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THE DEFORMATION OF PAIRS (X, E) LIFTING FROM BASE FAMILY*

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Abstract. We study the holomorphic family of pairs $\{(X_t, E_t)\}$ by the calculation of Kuranishi data on $\{E_t\}$, where each E_t is a holomorphic vector bundle over the compact complex manifold X_t. The splitting of holomorphic cotangent bundle over E_0 via any integrable connection decomposes the Kuranishi data into the horizontal part and the vertical part. The horizontal part is the Kursnishi data of base family $\{X_t\}$. When the vertical part is vanishing under the decomposition by a Nakano semi-positive Chern connection ∇, i.e. $\{(X_t, E_t)\}$ is lifting from base family $\{X_t\}$ via ∇, we get a infinitesimal extension of $\bar{\partial}$-closed bundle valued (n, q)-form by the recursive method.

Key words. Kuranishi data, Deformation of pairs, Recursive method.

Mathematics Subject Classification. 32G05, 32G08.

1. Introduction. This article aims to study the holomorphic family of pairs $\{(X_t, E_t)\}$ through the calculation of Kuranishi data in the setting of clarified Kuranishi theory in [3, 4]. The family of pairs $\{(X_t, E_t)\}$ means that both the holomorphic structure on the vector bundle E_t and the complex structure on the base manifold X_t vary simultaneously.

As in [4], the Kuranishi data $\xi(t)$ (it is also called Beltrami differential in [9] and [12]) of a holomorphic family $\{X_t\}$ describes the varying of complex structures. It is a linear map defined by two projections from holomorphic cotangent bundle of X_t, i.e.,

$$\xi(t) := \pi^{0,1} \circ (\pi^{1,0})^{-1} : T^{1,0}(X_0) \to T^{1,0}(X_t) \to T^{0,1}(X_0),$$

$$\pi^{1,0} \oplus \pi^{0,1} : T^1(X_0) \to T^{1,0}(X_0) \oplus T^{0,1}(X_0),$$

where $T^1(X_0)$ is the direct sum of the holomorphic cotangent bundle $T^{1,0}(X_0)$ and the antiholomorphic cotangent bundle $T^{0,1}(X_0)$.

The main construction of Kuranishi data is recalled in Section 2, since we will use the analogous details to calculate the Kuranishi data of $\{E_t\}$ as a family of complex manifolds in Section 4. In this situation, the local trivialization of vector bundle E_0 determines a holomorphic coordinate $(U_\alpha, x_\alpha, v_\alpha)$ on E_0 as a holomorphic coordinate of complex manifold. The Kuranishi data of $\{E_t\}$ is represented in this coordinate.

In order to show the relation between the Kuranishi data of base family $\{X_t\}$ and the Kuranishi data of $\{E_t\}$, we introduce the splitting of the holomorphic cotangent bundle $T^{1,0}(E_0)$ in Section 3. More precisely, if ∇ is an integrable connection (i.e. $\nabla = \bar{\partial} + \nabla^{1,0}$) of holomorphic vector bundle $p : E \to X$, then the holomorphic cotangent bundle $T^{1,0}(E)$ of E can be split into two parts via ∇, i.e.,

$$T^{1,0}(E) = p^*(T^{1,0}(X)) \oplus \left(T^{1,0}(E) - q_\nabla(T^{1,0}(E))\right),$$

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where the projection q_∇ is defined by connection 1-forms, and p^* is the pull-back map. As a result, we get the following decomposition.

THEOREM 1.1. *The Kuranishi data* $\xi(t)$ *can be decomposed into two parts*

$$\xi(t) = \xi(t) \circ q_\nabla + \xi(t) \circ (Id - q_\nabla) =: \phi(t) + \psi(t), \tag{1.1}$$

in the sense of $\psi(t) \in A^{0,1}(X_0, End(E_0))$ *and* $\phi(t) \in A^{0,1}(X_0, T_{1,0}(X_0))$.

Conversely, we also prove that when the Kuranishi data $\phi(t) \in A^{0,1}(X_0, T_{1,0}(X_0))$ of base family and $\psi(t) \in A^{0,1}(X_0, End(E_0))$ are given, there is a necessary and sufficient condition to make $\xi(t)$ integrable.

THEOREM 1.2. *The Kuranishi data defined by (1.1) is integrable, i.e., it represents a complex structure of pair* (X_t, E_t) *for each* t, *if and only if*

$$\bar{\partial}\phi(t) = \frac{1}{2}[\phi(t), \phi(t)],$$

and

$$(\bar{\partial} - \mathcal{L}_{\phi(t)})\psi(t) - \psi(t) \wedge \psi(t) - i_{\phi(t)}\Theta_0^{1,1} - \frac{1}{2} i_{\phi(t)} \circ i_{\phi(t)}\Theta = 0.$$

Here Θ is the curvature of integrable connection ∇ and $\Theta_0^{1,1}$ is the $(1,1)$-type projection with respect to the complex structure of X_0, and $\mathcal{L}_{\phi(t)}$ is defined by (2.1). Particularly, if ∇ is a Chern connection, the second equation is

$$(\bar{\partial} - \mathcal{L}_{\phi(t)})\psi(t) - \psi(t) \wedge \psi(t) - i_{\phi(t)}\Theta = 0.$$

In the classical Kodaira-Spencer-Kuranishi deformation theory, Kuranishi data provides a criterion of holomorphic functions (see Lemma 2.2). Through the decomposition (1.1) we get an analogous criterion for holomorphic sections.

COROLLARY 1.3. *A smooth section* s *is holomorphic under the complex structure of* (X_t, E_t) *if and only if*

$$\left(\bar{\partial}_{E_0} - \mathcal{L}_{\phi(t)} + \psi(t)\right) s = 0.$$

In particular, we have

COROLLARY 1.4. *The decomposition (1.1) induced by a Chern connection* ∇ *on* E_0 *satisfies* $\psi(t) \equiv 0$ *if and only if* ∇ *is also a Chern connection on* E_t.

We call this type of pair deformation $\{(X_t, E_t)\}$ as *a lifting from base family by* ∇. In Section 5, we generalize the iteration method in [9] to construct an extension of bundle valued (n, q)-forms when the pair deformation $\{(X_t, E_t)\}$ is a lifting from base family $\{X_t\}$ by a Nakano semi-positive Chern connection ∇ of E_0.

THEOREM 1.5. *Let the deformation* $\{(X_t, E_t)\}$ *be a lifting from base holomorphic family* X_t *by a Chern connection* ∇ *on* E_0 *with Nakano semi-positive curvature* Θ, *and the central fiber* X_0 *be Kähler. Denote the Kuranishi data of base family* $\{X_t\}$ *as*

$$\phi(t) = \sum \varphi_{\gamma_1 \cdots \gamma_N} t_1^{\gamma_1} \cdots t_N^{\gamma_N},$$

for $|t| < \varepsilon$. Then for any $\bar{\partial}$-closed $s \in A^{n,q}(X_0, E_0)$, we can construct a convergent power series

$$s_t = s_0 + \sum_{|I|>0}^{\infty} s_I t^I \in A^{n,q}(X_0, E_0)$$

such that $s_0 = s$ with the following properties:
(a) $s_t^C := e^{i_{\phi(t)}} s_t$ *is $\bar{\partial}_t$-closed with respect to E_t,*
(b) s_I *is $\bar{\partial}^*$-exact for all $|I| \geq 1$,*
(c) *if s is an E_0-valued $(n, 0)$ form, s_t^C is a smooth extension of s.*

Finally, in Section 6, we compute the Kuranishi data of holomorphic tangent bundles as an application.

The analytic theory of pair deformation $\{(X_t, E_t)\}$ has been studied by [6], and recently by [1]. Differing from us, they emphasized the construction of DGLA and corresponding Maurer-Cartan equation to study the local moduli space and obstruction problems. Although we get the same results of the holomorphic criterion and integrable condition, we refine the relation between pair deformation and base family, and our key point is the extension of $\bar{\partial}$-closed bundle valued (n, q)-forms.

2. Preliminaries. In this section, some basic notations and results on analytic deformation theory in [9] and [4] are recalled. Let $\pi : X \to \Delta$ be a holomorphic family of compact complex manifolds with dimension n, where $\Delta = \left\{t \in \mathbb{C}^k : |t_i| << 1\right\}$ is a small neighborhood of 0. It means that $\pi : X \to \Delta$ is a proper holomorphic submersion, and we always denote the complex submanifold $\pi^{-1}(t)$ as X_t.

Definition 2.1 (Definition 4.1 in [4]). *A C^∞-diffeomorphism*

$$F_\sigma = (\sigma, \pi) : X \to X_0 \times \Delta$$

will be called transversely holomorphic trivialization of the holomorphic family X/Δ if $\sigma^{-1}(x_0)$ is an analytic polydisk for each $x_0 \in X_0$, and $\sigma|_{X_0} = Id$.

