5.5 applied with $\kappa = 3$, for all t, on A_n we have

$$\begin{aligned}\sup_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} |f_{n,w,m}(t) - 1| &\leq t^2 \left(\frac{(b-1)^{3/2}+1}{u}\right)^{2n/3} V_{3,n}^{2/3} \\ &\leq n^{2/3} t^2 \left(\frac{(b-1)^{3/2}+1}{u}\right)^{2n/3}.\end{aligned}$$

So, since $(b-1)^3 < (u-1)^2$, t being fixed, for n large enough we have

$$\sup_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} |f_{n,w,m}(t) - 1| \leq \frac{1}{2}$$

and therefore

$$|\log f_{n,w,m}(t) - (f_{n,w,m}(t) - 1)| \leq |f_{n,w,m}(t) - 1|^2.$$

But

$$\begin{aligned}&\left| f_{n,w,m}(t) - 1 + \frac{t^2}{2}(b-1)^{(n-\varsigma(m))} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j)\right| \\ &\leq \frac{|t|^3}{6}(b-1)^{3(n-\varsigma(m))/2} \prod_{j=0}^{n-1} a_{w_{j+1}}^3(w|_j, m|_j) \sup_{j\geq 0} \mathbb{E}\,|Z_j|^3.\end{aligned}$$

So, if we set $g_n(t) = \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} \log f_{n,w,m}(t)$ and $C = \sup_{j\geq 0} \mathbb{E}\,|Z_j|^3$, for fixed t, for n large enough, on A_n,

$$\left| g_n(t) + \frac{t^2 U_n}{2}\right| \leq \sum_{\substack{m\in\{0,1\}^n\\ w\in\mathscr{T}_n}} |f_{n,w,m}(t) - 1|^2 + C\left(\frac{(b-1)^{3/2}+1}{u}\right)^n |t|^3 V_{3,n}.$$

By writing $\sum |f_{n,w,m} - 1|^2 \leq \big(\sup |f_{n,w,m} - 1|\big) \sum |f_{n,w,m} - 1|$ one gets on A_n

$$\left| g_n(t) + \frac{t^2 U_n}{2}\right| \leq t^4 \left(\frac{(b-1)^{3/2}+1}{u}\right)^{2n/3} V_{3,n}^{2/3} U_n + C\left(\frac{(b-1)^{3/2}+1}{u}\right)^n |t|^3 V_{3,n}.$$

We have obtained

$$\mathbb{E}\left(e^{\mathrm{i}tT_{2,n}}\right) = \mathbb{E}\left(e^{-\frac{t^2U_n}{2}+r_n(t)}\mathbf{1}_{\{V_{3,n}\le n\}}\right) + \mathbb{E}\left(e^{\mathrm{i}tT_{2,n}}\mathbf{1}_{\{V_{3,n}>n\}}\right),$$

with $|r_n(t)| \le t^4\left(\frac{(b-1)^{3/2}+1}{u}\right)^{2n/3} n^{2/3}U_n + C\left(\frac{(b-1)^{3/2}+1}{u}\right)^n |t|^3 n$ on A_n.

Since both U_n and V_n converge almost surely and $\frac{(b-1)^{3/2}+1}{u} < 1$, we obtain

$$\lim_{n\to\infty} \mathbb{E}\left(e^{-\frac{t^2U_n}{2}+r_n(t)}\mathbf{1}_{\{V_{3,n}\le n\}}\right) = \mathbb{E}\left(e^{-\frac{t^2U}{2}}\right)$$

and

$$\lim_{n\to\infty} \mathbb{E}\left(e^{\mathrm{i}tT_{2,n}}\mathbf{1}_{\{V_{3,n}>n\}}\right) = 0$$

by the dominated convergence theorem.

6. Central limit theorem in the complex case when $q = 1$. In this section, we suppose $q = 1$ and $\beta > 1$, and study the convergence in law of $(b-1)^{n/2}\big(W_n - 1)\big)$ when W_0 is a complex valued random variable.

Recall that the fact $q = 1$ implies the relation $\dfrac{1}{u} + \dfrac{3}{v} + \dfrac{1}{w} = 1$.

We start with useful observations on the asymptotic behavior of the variances of $\Re W_n$ and $\Im W_n$, as well as that of their covariance.

6.1. Variances and covariances. According to (13) and (16), if $\mathbb{E}\,|W_0^2| < b-1$, we have

$$\mathbb{E}\,|W_{n+1}^2| = \frac{b-1}{b-\mathbb{E}\,|W_n^2|} \quad \text{and} \quad \mathbb{E}\,W_{n+1}^2 = \frac{b-1}{b-\mathbb{E}\,W_n^2},$$

so, both $\mathbb{E}\,|W_{n+1}^2|$ and $\mathbb{E}\,W_{n+1}^2$ converge to the fixed point 1 of φ. Due to (22), $(b-1)^n\big(\mathbb{E}\,W_n^2 - 1\big)$ and $(b-1)^n\big(\mathbb{E}\,|W_n^2| - 1\big)$ have explicit limits when n goes to ∞.

It results that, if we set

$$x_n = \mathbb{E}(\Re W_n)^2, \quad y_n = \mathbb{E}\big(\Im W_n\big)^2,$$

and

$$z_n = \mathbb{E}(\Re W_n)(\Im W_n) = \mathbb{E}(\Re W_n - 1)(\Im W_n)$$

there exist x, y, and z (depending on W_0) such that

$$\lim\,(b-1)^n(x_n-1) = x, \quad \lim\,(b-1)^n y_n = y, \quad \text{and} \quad \lim\,(b-1)^n z_n = z. \tag{44}$$

Set

$$u = \mathbb{E}(\Re W_0)^2 = \mathbb{E}(\Re W_0 - 1)^2 + 1, \quad v = \mathbb{E}(\Im W_0)^2,$$

and

$$w = \mathbb{E}(\Re W_0)(\Im W_0) = \mathbb{E}(\Re W_0 - 1)(\Im W_0).$$

Notice that by Cauchy-Schwarz inequality we must have $w^2 \le v(u-1)$. Using a formal computing software (e.g. Maple) shows that one has

$$(b-2)^{-3}\det\begin{pmatrix} x & z \\ z & y\end{pmatrix} = \frac{v(bu-u^2+v^2-b+1)-(b-2u-2v)w^2}{(b-1-u-v)^2\big(4w^2+(b-u+v-1)^2\big)}.$$

It is easily seen that the denominator of this last expression is positive. Its numerator, call it $R(u,v,w)$, assumes its minimum for $w=0$ if $u+v \geq 2b$, and for $w^2 = v(u-1)$ otherwise. We have

$$R(u,v,0) = \frac{1}{4}\, v(b-2)^3\big((2v-2u-b)(2v+2u+b)+(b-2)^2\big)$$

and

$$R(u,v,\pm\sqrt{v(u-1)}) = v(u+v-1)^2(b-2)^3.$$

It results that this determinant is positive, except if $v=0$. We thus have proven the following proposition.

