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SHARP UPPER ESTIMATE OF GEOMETRIC GENUS AND COORDINATE-FREE CHARACTERIZATION OF ISOLATED HOMOGENEOUS HYPERSURFACE SINGULARITIES*

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Dedicated to Ngaiming Mok on the occasion of his 60th birthday

Abstract. The subject of counting positive lattice points in n-dimensional simplexes has interested mathematicians for decades due to its applications in singularity theory and number theory. Enumerating the lattice points in a right-angled simplex is equivalent to determining the geometric genus of an isolated singularity of a weighted homogeneous complex polynomial. It is also a method to shed insight into large gaps in the sequence of prime numbers. Seeking to contribute to these applications, in this paper, we prove the Yau Geometric Conjecture in six dimensions, a sharp upper bound for the number of positive lattice points in a six-dimensional tetrahedron. The main method of proof is summing existing sharp upper bounds for the number of points in 5-dimensional simplexes over the cross sections of the six-dimensional simplex. Our new results pave the way for the proof of a fully general sharp upper bound for the number of lattice points in a simplex. It also sheds new light on proving the Yau Geometric and Yau Number-Theoretic Conjectures in full generality.

Key words. Sharp estimate, integral points, simplex.

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1. introduction. Let Δ_n be an n-dimensional real right-angled simplex defined by the inequality

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \le 1,$$

where $x_1, \cdots, x_n \ge 0$ and $a_1 \ge a_2 \ge \cdots \ge a_n \ge 1$. Define P_n to be the number of positive integral points in Δ_n, or

$$P_n = \# \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{Z}_+^n \;\middle|\; \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \le 1 \right\}.$$

Similarly, we define Q_n to be the number of non-negative integral points in Δ_n, or

$$Q_n = \# \left\{ (x_1, x_2, \cdots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n \;\middle|\; \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \le 1 \right\}.$$

According to Granville [3], the numbers P_n and Q_n are intimately related to a number theoretic function known as the *Dickman–de Bruijn function.*

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Definition 1.1. *The Dickman–de Bruijn function $\psi(x, y)$ is defined as the number of positive integers n such that $n \leq x$, and all of the prime factors of n are at most y, where x and y are positive integers.*

The connection, described by Luo, et. al. [10], is most readily observed by noting that, if $p_1 < p_2 < \cdots < p_k$ are the primes less than or equal to y, then,

$$p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \leq x,$$

is equivalent to,

$$e_1 \log p_1 + e_2 \log p_2 + \cdots + e_k \log p_k \leq \log x,$$

which can be rewritten as,

$$\frac{e_1}{\frac{\log x}{\log p_1}} + \frac{e_2}{\frac{\log x}{\log p_2}} + \cdots + \frac{e_k}{\frac{\log x}{\log p_k}} \leq 1,$$

an expression in the format of the condition in the definition of Q_n. Hence, enumerating the Dickman–de Bruijn function is equivalent to calculating Q_n.

Granville [3] also describes connections between P_n and Q_n and other areas of number theory, including primality testing, determining large gaps in the sequence of the primes, and discovering new algorithms for prime factorization. Furthermore, Lin, et. al. [9] describes how determining the values of P_n and Q_n leads to insights in singularity theory.

P_n and Q_n are intimately linked through the equation [8]

$$P_n(a_1, a_2, \cdots, a_n) = Q_n(a_1(1-a), a_2(1-a), \cdots, a_n(1-a)),$$

where a is defined as $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$. Hence, we can essentially treat the tasks of finding P_n and Q_n to be equivalent in general.

The quest to find and estimate P_n and Q_n dates back to 1899, when Pick [14] discovered the famous *Pick's theorem,* or a formula for Q_2.

$$Q_2 = area(\Delta) + \frac{\left|\partial\Delta \cap \mathbb{Z}^2\right|}{2} - 1,$$

where Δ is a 2-dimensional tetrahedron, or a triangle, $\partial\Delta$ represents the boundary of the triangle, and $\left|\partial\Delta \cap \mathbb{Z}^2\right|$ represents the number of integral points on the boundary. Mordell [13] continued by discovering a formula for Q_3 using Dedekind sums. Erhart [2] followed with the discovery of *Ehrhart polynomials*, which facilitate the calculation of Q_n. However, these polynomials are only useful if every coefficient is known, a condition that is extremely difficult to meet in the general case.

The difficulty of this problem eventually led mathematicians to start trying to bound P_n and Q_n instead of finding precise formulas. Lehmer [5] found that if $a = a_1 = a_2 = \cdots = a_n$, then

$$Q_n = \binom{\lfloor a \rfloor + n}{n},$$

where $\lfloor x \rfloor$ denotes the integral part of a real number x. This formula naturally yields a nice definition of sharpness of an estimate R_n of Q_n. We consider the estimate sharp if and only if

$$R_n|_{a_1=a_2=\cdots=a_n=a\in\mathbb{Z}_+} = \binom{a+n}{n}.$$

In other words, any upper or lower bound is sharp if and only if its estimate is exact when $a_1 = a_2 = \cdots = a_n \in \mathbb{Z}_+$.

