ASYMPTOTIC POLYBALANCED KERNELS ON EXTREMAL KÄHLER MANIFOLDS[*]

TOSHIKI MABUCHI[†]

In honor of Professor Ngaiming Mok's 60th birthday

Abstract. In this paper, improving a result in [12], we obtain asymptotic polybalanced kernels associated to extremal Kähler metrics on polarized algebraic manifolds. As a corollary, we strengthen a result in [16] on asymptotic relative Chow-polystability for extremal Kähler polarized algebraic manifolds. Finally, related to the Yau-Tian-Donaldson Conjecture for extremal Kähler metrics, we shall discuss the difference between strong relative K-stability (cf. [20]) and relative K-stability.

Key words. Asymptotic polybalanced kernels, extremal Kähler metrics, asymptotic relative Chow-stability.

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1. Introduction. In this paper, we fix once for all a *polarized algebraic manifold* (M, L) which is by definition a pair of a nonsingular irreducible complex projective variety M of dimension n and a very ample holomorphic line bundle L on M. By taking the identity component $\text{Aut}^0(M)$ of the group of all biholomorphisms of M, we consider the maximal connected linear algebraic subgroup H of $\text{Aut}^0(M)$. Hence $\text{Aut}^0(M)/H$ is an abelian variety. For the identity component Z of the center of a maximal compact connected subgroup K of H, we take its complexification $Z_{\mathbb{C}}$ in H. Let

$$\mathfrak{z} := \text{Lie}(Z) \quad \text{and} \quad \mathfrak{z}_{\mathbb{C}} := \text{Lie}(Z_{\mathbb{C}})$$

be the associated Lie algebras. Then the infinitesimal action of $\mathfrak{z}_{\mathbb{C}}$ on M lifts to an infinitesimal bundle action of $\mathfrak{z}_{\mathbb{C}}$ on L. For

$$V_m := H^0(M, \mathcal{O}(L^{\otimes m})), \qquad m = 1, 2, \ldots,$$

we view $\mathfrak{z}_{\mathbb{C}}$ as a Lie subalgebra of $\mathfrak{sl}(V_m)$ by taking the traceless part for each element of $\mathfrak{z}_{\mathbb{C}}$. In view of the infinitesimal action of $\mathfrak{z}_{\mathbb{C}}$, V_m is expressible as a direct sum of K-invariant subspaces,

$$V_m = \bigoplus_{\alpha=1}^{\nu_m} V_{m,\alpha},$$

where $V_{m,\alpha} := \{ \tau \in V_m \, ; Y\tau = \chi_{m,\alpha}(Y)\tau \text{ for all } Y \in \mathfrak{z}_{\mathbb{C}} \}$ with mutually distinct characters $\chi_{m,\alpha} \in \mathfrak{z}_{\mathbb{C}}^*$, $\alpha = 1, 2, \ldots, \nu_m$. Let h be a K-invariant Hermitian metric for L such that the associated first Chen form $\omega = c_1(L; h)$ is Kähler. Define a Hermitian metric $\rho_m(h)$ for V_m by

$$\rho_m(h)(\tau, \tau') := \int_X (\tau, \tau')_h \, \omega^n \qquad \tau, \tau' \in V_m,$$

[†]Mathematics Department, Osaka University, Toyonaka, Osaka, 560-0043 Japan (mabuchi@math.sci.osaka-u.ac.jp). Supported by JSPS Grant-in-Aid for Scientific Research (B) No. 25287010.

where $(\tau, \tau')_h$ denotes the pointwise Hermitian pairing of τ, τ' in terms of the Hermitian metric h. If τ and τ' coincide, then we write $(\tau, \tau)_h$ simply as $|\tau|_h^2$. By setting $q := 1/m$, we now define

$$B_{m,\alpha}(h) := n!\, q^n \sum_{i=1}^{n_\alpha} |\tau_{\alpha,i}|_h^2, \qquad B_m(h) := \sum_{\alpha=1}^{\nu_m} B_{m,\alpha}(h),$$

where $\{\tau_{\alpha,i}\,;\, i = 1, 2, \ldots, n_\alpha\}$ is an orthonormal basis for the subspace $V_{m,\alpha}$ of the Hermitian vector space $(V_m, \rho_m(h))$. Note that, if M admits an extremal Kähler metric ω_0 in the class $c_1(L)$, then in view of [2], by choosing K to be the identity component of the group of isometries for (M, ω_0) in H, we may assume that the extremal Kähler vector field $\mathcal{V}$ (cf. [8]) belongs to $\mathfrak{z}$. Let σ_{ω_0} be the scalar curvature of ω_0, and define a real constant C_0 by

$$C_0 := \{2c_1(L)^n[M]\}^{-1} \left\{ \int_M \sigma_{\omega_0}\, \omega_0^n + \sqrt{-1} \int_M h_0^{-1}(\mathcal{V}h_0)\, \omega_0^n \right\}.$$

Then by setting $\mathcal{Y}_0 := \sqrt{-1}\mathcal{V}/2$ and $\varphi_0 := 0$, we obtain

MAIN THEOREM. *Suppose that L admits a Hermitian metric h_0 such that $\omega_0 := c_1(L; h_0)$ is an extremal Kähler metric with extremal Kähler vector field $\mathcal{V} \in \mathfrak{z}$. Then there exist vector fields $\mathcal{Y}_k \in \sqrt{-1}\mathfrak{z}$, smooth real-valued K-invariant functions φ_k, real constants C_k, $k = 1, 2, \ldots$, on M such that*

$$\sum_{\alpha=1}^{\nu_m} \{1 - \chi_{m,\alpha}(\mathcal{Y}(\ell))\} B_{m,\alpha}(h(\ell)) = 1 + C(\ell) + O(q^{\ell+2}),\ \ell = 0, 1, 2, \ldots, \qquad (1.1)$$

with $\mathcal{Y}_k$, φ_k, C_k independent of q and ℓ, where $\mathcal{Y}(\ell) := \Sigma_{k=0}^{\ell} q^{k+2}\mathcal{Y}_k$, $h(\ell) := h_0 \exp(-\Sigma_{k=0}^{\ell} q^k \varphi_k)$, and $C(\ell) := \Sigma_{k=0}^{\ell} C_k q^{k+1}$.

Here for every integer r, we mean by $O(q^r)$ a quantity whose C^j-norm for every nonnegative integer j is bounded by $\kappa_j q^r$ for some positive constant κ_j independent of q and α. In the above Main Theorem, let ℓ be sufficiently large. Then the formal expression of the left-hand side of (1.1) is called the *asymptotic polybalaced kernel* for (M, L).

Let T be an arbitrary algebraic torus in H satisfying $Z_{\mathbb{C}} \subset T$. As a corollary of Main Theorem, we obtain

COROLLARY. *If M admits an extremal Kähler metric in the class $c_1(L)$, then (M, L) is asymptotically Chow-polystable relative to T, i.e., $(M, L^{\otimes m})$, $m \gg 1$, are Chow-polystable relative to T.*

In the last section, in view of [20] and a recent result of Yotsutani-Zhou [30], we shall discuss extremal Kähler versions of the Yau-Tian-Donaldson Conjecture from various points of view.

For preceding related works, see Apostolov-Huang [1], Donaldson [5], [6], Futaki [7], Hashimoto [9], Lu [10], Ono-Sano-Yotsutani [22], Phong-Sturm [23], Sano-Tipler [24], Székelyhidi [25], [26], Tian [27], [28], Zelditch [31] and Zhang [32]. I owe much to these works.

Parts of this paper were announced in Pacific Rim Conference on Complex and Simplectic Geometry XI at Heifei in July, 2016. Afterwards, during the preparation of this paper, I heard that R. Seyyedali showed asymptotic Chow-stability, relative to a maximal algebraic torus in $\mathrm{Aut}^0(M)$, for an extremal Kähler polarized algebraic manifold (M, L).

2. Proof of Main Theorem. Before getting into the proof, let us briefly discuss how the proof is obtained. Note that, if $\mathrm{Aut}^0(M)$ is discrete, then by the vanishing $\mathcal{Y}(\ell) = 0$ our equation (1.1) reduces to the formal equation of Donaldson (see Theorem 26 in [5]). As in Donaldson's proof, our proof is based on the Tian-Yau-Zelditch expansion and also on the variation formula for the scalar curvature in terms of the Lichnérowicz operator (cf. [2], (3.3); see also [5], (49)). In the non-discrete $\mathrm{Aut}^0(M)$ case, the difficulty arises from the nontriviality of the kernel of the Lichnérowicz operator which prevents us from directly applying Donaldson's method. However, for the formal equation of Donaldson replaced by our equation (1.1) above, a slight modification of the proof of Theorem B in [12] allows us to complete the proof.