PROPOSITION 6.1. *The matrix* $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$ *is definite positive if and only if and only if* W_0 *is not almost surely real.*

6.2. Complex version of Theorem 5.3. We need to adapt our previous discussion on the moments of order 3 to the complex case. In this context, if $Y = \mathsf{T}W$, we have

$$\mathbb{E}\, Y^3 = \mathbb{E}\, W^3\, \mathbb{E}\, Y^3/u + 3\,\mathbb{E}\, W^2\, \mathbb{E}\, Y^2/v + 1/w$$

and

$$\begin{aligned}\mathbb{E}\,|Y|^3 &\leq \mathbb{E}\,|W|^3\,\mathbb{E}\,|Y|^3/u + 3\,\mathbb{E}\,|W|^2\,\mathbb{E}\,|Y|^2\,\mathbb{E}\,|W|\,\mathbb{E}\,|Y|/v + \big(\mathbb{E}\,|W|\,\mathbb{E}\,|Y|\big)^3/w \\ &\leq \frac{1}{u}\,\mathbb{E}\,|W|^3\,\mathbb{E}\,|Y|^3 + \frac{u-1}{u}\big(\mathbb{E}\,|W|^2\,\mathbb{E}\,|Y|^2\big)^{3/2}.\end{aligned}$$

Finally

$$\mathbb{E}\,|Y|^3 \leq \frac{(u-1)\big(\mathbb{E}\,|W|^2\,\mathbb{E}\,|Y|^2\big)^{3/2}}{u-\mathbb{E}\,|W^3|}. \tag{45}$$

This time for $1 \leq \theta \leq \beta$ we use the function

$$\psi_\theta(t) = \frac{(u-1)\big(\theta\varphi(\theta)\big)^{3/2}}{u-t}. \tag{46}$$

It has fixed points if $\theta\varphi(\theta) \leq \left(\frac{u^2}{4(u-1)}\right)^{2/3}$, i.e., if $\theta \leq \theta_a$ for some critical real number θ_a. As ψ_1 has two fixed points 1 and $u-1$, one has $\theta_a > 1$. If $\theta \leq \theta_a$, let $\gamma_\pm(\theta)$ be the fixed points. Consider the sets

$$\Omega = \big\{(\theta,t)\ : \theta < \min(\beta,\theta_a), t \leq \gamma_+(\theta)\big\},$$

and

$$\mathscr{D} = \{\mu \in \mathscr{P}\ :\ \mathbf{m}_1(\mu) = 1,\ (\mathbf{m}_2(\mu), \mathbf{m}_3(\mu) \in \Omega)\}.$$

Arguments similar to previous ones show that if W_0 is distributed according to $\mu \in \mathscr{D}$, then, $\mathbb{E}\,|W_n|^3 < \infty$ for all n, and $\limsup \mathbb{E}\,|W_n|^3 \leq 1$. Since $\mathbb{E}\,|W_n|^3 \geq 1$, we have

$$\lim \mathbb{E}\,|W_n|^3 = \limsup \mathbb{E}\,|W_n|^3 = 1.$$

Lemma 5.6 and its corollary have the following counterparts.

LEMMA 6.2. *There exists C such that for all W whose distribution is in $\mathscr{D}\setminus\{\delta_1\}$ one has*

$$(u-\mathbb{E}\,|W|^3)\,\mathbb{E}\,Z_Y^3 \le (b-1)^{3/2}\,\mathbb{E}\,Z_W^3 + C\big((\mathbb{E}\,Z_W^3)^{2/3}+(\mathbb{E}\,Z_W^3)^{1/3}+1\big),$$

where $Y=\mathsf{T}W$, $Z_W=|W-1|/\sigma_W$, and $Z_Y=|Y-1|/\sigma_Y$.

COROLLARY 6.3. *If $(b-1)^3<(u-1)^2$ and $\mu\in\mathscr{D}\setminus\{\delta_1\}$, then*

$$\sup_n \int (b-1)^{3n/2}|x-1|^3\mathsf{T}^n\mu(\mathrm{d}x)<\infty.$$

Estimates similar to those used in the nonnegative case now yield:

THEOREM 6.4. *Suppose that $(b-1)^3<(u-1)^2$ and the distribution of W_0 lies in $\mathscr{D}\setminus\{\delta_1\}$. Then $(b-1)^{n/2}\big(W_n-1\big)$ converges in law to $\sqrt{U}\xi$, where ξ is a centered normal vector independent of U whose covariance matrix is the matrix $\begin{pmatrix} x & z\\ z & y\end{pmatrix}$ determined in Section 6.1.*

7. Higher order smoothing transformations. Now, we consider the general case: N is a given nonnegative integer valued random variable such that the radius of convergence of its probability generating function $f_N(t)=f(t)=\sum_{n\ge 0}\mathbb{P}(N=n)\,t^n$ is larger than 1, and f_N is not constant, i.e., $\mathbb{P}(N=0)<1$.

If W is a square integrable random variable of expectation 1, we already defined $W_{(N)}$ to be the product of N independent random variables equidistributed with W and independent of N:

$$W_{(N)}=\prod_{1\le k\le N} W_k. \tag{47}$$

It is convenient to adopt the simpler notation:

$$\tilde{W}=W_{(N)}.$$

Then we consider the martingale

$$Y_n=\sum_{w=j_1j_2\dots j_n\in\mathscr{T}_n}\prod_{k=0}^{n-1} a_{j_{k+1}}(w|_k)\tilde{W}(w|_{k+1}).$$

According to Section 2, this martingale is bounded in L^2 if and only if $f(\mathbb{E}\,W^2)<bq$ and, if it is so, its limit Y satisfies

$$\mathbb{E}\,Y^2=\frac{b-1}{bq-f(\mathbb{E}\,W^2)} \quad\text{and}\quad \mathbb{E}\,|Y|^2=\frac{b-1}{bq-f(\mathbb{E}\,|W|^2)}.$$

Recall that T_N stands for the map which sends the distribution of W to the one of Y. Indeed, T_N depends only on the distribution of N.