Another important estimate is the two-part GLY Conjecture, an upper bound for P_n formulated by Lin, et. al. [8]. However, to state the GLY Conjecture, we need to first introduce the *signed Stirling numbers of the first kind* and a notation A_k^n for elementary symmetric polynomials [21].

Definition 1.2. *The (signed) Stirling numbers of the first kind $s(n,k)$ are defined by the following property:*

$$\prod_{i=0}^{n-1}(x-i) = \sum_{k=0}^{n} s(n,k)x^k,$$

where $s(n,0) = 0$, $s(n,n) = 1$.

Definition 1.3. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers. We denote*

$$A_k^n = \left(\prod_{i=1}^{n} a_i\right) \sum_{1\le i_1<i_2<\cdots<i_k\le n} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}}.$$

Thus, A_k^n is the elementary symmetric polynomial of $a_1, a_2, \cdots, a_n$ with degree $n-k$.

Conjecture 1.4 (GLY Conjecture). *Let*

$$P_n = \#\left\{(x_1, x_2, \cdots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \le 1\right\},$$

where $a_1 \ge a_2 \ge \cdots \ge a_n \ge 1$ are real numbers. If $n \ge 3$, then:

(1) *Rough (general) upper estimate: For all $a_n > 1$,*

$$n!\,P_n < q_n := \prod_{i=1}^{n}(a_i - 1).$$

(2) *Sharp upper estimate: For $n \ge 3$, if $a_1 \ge a_2 \ge \cdots \ge a_n \ge n-1$, then*

$$n!\,P_n \le\ A_0^n + \frac{s(n,n-1)}{n}A_1^n + \sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A_l^{n-1},$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n \in \mathbb{Z}_+$.

The sharp GLY Conjecture has been proven to be true for $3 \le n \le 6$ [19, 4, 17, 7]. The rough GLY upper estimate for all n was proven by Yau and Zhang [20]. In this paper, we will use the following theorem (the sharp GLY conjecture for $n = 5$):

Theorem 1.5 (GLY Conjecture for $n = 5$). *Let $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ge 4$ be real numbers and P_5 be the number of positive integral solutions of*

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} \le 1.$$

Then,

$$120P_5 \le a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5)$$

$$+\frac{35}{4}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)$$
$$-\frac{50}{6}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4) + 6(a_1 + a_2 + a_3 + a_4).$$

The quest to determine the validity of the sharp GLY Conjecture led to another conjecture, namely the *Yau Number-Theoretic Conjecture* (Conjecture 1.6).

CONJECTURE 1.6 (Yau Number-Theoretic Conjecture). *Let $n \geq 3$ be a positive integer, and let $a_1 \geq a_2 \cdots \geq a_n > 1$ be real numbers. If $P_n > 0$, then*

$$n!\,P_n \leq (a_1 - 1)\cdots(a_n - 1) - (a_n - 1)^n + a_n(a_n - 1)\cdots(a_n - (n-1)),$$

and equality holds if and only if $a_1 = \cdots = a_n \in \mathbb{Z}$.

In this paper, we will use the $n = 5$ case of Conjecture 1.6, proven by Chen, et. al. [1], extensively. We reproduce it as a theorem below for easy reference.

THEOREM 1.7 (Yau Number-Theoretic Conjecture for $n = 5$). *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 > 1$ be real numbers. If $P_5 > 0$, then*

$$120P_n \leq (a_1 - 1)(a_2 - 1)\cdots(a_5 - 1) - (a_5 - 1)^5 + a_5(a_5 - 1)(a_5 - 2)(a_5 - 3)(a_5 - 4).$$

Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 \in \mathbb{Z}$.

For recent progress of Yau Number-Theoretic Conjecture, one can see [6, 24]. Similar to the Yau Number-Theoretic Conjecture is Conjecture 1.9, or the Yau Geometric Conjecture. In order to state the Yau Geometric Conjecture, we must first define a *weighted homogeneous polynomial*:

DEFINITION 1.8. *A polynomial $f(x_1, x_2, \cdots, x_n)$ is a weighted homogeneous polynomial if it is a sum of monomials $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ such that, for some fixed positive rational numbers $w_1, w_2, \cdots, w_n$,*

$$\frac{i_1}{w_1} + \frac{i_2}{w_2} + \cdots + \frac{i_n}{w_n} = 1,$$

for every monomial of f. The numbers $w_1, w_2, \cdots, w_n$ are known as the weights of the polynomial.