In this section, we prove Main Theorem by induction on ℓ. For each K-invariant Hermitian metric h for L, by setting $|\tau_{\alpha,i}|^2 := \tau_{\alpha,i}\,\bar{\tau}_{\alpha,i}$, we consider

$$\Psi_m(h) \; := \; n!\, q^n \sum_{\alpha=1}^{\nu_m} \sum_{i=1}^{n_\alpha} |\tau_{\alpha,i}|^2, \tag{2.1}$$

where $\{\tau_{\alpha,i}\,;\, i = 1, 2, \ldots, n_\alpha\}$ is an orthonormal basis for $(V_m, \rho_m(h))$ as in the introduction. Then the left-hand side of (1.1) is the real-valued function on X obtained as the contraction

$$h(\ell)^m \cdot \{(1 - \mathcal{Y}(\ell))\Psi_m(h(\ell))\}$$

of $(1 - \mathcal{Y}(\ell))\Psi_m(h(\ell))$ with $h(\ell)^m$ (see [11], (1.4.1), for the definition of $\mathcal{Y}(\ell)|\tau_{\alpha,i}|^2$). Hence the proof of (1.1) (and hence Main Theorem) is reduced to showing the following for all nonnegative integers ℓ:

$$h(\ell)^m \cdot \{(1 - \mathcal{Y}(\ell))\Psi_m(h(\ell))\} \; = \; 1 + C(\ell) + O(q^{\ell+2}). \tag{2.2}$$

Note that $\mathcal{Y}(\ell)$ acts on the antiholomorphic section $\bar{\tau}_{\alpha,i}$ trivially. Let $\mathcal{D}_0$ be the Lichnérowicz operator as defined in [2], (2.1), for the extremal Kähler manifold (M, ω_0), where we write ω_0 as

$$\omega_0 \; = \; \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}}\, dz^i \wedge d\overline{z^j}$$

in terms of a system $(z^1, \ldots, z^n)$ of holomorphic local coordinates on M. Let $\mathcal{S}$ be the space of all real-valued smooth K-invariant functions φ on M such that $\int_M \varphi\omega_0^n = 0$. Since $\mathcal{D}_0$ maps $\mathcal{S}$ into itself, the restricted operator

$$\mathcal{D}_0 : \mathcal{S} \to \mathcal{S}$$

is denoted also by $\mathcal{D}_0$ whose kernel in $\mathcal{S}$ is written simply as $\mathrm{Ker}\,\mathcal{D}_0$. Then we have an isomorphism

$$e_0 : \; \mathrm{Ker}\,\mathcal{D}_0 \; \cong \; \mathfrak{z}, \qquad \varphi \leftrightarrow e_0(\varphi) := \mathrm{grad}_{\omega_0}^{\mathbb{C}}\, \varphi, \tag{2.3}$$

where $\mathrm{grad}_{\omega_0}^{\mathbb{C}}\, \varphi := (1/\sqrt{-1})\Sigma_{i,j} g^{\bar{j}i}(\partial\varphi/\partial z^{\bar{j}})\partial/\partial z^i$. By the inner product

$$(\varphi, \psi)_{\omega_0} \; := \; \int_M \varphi\psi\,\omega_0^n, \qquad \varphi, \psi \in \mathcal{S},$$

we write $\mathcal{S}$ as an orthogonal direct sum $\operatorname{Ker} \mathcal{D}_0 \oplus (\operatorname{Ker} \mathcal{D}_0)^{\perp}$. We then consider the orthogonal projection $\operatorname{pr}_1 : \mathcal{S} \to \operatorname{Ker} \mathcal{D}_0$ to the first factor. The proof of (2.2) is divided into two steps:

Step 1. In this step, we shall show that (2.2) is true for $\ell = 0$. Note that $h(0) = h_0$. In view of Lu [10], the Tian-Yau-Zelditch asymptotic expansion ([27], [31]; see also [3]) is written in the form

$$h_0^m \cdot \Psi_m(h(0)) \; (= B_m(h(0))\,) \; = \; 1 + \frac{\sigma_{\omega_0}}{2} q + O(q^2). \tag{2.4}$$

By $\mathcal{Y}(0) = q^2 \sqrt{-1}\mathcal{V}/2$, we have $m\mathcal{Y}(0) = q\sqrt{-1}\mathcal{V}/2$. Take the infinitesimal action of $\mathcal{Y}(0)$ on (2.4). Dividing it further by (2.4), we obtain

$$q\, h_0^{-1} \sqrt{-1}\,(\mathcal{V}/2)\, h_0 \; + \; \Psi_m(h(0))^{-1}\{\mathcal{Y}(0)\Psi_m(h(0))\} \; = \; O(q^3). \tag{2.5}$$

On the other hand, by [12], p.579, we have $h_0^{-1}\sqrt{-1}(\mathcal{V}/2)h_0 = C_0 - (\sigma_{\omega_0}/2)$. Hence (2.5) is rewritten as

$$\Psi_m(h(0))^{-1}\{\mathcal{Y}(0)\Psi_m(h(0))\} \; = \; -q\{C_0 - (\sigma_{\omega_0}/2)\} + O(q^3).$$

This together with (2.4) implies that

$$h_0^m \cdot \{\mathcal{Y}(0)\Psi_m(h(0))\} \; = \; -q\{C_0 - (\sigma_{\omega_0}/2)\} + O(q^2).$$

Subtracting this from (2.4), we obtain the required equality:

$$h(0)^m \cdot \{(1 - \mathcal{Y}(0))\Psi_m(h(0))\} \; = \; (1 + qC_0) + O(q^2) \; = \; 1 + C(0) + O(q^2).$$

Step 2. Note that the left-hand side of (2.2) is

$$\{h(\ell)^m \cdot \Psi_m(h(\ell))\}\frac{(1 - \mathcal{Y}(\ell))\Psi_m(h(\ell))}{\Psi_m(h(\ell))} \; = \; B_m(h(\ell))\left\{1 - \frac{\mathcal{Y}(\ell)\Psi_m(h(\ell))}{\Psi_m(h(\ell))}\right\}.$$

For a positive integer ℓ, by assuming the induction hypothesis

$$B_m(h(\ell-1))\left\{1 - \frac{\mathcal{Y}(\ell-1)\Psi_m(h(\ell-1))}{\Psi_m(h(\ell-1))}\right\} \; = \; 1 + C(\ell-1) + O(q^{\ell+1}), \tag{2.6}$$

we have only to find $\mathcal{Y}_\ell$, φ_ℓ and C_ℓ such that $\mathcal{Y}(\ell) := \mathcal{Y}(\ell-1) + q^{\ell+2}\mathcal{Y}_\ell$, $h(\ell) := h(\ell-1)e^{-q^\ell \varphi_\ell}$ and $C(\ell) := C(\ell-1) + C_\ell q^{\ell+1}$ satisfy

$$B_m(h(\ell))\left\{1 - \frac{\mathcal{Y}(\ell)\Psi_m(h(\ell))}{\Psi_m(h(\ell))}\right\} \; = \; 1 + C(\ell) + O(q^{\ell+2}). \tag{2.7}$$

By setting $\omega(\ell) := c_1(L; h(\ell))$ and $\omega(\ell-1) := c_1(L; h(\ell-1))$, we have $\omega(\ell) = \omega(\ell-1) + (\sqrt{-1}/2\pi)q^\ell \partial\bar{\partial}\varphi_\ell$. Let $C^\infty(M)_{\mathbb{R}}^K$ be the space of all real-valued K-invariant functions on M. To each $(\mathcal{Y}_\ell, \varphi_\ell, C_\ell)$ in $\sqrt{-1}\mathfrak{z} \times C^\infty(M)_{\mathbb{R}}^K \times \mathbb{R}$, we assign a real-valued K-invariant function $\Phi(q; \mathcal{Y}_\ell, \varphi_\ell, C_\ell)$ by

$$\Phi(q; \mathcal{Y}_\ell, \varphi_\ell, C_\ell) := B_m(h(\ell))\left\{1 - \frac{\mathcal{Y}(\ell)\Psi_m(h(\ell))}{\Psi_m(h(\ell))}\right\}$$

$$= B_m(h(\ell-1)e^{-q^\ell \varphi_\ell})\left\{1 - \frac{(\mathcal{Y}(\ell-1) + q^{\ell+2}\mathcal{Y}_\ell)\Psi_m(h(\ell-1)e^{-q^\ell \varphi_\ell})}{\Psi_m(h(\ell-1)e^{-q^\ell \varphi_\ell})}\right\}.$$

By the induction hypothesis (2.6), there exists a real-valued K-invariant function u_ℓ on M such that

$$\Phi(q; 0, 0, 0) \equiv 1 + C(\ell - 1) + u_\ell q^{\ell+1}, \qquad \text{modulo } q^{\ell+2}. \tag{2.8}$$

In view of the variation formula for the scalar curvature (see for instance [2], (2.5)), we see that, modulo $q^{\ell+2}$,

$$\begin{cases} B_m(h(\ell)) - B_m(h(\ell - 1)) & \equiv (q/2)\{\sigma_{\omega(\ell)} - \sigma_{\omega(\ell-1)}\} \\ & \equiv q^{\ell+1}(-\mathcal{D}_0 + \sqrt{-1}\,\mathcal{V})(\varphi_\ell/2). \end{cases} \tag{2.9}$$

Put $I_1 := \{\mathcal{Y}(\ell)\Psi_m(h(\ell))\}/\Psi_m(h(\ell))$, $J := \{\mathcal{Y}(\ell - 1)\Psi_m(h(\ell))\}/\Psi_m(h(\ell))$ and $I_2 := \{\mathcal{Y}(\ell - 1)\Psi_m(h(\ell - 1))\}/\Psi_m(h(\ell - 1))$. In view of (2.4), we have $h_0^m \cdot \Psi_m(h(\ell)) \equiv 1$ modulo q. Hence, modulo $q^{\ell+2}$,

$$I_1 - J \ (= q^{\ell+2}\{\mathcal{Y}_\ell\Psi_m(h(\ell))\}/\Psi_m(h(\ell))) \equiv -q^{\ell+1}h_0^{-1}(\mathcal{Y}_\ell h_0). \tag{2.10}$$