To iterate this operation, this time we have to deal with the function

$$\varphi(x)=\frac{b-1}{bq-f(x)}.$$

In the interval $[0, f^{-1}(bq)]$ this mapping has at most two positive fixed points, the roots of the strictly convex function $p(x) = xf(x) - bqx + b - 1$. Observe that $p(0) = p\big(f^{-1}(bq)\big) = b - 1 > 0$. Then, φ has two fixed points α and β such that $0 \le \alpha \le \beta < f^{-1}(bq)$ if and only if the minimum of p on this interval is nonpositive. This happens if and only if $x_0^2 f'(x_0) \ge b - 1$, where x_0 is the solution to equation $x_0 f'(x_0) + f(x_0) = bq$. But, as previously, we wish that $\alpha \ge 1$. This means $x_0 \ge 1$, i.e., $f'(1) + f(1) \le bq$. As $0 < q \le 1$, this gives $b \ge 1 + \mathbb{E}\, N$. When this condition is fulfilled, then q is subject to the restriction

$$\frac{1 + \mathbb{E}\, N}{b} \le q \le 1.$$

From now on we suppose that

$$b > 1 + \mathbb{E}\, N \quad \text{and} \quad \frac{1 + \mathbb{E}\, N}{b} \le q \le 1 \tag{48}$$

(as previously we discard the trivial case $b = 1 + \mathbb{E}\, N$ which yields $q = \alpha = \beta = 1$).

7.1. Fixed points. In the introduction we also defined, see (1) the smoothing transformation S_{N+1} which associates with a probability measure μ on $\mathbb{C}$ the measure

$$\mathsf{S}_{N+1}(\mu) = \mathscr{L}\left(\sum_{j\ge 1} a_j \prod_{0\le k\le N} W_k(j)\right), \tag{49}$$

where the random variables $\{W_k(j)\}_{j\ge 1, k\ge 0}$ are distributed according to μ, independent and independent of a. As for T_N there is a slight abuse of notation: obviously S_{N+1} depends only on the distribution of N and not on its realization.

Then, as previously, a fixed point of T_N is also a fixed point of S_{N+1}. In the same way as when $N = 1$ almost surely, a fixed point of S_{N+1} is constructed as the law of the limit of a martingale:

Consider the Galton-Watson tree $\mathcal{T}$ defined by the variable $N+1$. Let $\mathcal{T}_n$ stand for the nodes of generation n. Consider $\{a(w,m)\}_{\substack{k\ge 1\\ (w,m)\in \mathscr{T}_k\times \mathcal{T}_k}}$ a sequence of independent copies of a and $\{N(w,m)\}_{\substack{k\ge 1\\ (w,m)\in \mathscr{T}_k\times \mathcal{T}_k}}$ a sequence of independent copies of N also independent of a and of $\{a(w,m)\}$. For all (w,m) we define $\Xi_1(w,m) = \sum_{j\ge 1} a_j(w,m)$. It has $\mathsf{S}_{N+1}(\delta_1)$ as distribution. Then we define recursively, for all $k \ge 2$ and (w,m),

$$\Xi_k(w,m) = \sum_{j\ge 1} a_j(w,m) \prod_{\ell=0}^{N(w,m)} \Xi_{k-1}(wj, m\ell),$$

which by induction is clearly distributed according to $\mathsf{S}_{N+1}^k(\delta_1)$.

Set $\Xi_k = \Xi_k(\epsilon,\epsilon)$. As previously, the sequence $(\Xi_k)_{k\ge 1}$ is a nonnegative martingale. The law M of its limit is a fixed point of S_{N+1} as well as T_N and the same analysis as in Section 3 can be performed: if (48) holds, M is the unique fixed point of T_N belonging to $\mathscr{P}_\alpha \cap \mathcal{P}(\mathbb{R}^+)$, and for all $\mu \in \mathcal{P}(\mathbb{R}^+) \cap \bigcup_{\alpha\le\gamma<\beta} \mathscr{P}_\gamma$ if $\beta > \alpha$ and all $\mu \in \mathcal{P}(\mathbb{R}^+) \cap \mathscr{P}_\alpha$ if $\beta = \alpha$, the sequence $\mathsf{T}_N^j \mu$ converges to M. Moreover, if the $W_k(j)$ are independent and independent of a and N, and all distributed according to M, then

$$W = \sum_{j\geq 1} a_j \prod_{k=0}^{N} W_k(j)$$

is distributed according to M.

If we wish to deal with measures not supported on $\mathbb{R}^+$, we have to make the extra assumption that $\mathbb{E}\,\overline{W}_0 W_0'$ lies in the basin of attraction of the fixed point α to get the analog of Lemma 3.3. This assumption is automatically fulfilled when N is the constant 1. With $\mathbb{P}(N=2)>0$, the result holds without this assumption if we restrict ourselves to probability supported in $\mathbb{R}$. But in general we do not know whether it may happen that the mapping φ has other attractive fixed points.

7.2. Central limit theorem. In this section we suppose that $q=1$ and still assume (48) holds. We just outline modifications to be brought to Sections 6.1 and 6.2. When starting from $\mu \in \bigcup_{1<\gamma<\beta} \mathscr{P}_\gamma$, one has $\lim \mathbb{E}\,|W_n|^2 = 1$ and, since $\mathbb{E}\,W_n = 1$, $\lim \mathbb{E}\,W_n^2 = 1$. Since $\mathbb{E}\,N/(b-1) = \varphi'(1)$, the following limits exist

$$\lim_{k\to\infty} \big((\mathbb{E}\,N)^{-1}(b-1)\big)^k \left(\mathbb{E}\,W_k^2 - 1\right) \text{ and } \lim_{k\to\infty} \big((\mathbb{E}\,N)^{-1}(b-1)\big)^k \left(\mathbb{E}\,|W_k|^2 - 1\right),$$

since 1 is an attracting fixed point of φ and our assumptions on f imply that $\varphi'(1)\neq 0$. It results that, if we set

$$x_n = \mathbb{E}(\Re W_n)^2, \quad y_n = \mathbb{E}\big(\Im W_n\big)^2, \quad \text{and} \quad z_n = \mathbb{E}(\Re W_n)(\Im W_n)$$

there exist x, y, and z (depending on W_0) such that

$$\lim \frac{x_n - 1}{\big((\mathbb{E}\,N)^{-1}(b-1)\big)^n} = x, \quad \lim \frac{y_n}{\big((\mathbb{E}\,N)^{-1}(b-1)\big)^n} = y,$$

and

$$\lim \frac{z_n - 1}{\big((\mathbb{E}\,N)^{-1}(b-1)\big)^n} = z.$$

If $Y = \mathsf{T}_N W$, Formulae (31), (45), and (46) become

$$\mathbb{E}\,Y^3 = \frac{u\big(3wf(\mathbb{E}\,W^2)\,\mathbb{E}\,Y^2 + v\big)}{vw\big(u - f(\mathbb{E}\,W^3)\big)},$$

$$\mathbb{E}\,|Y|^3 \leq \frac{(u-1)\big(f(\mathbb{E}\,|W^2|)\,\mathbb{E}\,|Y|^2\big)^{3/2}}{u - f(\mathbb{E}\,|W|^3)},$$

$$\psi_\theta(t) = \frac{(u-1)\big(f(\theta)\varphi(\theta)\big)^{3/2}}{u - f(t)}.$$