CONJECTURE 1.9 (Yau Geometric Conjecture). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g, and v be the Milnor number, geometric genus, and multiplicity of the singularity $V = \{z : f(z) = 0\}$. Then,*

$$\mu - p(v) \geq n!\,p_g,$$

where $p(v) = (v-1)^n - v(v-1)\cdots(v-n+1)$. Equality holds if and only if f is a homogeneous polynomial.

Note that p_g counts the number of positive lattice points in the simplex

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1,$$

where the a_i are the weights of the *weighted* homogeneous polynomial f and $a_1 \geq a_2 \geq \cdots \geq a_n > 1$ (cf. [11]). Thus, the equality case of Conjecture 1.9 is $a_1 =$

$a_2 = \cdots = a_n \in \mathbb{Z}$. Furthermore, $v = \lceil a_n \rceil$ (round up a_n) by Saeki [15] and it is known that $\mu = (a_1 - 1)(a_2 - 1)\cdots(a_n - 1)$ (cf. [12]). The fractional part of a_n has to be one of $\frac{a_n}{a_1}, \frac{a_n}{a_2}, \cdots,$ or $\frac{a_n}{a_{n-1}}$ [1]. Finally, we can also define the polynomial $p_n(v) = (v-1)^n - v(v-1)\cdots(v-n+1)$. Thus, we have

$$p_5(v) = 5v^4 - 25v^3 + 40v^2 - 19v - 1, \tag{1}$$

$$p_6(v) = 1 + 114v - 259v^2 + 205v^3 - 70v^4 + 9v^5. \tag{2}$$

Conjecture 1.9 has been proven for $3 \leq n \leq 5$ [1, 18, 8]. In this paper, we justify Conjecture 1.9 for $n = 6$, extending the proof Yau Geometric Conjecture to a higher dimension. In and of itself, this is difficult because the number of cases has increased from 4 in the 5-dimensional case to 6 in the 6-dimensional one, and each case has increased in complexity as well. The number of subcases also posed difficulty in the research, and efforts were made to simplify the proof as much as possible. Nevertheless, much complexity remains. Some new delicate analysis techniques are used to prove our main theorem. This is a significant improvement and it may shed a new light on the solution of arbitrary dimensional Yau Geometric Conjecture.

Because of the difficulty of the proof and the sheer size of the algebraic expressions involved, all computations in this paper were done using Mathematica 10.1 and Maple 2015 for simplicity and accurate calculation. Hence, our main theorem is:

Theorem 1.10 (Main Theorem). *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$, then*

$$6!\, P_6 \leq (a_1 - 1)(a_2 - 1)\cdots(a_6 - 1) - (1 + 114v - 259v^2 + 205v^3 - 70v^4 + 9v^5),$$

where v is calculated as $v = \lceil a_6 \rceil$. Note that the fractional part β of a_6 is one of $\frac{a_6}{a_1}, \frac{a_6}{a_2}, \frac{a_6}{a_3}, \frac{a_6}{a_4},$ or $\frac{a_6}{a_5}$. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \in \mathbb{Z}$.

In fact the condition $P_6 > 0$ can be removed from the above main theorem, please see section 4.

2. Two lemmas. We will frequently use the following two lemmas to decide the positivity of polynomials in some restricted domains.

Lemma 2.1 ([17] Lemma 3.1). *Let $f(\beta)$ be a polynomial defined by*

$$f(\beta) = \sum_{i=0}^{n} c_i \beta^i$$

where $\beta \in (0,1)$. If for any $k = 0, 1, \cdots, n$

$$\sum_{i=0}^{k} c_i \geq 0$$

then $f(\beta) \geq 0$ for $\beta \in (0,1)$.

Lemma 2.1 is easy to use. However, the condition of Lemma 2.1 may not be satisfied in some situation. In that case, we shall make use of the following lemma.

LEMMA 2.2 (Sturm's Theorem). *Starting from a given polynomial* $X_0 = f(x)$, *let* $X_1 = f'(x)$ *and the polynomials* $X_2, X_3, \cdots, X_r$ *be determined by Euclidean algorithm as follows:*

$$\begin{aligned} X_0 &= Q_1X_1 - X_2, \\ X_1 &= Q_2X_2 - X_3, \\ &\cdots\cdots\cdots\cdots \\ X_{r-2} &= Q_{r-1}X_{r-1} - X_r, \\ X_{r-1} &= Q_rX_r \end{aligned}$$

where $\deg X_k > \deg X_{k+1}$ *for* $k = 1, \cdots, r-1$. *For every real number* a *which is not a root of* $f(x)$ *let* $w(a)$ *be the number of variations in sign in the number sequence*

$$X_0(a), X_1(a), \cdots, X_r(a)$$

in which all zeros are omitted. If b *and* c *are any numbers* $(b < c)$ *for which f(x) does not vanish, then the number of the various roots in the interval* $b \leq x \leq c$*(multiple roots to be counted only once) is equal to*

$$w(b) - w(c).$$

Proof. See [16]. □

The condition of Lemma 2.2 is necessary and sufficient, so it can be applied to judge the positivity of any such polynomials in some intervals. The computation in Lemma 2.2 is more complicated than that in Lemma 2.1. Therefore, we prefer Lemma 2.1 when it works.