Note here that, by setting

$$C := \{c_1(L)^n[M]\}^{-1} \int_M h_0^{-1}(\mathcal{Y}_\ell h_0)\,\omega_0^n,$$

we obtain $h_0^{-1}(\mathcal{Y}_\ell h_0) = C + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell)$ (see for instance [11]). On the other hand, we have the following:

$$\begin{aligned} J - I_2 &= \mathcal{Y}(\ell - 1)\log\left\{\frac{\Psi_m(h(\ell))}{\Psi_m(h(\ell - 1))}\right\} \\ &= \mathcal{Y}(\ell - 1)\left(q^{\ell-1}\varphi_\ell + \log\left\{\frac{h(\ell)^m \cdot \Psi_m(h(\ell))}{h(\ell - 1)^m \cdot \Psi_m(h(\ell - 1))}\right\}\right) \\ &= \mathcal{Y}(\ell - 1)\left(q^{\ell-1}\varphi_\ell + \log\{B_m(h(\ell))/B_m(h(\ell - 1))\}\right). \end{aligned}$$

Since $B_m(h(\ell)) \equiv B_m(h(\ell - 1)) \equiv 1$ modulo q, we see from (2.9) that $\log\{B_m(h(\ell))/B_m(h(\ell - 1))\} \equiv 0$ modulo $q^{\ell+1}$. Moreover, by $\ell \geq 1$, we obtain $\mathcal{Y}(\ell - 1) \equiv q^2(\sqrt{-1}/2)\mathcal{V}$ modulo q^3. It then follows that

$$J - I_2 \equiv q^{\ell+1}\sqrt{-1}\mathcal{V}(\varphi_\ell/2), \quad \text{modulo } q^{\ell+2}. \tag{2.11}$$

On the other hand, since $h_0^m \cdot \Psi_m(h(\ell)) \equiv 1$ modulo q, we obtain

$$\begin{cases} I_1 & \equiv q^2(\sqrt{-1}\mathcal{V}/2)\Psi_m(h(\ell))/\Psi_m(h(\ell)) \\ & \equiv -(\sqrt{-1}/2)\,qh_0^{-1}(\mathcal{V}h_0) \equiv 0, \qquad \text{modulo } q. \end{cases} \tag{2.12}$$

Now by (2.10) and (2.11), $I_1 - I_2 \equiv q^{\ell+1}\{\sqrt{-1}\mathcal{V}(\varphi_\ell/2) - h_0^{-1}(\mathcal{Y}_\ell h_0)\} \equiv q^{\ell+1}\{\sqrt{-1}\mathcal{V}(\varphi_\ell/2) - C - e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell)\}$ modulo $q^{\ell+2}$. Thus by setting $B_1 := B_m(h(\ell))$ and $B_2 := B_m(h(\ell - 1))$, we see from (2.9) and (2.12) the following:

$$\begin{cases} \Phi(q; \mathcal{Y}_\ell, \psi_\ell, C_\ell) - \Phi(q; 0, 0, 0) = B_1(1 - I_1) - B_2(1 - I_2) \\ = (B_1 - B_2)(1 - I_1) - B_2(I_1 - I_2) \\ \equiv q^{\ell+1}\{(-\mathcal{D}_0 + \sqrt{-1}\mathcal{V})(\varphi_\ell/2) \\ \qquad\qquad -\sqrt{-1}\mathcal{V}(\varphi_\ell/2) + C + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell)\} \\ \equiv q^{\ell+1}\{-\mathcal{D}_0(\varphi_\ell/2) + C + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell)\}, \quad \text{modulo } q^{\ell+2}. \end{cases} \tag{2.13}$$

Since the function $u_\ell - \mu_\ell$ belongs to $\mathcal{S}$ for $\mu_\ell := \{c_1(L)^n[M]\}^{-1} \int_M u_\ell \, \omega_0^n$, we can write u_ℓ as a sum

$$u_\ell \;=\; \mu_\ell + u_\ell' + u_\ell'', \tag{2.14}$$

where $u_\ell' := (1 - \mathrm{pr}_1)(u_\ell - \mu_\ell) \in (\mathrm{Ker}\,\mathcal{D}_0)^\perp$ and $u_\ell'' := \mathrm{pr}_1(u_\ell - \mu_\ell) \in \mathrm{Ker}\,\mathcal{D}_0$. Let φ_ℓ be the unique element of $(\mathrm{Ker}\,\mathcal{D}_0)^\perp$ such that $\mathcal{D}_0(\varphi_\ell/2) = u_\ell'$. Put

$$\mathcal{Y}_\ell \;:=\; \sqrt{-1}\, e_0(u_\ell'') \qquad \text{and} \qquad C_\ell \;:=\; \mu_\ell + C. \tag{2.15}$$

Then by (2.3), $\mathcal{Y}_\ell \in \sqrt{-1}\mathfrak{z}$, while C_ℓ is a real constant. Since $\mathcal{D}_0(\varphi_\ell/2) = u_\ell'$, it follows from (2.8), (2.13), (2.14) and (2.15) that, modulo $q^{\ell+2}$,

$$\begin{aligned}
\Phi(q; \mathcal{Y}_\ell, \psi_\ell, C_\ell) &\equiv \Phi(q; 0, 0, 0) + q^{\ell+1}\{-\mathcal{D}_0(\varphi_\ell/2) + C + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_\ell)\} \\
&\equiv 1 + C(\ell - 1) + q^{\ell+1}\{u_\ell - \mathcal{D}_0(\varphi_\ell/2) + C + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_\ell)\} \\
&= 1 + C(\ell - 1) + q^{\ell+1}\{(u_\ell' - \mathcal{D}_0(\varphi_\ell/2)) + (\mu_\ell + C) + (u_\ell'' + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_\ell))\} \\
&= 1 + C(\ell - 1) + q^{\ell+1}C_\ell \equiv 1 + C(\ell),
\end{aligned}$$

which shows (2.7), as required. □

REMARK 2.16. The preceding work in [12], Theorem B, is obtained from Main Theorem above by replacing the left-hand side of (1.1) by

$$\sum_{\alpha=1}^{\nu_m} \exp\{-\chi_{m,\alpha}(\mathcal{Y}(\ell))\}\, B_{m,\alpha}(h(\ell)). \tag{2.17}$$

The point is the following: The coefficients $1 - \chi_{m,\alpha}(\mathcal{Y}(\ell))$, $\alpha = 1, 2, \ldots, \nu_m$, in (1.1) are linear in $\mathcal{Y}(\ell)$, while the coefficients $\exp\{-\chi_{m,\alpha}(\mathcal{Y}(\ell))\}$, $\alpha = 1, 2, \ldots, \nu_m$, in (2.17) aren't. This linearity is essential in the proof of the asymptotic relative Chow-polystability.

3. Relative Chow-polystability. In this section, we fix an algebraic torus T in $\mathrm{Aut}^0(M)$. Then for K in the introduction, replacing T by its conjugate group, we may assume that the maximal compact subgroup T_c of T sits in K. Put $\mathfrak{t}_c := \mathrm{Lie}(T_c)$. Note that the infinitesimal action of the Lie algebra $\mathfrak{t}$ of T lifts to an infinitesimal bundle action of $\mathfrak{t}$ on L. For each positive integer m, let $V_m := H^0(M, \mathcal{O}(L^{\otimes m}))$, and we view $\mathfrak{t}$ as a Lie subalgebra of $\mathfrak{sl}(V_m)$ by considering the traceless part. Define a symmetric bilinear form $\langle \, , \, \rangle_m$ on $\mathfrak{sl}(V_m)$ by

$$\langle X, Y \rangle_m \;:=\; \mathrm{Tr}(XY)/m^{n+2}, \qquad X, Y \in \mathfrak{sl}(V_m).$$

Let $\mathfrak{z}_m$ be the centralizer of $\mathfrak{t}$ in $\mathfrak{sl}(V_m)$, and $\mathfrak{g}_m$ be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{z}_m$, i.e.,

$$\begin{cases} \mathfrak{z}_m &:= \{X \in \mathfrak{sl}(V_m)\,;\, [X, Y] = 0 \ \text{ for all } Y \in \mathfrak{t}\}, \\ \mathfrak{g}_m &:= \{X \in \mathfrak{z}_m\,;\, \langle X, Y \rangle_m = 0 \ \text{ for all } Y \in \mathfrak{t}\}. \end{cases} \tag{3.1}$$