The function ψ_θ has two fixed points between 0 and u if and only if $\theta \leq \theta_{a,N}$ for some critical real number $\theta_{a,N}$. But as ψ_1 has two fixed points 1 and $u-1$, one has $\theta_{a,N} > 1$. If $1 \leq \theta \leq \theta_{a,N}$, let $\gamma_\pm(\theta)$ be the fixed points. Consider the sets

$$\Omega = \left\{(\theta, t) \ : \theta < \min(\beta, \theta_{a,N}), t \leq \gamma_+(\theta)\right\},$$

and

$$\mathscr{D} = \{\mu \in \mathscr{P} \ : \ \mathbf{m}_1(\mu) = 1, \, (\mathbf{m}_2(\mu), \mathbf{m}_3(\mu) \in \Omega)\} \, .$$

Arguments similar to previous ones show that if W_0 is distributed according to $\mu \in \mathscr{D}$, then, $\mathbb{E}\,|W_n|^3 < \infty$ for all n, and $\limsup \mathbb{E}\,|W_n|^3 \leq 1$. Since $\mathbb{E}\,|W_n|^3 \geq 1$, we have

$$\lim \mathbb{E}\,|W_n|^3 = \limsup \mathbb{E}\,|W_n|^3 = 1.$$

Lemma 5.6 and its corollary have the following counterparts.

LEMMA 7.1. *There exists C such that for all W_0 whose distribution is in $\mathscr{D} \setminus \{\delta_1\}$ one has*

$$\big(u - f(\mathbb{E}\,|W_n|^3)\big)\ \mathbb{E}\, Z^3_{W_{n+1}} \leq C_n (\mathbb{E}(N))^{-1/2} (b-1)^{3/2}\, \mathbb{E}\big(N^3 (\mathbb{E}\,|W|^3)^{N-1}\big)\, \mathbb{E}\, Z^3_{W_n}$$
$$+ C\big((\mathbb{E}\, Z^3_{W_n})^{2/3} + (\mathbb{E}\, Z^3_{W_n})^{1/3} + 1\big),$$

where $W_n = \mathsf{T}^n_N W_0$, $Z_{W_n} = |W_n - 1|/\sigma_{W_n}$, and $C_n = 1 + \mathrm{o}(1)$.

COROLLARY 7.2. *Suppose that $(b-1)^{3/2}(\mathbb{E}(N))^{-1/2} < u - 1$ and $\mu \in \mathscr{D} \setminus \{\delta_1\}$. Then*

$$\sup_n \int \sigma_n^{-3} |x-1|^3 \mathsf{T}^n_N \mu(\mathrm{d}x) < \infty,$$

where σ_n is the standard deviation of variables distributed according to $\mathsf{T}^n_N(\mu)$.

We now have to mention the analog of the discussion following Equation (38).

By iteration, we get a sequence $\{W_n\}$ of variables. We set $\sigma_n^2 = \operatorname{Var} W_n$, $\tilde\sigma_n^2 = \operatorname{Var} \tilde W_n$, $Z_n = \sigma_n^{-1}(W_n - 1)$, and $\tilde Z_n = \tilde\sigma_n^{-1}(\tilde W_n - 1)$.

The following facts are easily proven:

- $\dfrac{\sigma_{n+1}}{\sigma_n} \sim \dfrac{\sqrt{\mathbb{E}\, N}}{\sqrt{b-1}}$,
- $\tilde\sigma_n^2 = f(\sigma_n^2 + 1) - 1 \sim \sigma_n^2\, \mathbb{E}\, N$,

Equation (38) becomes

$$Z_{n+1} = \sum_{j \geq 1} a_j \left[\tilde\sigma_n\, \tilde Z_n(j)\, Z_{n+1}(j) + \frac{\tilde\sigma_n}{\sigma_{n+1}}\, \tilde Z_n(j) + Z_{n+1}(j) \right]. \tag{50}$$

But as $\tilde W - 1 = (W_1 - 1) W_2 \cdots W_N + (W_2 - 1) W_3 \cdots W_N + \cdots + (W_N - 1)$ (with $W_1, W_2, \cdots$ i.i.d.) we have $\tilde W - 1 = \sum_{1 \leq j \leq N} W_j - 1$ with a L^2-error of the same order of magnitude as $\operatorname{Var} W$. So Equation (50) can be rewritten as

$$Z_{n+1} = R_{n+1} + \sum_{j \geq 1} a_j Z_{n+1}(j) + \frac{\sqrt{b-1}}{\sqrt{\mathbb{E}\, N}} \sum_{j \geq 1} a_j \sum_{1 \leq k \leq N} Z_n(j,k), \tag{51}$$

where R_{n+1} is a sum of 'error terms'.

As previously, we iterate this formula, but using a Galton-Watson tree associated with the variable $N+1$ instead of using a binary tree. Finally, we get $Z_n = T_{1,n} + T_{2,n}$, with

$$T_{1,n} = \sum_{k=0}^{n-1} \sum_{\substack{m\in\mathcal{T}_k\\ w\in\mathscr{T}_k}} \left(\frac{b-1}{\mathbb{E}\,N}\right)^{\frac{k-\varsigma(m)}{2}} R_{n-k+\varsigma(m)}(w,m) \prod_{j=0}^{k-1} a_{w_{j+1}}(w|_j, m|_j) \tag{52}$$

$$T_{2,n} = \sum_{\substack{m\in\mathcal{T}_n\\ w\in\mathscr{T}_n}} \left(\frac{b-1}{\mathbb{E}\,N}\right)^{\frac{n-\varsigma(m)}{2}} Z_{\varsigma(m)}(w,m) \prod_{j=0}^{n-1} a_{w_{j+1}}(w|_j, m|_j), \tag{53}$$

where $\mathcal{T}_n$ stands for the n-th generation nodes of the Galton-Watson tree and $\varsigma(m)$ stands for the number of zeroes in m. Moreover, all variables in Equation (53) are independent, and in Equation (52), the variables corresponding to the same k are independent.

