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NON-DIAGONAL HOLOMORPHIC ISOMETRIC EMBEDDINGS OF THE POINCARÉ DISK INTO THE SIEGEL UPPER HALF-PLANE*

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Abstract. In this paper, we give new examples of non-diagonal holomorphic isometric embeddings of the Poincaré disk into the Siegel upper half-plane.

Key words. Holomorphic isometric embedding, Poincaré disk, Siegel upper half-plane, Bergman metric.

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1. Introduction. Let Ω be an irreducible bounded symmetric domain equipped with the Bergman metric and Ω^p, $p \geq 2$, the product domain equipped with the product metric. Motivated by questions from arithmetic geometry, Clozel and Ullmo studied holomorphic isometric embeddings of Ω into Ω^p, up to normalizing constants, in [5]. By the Hermitian metric rigidity (see e.g. [7]), they showed that such embeddings must be totally geodesic when the rank of Ω is greater than one. In [8], Mok showed that such embeddings also must be totally geodesic when the rank of Ω is equal to one, i.e. $\Omega = \mathbb{B}^n$ the complex unit ball, with $n \geq 2$. When $n = 1$, i.e. $\Omega = \Delta$ the Poincaré disk, non-standard (i.e., not totally geodesic) holomorphic isometric embeddings of Δ into Δ^p were constructed in [10] by Mok. Further studies of these non-standard embeddings were carried out in [9, 11, 12, 1, 2].

The Poincaré disk Δ is biholomorphically equivalent to the upper half-plane $\mathcal{H} = \{\tau \in \mathbb{C};\ \mathsf{Im}\tau > 0\}$, equipped with the Poincaré metric $ds^2_{\mathcal{H}} = 2\mathsf{Re}\frac{d\tau\otimes d\bar{\tau}}{2(\mathsf{Im}\tau)^2}$ of constant Gaussian curvature -1. Equip $\mathcal{H}^p$ with the product metric $ds^2_{\mathcal{H}^p}$. The non-standard holomorphic isometric embeddings of Δ into Δ^p constructed in [10] were actually in the form of holomorphic isometric embeddings of $\mathcal{H}$ into $\mathcal{H}^p$ as follows ([10, Proposition 3.2.1]).

$$f : \mathcal{H} \to \mathcal{H}^p;\ \tau \mapsto (\tau^{1/p}, \gamma\tau^{1/p}, \cdots, \gamma^{p-1}\tau^{1/p}), \quad \tau = re^{i\varphi} \in \mathcal{H},\ \gamma = e^{i\pi/p}.$$

We call such maps *p-th root embeddings*. Note that f has a singularity at the origin, thus must be non-standard.

The key of Mok's construction is the following *p-th root identity* ([10, Lemma 3.2.2])

$$\prod_{j=0}^{p-1} \sin[\frac{1}{p}(\varphi + j\pi)] = c_p \sin\varphi, \quad c_p > 0. \tag{1.1}$$

Let $\mathcal{H}_p$ be the Siegel upper half-plane, which is biholomorphically equivalent to the classical domain of type three. Denote by $S_p(\mathbb{C})$ the vector space of symmetric p-by-p complex matrices. Then $\mathcal{H}_p$ is defined as

$$\mathcal{H}_p = \{Z \in S_p(\mathbb{C});\ \mathsf{Im}Z > 0\}.$$

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Equip $\mathcal{H}_p$ with the Bergman metric $ds^2_{\mathcal{H}p}$, for which the Bergman kernel is of the form $c(\det \mathsf{Im} Z)^{-(p+1)}$ for some positive constant c (see e.g. [6]).

In [10], Mok also constructed a non-standard holomorphic isometric embedding G of $\mathcal{H}$ into $\mathcal{H}_p$ for $p = 3$. Note that $\mathcal{H}^p$ can be canonically embedded into $\mathcal{H}_p$ via the *diagonal embedding* ι, and Mok showed that $\iota \circ f$ and G are not congruent (cf. [11, Proposition 3]). Recall that two maps F and G between bounded symmetric domains Ω_1 and Ω_2 are said to be *congruent* if there exist $\varphi \in \mathrm{Aut}(\Omega_1)$ and $\psi \in \mathrm{Aut}(\Omega_2)$ such that $G = \psi \circ F \circ \varphi$. In [11], Mok and Ng constructed more examples of non-standard embeddings of $\mathcal{H}$ into $\mathcal{H}_3$, which turn out to be congruent to G. Since these embeddings are not diagonal, we will call them *non-diagonal embeddings.*