Let Z_m and G_m be the connected reductive algebraic subgroups in $\mathrm{SL}(V_m)$ associated to $\mathfrak{z}_m$ and $\mathfrak{g}_m$, respectively. By the infinitesimal $\mathfrak{t}$-action on V_m, we can write V_m as a direct sum of T_c-invariant subspaces,

$$V_m \;=\; \bigoplus_{\gamma=1}^{\eta_m} \hat{V}_{m,\gamma}, \tag{3.2}$$

where $\hat{V}_{m,\gamma} = \{\sigma \in V_m \, ; \, Y\sigma = \hat{\chi}_{m,\gamma}(Y)\sigma \; \text{ for all } Y \in \mathfrak{t}\}$ with mutually distinct characters $\hat{\chi}_{m,\gamma} \in \mathfrak{t}^*$, $\gamma = 1, 2, \ldots, \eta_m$. We now consider the algebraic subgroup R_m of $\mathrm{SL}(V_m)$ defined by

$$R_m \; := \; \prod_{\gamma=1}^{\eta_m} \mathrm{SL}(\hat{V}_{m,\gamma}),$$

where each $\mathrm{SL}(\hat{V}_{m,\gamma})$ fixes $\hat{V}_{m,\gamma'}$ if $\gamma' \neq \gamma$. Then the centralizer H_m of R_m in $\mathrm{SL}(V_m)$ consists of all diagonal matrices in $\mathrm{SL}(V_m)$ acting on each $\hat{V}_{m,\gamma}$, $\gamma = 1, 2, \ldots, \eta_m$, by constant scalar multiplication. Note that $\mathfrak{t}$ viewed as a Lie subalgebra of $\mathfrak{sl}(V_m)$ sits in the Lie algebra $\mathfrak{h}_m$ of H_m. Let

$$\mathfrak{t}_m^\perp \; := \; \{X \in \mathfrak{h}_m \, ; \, \langle X, Y \rangle_m = 0 \; \text{ for all } Y \in \mathfrak{t}\}$$

be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}_m$. Let $T_m^\perp$ denote the corresponding algebraic torus sitting in H_m. Since $Z_m = H_m \cdot R_m$, it follows that

$$G_m \; = \; T_m^\perp \cdot R_m.$$

Let M_m be the image of M under the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$ associated to the complete linear system $|L^{\otimes m}|$ on M. For the degree d_m of M_m in $\mathbb{P}^*(V_m)$, we consider the space

$$W_m \; := \; \{\mathrm{Sym}^{d_m}(V_m)\}^{\otimes n+1},$$

where $\mathrm{Sym}^{d_m}(V_m)$ denotes the d_m-th symmetric tensor product of V_m. For the dual space W_m^* of W_m, let $0 \neq \tilde{M}_m \in W_m^*$ denote the Chow form for the irreducible reduced algebraic cycle M_m on $\mathbb{P}^*(V_m)$, so that the associated point $[\tilde{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point for M_m.

For relative stability, we showed in [13] (see also [16]) that, if $c_1(L)$ admits an extremal Kähler metric, then the orbit $R_m \cdot \tilde{M}_m$ is closed in W_m^* for all sufficiently large m. In [25] (see also [26]), Székelyhidi introduced the following stronger stability concept:

DEFINITION 3.2. (1) A polarized algebraic manifold $(M, L^{\otimes m})$ is called *Chow-polystable relative to* T, if $G_m \cdot \tilde{M}_m$ is closed in W_m^*.

(2) (M, L) is called *asymptotically Chow-polystable relative to* T, if $(M, L^{\otimes m})$ is Chow-polystable relative to T for $m \gg 1$.

DEFINITION 3.3. A polarized algebraic manifold $(M, L^{\otimes m})$ is called *Chow-stable relative to* T, if the following conditions are satisfied:

(1) $G_m \cdot \tilde{M}_m$ is closed in W_m^*.
(2) The isotropy subgroup of G_m at $[\tilde{M}_m]$ is finite.

4. Proof of Corollary. In this section, using the same notation as in the preceding sections, we assume that M admits an extremal Kähler metric $\omega_0 = c_1(L; h_0)$ in the class $c_1(L)$, where h_0 is a Hermitian metric for L. The rough idea of the proof is that, by the implicit function theorem (see for instance [15]), the perturbation of the extremal Kähler metric ω_0 at $q = 0$ allows us to obtain polybalanced metrics (cf. Section 5) for $0 < q \ll 1$, and hence the asymptotic Chow-polystability for (M, L) relative to T is obtained. However, the precise arguments proceed as follows.

Following the arguments in [14], we shall show that the polarized algebraic mani-folds $(M, L^{\otimes m})$ are Chow-polystable relative to T for $m \gg 1$. As in the introduction, we may assume that K is the identity component of the group of isometries for (M, ω_0). Put $\mathfrak{k} := \mathrm{Lie}(K)$. Let $[n/2]$ be the largest integer which does not exceed $n/2$. By applying Main Theorem to

$$\ell := [n/2] + 3,$$

we obtain a K-invariant Kähler metric $\omega(\ell) = c_1(L; h(\ell))$ in the class $c_1(L)$ such that (1.1) holds. For the compact group

$$K_m := \mathrm{SU}(V_m; \rho_m(h(\ell))) \cap G_m,$$

we can view G_m as its complexification. Then for the G_m-action on W_m^*, the isotropy subgroup of K_m at $\tilde{M}_m$ has the Lie algebra $\mathfrak{k}_0$ sitting in $\mathfrak{k}$. Since $Z \subset T$, the isotropy subgroup of K_m at $[\tilde{M}_m]$ has the same Lie algebra $\mathfrak{k}_0$. For $\mathfrak{g}_m := \mathrm{Lie}(G_m)$, we define the Lie subalgebras $\mathfrak{p}_m$ and $\mathfrak{p}$ by

$$\mathfrak{p}_m := \sqrt{-1}\,\mathfrak{k}_m \quad \text{and} \quad \mathfrak{p} := \sqrt{-1}\,\mathfrak{k}_0,$$

where $\mathfrak{k}_m := \mathrm{Lie}(K_m)$. Put $n_\gamma := \dim \hat{V}_{m,\gamma}$. By choosing an orthonormal basis $\{\sigma_{\gamma,i}; i = 1, 2, \ldots, n_\gamma\}$ for $(\hat{V}_{m,\gamma}, \rho_m(h(\ell)))$, we set

$$j(\gamma, i) := i + \sum_{\beta=1}^{\gamma-1} n_\beta, \qquad i = 1, 2, \ldots, n_\gamma; \; \gamma = 1, 2, \ldots, \eta_m, \qquad (4.1)$$

where the right-hand side denotes i in the special case $\gamma = 1$. Let $m \gg 1$. Then for each γ and i as above, we put

$$\hat{\sigma}_{\gamma,i} := \sqrt{1 - \hat{\chi}_{m,\gamma}(\mathcal{Y}(\ell))}\,\sigma_{\gamma,i}. \qquad (4.2)$$

By writing $\hat{\sigma}_{\alpha,i}, \sigma_{\alpha,i}$ as $\hat{\sigma}_{j(\alpha,i)}, \sigma_{j(\alpha,i)}$, by abuse of terminology, we have bases

$$\{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N_m}\} \quad \text{and} \quad \{\sigma_1, \sigma_2, \ldots, \sigma_{N_m}\}, \qquad (4.3)$$

respectively, for $(V_m, \rho_m(h(\ell)))$. Let $\Phi_m : M \hookrightarrow \mathbb{P}^{N_m-1}(\mathbb{C}) \; (= \mathbb{P}^*(V_m))$ be the associated Kodaira embedding defined by

$$\Phi_m(p) := (\hat{\sigma}_1(p) : \hat{\sigma}_2(p) : \cdots : \hat{\sigma}_{N_m}(p)), \qquad p \in M. \qquad (4.4)$$

Here V_m and $\mathbb{C}^{N_m}$ are identified by the basis $\{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N_m}\}$. Let g_{euc} be the Euclidean metric for the space $\mathbb{C}^{N_m} = \{(z_1, z_2, \ldots, z_{N_m})\}$. Define the Fubini-Study form ω_{FS} on $\mathbb{P}^*(V_m) \; (= \{(z_1 : z_2 : \cdots : z_{N_m})\})$ by

$$\omega_{\mathrm{FS}} := (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\Sigma_{j=1}^{N_m}|z_j|^2).$$

For each $X \in \mathfrak{p}_m$, let $\mathcal{V}_X$ be the associated holomorphic vector field on $\mathbb{P}^*(V_m)$. We then have a unique real-valued function φ_X on $\mathbb{P}^*(V_m)$ satisfying

$$\int_{\mathbb{P}^{N_m-1}(\mathbb{C})} \varphi_X\, \omega_{\mathrm{FS}}^{N_m-1} = 0 \quad \text{and} \quad i_{\mathcal{V}_X}(\omega_{\mathrm{FS}}/m) = (\sqrt{-1}/2\pi)\bar{\partial}\varphi_X.$$

Let us consider the real-valued function $\zeta = \zeta(x)$ on $\mathbb{R}$ defined by $\zeta(x) := x(e^x + e^{-x})/(e^x - e^{-x})$, $x \in \mathbb{R}$. In view of $M_m = \Phi_m(M)$, we define a positive semidefinite K-invariant inner product $(\, , \,)_m$ on $\mathfrak{p}_m$ by

$$(X, Y)_m = \sqrt{-1} \int_{M_m} \partial \varphi_Y \wedge \bar{\partial} \varphi_X \wedge n \omega_{\mathrm{FS}}^{n-1}, \qquad X, Y \in \mathfrak{p}_m.$$