The existence of transversely holomorphic trivialization on a small polydisc has been proved in Appendix A of [4](also see Propostion 9.5 in [14] and Theorem IV.31 in [10]). Through the trivialization we can treat the holomorphic family $\pi : X \to \Delta$ as an analytic family of complex structures on X_0. Given any transversely holomorphic trivialization F_σ, there is a family of C^∞-isomorphism

$$\sigma_t : X_t \to X_0, \quad X_t := \pi^{-1}(t)$$

induced by σ. Denote $T_{1,0}(X_t)$ and $T^{1,0}(X_t)$ as the holomorphic tangent bundle and holomorphic cotangent bundle of X_t respectively. Then $T^{1,0}(X_t)$ corresponds to a smooth subbundle

$$T_t^{1,0} = \sigma_t^{-1*}\left(T^{1,0}(X_t)\right) \subset T^1(X_0).$$

If

$$\pi^{1,0} \oplus \pi^{0,1} : T^1(X_0) \to T^{1,0}(X_0) \oplus T^{0,1}(X_0)$$

are two projections of complexificated cotangent bundle $T^1(X_0)$. The $(1, 0)$ projection

$$\pi^{1,0} : T_t^{1,0} \to T^{1,0}(X_0)$$

is an isomorphism for small t, so that the composition

$$\pi^{0,1} \circ (\pi^{1,0})^{-1} : T^{1,0}(X_0) \to T_t^{1,0} \to T^{0,1}(X_0)$$

defines a smooth mapping

$$\xi(t) : T^{1,0}(X_0) \to T^{0,1}(X_0),$$

and

$$T_t^{1,0} = \{u + \xi(t)u | u \in T^{1,0}(X_0)\}.$$

The linear map can be regarded as an element $\xi(t) \in A^{0,1}\left(X_0, T_{1,0}(X_0)\right)$, i.e, holomorphic tangent bundle valued form of X_0. It is called *Kuranishi data* for the transversely holomorphic trivialization F_σ.

Conversely, the Kuranishi data represents a family of complex structures on X_0 if and only if the Maurer-Cartan equation

$$\bar{\partial}\xi(t) = \frac{1}{2}[\xi(t), \xi(t)]$$

is valid by the integrable condition in Appendix C of [4]. Notice that the Lie bracket of $\phi \in A^{0,p}\left(X, T_{1,0}(X)\right)$ and $\psi \in A^{0,q}\left(X, T_{1,0}(X)\right)$ is defined by

$$[\phi, \psi] = \sum_{i,j} \left(\phi^i \wedge \partial_i \psi - (-1)^{pq} \psi^i \wedge \partial_j \phi^j\right) \otimes \partial_j.$$

For $\varphi \in A^{0,p}\left(X, T_{1,0}(X)\right)$ we denote the action via contraction as i_φ, and the Lie derivative as

$$L_\varphi := i_\varphi \circ d + (-1)^p d \circ i_\varphi.$$

In [9] the Lie derivative is also defined on vector bundle valued forms by

$$\mathcal{L}_\varphi := i_\varphi \circ \nabla + (-1)^p \nabla \circ i_\varphi \tag{2.1}$$

where ∇ is a connection of vector bundle.

The following criterion of holomorphic functions has been proved by many authors in different view points(see Proposition 1.2 in [11], Lemma 4.2 in [4], and Proposition 1.5 in [15]). We recall the proof to see that the criterion is valid when the Kuranishi data is given by a suitable trivialization of vector bundle. In Section 4, we will construct the compatible trivialization from vector bundle E_t to the central vector bundle E_0. The criterion will be utilized to prove the criterion of holomorphic sections.

Lemma 2.2. *Let $\sigma_t : X_t \to X_0$ be the diffeomorphism induced by holomorphic transversely trivialization F_σ. Then for any smooth function $f \in A^0(X_t)$,*

$$\bar{\partial}_t f = 0$$

if and only if

$$\left(\bar{\partial}_0 - L_{\xi(t)}\right)(f \circ \sigma_t^{-1}) = 0. \tag{2.2}$$

Proof. Since

$$\sigma_t^{-1*} \circ \partial_t f = \left(\frac{\partial f}{\partial x_t} \circ \sigma_t^{-1} \right) \cdot \frac{\partial x_t}{\partial x_0} (dx_0 + i_{\xi(t)} dx_0)$$

by the definition of Kuranishi data, we have

$$\begin{aligned} \sigma_t^{-1*} \circ \bar{\partial}_t f &= \sigma_t^{-1*} \circ df - \sigma_t^{-1*} \circ \partial_t f \\ &= \bar{\partial}_0 (f \circ \sigma_t^{-1}) - L_{\xi(t)} (f \circ \sigma_t^{-1}) \\ &\quad + \left(\frac{\partial f}{\partial \bar{x}_t} \circ \sigma_t^{-1} \right) \cdot \frac{\partial \bar{x}_t}{\partial x_0} dx_0 + i_{\xi(t)} \left(\frac{\partial f}{\partial \bar{x}_t} \circ \sigma_t^{-1} \right) \cdot \frac{\partial \bar{x}_t}{\partial x_0} dx_0. \end{aligned}$$

When $\bar{\partial}_t f = 0$, (2.2) is valid obviously.

Conversely, when

$$\left(\bar{\partial}_0 - L_{\xi(t)} \right) (f \circ \sigma_t^{-1}) = 0,$$

$$\sigma_t^* \left(\left(\frac{\partial f}{\partial \bar{x}_t} \circ \sigma_t^{-1} \right) \cdot \frac{\partial \bar{x}_t}{\partial x_0} dx_0 + i_{\xi(t)} \left(\frac{\partial f}{\partial \bar{x}_t} \circ \sigma_t^{-1} \right) \cdot \frac{\partial \bar{x}_t}{\partial x_0} dx_0 \right)$$

is a $(1, 0)$ form of X_t, then $\bar{\partial}_t f = 0$. ▯

We end this section by some basic formulas of the Lie derivative in [9] and [4] which will be used in the later. The Leibniz rule of Lie derivative for $\xi \in A^{0,1}(X, T_{1,0}(X))$ is

$$L_\xi(\omega \wedge \eta) = (L_\xi \omega) \wedge \eta + (-1)^{deg\omega} \omega \wedge (L_\xi \eta), \tag{2.3}$$

and similarly

$$\mathcal{L}_\xi(\omega \wedge \eta) = (\mathcal{L}_\xi \omega) \wedge \eta + (-1)^{deg\omega} \omega \wedge (\mathcal{L}_\xi \eta). \tag{2.4}$$

Commute with contraction and $\bar{\partial}$

$$[L_\xi, i_{\xi'}] = L_\xi \circ i_{\xi'} - i_{\xi'} \circ L_\xi = i_{[\xi, \xi']}, \tag{2.5}$$

$$\left[\bar{\partial}, L_\xi \right] = \bar{\partial} \circ L_\xi + L_\xi \circ \bar{\partial} = L_{\bar{\partial}\xi}. \tag{2.6}$$

So that (Lemma 7.2 (ii) in [4])

$$(\bar{\partial} - L_\xi) \circ (\bar{\partial} - L_\xi) = -L_{(\bar{\partial}\xi - \frac{1}{2}[\xi, \xi])}. \tag{2.7}$$

Here we see the integrable condition is equivalent to

$$(\bar{\partial} - L_\xi) \circ (\bar{\partial} - L_\xi) = 0. \tag{2.8}$$

Define

$$e^{i_\xi} = \sum_{k=0}^{\infty} \frac{1}{k!} i_\xi^k, \quad \text{where} \quad i_\xi^k = \underbrace{i_\xi \circ \cdots \circ i_\xi}_{k \quad \text{times}}.$$

Theorem 1.3 in [9] said that

$$e^{-i_\xi} \circ \nabla \circ e^{i_\xi} = \nabla - \mathcal{L}_\xi^{1,0} + i_{\bar{\partial}\xi - \frac{1}{2}[\xi,\xi]}, \tag{2.9}$$

where

$$\mathcal{L}_\xi^{1,0} = i_\xi \circ \nabla^{1,0} - \nabla^{1,0} \circ i_\xi.$$

When $\bar{\partial}\xi = \frac{1}{2}[\xi,\xi]$, $\sigma \in A^{n,q}(X,E)$,

$$\left(e^{-i_\xi} \circ \nabla \circ e^{i_\xi}\right) \sigma = \bar{\partial}\sigma + \nabla^{1,0} \circ i_\xi \sigma. \tag{2.10}$$

In the context, we adopt the Einstein summation convention. When $E \to X$ is a holomorphic vector bundle on the compact complex manifold X, we also view E as a complex manifold in some situations. To distinguish from $A^k(E)$ as the space of k-forms on complex manifold E, we denote $A^k(X,E)$ as the space of E-valued k-forms.

3. Splitting of cotangent bundle. Let X be a compact complex manifold, and $p : E \to X$ a holomorphic vector bundle on X. The holomorphic local frame (U_α, e_α) of E and the holomorphic coordinate chart (U_α, x_α) on $U_\alpha \subset X$ induce a holomorphic coordinate (x_α, v_α) on E. That is,

$$v \in E, p(v) \in U_\alpha, \quad v = v_\alpha^i e_\alpha^i|_{p(v)} \mapsto (x_\alpha(p(v)), v_\alpha).$$

We call it *holomorphic frame coordinate* of E, and it is denoted by $(U_\alpha, e_\alpha, x_\alpha, v_\alpha)$.