The purpose of this note is to give new examples of non-diagonal holomorphic isometric embeddings of $\mathcal{H}$ into $\mathcal{H}_p$ for an arbitrary $p \geq 3$. Surprisingly, these examples are also constructed by using the p-th root identity (1.1). Therefore, we call them the *non-diagonal p-th root embeddings.*

THEOREM 1.1. *Let $n \in \mathbb{N}$ and $p > 2^n$ such that p is divisible by 2^{n-1}. Then there exist non-diagonal p-th root embeddings of $\mathcal{H}$ into $\mathcal{H}_p$ (with $2^n \times 2^n$ blocks).*

For $p = 3$, our examples are congruent to those given by Mok [10] and Mok-Ng [11]. For $q < p$, there also exist (degenerate) non-diagonal p-th root embeddings of $\mathcal{H}$ into $\mathcal{H}_q$ (see Remark 2.1).

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2. The non-diagonal p-th root embeddings. In this section, we construct non-diagonal holomorphic isometric embeddings of the Poincaré disk $\mathcal{H}$ into the Siegel upper half-plane $\mathcal{H}_p$ using the p-th root identity (1.1). For simplicity, set $\alpha = \varphi/p$, $\beta = \pi/p$ and $\gamma = e^{i\beta}$.

Let $0 \leq j < k \leq p-1$ with $k - j > p/2$. Then we have

$$\begin{aligned} &\sin(\alpha + j\beta)\sin(\alpha + k\beta) \\ =&\sin(\alpha + j\beta)[\cos(\alpha + j\beta)\sin(k-j)\beta + \sin(\alpha + j\beta)\cos(k-j)\beta] \\ =&\frac{1}{2}\sin(k-j)\beta\sin 2(\alpha + j\beta) + \cos(k-j)\beta\sin^2(\alpha + j\beta). \end{aligned} \tag{2.1}$$

Let $\kappa > 0$ and choose $a, d \in \mathbb{R}^+$ and $b \in \mathbb{R}$ such that

$$ad = \frac{\kappa}{2}\sin(k-j)\beta, \quad b^2 = -\kappa\cos(k-j)\beta.$$

Consider the following matrix

$$B^{(2)}_{j,k}(\tau) = \begin{bmatrix} a\gamma^{2j}\tau^{2/p} & b\gamma^j\tau^{1/p} \\ b\gamma^j\tau^{1/p} & di \end{bmatrix}. \tag{2.2}$$

Then,

$$\mathsf{Im}B^{(2)}_{j,k}(\tau) = \begin{bmatrix} ar^{2/p}\sin 2(\alpha + j\beta) & br^{1/p}\sin(\alpha + j\beta) \\ br^{1/p}\sin(\alpha + j\beta) & d \end{bmatrix}, \tag{2.3}$$

By (2.1), we get

$$\det\mathsf{Im}B^{(2)}_{j,k}(\tau) = \kappa r^{2/p}\sin(\alpha + j\beta)\sin(\alpha + k\beta) > 0.$$

Since $0 \le j < k - p/2 \le p/2 - 1$, we also have $\sin 2(\alpha + j\beta) > 0$. Therefore, we have $\mathsf{Im} B^{(2)}_{j,k}(\tau) > 0$ for $\tau \in \mathcal{H}$.

Assume now that $p > 4$ is even. Let $0 \le j < k \le p/2 - 1$ with $k - j > p/4$. Then we have

$$
\begin{aligned}
&\sin(\alpha + j\beta)\sin(\alpha + k\beta)\sin(\alpha + j\beta + \frac{\pi}{2})\sin(\alpha + k\beta + \frac{\pi}{2}) \\
=&\frac{1}{4}\sin 2(\alpha + j\beta)\sin 2(\alpha + k\beta) \\
=&\frac{1}{4}\sin 2(\alpha + j\beta)[\cos 2(\alpha + j\beta)\sin 2(k - j)\beta + \sin 2(\alpha + j\beta)\cos 2(k - j)\beta] \\
=&\frac{1}{8}\sin 2(k - j)\beta \sin 4(\alpha + j\beta) + \frac{1}{4}\cos 2(k - j)\beta \sin^2 2(\alpha + j\beta).
\end{aligned}
\tag{2.4}
$$