Then this inner product is positive definite when restricted to $\mathfrak{p}$. Hence as a vector space, $\mathfrak{p}_m$ is written as an orthogonal direct sum $\mathfrak{p} \oplus \mathfrak{p}^\perp$, where $\mathfrak{p}^\perp$ is the orthogonal complement of $\mathfrak{p}$ in $\mathfrak{p}_m$. Define an open neighborhood

$$U_m := \{ X \in \mathfrak{p}^\perp \, ; \, \zeta(\mathrm{ad}\, X)\mathfrak{p} \cap \mathfrak{p}^\perp = \{0\} \}$$

of the origin in $\mathfrak{p}^\perp$. Let $0 \neq X \in \mathfrak{p}^\perp$. Since X belongs to $\mathfrak{g}_m$, by choosing a suitable orthonormal basis $\{ \sigma_{\gamma,i} \, ; \, i = 1, 2, \dots, n_\gamma \}$ for $(\hat{V}_{m,\gamma}, \rho_m(h(\ell)))$, we obtain real constants b_j such that

$$X\hat{\sigma}_j = b_j \hat{\sigma}_j, \qquad j = 1, 2, \dots, N_m, \tag{4.5}$$

where $\{\hat{\sigma}_1, \hat{\sigma}_2, \dots, \hat{\sigma}_{N_m}\}$ is the basis for $(V_m, \rho_m(h(\ell)))$ as in (4.3). Define a real one-parameter subgroup $\lambda_X : \mathbb{R}_+ \to G_m$ of G_m by $\lambda_X(e^t) := \exp(tX)$, $t \in \mathbb{R}$. We then consider a real-valued function $f_{X,m}(t)$ on $\mathbb{R}$ defined by

$$f_{X,m}(t) := \log \|\lambda_X(e^t) \cdot \tilde{M}_m\|_{\mathrm{CH}(g_{\mathrm{euc}})}, \qquad t \in \mathbb{R}, \tag{4.6}$$

where $W_m^* \ni w \mapsto \|w\|_{\mathrm{CH}(g_{\mathrm{euc}})} \in \mathbb{R}_{\geq 0}$ is the Chow norm by Zhang [32] (see also [12]). Put $\dot{f}_{X,m}(t) := (d/dt)f_{X,m}$ and $\ddot{f}_{X,m}(t) := (d^2/dt^2)f_{X,m}$. Let

$$\delta_0 := \frac{q}{\sqrt{\Sigma_{j=1}^{N_m} b_j^2}}.$$

Then by Lemma 3.4 in [14], the proof of Corollary is reduced to showing that, if $m \gg 1$, then for every $0 \neq X \in \mathfrak{p}^\perp$,

$$\dot{f}_{X,m}(t_m) = 0 < \ddot{f}_{X,m}(t_m) \qquad \text{and} \qquad t_m \cdot X \in U_m, \tag{4.7}$$

where t_m is a suitable real number satisfying $|t_m| < \delta_0$. Now we divide the proof of (4.7) into three steps (see Steps 1 $\sim$ 3 below). Here, we briefly explain these steps. In Step 1, the work of Phong and Sturm [23] (see also [5]) gives us a lower bound for the second derivative $\ddot{f}_{X,m}(t)$ (see (4.9) below), while the first derivative $\dot{f}_{X,m}(0)$ is expressible as (4.10) below. Then in Step 2, by using (1.1), we reduce the proof of (4.7) to showing (4.16) below. Finally in Step 3, (4.16) can be shown by the choice $\ell := [n/2] + 3$.

Step 1. Put $\bar{b} := \max_j |b_j|$. Let t be an arbitrary real number satisfying

$$|t| \leq \delta_0.$$

It then follows that $|t| \leq q/\bar{b}$. Put $\lambda_t := \lambda_X(e^t)$ and $M_{m,t} := \lambda_t(M_m)$. Let $T\mathbb{P}^*(V_m)$ and $TM_{m,t}$ denote the holomorphic tangent bundles of $\mathbb{P}^*(V_m)$ and $M_{m,t}$, respectively. From now on, by C_i, $i = 0, 1, 2, \dots$, we mean positive real constants independent of the choice of the triple (m, t, X). Let $m \gg 1$. Then for each integer $k \geq 0$, the argument in [14], Step 1, shows that

$$\|\omega_0 - (1/m)\Phi_m^* \lambda_t^* \omega_{\mathrm{FS}}\|_{C^k(M, \omega_0)} \leq C_0. \tag{4.8}$$

Here C_0 possibly depends on k. From now on, X viewed as a holomorphic vector field on $\mathbb{P}^*(V_m)$ will be denoted by $\mathcal{X}$. Metrically, we identify the normal bundle of $M_{m,t}$ in $\mathbb{P}^*(V_m)$ with the subbundle $TM_{m,t}^\perp$ of $T\mathbb{P}^*(V_m)_{|M_{m,t}}$ obtained as the the orthogonal complement of $TM_{m,t}$ in $T\mathbb{P}^*(V_m)_{|M_{m,t}}$. Hence $T\mathbb{P}^*(V_m)_{|M_{m,t}}$ is differentiably written as the direct sum $TM_{m,t} \oplus TM_{m,t}^\perp$. Associated to this, the restriction $\mathcal{X}_{|M_{m,t}}$ of $\mathcal{X}$ to $M_{m,t}$ is written as

$$\mathcal{X}_{|M_{m,t}} \;=\; \mathcal{X}_{TM_{m,t}} \oplus \mathcal{X}_{TM_{m,t}^\perp},$$

where $\mathcal{X}_{TM_{m,t}}$ and $\mathcal{X}_{TM_{m,t}^\perp}$ are C^∞ sections of $TM_{m,t}$ and $TM_{m,t}^\perp$, respectively. Then the second derivative $\ddot{f}_{X,m}(t)$ is given by

$$\ddot{f}_{X,m}(t) \;=\; \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}|^2_{\omega_{\mathrm{FS}}} \, \omega_{\mathrm{FS}}^n \;\geq\; 0.$$

In view of (4.8), it follows from the argument of Phong and Sturm [23] that (cf. [14], p.235; see also [5])

$$\begin{cases} \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}|^2_{\omega_{\mathrm{FS}}} \, \omega_{\mathrm{FS}}^n \;\geq\; C_1 q \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}}|^2_{\omega_{\mathrm{FS}}} \, \omega_{\mathrm{FS}}^n, \\ \ddot{f}_{X,m}(t) \;\geq\; C_2 \, q \int_{M_m} |\mathcal{X}_{|M_m}|^2_{\omega_{\mathrm{FS}}} \, \omega_{\mathrm{FS}}^n \;\geq\; C_2 \, q \int_M \Theta \, \Phi_m^* \omega_{\mathrm{FS}}^n, \end{cases} \tag{4.9}$$

where $\Theta := (\Sigma_{j=1}^{N_m} |\hat{\sigma}_j|^2)^{-2} \{ (\Sigma_{j=1}^{N_m} |\hat{\sigma}_j|^2)(\Sigma_{j=1}^{N_m} b_j^2 |\hat{\sigma}_j|^2) - (\Sigma_{j=1}^{N_m} b_j |\hat{\sigma}_j|^2)^2 \} \geq 0$. Moreover, by (3.4.2) in [32] (see also [12]),

$$\dot{f}_{X,m}(0) \;=\; (n+1) \int_M \frac{\Sigma_{j=1}^{N_m} b_j |\hat{\sigma}_j|^2_{h(\ell)}}{\Sigma_{j=1}^{N_m} |\hat{\sigma}_j|^2_{h(\ell)}} \, \Phi_m^* \omega_{\mathrm{FS}}^n. \tag{4.10}$$

Step 2. For the basis $\{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N_m}\}$ as above satisfying (4.2), we can write $\Psi_m(h(\ell))$ in (2.1) applied to $h = h(\ell)$ in the following form:

$$\Psi_m(h(\ell)) \;=\; n! \, q^n \sum_{j=1}^{N_m} |\sigma_j|^2 \;=\; n! \, q^n \sum_{\gamma=1}^{\eta_m} \sum_{i=1}^{n_\gamma} |\sigma_{\gamma,i}|^2.$$

Then the left-hand side of (1.1) is expressible as

$$h(\ell)^m \cdot \{(1 - \mathcal{Y}(\ell))\Psi_m(h(\ell))\} \;=\; n! \, q^n \sum_{\gamma=1}^{\eta_m} \sum_{i=1}^{n_\gamma} \{1 - \hat{\chi}_{m,\gamma}(\mathcal{Y}(\ell))\} \, |\sigma_{\gamma,i}|^2_{h(\ell)}$$

$$=\; n! \, q^n \sum_{j=1}^{N_m} |\hat{\sigma}_j|^2_{h(\ell)},$$

and hence (1.1) is written as

$$n! \, q^n \sum_{j=1}^{N_m} |\hat{\sigma}_j|^2_{h(\ell)} \;=\; 1 + C(\ell) + O(q^{\ell+2}). \tag{4.11}$$

By taking $(\sqrt{-1}/2\pi)\partial\bar{\partial} \log$ of both sides of (4.11), we obtain

$$\omega_{\mathrm{FS}} - m\omega(\ell) \;=\; O(q^{\ell+2}). \tag{4.12}$$

For each γ and i, we put $a_{\gamma,i} := \hat{\chi}_{m,\gamma}(\mathcal{Y}(\ell))$, where $a_{\gamma,i}$ is obviously independent of the choice of i. Then by $\hat{\chi}_{m,\gamma} = O(q^{-1})$ and $\mathcal{Y}(\ell) = O(q^2)$, we see that $a_{\gamma,i} = O(q)$, i.e., $|a_{\gamma,i}| \leq \kappa q$ for some positive constant κ independent of the choice of (m,γ,i). We also obtain

$$|\hat{\sigma}_{\gamma,i}|^2_{h(\ell)} \;=\; \{1 - \hat{\chi}_{m,\gamma}(\mathcal{Y}(\ell))\}\,|\sigma_{\gamma,i}|^2_{h(\ell)} \;=\; (1 - a_{\gamma,i})\,|\sigma_{\gamma,i}|^2_{h(\ell)}. \tag{4.13}$$