On the other hand, for any given holomorphic vector bundle we can define its holomorphic dual bundle $p^\vee : E^\vee \to X$ canonically by $(E^\vee)_x = (E_x)^\vee$. Let $\check{e}_\alpha$ be the dual frame of e_α, then $(U_\alpha, \check{e}_\alpha)$ is a local frame of $E^\vee$. We denote $(U_\alpha, \check{e}_\alpha, x_\alpha, \check{v}_\alpha)$ and $\check{g}_{\alpha\beta} = (g_{\alpha\beta}^{-1})^T$ as the corresponding frame coordinate and transition matrix of $E^\vee$ respectively.

For convenience, we introduce the following concept.

Definition 3.1. *A linear connection on holomorphic vector bundle E is called integrable connection if it satisfies $\nabla = \bar{\partial} + \nabla^{1,0}$.*

Note that the Chern connection is an integrable connection which is compatible with a hermitian metric on E.

Under the local frame (U_α, e_α) the connection 1-form is given by $\nabla e_\alpha^i = \theta_\alpha^{ik} e_\alpha^k$, and $\nabla \check{e}_\alpha^i = (\theta_\alpha^\vee)^{ik} \check{e}_\alpha^k$ for the dual bundle. Canonically,

$$0 = \nabla(e_\alpha^i(\check{e}_\alpha^j)) = \nabla e_\alpha^i(\check{e}_\alpha^j) + \nabla \check{e}_\alpha^j(e_\alpha^i) = \theta_\alpha^{ij} + (\theta_\alpha^\vee)^{ji}.$$

The connection 1-forms of an integrable connection are always $(1,0)$-type. Hence we can utilize it to construct a smooth projection on holomorphic cotangent bundle of E.

Proposition 3.2. *Given an integrable connection ∇ on the holomorphic vector bundle $p : E \to X$, there exists a smooth projection*

$$q_\nabla : T^{1,0}(E) \to p^*(T^{1,0}(X))$$

on the holomorphic cotangent bundle $T^{1,0}(E)$ of E such that

$$q_\nabla|_{p^*(T^{1,0}(X))} = Id,$$

where p^* is the pull-back of $p : E \to X$.

Proof. If the transition matrix on $U_\alpha \cap U_\beta$ between the holomorphic local frames (U_α, e_α) and (U_β, e_β) is defined by

$$e^i_\alpha = g^{ik}_{\alpha\beta} e^k_\beta.$$

Then the transition matrix on $p^{-1}(U_\alpha) \cap p^{-1}(U_\beta)$ between frame coordinate charts

$$(U_\alpha, e_\alpha, x_\alpha, v_\alpha) \quad \text{and} \quad (U_\beta, e_\beta, x_\beta, v_\beta)$$

is

$$\begin{pmatrix} \frac{\partial x_\alpha}{\partial x_\beta} & \frac{\partial v_\alpha}{\partial x_\beta} \\ \frac{\partial x_\alpha}{\partial v_\beta} & \frac{\partial v_\alpha}{\partial v_\beta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_\alpha}{\partial x_\beta} & v^k_\beta \frac{\partial \check{g}^{ik}_{\alpha\beta}}{\partial x^j_\beta} \\ 0 & \check{g}_{\alpha\beta} \end{pmatrix}.$$

Let q_∇ be the linear extension from

$$dx^i_\alpha \mapsto dx^i_\alpha, \quad dv^i_\alpha \mapsto (\theta^\vee_\alpha)^{ik} v^k_\alpha,$$

where $\theta^\vee_\alpha$ is the connection $(1,0)$-form of ∇ on $E^\vee$ under the frame coordinate chart $(U_\alpha, \check{e}_\alpha, x_\alpha, \check{v}_\alpha)$.

In the other frame coordinate chart $(U_\beta, e_\beta, x_\beta, v_\beta)$ we have

$$dx^i_\alpha = \frac{\partial x^i_\alpha}{\partial x^k_\beta} dx^k_\beta \mapsto \frac{\partial x^i_\alpha}{\partial x^k_\beta} dx^k_\beta = dx^i_\alpha,$$

and

$$dv^i_\alpha = v^k_\beta \frac{\partial \check{g}^{ik}_{\alpha\beta}}{\partial x^j_\beta} dx^j_\beta + \check{g}^{ik}_{\alpha\beta} dv^k_\beta \mapsto v^k_\beta \frac{\partial \check{g}^{ik}_{\alpha\beta}}{\partial x^j_\beta} dx^j_\beta + \check{g}^{ik}_{\alpha\beta} (\theta^\vee_\beta)^{kl} v^l_\beta = (\theta^\vee_\alpha)^{ik} v^k_\alpha.$$

Hence, the map q_∇ is well-defined. □

By the above smooth projection, the holomorphic cotangent bundle $T^{1,0}(E)$ can be represented as a smooth direct sum

$$T^{1,0}(E) = p^*(T^{1,0}(X)) \oplus \left(T^{1,0}(E) - q_\nabla(T^{1,0}(E))\right).$$

In general, the projection q_∇ can be extended naturally to the whole cotangent bundle $T^1(E)$ and $\wedge^k T^1(E)$ by conjugation and wedge product, i.e.,

$$q_\nabla(\alpha) := \overline{q_\nabla(\bar{\alpha})}, \quad \text{for} \quad \alpha \in T^{0,1}(E).$$
$$q_\nabla(\beta_1 \wedge \cdots \wedge \beta_k) := q_\nabla(\beta_1) \wedge \cdots \wedge q_\nabla(\beta_k), \quad \text{for} \quad \beta_1, \ldots, \beta_k \in T^1(E).$$

Similarly, the following smooth decompositions of cotangent bundle are valid,

$$T^1(E) = p^*(T^1(X)) \oplus \left(T^1(E) - q_\nabla(T^1(E))\right),$$
$$T^{0,1}(E) = p^*(T^{0,1}(X)) \oplus \left(T^{0,1}(E) - q_\nabla(T^{0,1}(E))\right),$$
$$\wedge^k T^1(E) = p^*(\wedge^k T^1(X)) \oplus \left(\wedge^k T^1(E) - q_\nabla(\wedge^k T^1(E))\right),$$

since

$$q_\nabla|_{p^*(\wedge^k T^1(X))} = Id.$$

In [5] and [3], Clemens treats the section of line bundle as a function on its dual bundle, so that the $\bar{\partial}$ operator of line bundle varying with holomorphic deformation can be represented by Kuranishi data. We utilize the same notation to holomorphic vector bundles and generalize the correspondence to bundle valued forms.

Let $f : X \to E$ be a smooth section of holomorphic vector bundle $p : E \to X$, and $\check{f} : E^\vee \to \mathbb{C}$ the corresponding smooth function on the dual bundle. Under the frame coordinate $(U_\alpha, e_\alpha, x_\alpha, v_\alpha)$ on E and $(U_\alpha, \check{e}_\alpha, x_\alpha, \check{v}_\alpha)$ on $E^\vee$ defined above, the correspondence can be represented as

$$\vee : f = f^i_\alpha e^i_\alpha \mapsto \check{f} = f^i_\alpha(x_\alpha)\check{v}^i_\alpha.$$

It is easy to check that $\check{f}$ is a well-defined global function and the map $\vee$ is injective. For a bundle valued form $g \in A^k(X, E)$, the map $\vee$ is defined in the same way,

$$\vee : A^k(X, E) \to A^k(E^\vee), \quad g = g^i_\alpha e^i_\alpha \mapsto \check{g} = g^i_\alpha(x_\alpha)\check{v}^i_\alpha.$$

Now we consider about the holomorphic cotangent bundle of $E^\vee$. As in Proposition 3.2, the integrable connection ∇ induces a smooth decomposition on the holomorphic cotangent bundle of dual bundle $p^\vee : E^\vee \to X$, i.e.,

$$T^{1,0}(E^\vee) = (p^\vee)^*(T^{1,0}(X)) \oplus \left(T^{1,0}(E^\vee) - q^\vee_\nabla(T^{1,0}(E^\vee))\right),$$

and

$$q^\vee_\nabla(d\check{v}^i_\alpha) = \theta^{ik}_\alpha v^k_\alpha,$$

under the frame coordinate. Then, we have the following properties of the map $\vee$.