Let $\kappa > 0$ and choose $a, d_1, d_2, d_3 \in \mathbb{R}^+$ and $b_1, b_2, b_3 \in \mathbb{R}$ such that

$$
ad_1d_2d_3 = \frac{\kappa}{8}\sin 2(k - j)\beta, \quad b_1^2 d_2 d_3 + b_2^2 d_1 d_3 + b_3^2 d_1 d_2 = -\frac{\kappa}{4}\cos 2(k - j)\beta.
$$

Consider the following matrix

$$
B^{(4)}_{j,k}(\tau) = \begin{bmatrix} a\gamma^{4j}\tau^{4/p} & b_1\gamma^{2j}\tau^{2/p} & b_2\gamma^{2j}\tau^{2/p} & b_3\gamma^{2j}\tau^{2/p} \\ b_1\gamma^{2j}\tau^{2/p} & d_1 i & 0 & 0 \\ b_2\gamma^{2j}\tau^{2/p} & 0 & d_2 i & 0 \\ b_3\gamma^{2j}\tau^{2/p} & 0 & 0 & d_3 i \end{bmatrix}. \tag{2.5}
$$

Then,

$$
\mathsf{Im} B^{(4)}_{j,k}(\tau) = \begin{bmatrix} ar^{4/p}\sin 4\rho & b_1 r^{2/p}\sin 2\rho & b_2 r^{2/p}\sin 2\rho & b_3 r^{2/p}\sin 2\rho \\ b_1 r^{2/p}\sin 2\rho & d_1 & 0 & 0 \\ b_2 r^{2/p}\sin 2\rho & 0 & d_2 & 0 \\ b_3 r^{2/p}\sin 2\rho & 0 & 0 & d_3 \end{bmatrix}, \tag{2.6}
$$

where $\rho = \alpha + j\beta$. By (2.4), we get

$$
\begin{aligned}
&\det \mathsf{Im} B^{(4)}_{j,k}(\tau) \\
=&r^{4/p}[ad_1d_2d_3\sin 4\rho - (b_1^2 d_2 d_3 + b_2^2 d_1 d_3 + b_3^2 d_1 d_2)\sin^2 2\rho] \\
=&\kappa r^{4/p}\sin(\alpha + j\beta)\sin(\alpha + k\beta)\sin(\alpha + j\beta + \frac{\pi}{2})\sin(\alpha + k\beta + \frac{\pi}{2}) \\
>&0.
\end{aligned}
\tag{2.7}
$$

Denote by A_l, $l = 2, 3$, the l-th leading principal minor of $\mathsf{Im} B^{(4)}_{j,k}$. Then, by (2.4) and (2.7), we get

$$
\begin{aligned}
A_3 &= r^{4/p}[ad_1d_2\sin 4\rho - (b_1^2 d_2 + b_2^2 d_1)\sin^2 2\rho] \\
&= [\det \mathsf{Im} B^{(4)}_{j,k}(\tau) + r^{4/p} b_3^2 d_1 d_2 \sin^2 2\rho]/d_3 > 0,
\end{aligned}
$$

and

$$
A_2 = r^{4/p}[ad_1\sin 4\rho - b_1^2\sin^2 2\rho] = [A_3 + r^{4/p} b_2^2 d_1 \sin^2 2\rho]/d_2 > 0.
$$

Since $0 \le j < k - p/4 \le p/4 - 1$, we also have $\sin 4(\alpha + j\beta) > 0$. Therefore, we have $\mathsf{Im} B^{(4)}_{j,k}(\tau) > 0$ for $\tau \in \mathcal{H}$.

In general, assume that $p > 2^n$ is divisible by 2^{n-1} for $n \geq 3$. Let $0 \leq j < k \leq p/2^{n-1} - 1$ with $k - j > p/2^n$. Then we have

$$\begin{aligned}
&\prod_{l=0}^{2^{n-1}-1} \sin(\alpha + j\beta + \frac{l\pi}{2^{n-1}}) \sin(\alpha + k\beta + \frac{l\pi}{2^{n-1}}) \\
=&\frac{1}{2^n} \sin 2^{n-1}(\alpha + j\beta) \sin 2^{n-1}(\alpha + k\beta) \\
=&\frac{1}{2^{n+1}} \sin 2^{n-1}(k-j)\beta \sin 2^n(\alpha + j\beta) + \frac{1}{2^n} \cos 2^{n-1}(k-j)\beta \sin^2 2^{n-1}(\alpha + j\beta).
\end{aligned} \tag{2.8}$$