In terms of (4.1), we write $a_{\gamma,i}$ as $a_{j(\gamma,i)}$. Note that X, as an element of $\mathfrak{p}^\perp$, belongs to $\mathfrak{g}_m$, so that X sits in $\mathfrak{sl}(V_m)$. This together with (3.1) implies

$$\sum_{j=1}^{N_m} b_j \;=\; \sum_{j=1}^{N_m} b_j a_j \;=\; 0.$$

Hence, by setting

$$I_m \;:=\; \Sigma_{j=1}^{N_m} b_j(1 - a_j)|\sigma_j|^2_{h(\ell)} \;\;(= \Sigma_{j=1}^{N_m} b_j|\hat{\sigma}_j|^2_{h(\ell)}) \tag{4.14}$$

and $\xi_1 := (n+1)!\{1 + C(\ell)\}^{-1}$, we see from (4.10), (4.11), (4.12) and (4.13) that $\dot{f}_{X,m}(0)$ is expressible as

$$\left\{
\begin{aligned}
& (n+1)\int_M \{n!q^n \Sigma_{j=1}^{N_m}|\hat{\sigma}_j|^2_{h(\ell)}\}^{-1}(n!q^n I_m)\,\Phi_m^*\omega_{\mathrm{FS}}^n \\
& = (n+1)!\int_M \{1 + C(\ell) + O(q^{\ell+2})\}^{-1} I_m\,\{\omega(\ell) + O(q^{\ell+3})\}^n \\
& = \xi_1 \int_M \{1 + O(q^{\ell+2})\}\, I_m\,\omega(\ell)^n \\
& = \xi_1\{(\Sigma_{j=1}^{N_m} b_j - \Sigma_{j=1}^{N_m} b_j a_j) + \int_M O(q^{\ell+2})\, I_m\,\omega(\ell)^n\} \\
& = \int_M O(q^{\ell+2})\, I_m\,\omega(\ell)^n.
\end{aligned}
\right. \tag{4.15}$$

Then by (4.12) and (4.15) together with the second line of (4.9), we see that

$$\left\{
\begin{aligned}
\dot{f}_{X,m}(\delta_0) & \geq \dot{f}_{X,m}(0) + C_2\,\delta_0\,q \int_M \Theta\,\Phi_m^*\omega_{\mathrm{FS}}^n \\
& \geq \int_M \{O(q^{\ell+2})\, I_m + C_3\,\delta_0\,q^{1-n}\,\Theta\}\,\omega(\ell)^n, \\
\dot{f}_{X,m}(-\delta_0) & \leq \dot{f}_{X,m}(0) - C_2\,\delta_0\,q \int_M \Theta\,\Phi_m^*\omega_{\mathrm{FS}}^n \\
& \leq \int_M \{O(q^{\ell+2})\, I_m - C_3\,\delta_0\,q^{1-n}\,\Theta\}\,\omega(\ell)^n.
\end{aligned}
\right.$$

Now as in [14], Remark 4.25, the inclusion $t_m X \in U_m$ follows from $|t_m| < \delta_0$, where in the proof, we use the basis $\{\hat{\sigma}_1, \ldots, \hat{\sigma}_{N_m}\}$ in place of $\{\sigma_1, \ldots, \sigma_{N_m}\}$. Hence, in order to prove (4.7), it suffices to show the following for $m \gg 1$:

$$\int_M \Theta\,\omega(\ell)^n > 0 \quad \text{and} \quad \frac{\int_M q^{\ell+2}\,|I_m|\,\omega(\ell)^n}{\int_M \delta_0\,q^{1-n}\,\Theta\,\omega(\ell)^n} \ll 1. \tag{4.16}$$

Step 3. Put $B_0 := \Sigma_{j=1}^{N_m}|\hat{\sigma}_j|^2_{h(\ell)}$, $B_1 := \Sigma_{j=1}^{N_m} b_j^2|\hat{\sigma}_j|^2_{h(\ell)}$, $B_2 := \Sigma_{j=1}^{N_m} b_j|\hat{\sigma}_j|^2_{h(\ell)}$. By setting $\theta_1 := \int_M (B_1/B_0)\,\omega(\ell)^n$ and $\theta_2 := \int_M (B_2/B_0)^2\,\omega(\ell)^n$, we obtain

$$\int_M \Theta\,\omega(\ell)^n \;=\; \theta_1 - \theta_2,$$

where both θ_1 and θ_2 are obviously nonnegative. Moreover, for $m \gg 1$,

$$\left\{
\begin{aligned}
\theta_1 & = \int_M (n!q^n B_0)^{-1} n!q^n B_1\,\omega(\ell)^n \\
& = \int_M \{1 + C(\ell) + O(q^{\ell+2})\}^{-1} n!q^n B_1\,\omega(\ell)^n \\
& \geq n!q^n(1 - \varepsilon)\Sigma_{j=1}^{N_m} b_j^2,
\end{aligned}
\right. \tag{4.17}$$

where $\varepsilon > 0$ is a small real constant independent of m. On the other hand, we see from (4.14) that

$$\int_M q^{\ell+2} |I_m| \, \omega(\ell)^n \; \leq \; q^{\ell+2} \, \Sigma_{j=1}^{N_m} |b_j(1-a_j)|. \tag{4.18}$$

Now the following cases are possible:

Case 1: $\theta_1 > 2\theta_2$,   Case 2: $\theta_1 \leq 2\theta_2$.

Suppose Case 1 occurs. Then $\theta_1 - \theta_2 = (1/2)\theta_1 + (1/2)(\theta_1 - 2\theta_2) > 0$, i.e., the first inequality in (4.16) holds. Let $m \gg 1$. Then by $a_j = O(q)$,

$$0 \; < \; 1 - a_j \; < \; 1 + \varepsilon,$$

where ε is as above. Let L.H.S. denotes the left-hand side of the second inequality in (4.16). Then in view of (4.17) and (4.18), we obtain

$$\begin{aligned}
\text{L.H.S.} \; & \left(= \; \frac{\int_M q^{\ell+2} |I_m| \, \omega(\ell)^n}{\int_M \delta_0 \, q^{1-n} \, \Theta \, \omega(\ell)^n} \right) \; \leq \; \frac{q^{\ell+2} \, \Sigma_{j=1}^{N_m} |b_j(1-a_j)|}{\delta_0 \, q^{1-n} \, (\theta_1 - \theta_2)} \\
& \leq \; \frac{q^{\ell+2} \, \Sigma_{j=1}^{N_m} |b_j(1-a_j)|}{(1/2) \, \delta_0 \, q^{1-n} \, \theta_1} \; \leq \; \frac{q^{\ell+2} (1+\varepsilon) \Sigma_{j=1}^{N_m} |b_j|}{(1/2) \, \delta_0 \, q \, n! (1-\varepsilon) \Sigma_{j=1}^{N_m} b_j^2} \\
& = \; \frac{q^\ell (1+\varepsilon) \Sigma_{j=1}^{N_m} |b_j|}{(1/2) \, n! (1-\varepsilon) \sqrt{\Sigma_{j=1}^{N_m} b_j^2}} \; \leq \; \frac{q^\ell (1+\varepsilon) N_m^{1/2}}{(1/2) \, n! (1-\varepsilon)},
\end{aligned}$$

where the Schwarz inequality $\Sigma_{j=1}^{N_m} |b_j| \leq N_m^{1/2} \sqrt{\Sigma_{j=1}^{N_m} b_j^2}$ is used in the last inequality. Hence by $N_m = O(m^n)$, it follows that

$$\text{L.H.S.} \; \leq \; O(q^{\ell - \frac{n}{2}}).$$

In view of the definition $\ell := [n/2] + 3$, we have $\ell - \frac{n}{2} > 0$, and therefore L.H.S. $\ll 1$, as required. Thus in this case (4.7) holds.

Next we consider the situation where Case 2 occurs. Note that, for $0 \neq X \in \mathfrak{p}^\perp$, we can write

$$\Phi_m^* \varphi_X \; = \; \frac{\Sigma_{j=1}^{N_m} b_j |\hat{\sigma}_j|^2}{m \Sigma_{j=1}^{N_m} |\hat{\sigma}_j|^2} \; = \; \frac{B_2}{m B_0}. \tag{4.19}$$

Let c_X be the real constant such that the function $\phi_X := c_X + \Phi_m^* \varphi_X$ on M satisfies $\int_M \phi_X \, \tilde{\omega}^n = 0$, where $\tilde{\omega} := \Phi_m^*(q\omega_{\text{FS}})$. Then

$$\|\phi_X\|_{L^2(M,\tilde{\omega})}^2 \; \leq \; C_4 \|\bar{\partial}\phi_X\|_{L^2(M,\tilde{\omega})}^2 \; = \; C_4 \|\Phi_m^* \mathcal{X}_{TM_m}\|_{L^2(M,\tilde{\omega})}^2$$

for some C_4, while by the first inequality in (4.9) applied to $t = 0$,

$$\|\Phi_m^* \mathcal{X}_{TM_m}\|_{L^2(M,\tilde{\omega})}^2 \; \leq \; C_1^{-1} q^{-1} \|\Phi_m^* \mathcal{X}_{TM_m^\perp}\|_{L^2(M,\tilde{\omega})}^2.$$

In view of these inequalities, we obtain

$$\|\phi_X\|_{L^2(M,\tilde{\omega})}^2 \; \leq \; C_4 \, C_1^{-1} q^{-1} \|\Phi_m^* \mathcal{X}_{TM_m^\perp}\|_{L^2(M,\tilde{\omega})}^2. \tag{4.20}$$

Let $m \gg 1$. By (4.10) and (4.19),

$$|c_X| = \left(\int_M \tilde{\omega}^n\right)^{-1} \left|\int_M (\Phi_m^* \varphi_X)\,\tilde{\omega}^n\right| = \frac{q^{n+1}|\dot{f}_{X,m}(0)|}{(n+1)c_1(L)^n[M]}. \tag{4.21}$$