LEMMA 3.3. *Let $g \in A^k(X, E)$, then*

$$\bar{\partial}\check{g} = \vee(\bar{\partial}g).$$

Assume that ∇ is an integrable connection on vector bundle E and $q^\vee_\nabla$ is the projection defined above, then

$$q^\vee_\nabla(d\check{g}) = \vee(\nabla g).$$

Proof. Under the frame coordinate $(U_\alpha, x_\alpha, v_\alpha, \check{v}_\alpha)$, we have

$$\check{g} = g^i_\alpha(x_\alpha)\check{v}^i_\alpha.$$

Then

$$\bar{\partial}\check{g} = \bar{\partial}_x g^i_\alpha(x_\alpha)\check{v}^i_\alpha = \vee(\bar{\partial}g).$$

On the other hand,

$$d\check{g} = \bar{\partial}_x g^i_\alpha(x_\alpha)\check{v}^i_\alpha + \partial_x g^i_\alpha(x_\alpha)\check{v}^i_\alpha + (-1)^k g^i_\alpha(x_\alpha) \wedge d\check{v}^i_\alpha.$$

It implies

$$q^\vee_\nabla(d\check{g}) = d_x g^i_\alpha(x_\alpha)\check{v}^i_\alpha + (-1)^k g^i_\alpha(x_\alpha) \wedge \theta^{ik}_\alpha \check{v}^k_\alpha = \vee(\nabla g).$$

∎

4. Kuranishi data of pair. When we consider the holomorphic deformation of pair, it is natural for us to think about a holomorphic vector bundle $E \to X \to \Delta$ over total space X of holomorphic family $X \to \Delta$. For example in [5] and [3], the deformation of line bundle means a holomorphic line bundle $L \to X \to \Delta$ over total space X. In that case, Clemens provides a compatible transversely holomorphic trivialization of line bundle to prove that each fiber (X_t, L_t) is smoothly isomorphic to central fiber (X_0, L_0). To be specific,

Proposition 4.1. *Let $L \xrightarrow{p} X \xrightarrow{\pi} \Delta$ be a holomorphic line bundle over the total space of a holomorphic family $\pi : X \to \Delta$. Given a transversely holomorphic trivialization F_σ of the family, we can make a compatible trivialization respects the structure of complex line bundles*

$$\begin{array}{ccc} L & \overset{F_\lambda=(\lambda,\pi\circ p)}{\longrightarrow} & L_0 \times \Delta \\ \downarrow p & & \downarrow (p_0, id.) \\ X & \overset{F_\sigma=(\sigma,\pi)}{\longrightarrow} & X_0 \times \Delta \\ \downarrow \pi & & \downarrow \\ \Delta & = & \Delta \end{array}$$

such that the diagram is commutative and for each $x_0 \in X_0$ and the restricted map

$$F_\lambda : (p_0 \circ \lambda)^{-1}(x_0) \to p_0^{-1}(x_0) \times \Delta$$

is a holomorphic isomorphism.

The key point to prove this Proposition is to find a family of smooth functions $\mu_\alpha(x_0, t)$ on open covering sets $U_\alpha \subset X$ such that

$$\exp \mu_\alpha(x_0, t) = \frac{g_{\alpha\beta}(\sigma_t^{-1}(x_0))}{g_{\alpha\beta}(x_0)} \exp \mu_\beta(x_0, t),$$

where $g_{\alpha\beta}$ is the transition function on $U_\alpha \cap U_\beta$. Then

$$F_\lambda : (x_\alpha, v_\alpha) \mapsto (x_0, t, \exp \mu_\alpha(x_0, t) v_\alpha), \quad x_0 = \sigma(x), \quad t = \pi(x)$$

is a well-defined smooth isomorphism under the local frame e_α and $e_\alpha|_{X_0}$.
In [3] and [5], it is constructed by partition of unity $(\rho_\alpha(x_0), U_\alpha \cap X_0)$ on X_0 as

$$\mu_\alpha(x_0, t) := \left(\sum_\beta \rho_\beta(x_0) \log \frac{g_{\alpha\beta}(\sigma_t^{-1}(x_0))}{g_{\alpha\beta}(x_0)} \right).$$

Denote

$$\sigma_t : X_t \to X_0 \qquad \text{and} \qquad \lambda_t : L_t \to L_0$$

as the diffeomorphisms induced by σ and λ respectively. The family of functions $\exp \mu_\alpha(x_0, t)$ represents a smooth isomorphism between $\sigma_t^{-1*} L_t$ and L_0 under the local frame $\sigma_t^{-1*}(e_\alpha|_{X_t})$ and $e_\alpha|_{X_0}$. λ_t means that

$$\lambda_t : L_t \to \sigma_t^{-1*} L_t \to L_0.$$

For the general case $p : E \to X$, the vector bundle $(F_\sigma^{-1})^* E$ over $X_0 \times \Delta$ is smoothly equivalent to $E_0 \times \Delta$ by Lemma 7.1 in [7] p. 327. Then, there exists a compatible trivialization $F_\lambda : E \simeq E_0 \times \Delta$. Hence we just consider a family of complex structures (X_t, E_t) on the same smooth pair (X_0, E_0) as deformation of pairs, which is also used in Huang's work [6]. Under the holomorphic frame coordinate $(U_\alpha, (e_\alpha)_t, (x_\alpha)_t, (v_\alpha)_t)$ of (X_t, E_t), we have

$$(v_\alpha)_t^j = w_\alpha^{jl} (v_\alpha)_0^l,$$

since $(e_\alpha)_t$ is also a smooth local frame on E_0. The family of complex structures (X_t, E_t) corresponds to a Kuranishi data $\xi(t)$ which can be calculated as following.

Proposition 4.2. *Let $\phi(t)$ be the Kuranishi data corresponds to the complex structure of X_t. The Kuranishi data $\xi(t)$ is represented as a linear map*

$$\xi(t) : T^{1,0}(E_0) \to T^{0,1}(E_0),$$

which satisfies

$$d(x_\alpha)_0^i \mapsto i_{\phi(t)} d(x_\alpha)_0^i, \quad d(v_\alpha)_0^k \mapsto (w_\alpha^{-1})^{kj} \left(\left(\bar{\partial}_{x_0} - L_{\phi(t)} \right) w_\alpha^{jl} \right) (v_\alpha)_0^l.$$

Proof.

$$d(x_\alpha)_t^i = \frac{\partial (x_\alpha)_t^i}{\partial (x_\alpha)_0^j} d(x_\alpha)_0^j + \frac{\partial (x_\alpha)_t^i}{\partial (\bar{x}_\alpha)_0^j} d(\bar{x}_\alpha)_0^j,$$

and

$$d(v_\alpha)_t^k = dw_\alpha^{kl} (v_\alpha)_0^l + w_\alpha^{kl} d(v_\alpha)_0^l.$$

By the definition of Kuranishi data,

$$d(x_\alpha)_0^k \mapsto \left(\frac{\partial (x_\alpha)_t}{\partial (x_\alpha)_0} \right)^{-1}_{kj} \frac{\partial (x_\alpha)_t^j}{\partial (\bar{x}_\alpha)_0^l} d(\bar{x}_\alpha)_0^l = i_{\phi(t)} d(x_\alpha)_0^k,$$

and

$$\begin{aligned} d(v_\alpha)_0^k \mapsto & (w_\alpha^{-1})^{kj} \frac{\partial (w_\alpha)^{ji}}{\partial (\bar{x}_\alpha)_0^l} d(\bar{x}_\alpha)_0^l (v_\alpha)_0^i \\ & - (w_\alpha^{-1})^{kj} \frac{\partial (w_\alpha)^{ji}}{\partial (x_\alpha)_0^l} \left(\frac{\partial (x_\alpha)_t}{\partial (x_\alpha)_0} \right)^{-1}_{lm} \frac{\partial (x_\alpha)_0^m}{\partial (\bar{x}_\alpha)_0^u} d(\bar{x}_\alpha)_0^u (v_\alpha)_0^i \\ = & (w_\alpha^{-1})^{kj} \left(\left(\bar{\partial} - L_{\phi(t)} \right) (w_\alpha)^{ji} \right) (v_\alpha)_0^i. \end{aligned}$$

□

Given any integrable connection ∇ on E_0, we have known that it corresponds to a smooth decomposition

$$T^{1,0}(E_0) = p^*(T^{1,0}(X)) \oplus \left(T^{1,0}(E_0) - q_\nabla (T^{1,0}(E_0)) \right)$$

of holomorphic cotangent bundle on E_0 through Proposition 3.2. Then we decompose the Kuranishi data $\xi(t)$ by the direct sum.

Theorem 4.3. *The Kuranishi data $\xi(t)$ can be decomposed into two part*

$$\xi(t) = \xi(t) \circ q_\nabla + \xi(t) \circ (Id - q_\nabla) =: \phi(t) + \psi(t),$$

where $\phi(t)$ can be represented by the Kuranishi data of the base manifold, and $\psi(t)$ can be represented by an element in $A^{0,1}(X_0, End(E_0))$.

Proof. The part of $\phi(t)$ is obviously valid by above proposition. We need to find the representation of $\psi(t)$ such that

$$\psi(t)(d(v_\alpha)_0^k) = \xi(t)\left(d(v_\alpha)_0^k - (\theta_\alpha^\vee)^{kl} v_\alpha^l\right).$$

Let

$$\psi_\alpha^{kl}(t) := (w_\alpha^{-1})^{lj}\left(\left(\bar\partial - L_{\phi(t)}\right)(w_\alpha)^{jk}\right) - i_{\phi(t)}(\theta_\alpha^\vee)^{lk}.$$

Then

$$\psi(t)(d(v_\alpha)_0^k) = \psi_\alpha^{lk}(t) v_\alpha^l,$$

and

$$\psi_\alpha^{kl} = (g_{\alpha\beta})_0^{ki}\psi_\beta^{ij}(g_{\alpha\beta}^{-1})_0^{jl}.$$

Hence,

$$\psi(t)(e_\alpha)_0^k := \psi_\alpha^{kl}(t)(e_\alpha)_0^l$$

is a well-defined element in $A^{0,1}(X_0, End(E_0))$. □

Use the same calculation under the given frame coordinate, the Kuranishi data of dual bundle and tensor product bundle on the same base manifold is represented as following.