Let $\kappa > 0$ and choose $a, \{d_l\}_{l=1}^{2^n-1} \in \mathbb{R}^+$ and $\{b_l\}_{l=1}^{2^n-1} \in \mathbb{R}$ such that

$$\begin{aligned}
a \prod_{l=1}^{2^n-1} d_l &= \frac{\kappa}{2^{n+1}} \sin 2^{n-1}(k-j)\beta, \\
\sum_{l=1}^{2^n-1} \left(b_l^2 \prod_{\substack{1 \leq m \leq 2^n-1 \\ m \neq l}} d_m \right) &= -\frac{\kappa}{2^n} \cos 2^{n-1}(k-j)\beta.
\end{aligned} \tag{2.9}$$

Consider the following matrix

$$B_{j,k}^{(2^n)}(\tau) = \begin{bmatrix} a\gamma^{2^n j}\tau^{2^n/p} & b_1\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} & \cdots & b_{2^n-1}\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} \\ b_1\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} & d_1 i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{2^n-1}\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} & 0 & \cdots & d_{2^n-1} i \end{bmatrix}. \tag{2.10}$$

Then,

$$\mathsf{Im} B_{j,k}^{(2^n)}(\tau) = \begin{bmatrix} ar^{2^n/p} \sin 2^n\rho & b_1 r^{2^{n-1}/p} \sin 2^{n-1}\rho & \cdots & b_{2^n-1} r^{2^{n-1}/p} \sin 2^{n-1}\rho \\ b_1 r^{2^{n-1}/p} \sin 2^{n-1}\rho & d_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{2^n-1} r^{2^{n-1}/p} \sin 2^{n-1}\rho & 0 & \cdots & d_{2^n-1} \end{bmatrix}. \tag{2.11}$$

By (2.8), we get

$$\begin{aligned}
\det \mathsf{Im} B_{j,k}^{(2^n)}(\tau) &= r^{2^n/p} \left[a(\prod_{l=1}^{2^n-1} d_l) \sin 2^n\rho - \sum_{l=1}^{2^n-1} (b_l^2 \prod_{\substack{1 \leq m \leq 2^n-1 \\ m \neq l}} d_m) \sin^2 2^{n-1}\rho \right] \\
&= \kappa r^{2^n/p} \prod_{l=0}^{2^{n-1}-1} \sin(\alpha + j\beta + \frac{l\pi}{2^{n-1}}) \sin(\alpha + k\beta + \frac{l\pi}{2^{n-1}}) > 0.
\end{aligned} \tag{2.12}$$

Denote by A_l, $2 \leq l \leq 2^n$, the l-th leading principal minor of $\mathsf{Im} B_{j,k}^{(2^n)}$. Then, by (2.12), we know that $A_{2^n} = \det \mathsf{Im} B_{j,k}^{(2^n)}(\tau) > 0$. Inductively, assume that we already have $A_l > 0$ for $l = m+1$. Then for $l = m$, we have

$$A_m = r^{2^n/p}\left[a(\prod_{s=1}^{m-1} d_s)\sin 2^n\rho - \sum_{s=1}^{m-1}(b_s^2 \prod_{\substack{1\le t\le m-1\\ t\neq s}} d_t)\sin^2 2^{n-1}\rho\right]$$
$$= \left[A_{m+1} + r^{2^n/p} b_m^2 (\prod_{t=1}^{m-1} d_t)\sin^2 2^{n-1}\rho\right]/d_m > 0.$$

Since $0 \le j < k - p/2^n \le p/2^n - 1$, we also have $\sin 2^n(\alpha + j\beta) > 0$. Therefore, we have $\mathsf{Im} B_{j,k}^{(2^n)}(\tau) > 0$ for $\tau \in \mathcal{H}$.

Recall that the automorphisms of $\mathcal{H}_p$ are of the form

$$Z \mapsto (AZ+B)(CZ+D)^{-1}, \quad Z \in \mathcal{H}_p,$$

where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a $2p$-by-$2p$ real symplectic matrix.