Then, arguing as previously yields the convergence in distribution of $((\mathbb{E}\,N)^{-1}(b-1))^{n/2}\big(W_n - 1\big)$ as in Section 6, with U the limit of the Mandelbrot multiplicative martingale built on the tree $\bigcup_{n\geq 1} \mathscr{T}_n \times \mathcal{T}_n$ with the random vectors $A(w,m) = (A_{j,\varepsilon}(w,m))_{(j,\varepsilon)\in\mathscr{T}_1\times\{0,1,\cdots,N(w,m)\}}$, where

$$A_{j,0}(w,m) = a_j^2(w,m), \quad \text{and} \quad A_{j,\varepsilon}(w,m) = (\mathbb{E}\,N)^{-1}(b-1)a_j^2(w,m)$$

for $1 \leq \varepsilon \leq N(w,m)$, and the

$$\big((a_j(w,m))_{j\geq 1}, N(w,m)\big), \text{ for } (w,m) \in \bigcup_{n=0}^{\infty} \mathscr{T}_n \times \mathcal{T}_n$$

are independent copies of $((a_j)_{j\geq 1}, N)$.

8. A functional central limit theorem in the quadratic case. We suppose that $N \equiv 1$ and that $q = 1$. We are going to use words on two alphabets. The ones, denoted by w, v, $v' \ldots$ are finite sequences of positive integers. The others, denoted by m, m' are finite sequences of 0 and 1. It will be convenient to denote 0^j the word composed of j zeroes. Also, the concatenation will be either denoted simply by juxtaposition or by a dot. The expression $v \cdot w|_j$ should be understood as $v \cdot (w|_j)$.

8.1. Another writing for the martingale $(U_n)_{n\geq 1}$ and its limit. We assume that $q = 1$ and that the Mandelbrot martingale $(U_n)_{n\geq 1}$ of the Section 5 converges almost surely and in L^1 to its limit, i.e., is non degenerate. This is the case for instance under the assumptions of Theorems 5.3 or 6.4. We have

$$U_n = \sum_{|w|=|m|=n} (b-1)^{n-\varsigma(m)} \prod_{0\leq j<n} a_{w_{j+1}}^2(w|_j, m|_j). \tag{54}$$

We also consider the following copies of U_n: for w and m with the same length, we set

$$U_n(w,m) = \sum_{|w'|=|m'|=n} (b-1)^{n-\varsigma(m')} \prod_{0\leq j<n} a_{w'_{j+1}}^2(w\cdot w'|_j, m\cdot m'|_j).$$

In Formula (54), we split the summation according to the length of the prefix of maximal length of the form 0^k of m. We obtain

$$U_n = \sum_{|v|=n} \prod_{0\le j<n} a^2_{v_{j+1}(v|_i,0^j)}$$
$$+ \sum_{\substack{0\le k<n \\ |v|=n \\ |m'|=n-k-1}} (b-1)^{n-k-\varsigma(m')} \prod_{0\le j\le k} a^2_{v_{j+1}}(v|_j,0^j) \prod_{0\le j<n-k-1} a^2_{v_{k+j+2}}(v|_{j+k+1},0^k1\cdot m'|_j).$$

If we write $v = w\cdot w'$ with $|w| = k+1$, the second term of the right hand side of the last formula rewrites as

$$\sum_{0\le k<n}\sum_{|w|=k+1} (b-1) \prod_{0\le j\le k} a^2_{w_{j+1}}(w|_j,0^j)$$
$$\times \sum_{|w'|=|m'|=n-k-1} (b-1)^{n-k-1-\varsigma(m')} \prod_{0\le j<n-k-1} a^2_{w'_{j+1}}(w\cdot w'|_j,0^k1\cdot m'|_j).$$

Finally, we get

$$U_n = \sum_{|w|=n}\prod_{0\le j<n} a^2_{w_{j+1}(w|_j,0^j)}$$
$$+ \sum_{0\le k<n}\sum_{|w|=k+1}(b-1)\Big(\prod_{0\le j\le k} a^2_{w_{j+1}}(w|_j,0^j)\Big)U_{n-k-1}(w,0^k1).$$

It is worth noticing that the variables $U_{n-k-1}(w,0^k1)$, are independent.

We wish to prove the following formula:

$$U = \sum_{k\ge 0}\sum_{|w|=k+1}(b-1)\Big(\prod_{0\le j\le k} a^2_{w_{j+1}}(w|_j,0^j)\Big)U(w,0^k1),$$

where $U(w,0^k1)$ is the almost sure limit of $U_n(w,0^k1)$. To do so, denote by U' the right hand side in the above equality. We have

$$\|U'-U_n\|_1 \le (b-1)\sum_{k=1}^{n}\Big(\mathbb{E}\sum_{i\ge 1} a_i^2\Big)^k \|U-U_{n-k}\|_1$$
$$+(b-1)\sum_{k\ge n}\Big(\mathbb{E}\sum_{i\ge 1} a_i^2\Big)^k + \Big(\mathbb{E}\sum_{i\ge 1} a_i^2\Big)^n.$$

But U_k converges to U almost surely and in L^1, and $\mathbb{E}\sum_{i\ge 1} a_i^2 < 1$, it follows that $\|U'-U_n\|_1$ tends to 0 as $n\to\infty$, so that $U = U'$ almost surely.

Notice that all the copies of U involved in U' are independent, so the above relation rewrites

$$U = (b-1)\sum_{k=1}^{\infty}\sum_{v\in\mathscr{T}_k}\left(\prod_{j=0}^{k-1} a^2_{v_{j+1}}(v|_j)\right) U(v),$$

where $(U(v))_{v\in\bigcup_{n\ge 1}\mathbb{N}^n)}$ is a family of independent copies of U, which is independent of the family of independent copies of a, $(a(w))_{w\in\bigcup_{n\ge 0}\mathbb{N}_+^n}$.

Remark 8.1. In the general case (i.e., N is not identically 1) we have

$$U_n = \sum_{k=0}^{n-1} \sum_{\substack{v\in\mathscr{T}_k\\ 1\le m\le N(v,0^{(k)})\\ i\ge 1}} \left(\prod_{j=0}^{k-1} a^2_{v_{j+1}}(v|_j, 0^{(j)})\right)\left(\frac{b-1}{\mathbb{E}\,N}\right) a_i^2(v,0^{(k)})\,U_{n-k-1}(vi, 0^{(k)}\cdot m) + \sum_{v\in\mathscr{T}_n}\prod_{j=0}^{n-1} a^2_{v_{j+1}}(v|_j, 0^{(j)}),$$

where the random variables $U_{n-k-1}(vi, 0^{(k)}\cdot m)$ are independent copies of U_{n-k-1}.