COROLLARY 4.4. *Let (X_t, E_t) be a deformation of pairs on the base family X_t, its Kuranishi data decomposed by the integrable connection ∇ is*

$$\xi_E(t) = \phi(t) + \psi_E(t).$$

Then, for the dual bundle $(X_t, E_t^\vee)$ the Kuranishi data decomposed by ∇ is

$$\xi_{E^\vee}(t) = \phi(t) + \psi_{E^\vee}(t), \quad \psi_{E^\vee}(\check e_\alpha^k) = -\psi_E^{lk}\check e_\alpha^l.$$

For tensor product $(X_t, E_t \otimes E_t)$ the Kuranishi data decomposed by ∇ is

$$\xi_{E\otimes E}(t) = \phi(t) + \psi_{E\otimes E}(t), \quad \psi_{E\otimes E}(t)(e_\alpha^i \otimes e_\alpha^j) = \psi_E(e_\alpha^i)\otimes e_\alpha^j + e_\alpha^i \otimes \psi_E(e_\alpha^j).$$

In particular,

$$\xi_{\det E} = \phi(t) + \psi(t)_{\det E}, \quad \psi(t)_{\det E} = \sum_i \psi_\alpha^{ii}(t).$$

Proof. Under the frame coordinate $(v_\alpha)_t^j = w_\alpha^{jl}(v_\alpha)_0^l$, then

$$(\check v_\alpha)_t^j = (w_\alpha^{-1})^{lj}(\check v_\alpha)_0^l, \quad (v_\alpha^{E\otimes E})_t^{jk} = w_\alpha^{jl} w_\alpha^{km}(v_\alpha^{E\otimes E})_0^{lm}$$

$$(\psi_{E^\vee})_\alpha^{kl}(t) = w_\alpha^{jl}\left(\left(\bar\partial - L_{\phi(t)}\right)(w_\alpha^{-1})^{kj}\right) - i_{\phi(t)}(\theta_\alpha)^{lk} = -\psi_E^{lk}(t),$$

$$(\psi^{ijkl}_{E\otimes E})_\alpha(t) = (w_\alpha^{-1})^{lq}(w_\alpha^{-1})^{kp}\left(\left(\bar\partial - L_{\phi(t)}\right)(w_\alpha)^{pi}(w_\alpha)^{qj}\right) - i_{\phi(t)}(\theta_\alpha^\vee)^{ki} - i_{\phi(t)}(\theta_\alpha^\vee)^{lj}$$
$$= \psi^{ik}_E(t) + \psi^{jl}_E(t).$$

□

The Kuranishi data derives the criterion of holomorphic functions with respect to the varying complex structure such as in Lemma 2.2. In the case of pair deformation $\{(X_t, E_t)\}$ we can get the criterion of holomorphic sections by Kuranishi data $\xi_{E^\vee}(t)$ on the dual bundle $E^\vee$ as in [3], and the decomposition of $\xi_{E^\vee}(t)$ by integrable connection ∇ make it an operator of sections.

Corollary 4.5. *A smooth section s is holomorphic under the complex structure of (X_t, E_t) if and only if .*

$$\left(\bar\partial_{E_0} - \mathcal{L}_{\phi(t)} + \psi(t)\right) s = 0,$$

where $\xi(t) = \phi(t) + \psi(t)$ is the decomposition of Kuranishi data under the integrable connection ∇.

Proof. s is holomorphic section of (X_t, E_t) if and only if $\check s$ is a holomorphic function of $E_t^\vee$. By Lemma 2.2, the criterion is

$$\left(\bar\partial_0 - L_{\xi_{E^\vee}(t)}\right) \check s = 0.$$

By Lemma 3.3

$$\begin{aligned}\left(\bar\partial_0 - L_{\xi_{E^\vee}(t)}\right) \check s &= \vee \circ (\bar\partial_{E_0} s) - i_{\xi_{E^\vee}(t)} \circ q^\vee_\nabla(d\check s) - i_{\xi_{E^\vee}(t)} \circ (Id - q^\vee_\nabla)(d\check s)\\ &= \vee \circ (\bar\partial_{E_0} s) - \vee \circ i_{\phi(t)} \circ \nabla s - s^k_\alpha \psi_{E^\vee}(t)(d\check v^k_\alpha)\\ &= \vee \circ (\bar\partial_{E_0} s) - \vee \circ i_{\phi(t)} \circ \nabla s + s^k_\alpha \psi^{kl} \check v^l_\alpha\\ &= \vee \circ \left(\bar\partial_{E_0} - \mathcal{L}_{\phi(t)} + \psi(t)\right) s\end{aligned}$$

we get the assertion. □

We observe that $\psi(t)$ produces the main discrepancy between deformation of manifolds and pair deformation. If $\psi(t) = 0$, most of the results in deformation of manifolds can be generalized directly to the pair case.

Corollary 4.6. $\psi(t) = 0$ *under the decomposition*

$$\xi(t) = \phi(t) + \psi(t)$$

induced from integrable connection ∇ on E_0, if and only if ∇ is also an integrable connection on E_t, i.e.,

$$\nabla = \bar\partial_{E_0} + \nabla^{1,0}_{X_0} = \bar\partial_{E_t} + \nabla^{1,0}_{X_t}.$$

Proof. Under the frame coordinate $(U_\alpha, (e_\alpha)_t, (x_\alpha)_t, (v_\alpha)_t)$ of (X_t, E_t), as in Proposition 4.2

$$(e_\alpha)^k_t = (w_\alpha^{-1})^{lk}(e_\alpha)^l_0$$

$$\begin{aligned}\nabla(e_\alpha)_t^k &= d(w_\alpha^{-1})^{lk}(e_\alpha)_0^l + (w_\alpha^{-1})^{lk}\theta_\alpha^{lj}(e_\alpha)_0^j \\ &= \left(\partial(w_\alpha^{-1})^{jk} + (w_\alpha^{-1})^{lk}\theta_\alpha^{lj}\right)(e_\alpha)_0^j + i_{\phi(t)}\left(\partial(w_\alpha^{-1})^{jk} + (w_\alpha^{-1})^{lk}\theta_\alpha^{lj}\right)(e_\alpha)_0^j \\ &\quad + \left((\bar\partial - L_{\phi(t)})(w_\alpha^{-1})^{jk} - (w_\alpha^{-1})^{lk} i_{\phi(t)}\theta_\alpha^{lj}\right)(e_\alpha)_0^j \\ &= \left(\partial(w_\alpha^{-1})^{jk} + (w_\alpha^{-1})^{lk}\theta_\alpha^{lj}\right)(e_\alpha)_0^j + i_{\phi(t)}\left(\partial(w_\alpha^{-1})^{jk} + (w_\alpha^{-1})^{lk}\theta_\alpha^{lj}\right)(e_\alpha)_0^j \\ &\quad - (w_\alpha^{-1})^{lk}\psi_\alpha^{lj}(t)(e_\alpha)_0^j \\ &= \alpha + i_{\phi(t)}\alpha - \psi(t)((e_\alpha)_t^k),\end{aligned}$$

where α is a holomorphic bundle valued $(1,0)$-form on X_0. Hence, $\nabla + \psi(t)$ is an integrable connection on (E_t, X_t). □

An obvious example is $(X_t, X_t \times \mathbb{C}^k)$, since d is the integrable connection for all t. Especially, if we take ∇ as a Chern connection of Hermitian metric h on E_0, the compatible condition

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$$

is still true for (E_t, h), since they are the same as a smooth complex vector bundle.

Corollary 4.7. *$\psi(t) = 0$ under the decomposition*

$$\xi(t) = \phi(t) + \psi(t)$$

induced by Chern connection ∇ on E_0, if and only if ∇ is also a Chern connection on E_t.

In the next section, iteration formula of this special pair deformation will be constructed. For convenience, we introduce the following concept.

Definition 4.8. *If the pair deformation (X_t, E_t) satisfies $\psi(t) = 0$ under the decomposition of integrable connection ∇ on central fiber (X_0, E_0), we call it the lifting of base family by integrable connection ∇.*

In the above content the Kuranishi data of pair deformation $\{(X_t, E_t)\}$ has been calculated. Conversely, given an integrable connection ∇ on E_0, $\phi(t) \in A^{0,1}(X_0, T_{1,0}(X_0))$ and $\psi(t) \in A^{0,1}(X_0, End(E_0))$, we have the following integrable condition.

Theorem 4.9. *The Kuranishi data defined by*

$$\xi(t) = \xi(t) \circ q_\nabla + \xi(t) \circ (Id - q_\nabla) := \phi(t) + \psi(t)$$

is integrable i.e., represents a complex structure of pair (X_t, E_t) for each t, if and only if

$$\bar\partial\phi(t) = \frac{1}{2}[\phi(t), \phi(t)]$$

and

$$(\bar\partial - \mathcal{L}_{\phi(t)})\psi(t) - \psi(t) \wedge \psi(t) - i_{\phi(t)}\Theta_0^{1,1} - \frac{1}{2} i_{\phi(t)} \circ i_{\phi(t)}\Theta = 0,$$

where the equalities are valid in the sense of $\psi(t) \in A^{0,1}(X_0, End(E_0))$ and $\phi(t) \in A^{0,1}(T_{1,0}(X_0))$.