Consider M with $B = C = 0$, A diagonal and $\det A \neq 0$, and $D = A^{-1}$. If A is equal to the identity matrix except at the block corresponding to $B_{j,k}^{(2^n)}(\tau)$ where it is of the form $\mathrm{diag}\{1, -1, \cdots, -1\}$, then we can change b_l to $-b_l$, $1 \le l \le 2^n - 1$, in $B_{j,k}^{(2^n)}(\tau)$ while fixing a and d_l. If A is equal to the identity matrix except at the block corresponding to $B_{j,k}^{(2^n)}(\tau)$ where it is of the form $\mathrm{diag}\{c_0, c_1, \cdots, c_{2^n-1}\}$, with $c_l^{-1} = \sqrt{d_l}$, $1 \le l \le 2^n - 1$, and $\prod_{m=0}^{2^n-1} c_m = 1$, then we can change d_l in $B_{j,k}^{(2^n)}(\tau)$ to be equal to 1 while fixing $a\prod_{l=1}^{2^n-1} d_l$ and $\sum_{l=1}^{2^n-1}(b_l^2 \prod_{\substack{1\le m\le 2^n-1\\ m\neq l}} d_m)$. We can also of course multiply $B_{j,k}^{(2^n)}(\tau)$ by a positive constant.

Without loss of generality, we can assume that $|b_1| \ge |b_2| \ge \cdots \ge |b_{2^n-1}|$. Choose $\kappa = -\frac{2^n}{\cos 2^{n-1}(k-j)\beta}$ in (2.9). By the discussion in the last paragraph, we can assume that $d_l = 1$ for all $1 \le l \le 2^n - 1$. Then, we have $a = -\frac{1}{2}\tan 2^{n-1}(k-j)\beta$ and

$$\sum_{l=1}^{2^n-1} b_l^2 = 1, \quad b_1 \ge b_2 \ge \cdots \ge b_{2^n-1} \ge 0. \tag{2.13}$$

Therefore, we have the following normal form for $B_{j,k}^{(2^n)}(\tau)$:

$$\begin{bmatrix} -\frac{1}{2}\tan 2^{n-1}(k-j)\beta\gamma^{2^n j}\tau^{2^n/p} & b_1\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} & \cdots & b_{2^n-1}\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} \\ b_1\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} & i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{2^n-1}\gamma^{2^{n-1}j}\tau^{2^{n-1}/p} & 0 & \cdots & i \end{bmatrix}, \tag{2.14}$$

where $\{b_l\}_{l=1}^{2^n-1}$ satisfy (2.13).

Set

$$m := \max\{l+1;\ b_l \neq 0\}.$$

If $m = 2^n$, then we say that $B_{j,k}^{(2^n)}(\tau)$ is *non-degenerate*. If $2 \le m \le 2^n - 1$, then we say that $B_{j,k}^{(2^n)}(\tau)$ is *degenerate*. In the degenerate case, we will denote by $B_{j,k}^{(2^n;m)}(\tau)$ the m-th leading principal minor of $B_{j,k}^{(2^n)}(\tau)$.

Remark 2.1. When p is divisible by 2^n, we can also consider the "critical" case where $k-j=p/2^n$. Then we have all b_l equal to zero (thus $m=1$) and $B^{(2^n)}_{j,k}(\tau)$ is diagonal. We will then say that $B^{(2^n)}_{j,k}(\tau)$ is *totally degenerate*.

For a given $p>2^n$ which is divisible by 2^{n-1} for some $n\geq 1$, write $p=r2^{n-1}$ and set $s=\lfloor r/2\rfloor$. By (2.8), each pair of (j,k) with $0\leq j<k\leq p/2^{n-1}-1$ and $k-j>p/2^n$ takes up the index set

$$I_{j,k}:=\{j+lr,k+lr\}_{l=0}^{2^{n-1}-1}.$$

Let (j_μ,k_μ), $1\leq\mu\leq\nu\leq s$, be ν pairs with $0\leq j_\mu<k_\mu\leq p/2^{n-1}-1$ and $k_\mu-j_\mu>p/2^n$ for each μ. And, if $p-2^n\nu>0$, let l_σ, $1\leq\sigma\leq p-2^n\nu$, be the integers between 0 and $p-1$ outside of $\cup_{\mu=1}^{\nu}I_{j_\mu,k_\mu}$. For each (j_μ,k_μ), $1\leq\mu\leq\nu$, we get a block $B^{(2^n)}_{j_\mu,k_\mu}$. And for each l_σ, $1\leq\sigma\leq p-2^n\nu$, we put $c_\sigma\gamma^{l_\sigma}\tau^{1/p}$ with $c_l>0$ on the diagonal. Denote by $P(\tau)$ the resulting $p\times p$ matrix. It is clear from the construction that we have $\mathsf{Im}P(\tau)>0$ for $\tau\in\mathcal{H}$. And, by (1.1), we have that $\det\mathsf{Im}P(\tau)=c_p\mathsf{Im}\tau$. If every block $B^{(2^n)}_{j_\mu,k_\mu}$ is non-degenerate, then we get a *non-degenerate non-diagonal p-th root embedding* of $\mathcal{H}$ into $\mathcal{H}_p$. If at least one block $B^{(2^n)}_{j_\mu,k_\mu}$ is degenerate, then we get a *degenerate non-diagonal p-th root embedding* of $\mathcal{H}$ into $\mathcal{H}_q$, $q<p$. Here, we have $q<p$ since we only keep the block $B^{(2^n;m)}_{j_\mu,k_\mu}(\tau)$ and at least one m is smaller than 2^n.