8.2. A CLT for random finitely additive measures.

8.2.1. Some random finitely additive measures. We start by defining a class of finitely additive random measures. We will see that the limit object of the central limit theorem obtained in the next subsection belongs to this class.

Suppose again that the Mandelbrot martingale $(U_n)_{n\ge1}$ converges almost surely and in L^1.

Let ξ stand for a centered normal vector with covariance matrix A and independent of U. Then consider a family $(U(w),\xi(w))_{w\in\bigcup_{n\ge1}\mathbb{N}^n}$ of independent copies of (U,ξ). Also, consider $(a(w))_{w\in\bigcup_{n\ge0}\mathbb{N}^n}$ a family of independent copies of a, which is independent of $(U(w),\xi(w))_{w\in\bigcup_{n\ge1}\mathbb{N}^n}$. By the calculation of the above paragraph, for all $w\in\bigcup_{n\ge1}\mathbb{N}^n$, the sequence of random variables

$$X_n(w) = \sum_{k=1}^{n}\sum_{|v|=k}\left(\prod_{j=0}^{k-1} a_{v_{j+1}}(w\cdot v_{|j})\right)\sqrt{U(w\cdot v)}\,\xi(w\cdot v)\quad (n\ge 1),$$

which, conditionally on $\sigma(\{a(w),U(w)\})$, is a martingale bounded in L^2 converging almost surely to a random variable $X(w)$, which is a centered normal vector whose covariance matrix equals $(b-1)^{-1}\cdot A$ times a copy of U. In other words $X(w)=(b-1)^{-1/2}\sqrt{\widetilde{U}(w)}\widetilde{\xi}(w)$, where $\widetilde{U}(w)\sim U$, $\widetilde{\xi}(w)\sim\xi$, and $\widetilde{U}(w)$ and $\widetilde{\xi}(w)$ are independent. Moreover, the random vectors $(\widetilde{U}(w),\widetilde{\xi}(w))_{w\in\mathbb{N}^n}$ are independent, and also independent of $\sigma(\{a(w_{k-1}),U(w),\xi(w) : w\in\bigcup_{k=1}^n\mathbb{N}_+^k)\})$.

Also, we have the relation

$$X(w) = \sum_{|i|=1} a_i(w)\big(X(wi)+\sqrt{U(w\cdot i)}\,\xi(w\cdot i)\big). \tag{55}$$

Consequently,

$$M(w) = \left(\prod_{j=0}^{|w|-1} a_{w_{j+1}}(w_{|j})\right)\left(\sqrt{\widetilde{U}(w)}\widetilde{\xi}(w)+\sqrt{b-1}\sum_{j=1}^{|w|}\sqrt{U(w_{|j})}\xi(w_{|j})\right)$$

defines a $\mathbb{R}^2$-valued random finitely additive measure on $\mathscr{T}$. We notice that this measure is obtained as the limit of a mixture of additive and multiplicative cascades.

8.2.2. CLT for random measures. Now suppose that $\gamma \in (1, \beta)$ and $\mu \in \mathscr{P}_\gamma$.

Fix a sequence $\big(a(w)\big)_{w\in\mathscr{T}}$ of independent copies of a. For each $n \geq 0$, a complex or equivalently $\mathbb{R}^2$-valued random measure on $\mathscr{T}$ is naturally associated with $\mathsf{T}^n(\mu)$: consider a sequence $\big(W_n(w)\big)_{w\in\mathscr{T}}$ of independent variables equidistributed with $W_n \sim \mathsf{T}^n(\mu)$, and independent of $\big(a(w)\big)_{w\in\mathscr{T}}$. Then define

$$W_{n+1}(w) = \lim_{p\to\infty} \sum_{|v|=p} \prod_{k=0}^{p-1} a_{v_{k+1}}(w \cdot v|_k) W_n(w \cdot v|_{k+1})$$

and

$$\nu_n(w) = W_{n+1}(w) \prod_{k=0}^{|w|-1} a_{w_{k+1}}(w|_k) W_n(w|_{k+1}).$$

When μ is supported on $\mathbb{R}_+$ this measure coincides with the restriction to cylinders of so-called Mandelbrot measure supported on the boundary of $\mathscr{T}$ and associated with the family of vectors $(a_i(w), W_n(w \cdot i))_{i\geq 1}$, $w \in \mathscr{T}$.

Also, let ν be the conservative Mandelbrot measure built from the family of vectors $(a_i(w))_{i\geq 1}$, $w \in \mathscr{T}$, i.e., $\nu = \nu_0$ when $\mu = \delta_1$.

It is then almost direct to get the following result from Theorems 5.3 and 6.4 :

Theorem 8.2. *Suppose that either the assumptions of Theorem 6.4 are fulfilled or those of Theorem 5.3 are fulfilled if W_0 is non negative. For each $p \geq 0$, $\Big((b-1)^{(n+1)/2}(\nu_n(w) - \nu(w))\Big)_{w\in\bigcup_{k=0}^p \mathscr{T}_k}$ converges in law to $(M(w))_{w\in\bigcup_{k=0}^p \mathscr{T}_k}$ as $n \to \infty$, with $A = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$.*

Proof. Given $p \geq 0$, for $n \geq p$, and $w \in \mathscr{T}_p$, we can write

$$\nu_n(w) - \nu(w) = \left(\prod_{k=0}^{p-1} a_{j_{k+1}}\right)\left(W_{n+1}(w) \prod_{k=1}^{p} W_n(w|_k) - 1\right).$$

Moreover,

$$W_{n+1}(w) \prod_{k=1}^{p} W_n(w|_k) - 1 = \theta_{n+1}(W_{n+1}(w) - 1) + \sum_{k=1}^{p} \theta_{n,k}(W_n(w|_k) - 1),$$

where the random variables θ_{n+1} and $\theta_{n,k}$ are products of p independent random variables all converging to 1 in law as $n \to \infty$ and uniformly bounded in L^2. Also, due to Theorem 5.3, for each $0 \leq k \leq p$, $(b-1)^{n/2}(W_n(w_{|k}) - 1)$ converges in law to a copy $\sqrt{U(w_{|k})}\xi(w_{|k})$ of $\sqrt{U}\xi$, as well as $(b-1)^{(n+1)/2}(W_{n+1}(w) - 1)$ to such a $\sqrt{\widetilde{U}(w)}\widetilde{\xi}(w)$. Due to the independence properties of the random variables defining ν_n and the relation (55) associated with M, the conclusion follows by induction on p.