Proof. The first equation is obvious by the integrable condition of base complex manifold. The Kuranishi data $\xi(t)$ is integrable as a complex structure of manifold E_t if and only if

$$\bar{\partial}\xi(t) = \frac{1}{2}[\xi(t), \xi(t)]. \tag{4.1}$$

Under the frame coordinate $(U_\alpha, e_\alpha, x_\alpha, v_\alpha)$ on (X_0, E_0), by Proposition 4.2 denote

$$A(t) = \left(\psi_\alpha^{lk} - i_{\phi(t)}\theta_\alpha^{lk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k},$$

$$B(t) = \phi_k^i d\bar{x}_\alpha^k \otimes \frac{\partial}{\partial x_\alpha^i},$$

and

$$\xi(t) = A(t) + B(t) \in A^{0,1}(E_0, T_{1,0}(E_0)).$$

In this setting, (4.1) means that

$$\begin{aligned}
&\bar{\partial}\left(A(t) + B(t)\right) - \frac{1}{2}\left[A(t) + B(t), A(t) + B(t)\right] \\
=&\bar{\partial}A(t) - [B(t), A(t)] - \frac{1}{2}[A(t), A(t)] + \bar{\partial}B(t) - \frac{1}{2}[B(t), B(t)] \\
=&\bar{\partial}A(t) - [B(t), A(t)] - \frac{1}{2}[A(t), A(t)] \\
=&0.
\end{aligned}$$

For each term,

$$\begin{aligned}
\bar{\partial}A(t) &= \left(\bar{\partial}\psi_\alpha^{lk} - \bar{\partial} \circ i_{\phi(t)}\theta_\alpha^{lk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k} \\
&= \left(\bar{\partial}\psi_\alpha^{lk} - i_{\bar{\partial}\phi(t)}\theta_\alpha^{lk} - i_{\phi(t)} \circ \bar{\partial}\theta_\alpha^{lk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k},
\end{aligned}$$

$$\begin{aligned}
[B(t), A(t)] &= i_{\phi(t)} \circ \partial_x \left(\psi_\alpha^{lk} - i_{\phi(t)}\theta_\alpha^{lk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k} \\
&= \left(i_{\phi(t)} \circ \partial_x \psi_\alpha^{lk} - i_{\phi(t)} \circ \partial_x \circ i_{\phi(t)}\theta_\alpha^{lk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k},
\end{aligned}$$

$$\begin{aligned}
[A(t), A(t)] =& 2\left(\psi_\alpha^{lj} \wedge \psi_\alpha^{jk} - i_{\phi(t)}\theta_\alpha^{lj} \wedge \psi_\alpha^{jk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k} \\
&+ 2\left(i_{\phi(t)}\theta_\alpha^{lj} \wedge i_{\phi(t)}\theta_\alpha^{jk} - \psi_\alpha^{lj} \wedge i_{\phi(t)}\theta_\alpha^{jk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k}.
\end{aligned}$$

Since

$$2i_{\phi(t)}\theta_\alpha \wedge i_{\phi(t)}\theta_\alpha = i_{\phi(t)} \circ i_{\phi(t)}(\theta_\alpha \wedge \theta_\alpha) = i_{\phi(t)} \circ i_{\phi(t)}\left(\Theta - d\theta_\alpha\right),$$

$$i_{\phi(t)}\psi_\alpha = i_{\phi(t)} \circ \bar{\partial}\psi_\alpha = 0$$

by type of form and

$$i_{\frac{1}{2}[\phi(t),\phi(t)]}\theta_\alpha = -\frac{1}{2}i_{\phi(t)} \circ i_{\phi(t)} \circ d\theta_\alpha + i_{\phi(t)} \circ \partial_x \circ i_{\phi(t)}\theta_\alpha$$

by (2.5), we have

$$\begin{aligned}
&\bar{\partial}A(t) - [B(t), A(t)] - \frac{1}{2}[A(t), A(t)] \\
&= \left(\bar{\partial}\psi_\alpha^{lk} - i_{\phi(t)}(\Theta_0^{1,1})^{lk} - i_{\phi(t)} \circ \partial_x\psi_\alpha^{lk} - i_{\phi(t)}\theta_\alpha^{lj} \wedge \psi_\alpha^{jk} + i_{\phi(t)}\theta_\alpha^{jk} \wedge \psi_\alpha^{lj}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k} \\
&\left(-\frac{1}{2}i_{\phi(t)} \circ i_{\phi(t)}\Theta^{lk} + \psi_\alpha^{lj} \wedge \psi_\alpha^{jk}\right) v_\alpha^l \otimes \frac{\partial}{\partial v_\alpha^k}.
\end{aligned}$$

□

COROLLARY 4.10. *If $\{(X_t, E_t)\}$ is a holomorphic family of pair, which is a lifting of base family X_t by Chern connection ∇, then*

$$(\bar{\partial} - \mathcal{L}_{\phi(t)}) \circ (\bar{\partial} - \mathcal{L}_{\phi(t)}) = -i_{\phi(t)}\Theta = 0$$

Proof. Let $\xi(t) = \phi(t) + \psi(t)$ be the Kuranishi data of $\{(X_t, E_t)\}$.

$$-i_{\phi(t)}\Theta = 0$$

by the integrable condition directly, since $\psi(t) = 0$. By (2.7) in Section 2,

$$(\bar{\partial} - L_\xi) \circ (\bar{\partial} - L_\xi) = -L_{(\bar{\partial}\xi - \frac{1}{2}[\xi,\xi])} = 0.$$

On the other hand, for any smooth section s,

$$0 = (\bar{\partial} - L_\xi) \circ (\bar{\partial} - L_\xi)\check{s} = \vee \circ (\bar{\partial} - \mathcal{L}_{\phi(t)}) \circ (\bar{\partial} - \mathcal{L}_{\phi(t)})s.$$

Then we get the assertion. □

5. iteration formula. In [9] the first author, Rao and Yang construct a iteration formula to calculate the extension of holomorphic $(n, 0)$-form from central fiber to a small deformation. We generalize the formula to give an extension of bundle valued (n, q)-form on the lifting deformation we defined above. At first, we adjust some basic results in [9].

Let $\{(X_t, E_t)\}$ be a holomorphic deformation of pair, which is a lifting of base family $\{X_t\}$ by Chern connection ∇. Denote $\phi(t)$ as the Kuranishi data of base family $\{X_t\}$, then the operator defined in Section 2

$$e^{i_{\phi(t)}} : A^{n,q}(X_0, E_0) \to A^{n,q}(X_t, E_t)$$

is a linear isomorphism. Since for any $\alpha_i, \beta_i \in A^{1,0}(X_0)$ and $s \in A^0(X_0, E_0)$ the map

$$\begin{aligned}
e^{i_{\phi(t)}} &: \alpha_1 \wedge \cdots \wedge \alpha_n \wedge \bar{\beta}_1 \wedge \cdots \wedge \bar{\beta}_q \otimes s \mapsto \\
&(\alpha_1 + i_\phi\alpha_1) \wedge \cdots \wedge (\alpha_n + i_\phi\alpha_n) \wedge \bar{\beta}_1 \wedge \cdots \wedge \bar{\beta}_q \otimes s
\end{aligned}$$

keeps the total degree of forms invariant, and

$$(\alpha_1 + i_\phi \alpha_1) \wedge \cdots \wedge (\alpha_n + i_\phi \alpha_n)$$

is a $(n,0)$-form with respect to the complex structure of X_t.

Similar to Proposition 5.1 in [9], there is a $\bar{\partial}_t$-closed criterion.

Proposition 5.1. *For any $s \in A^{n,q}(X_0, E_0)$, the section $e^{i_{\phi(t)}}(s) \in A^{n,q}(X_t, E_t)$ is $\bar{\partial}_t$-closed with respect to the complex structure induced by $\phi(t)$ on E_t if and only if $\bar{\partial} s + \nabla^{1,0}\left(i_{\phi(t)} s\right) = 0$.*

Proof. Since $e^{i_{\phi(t)}}(s) \in A^{n,q}(X_t, E_t)$, and ∇ is also a Chern connection of E_t by Corollary 4.7. We have $\nabla_t^{1,0}\left(e^{i_{\phi(t)}}(s)\right) = 0$. By (2.10) in Section 2

$$\bar{\partial} s + \nabla^{1,0} \circ i_{\phi(t)}(s) = \left(e^{-i_\phi} \circ \nabla \circ e^{i_\phi}\right)(s) = \left(e^{-i_\phi} \circ \bar{\partial}_t \circ e^{i_\phi}\right)(s).$$

It implies the assertion. □

To get the solution in each step of iteration, we need to solve a $\bar{\partial}$-equation by the $\bar{\partial}$-inverse formula.