In view of (4.12) and (4.15), there exist C_5 and C_6 satisfying

$$|\dot{f}_{X,m}(0)| \le C_5\, q^{\ell+2}\|I_m\|_{L^1(M,\omega(\ell))} \le C_6\, q^{\ell+2}\|I_m\|_{L^2(M,\tilde{\omega})},$$

while by (4.11), (4.14) and (4.19), we obtain C_7 such that

$$\|I_m\|_{L^2(M,\tilde{\omega})} \le C_7\, m^{n+1}\|\Phi_m^*\varphi_X\|_{L^2(M,\tilde{\omega})}.$$

Hence for some C_8, it follows that

$$|\dot{f}_{X,m}(0)| \le C_8\, q^{\ell-n+1}\|\Phi_m^*\varphi_X\|_{L^2(M,\tilde{\omega})}. \tag{4.22}$$

In view of (4.21) and (4.22), $|c_X| \le C_9\, q^{\ell+2}\|\Phi_m^*\varphi_X\|_{L^2(M,\tilde{\omega})}$ for some C_9. Then by the definition of ϕ_X together with (4.12), we obtain

$$\begin{cases} \|\phi_X\|_{L^2(M,\tilde{\omega})} \ge \|\Phi_m^*\varphi_X\|_{L^2(M,\tilde{\omega})} - \|c_X\|_{L^2(M,\tilde{\omega})} \\ \qquad \ge (1 - C_9\, q^{\ell+2}\{c_1(L)^n[M]\}^{1/2})\,\|\Phi_m^*\varphi_X\|_{L^2(M,\tilde{\omega})} \\ \qquad \ge C_{10}\|\Phi_m^*\varphi_X\|_{L^2(M,\tilde{\omega})} \ge C_{11}\|\Phi_m^*\varphi_X\|_{L^2(M,\omega(\ell))} \end{cases} \tag{4.23}$$

for some C_{10} and C_{11}. Since $\theta_2 = \int_M (B_2/B_0)^2\,\omega(\ell)^n = m^2\|\Phi_m^*\varphi_X\|^2_{L^2(M,\omega(\ell))}$, we see from (4.20) and (4.23) that

$$\|\Phi_m^*\mathcal{X}_{TM_m^\perp}\|^2_{L^2(M,\tilde{\omega})} \ge C_{12}\, q^3\, \theta_2 \tag{4.24}$$

for some C_{12}. Note that $q^{1/2}|\mathcal{X}_{|M_m}|_{\omega_{\mathrm{FS}}} = |\mathcal{X}_{|M_m}|_{\tilde{\omega}} \ge |\mathcal{X}_{TM_m^\perp}|_{\tilde{\omega}}$. In view of the second line in (4.9), it follows from (4.24) that, for $|t| \le \delta_0$,

$$\begin{cases} \ddot{f}_{X,m}(t) \ge C_2 \int_{M_m} (q^{1/2}|\mathcal{X}_{|M_m}|_{\omega_{\mathrm{FS}}})^2\, \omega_{\mathrm{FS}}^n \\ \qquad \ge C_2\, q^{-n}\|\Phi_m^*\mathcal{X}_{TM_m^\perp}\|^2_{L^2(M,\tilde{\omega})} \ge C_{13}\, q^{3-n}\, \theta_2 \end{cases} \tag{4.25}$$

for some C_{13}. Therefore, by (4.22) and (4.25), we obtain

$$\dot{f}_{X,m}(\delta_0) \ge R \qquad \text{and} \qquad \dot{f}_{X,m}(-\delta_0) \ge -R,$$

where $R := C_{13}\,\delta_0\, q^{3-n}\theta_2 - C_8\, q^{\ell-n+1}\|\Phi_m^*\varphi_X\|_{L^2(M,\omega(\ell))}$. Moreover, if $R > 0$, then $\theta_2 > 0$, and hence by (4.25), $\ddot{f}_{X,m}(t) > 0$ for $|t| \le \delta_0$. It now suffices to show $R > 0$ for $m \gg 1$. By $\theta_2 = m^2\|\Phi_m^*\varphi_X\|^2_{L^2(M,\omega(\ell))}$, we can write R as

$$R = q^{3-n}\sqrt{\theta_2}\{-C_8\, q^{\ell-1} + C_{13}\,\delta_0\sqrt{\theta_2}\}.$$

Recall that, by our assumption of Case 2, we have $\theta_1 \le 2\theta_2$. Hence by (4.17) together with the definetion of δ_0, we obtain

$$\delta_0\sqrt{\theta_2} \ge \delta_0\sqrt{\theta_1}/\sqrt{2} \ge \sqrt{n!(1-\varepsilon)}\, q^{(n/2)+1} = C_{14}\, q^{(n/2)+1},$$

where $C_{14} := \sqrt{n!(1-\varepsilon)}$. In view of the definition $\ell := [n/2] + 3$,

$$(n/2) + 1 < [n/2] + 2 = \ell - 1.$$

Therefore $R > 0$ for $m \gg 1$, as required. $\square$

REMARK 4.26. Assume that $(M, L^{\otimes m})$ is Chow-polystable relative to a maximal algebraic torus $T_{\max}$ in $\mathrm{Aut}^0(M)$. Then the arguments in [20], Step 2, which uses [21] allow us to obtain finiteness of the isotropy subgroup of G_m at $[\tilde{M}_m]$. Hence in this case, $(M, L^{\otimes m})$ is Chow-stable relative to $T_{\max}$.

5. Polybalanced metrics. As in the introduction, we consider a K-invariant Hermitian metric h for L such that $\omega = c_1(L; h)$ is Kähler. Let m be a positive integer. In (3.2), we choose an orthonormal basis $\{\sigma_{\gamma,i} \, ; \, i = 1, 2, \dots, n_\gamma\}$ of the space $(\hat{V}_{m,\gamma}, \rho_m(h))$ for each γ. In this section, we discuss the result in [17] from a slightly different point of view. For recent related works, see [24] and [9].

Definition 5.1. For a polarized algebraic manifold (M, L), ω is called an *m-th poly-balanced metric relative to* T, if for some $\mathcal{Y} \in \sqrt{-1}\,\mathfrak{t}_c$ satisfying $1 - \hat{\chi}_{m,\gamma}(\mathcal{Y}) > 0$ for all γ, there exists a positive real constant C such that

$$\sum_{\gamma=1}^{\eta_m} \sum_{i=1}^{n_\gamma} \{1 - \hat{\chi}_{m,\gamma}(\mathcal{Y})\} \, |\sigma_{\gamma,i}|_h^2 \; = \; C. \tag{5.2}$$

This concept is closely related to relative Chow-polystability. For brevity, we use the notation in Section 4 freely until the end of this section.

THEOREM 5.3. $(M, L^{\otimes m})$ *is Chow-polystable relative to* T *if and only if* (M, L) *admits an m-th polybalanced metric relative to* T.

Proof. The proof of "only if" part follows from Theorem C and (3.7) in [17]. For "if" part, we give a proof as follows: Let

$$\{\hat{\sigma}_{\gamma,i} \, ; \, i = 1, 2, \dots, n_\gamma, \, \gamma = 1, 2, \dots, \eta_m\}$$

be the basis for V_m obtained from $\{\sigma_{\gamma,i} \, ; \, i = 1, 2, \dots, n_\gamma, \, \gamma = 1, 2, \dots, \eta_m\}$ by replacing $\mathcal{Y}(\ell)$ by $\mathcal{Y}$ in (4.2). Then by (4.4), we have the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^{N_m - 1}(\mathbb{C})$. Also by (4.6), we have the function $f_{X,m}(t)$ for the orbit through $\tilde{M}_m$ of the one-parameter group $\exp(tX)$, $t \in \mathbb{R}$, generated by $0 \neq X \in \mathfrak{p}^\perp$. Since (5.2) is written as $\Sigma_{j=1}^{N_m} |\hat{\sigma}_j|_h^2 = C$ in terms of the notation (4.1), by taking $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ of both sides of (5.2), we obtain

$$\Phi_m^* \omega_{\mathrm{FS}} \; = \; m\,\omega.$$

Put $a_{\gamma,i} := \hat{\chi}_{m,\gamma}(\mathcal{Y})$. Then by $\mathcal{Y} \in \sqrt{-1}\,\mathfrak{t}_c \subset \mathfrak{t}$ and $X \in \mathfrak{p}^\perp \subset \mathfrak{g}_m \subset \mathfrak{sl}(V_m)$, in view of (3.1), it follows that $\Sigma_{j=1}^{N_m} b_j = \Sigma_{j=1}^{N_m} b_j a_j = 0$, where b_j is as in (4.5). Now by replacing $h(\ell)$ by h in (4.10), we obtain

$$\begin{aligned}
\dot{f}_{X,m}(0) \; &= \; (n+1) \int_M \frac{\Sigma_{j=1}^{N_m} b_j |\hat{\sigma}_j|_h^2}{\Sigma_{j=1}^{N_m} |\hat{\sigma}_j|_h^2} \, (m\omega)^n \\
&= \; \frac{m^n(n+1)}{C} \int_M \Sigma_{j=1}^{N_m} b_j (1 - a_j) |\sigma_j|_h^2 \, \omega^n \\
&= \; \frac{m^n(n+1)}{C} \, \Sigma_{j=1}^{N_m} b_j (1 - a_j) \; = \; 0.
\end{aligned}$$

By this together with the convexity $\ddot{f}_{X,m}(t) \geq 0$, the function $f_{X,m}$ attains a minimum at the origin. Therefore every special one-parameter subgroup of G_m has a closed orbit through $\tilde{M}_m$ in W_m^*, and hence the orbit $G_m \cdot \tilde{M}_m$ is closed in W_m^* (cf. [12], p.568), as required. $\square$

REMARK 5.4. If (M, L) is asymptotically Chow-polystable relative to T, then for $m \gg 1$, there exists an m-th polybalanced metric ω such that

$$\hat{\chi}_{m,\gamma}(\mathcal{Y}) \; = \; O(q),$$

i.e., the inequality $|\hat{\chi}_{m,\gamma}(\mathcal{Y})| \leq C'q$ holds for some positive constant C' independent of m, γ and $\mathcal{Y}$ (see [17], Theorem A).