Remark 8.3. It seems not clear at the moment to associate a functional CLT with a general distribution for N.

8.3. A functional CLT for random continuous functions on $[0,1]$**.** We still assume that we are in the quadratic case. Moreover, we assume that $\mathscr{T}$ is a c-adic tree, with $c \geq 2$ (this means that $a_j = 0$ for $j > c$); it is easily seen that one must have $c \geq b$. If $w \in \mathscr{T}$, the closed c-adic interval naturally encoded by w is denoted by I_w, and given a complex or $\mathbb{R}^2$-valued function f defined over $[0,1]$, the increment of f over I_w is denoted $\Delta(f, I_w)$.

Suppose that $q = 1$ and the martingale $(U_n)_{n\geq 1}$ converges in $L^{1+\varepsilon}$. Then, with the notations of the previous section, it is direct that for all $p \geq 1$,

$$\mathbb{E} \sum_{w\in\mathscr{T}_p} |M(w)|^2 = \mathrm{O}\Big(p^2 \Big(\sum_{i=1}^{c} a_i^2\Big)^p\Big) = \mathrm{O}(c^{-p\gamma})$$

for some $\gamma > 0$. It follows that $\sup_{w\in\mathscr{T}_p} |M(w)|$ tends to 0 exponentially fast. Consequently, the process F defined on the c-adic numbers of $[0,1]$ by $F(0) = 0$ and $\Delta(F, I_w) = M(w)$ (this definition is consistent since M is a measure) extends to a unique complex-valued Hölder continuous function over $[0,1]$, still denoted by F.

Now suppose $b > 2$, $\gamma \in (1, \beta)$ and $\mu \in \mathscr{P}_\gamma$. At first, the complex-valued process defined for each $n \geq 1$, by $G_n(0) = 0$ and $\Delta(G_n, I_w) = \nu_n(w)$ for each $w \in \mathscr{T}$ can be shown to extend to a unique Hölder continuous function (see [5, Theorem 2.1]), hence if $\mu \neq \delta_1$, the relations $F_n(0) = 0$ and $\Delta(F_n, I_w) = (b-1)^{(n+1)/2}(\nu_n(w) - \nu(w))$ define a unique Hölder-continuous function.

THEOREM 8.4. *Suppose that either the assumptions of Theorem 6.4 are fulfilled or those of Theorem 5.3 are fulfilled if W_0 is non negative. Suppose also that $\mathscr{T}$ is a c-adic tree. The sequence $(F_n)_{n\geq 1}$ converges in law to F as $n \to \infty$.*

In [6] we obtained this result when $a_j = c^{-1}$ for all j, in which case $U = 1$ almost surely, and with μ supported on $\mathbb{R}_+$, which implies that $y = z = 0$ and so F is real valued.

Proof. Due to Theorem 8.2 we only have to prove the tightness of the sequence of laws of the functions F_n, $n \geq 1$. This is quite similar to the proof of Proposition 9 in [6], but for reader's convenience we include some details. The same arguments as those used to prove Theorem 8.2 imply that for all $n \geq 1$, for all $p \geq 1$

$$\mathbb{E}\Big(\sum_{w\in\mathscr{T}_p} |\Delta(F_n, I_w)|^2\Big) = \mathrm{O}\Big(p^2 \Big(\sum_{i=1}^{c} a_i^2\Big)^p\Big) = \mathrm{O}(p^2 c^{-p\gamma}),$$

where O is uniform with respect to n. For any $t > 0$ this yields

$$\mathbb{P}(\exists\, w \in \mathscr{T}_p,\ |\Delta(F_n, I_w)| \geq t\, c^{-p\gamma/4}) \leq \mathrm{O}(t^{-2} p^2 c^{-p\gamma/2}).$$

Let $\omega(F_n, \cdot)$ stand for the modulus of continuity of F_n. Fix $\varepsilon > 0$. It is standard that if $\delta \in (0,1)$ and $p_\delta = -\log_c(\delta)$,

$$\begin{aligned}\big\{\omega(F_n, \delta) \geq 2(c-1)\,\varepsilon\big\} &\subset \Big\{\sum_{p\geq p_\delta} \sup_{w\in\mathscr{T}_p} \Delta(F_n, I_w) > \varepsilon\Big\}\\ &\subset \bigcup_{p\geq p_\delta} \Big\{\sup_{w\in\mathscr{T}_p} \Delta(F_n, I_w) > (1 - c^{-\gamma/4})\, c^{p_\delta\gamma/4}\, \varepsilon\, c^{-p\gamma/4}\Big\},\end{aligned}$$

so

$$\mathbb{P}(\omega(F_n,\delta)\geq 2(c-1)\,\varepsilon)=\mathrm{O}\left(\frac{c^{-p_\delta\gamma/2}}{(1-c^{-\gamma/4})^2\varepsilon^2}\sum_{p\geq p_\delta}p^2c^{-p\gamma/2}\right)$$

uniformly in $n\geq 1$. Consequently,

$$\lim_{\delta\to 0}\sup_{n\geq 1}\mathbb{P}\big(\omega(F_n,\delta)>2(c-1)\,\varepsilon\big)=0,$$

which yields the desired tightness (see [10]). ☐

8.4. Multifractal analysis of the increments of the limit process F. We work under the assumptions of the previous section defining F as a non trivial Hölder continuous function. At first we notice that

$$\Delta(F,I_w)=\nu(w)\left(\sqrt{\widetilde{U}(w)}\widetilde{\xi}(w)+\sqrt{b-1}\sum_{j=1}^{n}\sqrt{U(w_{|j})}\xi(w_{|j})\right).$$

To simplify the purpose, we assume that the a_i, $i\geq 1$ do not vanish almost surely. If $x\in(0,1)$, denote by $w_n(x)$ the c-adic word w of generation n encoding the unique semi-open to the right c-dic interval which contains x.

The sequence $\left(\frac{\log(\nu(w_n(x)))}{n},\frac{\Delta(F,I_{w_n(x)}))}{\sqrt{b-1}n\,\nu(w_n(x))}\right)$, $n\geq 1$, provides a fine description of the asymptotic behavior of $\Delta(F,I_{w_n(x)})$. It is essentially the $\mathbb{R}^3$-valued branching random walk with independent components associated with the vectors $(\log(a_i(w)),\sqrt{U(wi)}\xi(wi))_{1\leq i\leq c}$, $w\in\mathscr{T}$.