Lemma 5.2. *(Proposition 2.3 in [9])Let E be a holomorphic vector bundle with Nakano semi-positive curvature Θ over the compact Kähler manifold X. Then, for any $g \in A^{n-1,q}(X, E)$,*

$$s = \bar{\partial}^* \mathbb{G} \nabla^{1,0} g$$

is a solution to the equation $\bar{\partial} s = \nabla^{1,0} g$ with $\bar{\partial} \nabla^{1,0} g = 0$, such that

$$\|s\|^2 \leq \langle \nabla^{1,0} g, \mathbb{G} \nabla^{1,0} g \rangle.$$

This solution is unique as long as it satisfies $\mathbb{H} s = 0$ and $\bar{\partial}^* s = 0$.

Notice that $\mathbb{G}$ and $\mathbb{H}$ are the Green operator and harmonic projection in the Hodge decomposition with respect to the operator $\bar{\partial}$. The curvature Θ is Nakano semi-positive if and only if $\langle \Theta_{ij} u^i, u^j \rangle \geq 0$ for all local sections u_i, u_j on the coordinate chart (z) such that $\Theta = \Theta_{ij} dz^i \wedge d\bar{z}^j$.

We can use the criterion to get the iteration formula.

Theorem 5.3. *Let (X_t, E_t) be a lifting of holomorphic base family X_t by Chern connection ∇ on E_0. The central fibre satisfies that X_0 is Kähler and ∇ has Nakano semi-positive curvature Θ. Denote the Kuranishi data of X_t as*

$$\phi(t) = \sum \varphi_{\gamma_1 \cdots \gamma_N} t_1^{\gamma_1} \cdots t_N^{\gamma_N},$$

for $|t| < \varepsilon$. Then for any $\bar{\partial}$-closed (n,q)-form $s \in A^{n,q}(X_0, E_0)$, we can construct a convergent power series

$$s_t = s_0 + \sum_{|I|>0}^{\infty} s_I t^I \in A^{n,q}(X_0, E_0)$$

such that $s_0 = s$ with the following properties:
(a) *$s_t^C := e^{i_{\phi(t)}} s_t$ is $\bar{\partial}_t$-closed with respect to E_t,*
(b) *s_I is $\bar{\partial}^*$-exact for all $|I| \geq 1$,*

(c) *if s is $(n,0)$ bundle valued form, s_t^C is a smooth extension of s.*

Proof. By Proposition 5.1, $e^{i_{\phi(t)}}s_t$ is $\bar{\partial}_t$-closed if and only if

$$\bar{\partial}s_t + \nabla^{1,0}\left(i_{\phi(t)}s_t\right) = 0.$$

If there exist such power series s_t, it must satisfy the equation formally, i.e.,

$$\begin{cases} \bar{\partial}s_0 = 0, \\ \bar{\partial}s_{\nu_1\cdots\nu_N} = -\nabla^{1,0}\left(\sum\limits_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right). \end{cases}$$

Let

$$\eta_{\nu_1\cdots\nu_N} = -\nabla^{1,0}\left(\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right).$$

If $\bar{\partial}\eta_{\nu_1\cdots\nu_N} = 0$, then by Lemma 5.2 we have $\eta_{\nu_1\cdots\nu_N}$ is $\bar{\partial}$-exact, which implies that $\eta_{\nu_1\cdots\nu_N} = \bar{\partial}\bar{\partial}^*\mathbb{G}\eta_{\nu_1\cdots\nu_N}$.
Now, we can construct a formal power series solution by induction.
For initial case $\sum \nu_i = 1$, one has

$$\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N} = i_{\varphi_{\nu_1\cdots\nu_N}} s_0,$$

and

$$-\bar{\partial}\nabla^{1,0}\left(i_{\varphi_{\nu_1\cdots\nu_N}} s_0\right) = (\nabla^{1,0}\bar{\partial} - \Theta)\left(i_{\varphi_{\nu_1\cdots\nu_N}} s_0\right) = \nabla^{1,0}\left(i_{\bar{\partial}\varphi_{\nu_1\cdots\nu_N}} s_0 + i_{\varphi_{\nu_1\cdots\nu_N}}\bar{\partial}s_0\right) = 0.$$

Here we should notice that

$$i_{\phi(t)}\Theta = 0,$$

by Corollary 4.10, and

$$\Theta\left(i_{\phi(t)}s\right) = i_{\phi(t)}(\Theta s) - (i_{\phi(t)}\Theta)s = 0$$

by type. Then

$$s_{\nu_1\cdots\nu_N} = -\bar{\partial}^*\mathbb{G}\nabla^{1,0}\left(i_{\varphi_{\nu_1\cdots\nu_N}} s_0\right)$$

solves the $\bar{\partial}$-equation. Suppose that the bundle valued (n,q)-form $s_{\nu_1\cdots\nu_N}$ with $\sum \nu_i \leq$

K are constructed. Then, for $\sum \nu_i = K+1$

$$\begin{aligned}
&-\bar{\partial}\nabla^{1,0}\left(\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right)\\
&=\nabla^{1,0}\left(\sum_{\gamma_i+\beta_i=\nu_i} i_{\bar{\partial}\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}+\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} \bar{\partial} s_{\beta_1\cdots\beta_N}\right)\\
&=\nabla^{1,0}\left(\sum_{\eta_i+\lambda_i+\beta_i=\nu_i} i_{\frac{1}{2}\left[\varphi_{\eta_1\cdots\eta_N},\varphi_{\lambda_1\cdots\lambda_N}\right]} s_{\beta_1\cdots\beta_N}\right.\\
&\qquad\qquad\left.+\sum_{\gamma_i+\lambda_i+\eta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}}\circ(-\nabla^{1,0})\circ i_{\varphi_{\lambda_1\cdots\lambda_N}} s_{\eta_1\cdots\eta_N}\right)\\
&=\nabla^{1,0}\left(-\nabla^{1,0}\sum_{\gamma_i+\lambda_i+\eta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}}\circ i_{\varphi_{\lambda_1\cdots\lambda_N}} s_{\eta_1\cdots\eta_N}\right)\\
&=0.
\end{aligned}$$

The third equality is given by Lemma 3.2 in [9]. Hence, the $\sum \nu_i = K+1$ case can be constructed in the same way by

$$s_{\nu_1\cdots\nu_N} = -\bar{\partial}^*\mathbb{G}\nabla^{1,0}\left(\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right).$$

Now, we need to prove the convergence of the formal power series. We have known that the power series of Kuranishi data $\phi(t)$ is convergent in Hölder norms $\|\cdot\|_{k,\alpha}$. Then, there exist constant $C>0$ and $0<\xi<\varepsilon$ such that

$$\sum_{\gamma_1+\cdots+\gamma_N=i} \|\varphi_{\gamma_1\cdots\gamma_N}\|_{k,\alpha}\xi^i \le C.$$

On the other hand, similar to the estimates in [11] p.160-162, we have

$$\begin{aligned}
\|s_{\nu_1\cdots\nu_N}\|_{k,\alpha} &= \left\|-\bar{\partial}^*\mathbb{G}\nabla^{1,0}\left(\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right)\right\|_{k,\alpha}\\
&\le C_1\left\|\mathbb{G}\nabla^{1,0}\left(\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right)\right\|_{k+1,\alpha}\\
&\le C_1C_2\left\|\nabla^{1,0}\left(\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right)\right\|_{k-1,\alpha}\\
&\le C_1C_2C_3\left\|\sum_{\gamma_i+\beta_i=\nu_i} i_{\varphi_{\gamma_1\cdots\gamma_N}} s_{\beta_1\cdots\beta_N}\right\|_{k,\alpha}\\
&\le C_1C_2C_3C_4\sum_{\gamma_i+\beta_i=\nu_i}\|\varphi_{\gamma_1\cdots\gamma_N}\|_{k,\alpha}\|s_{\beta_1\cdots\beta_N}\|_{k,\alpha}.
\end{aligned}$$

Denote

$$\|s_{\nu_1\cdots\nu_N}\|_{k,\alpha} \leq C_0 \sum_{\gamma_i+\beta_i=\nu_i} \|\varphi_{\gamma_1\cdots\gamma_N}\|_{k,\alpha} \|s_{\beta_1\cdots\beta_N}\|_{k,\alpha}.$$

Let $\|s_0\|_{k,\alpha} = 1$, we get the following estimate by induction. For $I = 1$

$$\sum_{\nu_1+\cdots+\nu_N=1} \|s_{\nu_1\cdots\nu_N}\|_{k,\alpha} \leq \sum_{\nu_1+\cdots+\nu_N=1} \|\varphi_{\nu_1\cdots\nu_N}\|_{k,\alpha}\|s_0\|_{k,\alpha} \leq C_0 C\xi^{-1}.$$

For general I

$$\begin{aligned}
\sum_{\nu_1+\cdots+\nu_N=I} \|s_{\nu_1\cdots\nu_N}\|_{k,\alpha} &\leq C_0 \sum_{\nu_1+\cdots+\nu_N=I} \sum_{\gamma_i+\beta_i=\nu_i} \|\varphi_{\gamma_1\cdots\gamma_N}\|_{k,\alpha} \|s_{\beta_1\cdots\beta_N}\|_{k,\alpha} \\
&= C_0 \sum_{\gamma_1+\cdots+\gamma_N+\beta_1+\cdots+\beta_N=I} \|\varphi_{\gamma_1\cdots\gamma_N}\|_{k,\alpha} \|s_{\beta_1\cdots\beta_N}\|_{k,\alpha} \\
&= C_0 \sum_{|\beta|=0}^{I-1} \left(\|s_{\beta_1\cdots\beta_N}\|_{k,\alpha} \sum_{|\gamma|=I-|\beta|} \|\varphi_{\gamma_1\cdots\gamma_N}\|_{k,\alpha} \right) \\
&\leq C_0 \sum_{|\beta|=0}^{I-1} \left(\|s_{\beta_1\cdots\beta_N}\|_{k,\alpha} C\xi^{|\beta|-I} \right) \\
&\leq \xi^{-I} \|s_0\|_{k,\alpha} \sum_{i=1}^{I} (C_0 C)^i.
\end{aligned}$$

Then the power series is convergent when $|t| < \frac{\xi}{C_0 C}$.