Let F be a non-degenerate non-diagonal p-th root embedding of $\mathcal{H}$ into $\mathcal{H}_p$. Denote by $\omega_{\mathcal{H}_p}$ the Kähler form on $\mathcal{H}_p$. Then

$$F^*\omega_{\mathcal{H}_p}=-(p+1)\sqrt{-1}\partial\bar\partial\log(\det\mathsf{Im}Z)=-(p+1)\sqrt{-1}\partial\bar\partial\log(\mathsf{Im}\tau)=\frac{p+1}{2}\omega_{\mathcal{H}}.$$

For the second equality above we have used the p-root identity (1.1). Thus F is a holomorphic isometric embedding $(\mathcal{H},\lambda ds^2_{\mathcal{H}})\to(\mathcal{H}_p,ds^2_{\mathcal{H}_p})$, with the isometric constant $\lambda=\frac{p+1}{2}$.

If G is a degenerate non-diagonal p-th root embedding of $\mathcal{H}$ into $\mathcal{H}_q$ with $q<p$, then the same computation shows that G is a holomorphic isometric embedding $(\mathcal{H},\lambda ds^2_{\mathcal{H}})\to(\mathcal{H}_q,ds^2_{\mathcal{H}_q})$, with the isometric constant $\lambda=\frac{p+1}{2}$. Thus, G is incongruent to a non-degenerate non-diagonal p-th root embedding of $\mathcal{H}$ into $\mathcal{H}_q$, which has the isometric constant $\frac{q+1}{2}$.

Remark 2.2. For $p=3$, choosing $j=0$, $k=2$ and $l=1$, we get the non-diagonal embeddings given by Mok [10] and Mok-Ng [11].

Remark 2.3. If every block $B^{(2^n)}_{j,k}$ is totally degenerate, then we get a diagonal embedding which is actually induced by a generalized p-th root embedding (see below for the definition). More precisely, if $p=2^nq$ and there are m totally degenerate blocks, $1\leq m\leq q-1$, then the generalized p-th root embedding is created by a q-th root embedding followed by $q-m$ 2^n-th root embedding applied to $q-m$ distinct factors which do not correspond to any of the blocks $B^{(2^n)}_{j,k}$.

Remark 2.4. Let $f:\mathcal{H}\to\mathcal{H}^p$ be the p-th root embedding and $g:\mathcal{H}\to\mathcal{H}^q$ the q-th root embedding, $p,q\geq 2$. If we apply f to one of the factors of $\mathcal{H}^q$, we get a new non-standard embedding of $\mathcal{H}$ into $\mathcal{H}^{p+q-1}$. Obviously, this procedure can be applied many times, using different p-th root embeddings and to different factors. We will call the resulting map the *generalized p-th root embeddings*. For such embeddings, the following *generalized p-th root identity* holds (cf. [12, Proposition 5.3]):

$$\prod_{j=1}^{p} \sin[\frac{1}{n_j}(\varphi + m_j\pi)] = c_p \sin\varphi, \quad \sum_{j=1}^{p} 1/n_j = 1. \tag{2.15}$$

Using (2.15) instead of (1.1), we can get *non-diagonal generalized p-th root embeddings* of $\mathcal{H}$ into $\mathcal{H}_p$ following a similar construction as above.

REMARK 2.5. As in the proof of [10, Theorem 3.2.1], one can get a real-analytic 1-parameter family of holomorphic isometric embeddings of $\mathcal{H}$ into $\mathcal{H}_p$, which are mutually incongruent to each other.

REMARK 2.6. Recently, in [3] and [4], Chan and Mok studied the structure of holomorphic isometric embeddings of complex unit balls into bounded symmetric domains. It would be interesting to see how our examples fit into this general framework.

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