For all subsets K of $\mathbb{R}^2$ set

$$E(K)=\left\{x\in(0,1):\bigcap_{N\geq 1}\overline{\left\{\left(\frac{\log(\nu(w_n(x)))}{n},\frac{\Delta(F,I_{w_n(x)}))}{\sqrt{b-1}n\,\nu(w_n(x))}\right):n\geq N\right\}}=K\right\}.$$

For $(q_1,q_2)\in\mathbb{R}\times\mathbb{R}^2$, set

$$P(q_1,q_2)=\log\left(\mathbb{E}\sum_{i=1}^{c}a_i^{q_1}\right)+\log\mathbb{E}\,e^{\langle q_2|\sqrt{U}\xi\rangle},$$

and for $(\gamma_1,\gamma_2)\in\mathbb{R}\times\mathbb{R}^2$, let

$$P^*(\gamma_1,\gamma_2)=\inf\{P(q_1,q_2)-\gamma_1q_1-\langle\gamma_2|q_2\rangle:(q_1,q_2)\in\mathbb{R}\times\mathbb{R}^2\}$$

be the concave Legendre transform of P at (γ_1,γ_2).

As a consequence of the general study of the multifractal behavior of vector valued branching random walks achieved in [4], we have:

THEOREM 8.5. *With probability 1, for all compact connected subsets K of $\mathbb{R}^3$, we have*

$$\dim E(K)=\frac{1}{\log(c)}\inf\{P^*(\gamma):\gamma\in K\},$$

where dim *stands for the Hausdorff dimension and a negative dimension means that* $E(K)$ *is empty.*

This is a non trivial extension of the result obtained in [6] in the case where $a_j = c^{-1} = b^{-1}$ for all j (hence ν is the Lebesgue measure and $U = 1$) and μ is supported on $\mathbb{R}_+$, which implies that the multifractal analysis reduces back to that of a centered Gaussian branching random walk in $\mathbb{R}$.

REMARK 8.6. It is worth specifying that $\mathbb{E}\big(e^{\langle q_2|\sqrt{U}\xi\rangle}|U\big) = e^{UQ(q_2)}$, where Q is a nonnegative non-degenerate quadratic form which is positive definite if and only if A is invertible. Moreover, we saw in Section 6.1 that A is invertible if and only if $\mathbb{P}(W \in \mathbb{C}\setminus\mathbb{R}) > 0$. In addition, by [16, Theorem 2.1] one has three situations for the behavior of the moment generating function of U:

(1) there exists $p > 1$ such that $\mathbb{E}\,U^p = \infty$, i.e. $((b-1)^p + 1)\,\mathbb{E}\sum_{i\geq 1} a_j^{2p} \geq 1$. Then, for $r \geq 0$ one has $\mathbb{E}\,e^{rU} < \infty$ only if $r = 0$. In particular, if Q is positive definite, then the domain of P reduces to $\mathbb{R}\times\{(0,0)\}$.

(2) If $\mathbb{E}\,U^p < \infty$ for all $p \geq 1$ and $\operatorname{ess\,sup}((b-1)^{p_0} + 1)\sum_{i=1}^{c} a_j^{2p_0} < 1$ for some $p_0 > 1$, then $\mathbb{E}\,e^{rU} < \infty$ for all $r \geq 0$, and the domain of P is $\mathbb{R}^3$.

(3) If $\mathbb{E}\,U^p < \infty$ and $\operatorname{ess\,sup}((b-1)^p + 1)\sum_{i\geq 1} a_j^{2p} \geq 1$ for all $p \geq 1$, then $\mathbb{E}\,e^{rU} < \infty$ for some $r > 0$, hence the domain of P contains $\mathbb{R}\times V$, where V is a neighborhood of $(0,0)$.

9. Some questions about fixed points of nonlinear smoothing transformations. Let us finish by addressing a remark and a few questions.

(1) One can wonder whether S_2 may have fixed points in the space $\mathcal{P}_+$ of probability measures on $\mathbb{R}_+$ with infinite first moment, like S_1 does (see [12, 17, 9, 1]). It turns out that this is not the case under mild conditions. Suppose that $\#\{i \geq 1 : a_i > 0\}$ is bounded and that $\mu \in \mathcal{P}_+$ is a fixed point of S_2 with $\mathbf{m}_1(\mu) = \infty$. Write the equality $\mu = \mathsf{S}_2(\mu)$ in the form $Y = \sum_{i\geq 1}(a_i Y_i)\widetilde{Y}_i$, so that μ is a fixed point of the mapping $\mathsf{U}_{\widetilde{a}}$ defined with $(\widetilde{a}_i = a_i Y_i)_{i\geq 1}$. We can use the theory of the fixed point of U [12, 2] to claim that there must exist a unique $\alpha \in (0,1]$ such that $\mathbb{E}\sum_{i\geq 1} a_i^\alpha Y_i^\alpha = 1$ and $\mathbb{E}\sum_{i\geq 1} a_i^\alpha Y_i^\alpha \log(a_i^\alpha Y_i^\alpha) \leq 0$. In particular, $\mathbb{E}\,Y^\alpha < \infty$, so $\alpha < 1$. Moreover, there exists a random variable Z, namely a fixed point of the mapping $\mathsf{U}_{a^{(\alpha)}}$ defined with $a_i^{(\alpha)} = a_i^\alpha Y_i^\alpha$, such that $Y = \mathscr{L}(Z^{1/\alpha}X)$, where X is a positive stable law of index α. In particular, $\mathbb{E}\,Y^\alpha = (\mathbb{E}\,Z)\,\mathbb{E}\,X^\alpha = \infty$, which is a contradiction.

(2) Under what necessary and sufficient condition on $(a_i)_{i\geq 1}$ does the martingale $(\Xi_n)_{n\geq 1}$ of Section 3.3 converge to a non degenerate random variable Ξ_∞?

At least we know that $\mathbb{E}\,\Xi_\infty = (\mathbb{E}\,\Xi_\infty)^2$, hence in case of non degeneracy the convergence holds in L^1.

(3) When Ξ_∞ is non degenerate as in Section 3.3 and $q < 1$, is it possible to tell something about its moments of negative order (recall that Theorem 3.4 concludes to the explosion of the moments of positive order)?

(4) How to develop an L^p theory ($1 \leq p \leq 2$) rather than the L^2 considered in this paper for the iteration of Mandelbrot cascades?

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