For regularity, we have known that $\frac{\partial}{\partial \bar{t}^j}\phi(t) = 0$ and $\frac{\partial}{\partial \bar{t}^j} s_t = 0$, then

$$\frac{\partial}{\partial \bar{t}^j} \left(e^{i_{\phi(t)}} s_t \right) = 0.$$

In addition,

$$\bar{\partial}_t \left(e^{i_{\phi(t)}} s_t \right) = 0$$

implies

$$\bar{\partial}_X \left(e^{i_{\phi(t)}} s_t \right) = 0 \quad \text{for total space} \quad X.$$

Hence,when s is $(n, 0)$-type, $e^{i_{\phi(t)}} s_t$ is smooth respect to (x, t) by hypoellipticity. □

6. The case of tangent bundle. In this section, we compute the Kuranishi data of these bundles associated with the complex structure along a family of compact complex manifold. Let $X \to \Delta$ be a holomorphic family of compact complex manifold. The holomorphic cotangent bundle $T^{1,0}(X_t)$ on each fiber X_t forms a family of holomorphic vector bundle, and there is a natural compatible trivialization on it

$$\pi^{1,0} : d(x_\alpha)_t^k \mapsto \frac{\partial (x_\alpha)_t^k}{\partial (x_\alpha)_0^j} d(x_\alpha)_0^j.$$

Denote $\xi(t)=\phi(t)+\psi(t)$ as the Kuranishi data of pair $(X_t, T^{1,0}(X_t))$ under the frame coordinate

$$\left(U_\alpha, (x_\alpha)_t, \frac{\partial (x_\alpha)_t^k}{\partial (x_\alpha)_0^j} d(x_\alpha)_0^j, (v_\alpha)_t\right).$$

Let ∇ be a Chern connection of holomorphic tangent bundle induced by a Hermitian metric h on X_0, then

$$\begin{aligned}
\psi^{kl}(t) &= (w_\alpha^{-1})^{lj}\left(\left(\bar\partial - L_{\phi(t)}\right)(w_\alpha)^{jk}\right) - i_{\phi(t)}(\theta_\alpha^\vee)^{lk} \\
&= \frac{\partial (x_\alpha)_t^j}{\partial (x_\alpha)_0^l}\left(\bar\partial - L_{\phi(t)}\right)\left(\frac{\partial (x_\alpha)_t}{\partial (x_\alpha)_0}\right)^{-1}_{kj} - i_{\phi(t)}(\theta_\alpha^h)^{lk} \\
&= -\frac{\partial^2 (x_\alpha)_t^j}{\partial (x_\alpha)_0^l \partial (\bar x_\alpha)_0^m}\left(\frac{\partial (x_\alpha)_t}{\partial (x_\alpha)_0}\right)^{-1}_{kj} d(\bar x_\alpha)_0^m \\
&\quad + \frac{\partial^2 (x_\alpha)_t^j}{\partial (x_\alpha)_0^l \partial (x_\alpha)_0^p}\left(\frac{\partial (x_\alpha)_t}{\partial (x_\alpha)_0}\right)^{-1}_{kj}\left(\frac{\partial (x_\alpha)_t}{\partial (x_\alpha)_0}\right)^{-1}_{pq}\frac{\partial (x_\alpha)_t^q}{\partial (\bar x_\alpha)_0^m} d(\bar x_\alpha)_0^m - i_{\phi(t)}(\theta_\alpha^h)^{lk} \\
&= -\partial_l \phi_m^k d(\bar x_\alpha)_0^m - i_{\phi(t)}(\theta_\alpha^h)^{lk} \\
&= -\partial_l \phi^k - \Gamma_{jl}^k \phi^j \\
&= \Gamma_{lj}^k \phi^j - \Gamma_{jl}^k \phi^j - (\nabla_l^{1,0}\phi)^k.
\end{aligned}$$

Corollary 4.4 implies that

Proposition 6.1. *If the central fibre is Kähler, and ∇ is the Chern connection of Kähler metric, the Kuranishi data of holomorphic tangent bundle can be represented as $\phi + \nabla^{1,0}\phi$.*

Example 4.12. and Example 4.13 in [2] provide similar formulas of $\psi(t)$ for general Hermitian central manifold. On the other hand, assume that K_{X_t} is the canonical bundle of fiber X_t, the compatible trivialization of holomorphic cotangent bundle induces its trivialization as a family of holomorphic line bundle, i.e.,

$$\mu_\alpha(x_0, t) = \log\det\left(\frac{\partial (x_\alpha)_t}{\partial (x_\alpha)_0}\right).$$

Then its Kuranishi data satisfies

$$\psi_{det}(t) = \sum_i \psi^{ii}(t) = -(\nabla_i^{1,0}\phi)^i = -\partial_i \phi_m^i(t) d\bar x_\alpha^m - i_{\phi(t)}\theta_\alpha^{det(h)}.$$

We hardly know whether $\psi(t)=0$ in general, but there are examples. In [13], Sun proved that a family of Kähler-Einstein manifolds of general type or a family of polarized Calabi-Yau manifolds satisfy that the divergence gauge is equivalent to the Kuranishi gauge. It means that if we take Kuranishi gauge to adjust the transversely holomorphic trivialization, $\psi_{det}(t)=0$ under the Chern connection of Kähler-Einstein metric.

In fact, $\psi(t)$ depends on the choice of connection, transversely trivialization and compatible trivialization. In our further work, we will continue to find out whether we can choose suitable trivialization and connection to make $\psi(t)=0$.

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REFERENCES

[1] K. Chan and Y. Suen, *A differential-geometric approach to deformations of pairs (X, E)*, Complex Manifolds, 3 (2016), Art. 2.

[2] K. Chan and Y. Suen, *On the jumping phenomenon of* $\dim_{\mathbb{C}} H^q(X_t, E_t)$, arXiv:1601.06472.

[3] H. Clemens, *A local proof of Petri's conjecture at the general curve*, J. Differential Geom., 54:1 (2000), pp. 139–176.

[4] H. Clemens, *Geometry of formal Kuranishi theory*, Adv. Math., 198:1 (2005), pp. 311–365.

[5] H. Clemens, C. Hacon, *Deformations of the trivial line bundle and vanishing theorems*, Amer. J. Math., 124:4 (2002), pp. 769–815.

[6] L. Huang, *On joint moduli spaces*, Math. Ann., 302:1 (1995), pp. 61–79.

[7] K. Kodaira, *Complex Manifolds and Deformations of Complex Structures*. Translated from the 1981 Japanese original by Kazuo Akao. Reprint of the 1986 English edition. Classics in Mathematics. Springer-Verlag, Berlin, 2005. x+465 pp.

[8] K. F. Liu and S. Rao, *Remarks on the Cartan formula and its applications*, Asian J. Math., 16:1 (2012), pp. 157–170.

[9] K. F. Liu, S. Rao, and X. K. Yang, *Quasi-isometry and deformations of Calabi-Yau manifolds*, Invent. Math., 199:2 (2015), pp. 423–453.

[10] M. Manetti, *Lectures on deformations of complex manifolds(deformations from differential graded viewpoint)*, Rend. Mat. Appl. (7), 24:1 (2004), pp. 1–183.

[11] J. Morrow and K. Kodaira, *Complex manifolds,Reprint of the 1971 edition with errata*, AMS Chelsea Publishing, Providence, RI, 2006. x+194 pp.

[12] Y. Shimizu and K. Ueno, *Advances in moduli theory*, Translated from the 1999 Japanese original. Translations of Mathematical Monographs, 206. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2002. xx+300 pp.

[13] X. Sun, *Deformation of canonical metrics I*, Asian. J. Math., 16:1 (2012), pp. 141–155.

[14] C. Vosin, *Hodge theory and complex algebraic geometry I*, Translated from the French by Leila Schneps. Reprint of the 2002 English edition. Cambridge Studies in Advanced Mathematics, 76. Cambridge University Press, Cambridge, 2007. x+322 pp.

[15] Q. Zhao and S. Rao, *Extension formulas and deformation invariance of Hodge numbers*, C. R. Math. Acad. Sci. Paris, 353:11 (2015), pp. 979–984.