6. Strong relative K-stability. In this section, for a polarized algebraic manifold (M, L), we consider an algebraic torus T in $\mathrm{Aut}^0(M)$. Let the group $\mathbb{C}^*$ act on the affine line $\mathbb{A}^1 := \{z \in \mathbb{C}\}$ by multiplication of complex numbers,

$$\mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1, \qquad (t, z) \mapsto tz.$$

By fixing a Hermitian metric h for L such that $\omega := c_1(L; h)$ is Kähler, we endow $V_m := H^0(X, L^{\otimes m})$ with the Hermitian metric $\rho_m(h)$ as defined in the introduction. We then consider the Kodaira embedding

$$\Phi_m : X \hookrightarrow \mathbb{P}^*(V_m), \qquad x \mapsto (\tau_1(x) : \tau_2(x) : \cdots : \tau_{N_m}(x)),$$

where $\{\tau_1, \tau_2, \ldots, \tau_{N_m}\}$ is an orthonormal basis for $(V_m, \rho_m(h))$. For G_m as in Section 3, we consider an algebraic group homomorphism

$$\psi : \mathbb{C}^* \to G_m$$

such that the maximal compact subgroup $S^1 \subset \mathbb{C}^*$ acts isometrically on the space $(V_m, \rho_m(h))$. Put $M_m := \Phi_m(M)$. Then by setting

$$\mathcal{M}_z^\psi := \{z\} \times \psi(z) M_m, \qquad z \in \mathbb{C}^*,$$

we consider the irreducible algebraic subvariety $\mathcal{M}^\psi$ of $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ obtained as the closure of the subset

$$\bigcup_{z \in \mathbb{C}^*} \mathcal{M}_z^\psi$$

in $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$, where $\psi(z)$ in G_m acts naturally on the space $\mathbb{P}^*(V_m)$ of all hyperplanes in V_m passing through the origin. Let

$$\pi : \mathcal{M}^\psi \to \mathbb{A}^1$$

be the map induced by the projection of $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ to the first factor $\mathbb{A}^1$. For the hyperplane bundle $\mathcal{O}_{\mathbb{P}^*(V_m)}(1)$ on $\mathbb{P}^*(V_m)$, we consider the pullback

$$\mathcal{L}^\psi := \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^*(V_m)}(1)_{|\mathcal{M}^\psi},$$

where $\mathrm{pr}_2 : \mathbb{A}^1 \times \mathbb{P}^*(V_m) \to \mathbb{P}^*(V_m)$ denotes the projection to the second factor. For the dual space V_m^* of V_m, the $\mathbb{C}^*$-action on $\mathbb{A}^1 \times V_m^*$ defined by

$$\mathbb{C}^* \times (\mathbb{A}^1 \times V_m^*) \to \mathbb{A}^1 \times V_m^*, \quad (t, (z, p)) \mapsto (tz, \psi(t)p),$$

induces $\mathbb{C}^*$-actions on $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ and $\mathcal{O}_{\mathbb{P}^*(V_m)}(-1)$, where G_m acts on V_m^* by the contragradient representation. This then induces $\mathbb{C}^*$-actions on $\mathcal{M}^\psi$ and $\mathcal{L}^\psi$, and hence $\pi : \mathcal{M}^\psi \to \mathbb{A}^1$ is a $\mathbb{C}^*$-equivariant projective morphism with a relatively very ample line bundle $\mathcal{L}^\psi$ satisfying

$$(\mathcal{M}_z^\psi, \mathcal{L}_z^\psi) \cong (M, L^{\otimes m}), \qquad z \neq 0,$$

where $\mathcal{L}_z^\psi$ is the restriction of $\mathcal{L}^\psi$ to $\mathcal{M}_z^\psi := \pi^{-1}(z)$. Then a triple $(\mathcal{M}, \mathcal{L}, \psi)$ is called a *test configuration for* (M, L), if we have both

$$\mathcal{M} = \mathcal{M}^\psi \quad \text{and} \quad \mathcal{L} = \mathcal{L}^\psi.$$

Here m is called the *exponent* of $(\mathcal{M}, \mathcal{L}, \psi)$. A test configuration $(\mathcal{M}, \mathcal{L}, \psi)$ is called *trivial*, if ψ is a trivial homomorphism. Let $\mathbf{M}$ be the set of all sequences $\{\mu_j\}$ of test configurations

$$\mu_j = (\mathcal{M}_j, \mathcal{L}_j, \psi_j), \qquad j = 1, 2, \ldots,$$

for (M, L) such that the exponent m_j of the test configuration μ_j satisfies the following growth condition:

$$m_j \to +\infty, \quad \text{as } j \to \infty.$$

In [18], to each $\{\mu_j\} \in \mathbf{M}$, we associated the Donaldson-Futaki invariant

$$F_1(\{\mu_j\}) \in \mathbb{R} \cup \{-\infty\},$$

which is viewed as a generalization of the Donaldson-Futaki invariant $DF(\mu)$ of a test configuration μ. We can also define the following strong version of K-stability and K-semistability:

DEFINITION 6.1. (1) A polarized algebraic manifold (M, L) is called *strongly K-semistable relative to T*, if $F_1(\{\mu_j\}) \leq 0$ for all $\{\mu_j\} \in \mathbf{M}$.

(2) A strongly K-semistable polarized algebraic manifold (M, L) is called *strongly K-stable relative to T*, if for every $\{\mu_j\} \in \mathbf{M}$ with $F_1(\{\mu_j\}) = 0$, there exists j_0 such that μ_j are trivial for all j with $j \geq j_0$.

7. Concluding Remarks. The Yau-Tian-Donaldson Conjecture for Kähler-Einstein cases was solved affirmatively by Chen-Donaldson-Sun [4] and Tian [29]. However, for general polarization cases or extremal Kähler cases, the conjecture is still open. In this paper, we discuss extremal Kähler versions of this conjecture by focussing on the difference between strong K-stability and K-stability. For an arbitrary polarized algebraic manifold (M, L) as in the introduction, recall the following definition of K-stability [6] (cf. [28]) :

DEFINITION 7.1. (1) (M, L) is called *K-semistable*, if $DF(\mu) \leq 0$ for all test configurations μ for (M, L).

(2) A K-semistable (M, L) is called *K-stable*, if every test configuration μ for (M, L) with $DF(\mu) = 0$ is trivial.

We now consider a maximal algebraic torus $T_{\max}$ in $\text{Aut}^0(M)$. As to the existence of extremal Kähler metrics, it is natural to ask the following:

CONJECTURE I. *A polarized algebraic manifold (M, L) is K-stable relative to $T_{\max}$ if and only if $c_1(L)$ admits an extremal Kähler metric.*

CONJECTURE II. *A polarized algebraic manifold (M, L) is strongly K-stable relative to $T_{\max}$ if and only if $c_1(L)$ admits an extremal Kähler metric.*

By a result of Donaldson [5], every constant scalar curvature Kähler metric is approximated by a sequence of balanced metrics. In other words, a balanced metric can be viewed as a quantized version of a constant scalar curvature Kähler metric. Similarly, a polybalanced metric can be viewed as a quantized version of an extremal Kähler metric. Since existence of polybalanced metrics corresponds to relative Chow-polystability, the following fact is viewed as a quantized version of the existence part of Conjecture II:

FACT (cf. [20]). *If a polarized algebraic manifold (M, L) is strongly K-stable relative to an algebraic torus T in $\text{Aut}^0(M)$, then $(M, L^{\otimes m})$, $m \gg 1$, are Chow-stable relative to T.*

Moreover, we expect that "if" part of Conjecture II is true. This will be discussed in a forthcoming paper [19]. In view of the above Fact, by assuming that "if" part of Conjecture II is true, we immediately obtain the case $T = T_{\max}$ of Corollary in the introduction.

For a polarized algebraic manifold (M, L), let T_{ex} be the algebraic torus in $\text{Aut}^0(M)$ generated by the extremal Kähler vector field. By a recent result of Yotsutani-Zhou [30], a smooth polarized toric Fano threefold

$$\Pi := (\mathcal{E}_4, K_{\mathcal{E}_4}^{-1})$$

is K-stable relative to $T_{\max}$, and is not asymptotically Chow-stable relative to T_{ex}. Let $Z_{\mathbb{C}}$ be as in the introduction. We finally pose the following:

PROBLEM. *Check whether or not Π is asymptotically Chow-stable relative to $Z_{\mathbb{C}}$. Or more generally, clarify whether there is an example of a polarized algebraic manifold (M, L) which is K-stable relative to $T_{\max}$ and is not asymptotically Chow-stable relative to $Z_{\mathbb{C}}$.*

If there is such an example of a polarized algebraic manifold (M, L), then by Corollary in the introduction, $c_1(L)$ admits no extremal Kähler metrics. In other words, this gives a counter-example to Conjecture I above.

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