3. Proof of Main Theorem 1.10. We prove Theorem 1.10 by estimating P_5 on hyperplanes with x_6 fixed and all other variables free. We then sum up these estimates to get an upper bound for $6!\,P_6$. We must then only show that this upper bound for $6!\,P_6$ is less than or equal to the RHS of Theorem 1.10.

In the rest of this paper, we shall refer to the intersection of the simplex in Theorem 1.10 with the hyperplane $x_6 = k$ as the *level* $x_6 = k$. Hence, in our simplex, $x_6 = k$ points are in the 5-dimensional tetrahedron defined by

$$\frac{x_1}{a_1\left(1-\frac{k}{a_6}\right)} + \frac{x_2}{a_2\left(1-\frac{k}{a_6}\right)} + \frac{x_3}{a_3\left(1-\frac{k}{a_6}\right)} + \frac{x_4}{a_4\left(1-\frac{k}{a_6}\right)} + \frac{x_5}{a_5\left(1-\frac{k}{a_6}\right)} \leq 1.$$

We shall break our proof up into cases based on the ceiling of a_6:

Case I: $1 < a_6 \leq 2$. Thus, $\lceil a_6 \rceil = 2$.
Case II: $2 < a_6 \leq 3$. Thus, $\lceil a_6 \rceil = 3$.
Case III: $3 < a_6 \leq 4$. Thus, $\lceil a_6 \rceil = 4$.
Case IV: $4 < a_6 \leq 5$. Thus, $\lceil a_6 \rceil = 5$.
Case V: $5 < a_6 \leq 6$. Thus, $\lceil a_6 \rceil = 6$.
Case VI: $6 < a_6$.

All of our cases will eventually reduce to proving some multivariate functions are positive over some domains. We will show a function is positive using a partial differentiation test, which involves calculating the partial derivative with respect to all the variables. We show this partial derivative is positive and then continue to partially differentiate with respect to one less variable for each consecutive step until only first-order partials remain. If we show that these are all positive through the domain of the function, and that the function is positive at the minimum, then we know that the function is positive throughout the domain.

3.1. Case I. In this case, $\lceil a_6 \rceil = 2$, so we can plug $v = 2$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

THEOREM 3.1. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $1 < a_6 \leq 2$, then*

$$6!\, P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 1.$$

Proof. In this case, $a_6 \in (1, 2]$, so the only level we have to consider is $x_6 = 1$. When $x_6 = 1$, $P_6 > 0$ implies that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.1. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in \left(0, \frac{1}{2}\right]$ because $a_6 \in (1, 2]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6!\, P_6 = 6!\, P_5(x_6 = 1) \leq 6[&(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 \\ &+ A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)]. \end{aligned}$$

If we let Δ_1 be the difference obtained by subtracting the right hand side (RHS) of the above inequality from the RHS of Theorem 3.1, substituting in $a_i = \frac{A_i}{\alpha}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_1 = \frac{1}{(1-\alpha)\alpha^4}\Big(&6\alpha^5 A_1A_2A_3A_4A_5 - 6\alpha^5 A_1A_2A_3A_4 - 6\alpha^5 A_1A_2A_3A_5 \\ &-6\alpha^5 A_1A_2A_4A_5 - 6\alpha^5 A_1A_3A_4A_5 - 6\alpha^5 A_2A_3A_4A_5 - 30\alpha^5 A_5^4 \\ &-6\alpha^4 A_1A_2A_3A_4A_5 + 6\alpha^5 A_1A_2A_3 + 6\alpha^5 A_1A_2A_4 + 6\alpha^5 A_1A_2A_5 + 6\alpha^5 A_1A_3A_4 \\ &+6\alpha^5 A_1A_3A_5 + 6\alpha^5 A_1A_4A_5 + 6\alpha^5 A_2A_3A_4 + 6\alpha^5 A_2A_3A_5 + 6\alpha^5 A_2A_4A_5 \\ &+6\alpha^5 A_3A_4A_5 + 150\alpha^5 A_5^3 + 6\alpha^4 A_1A_2A_3A_4 + 6\alpha^4 A_1A_2A_3A_5 + 6\alpha^4 A_1A_2A_4A_5 \\ &+6\alpha^4 A_1A_3A_4A_5 + 6\alpha^4 A_2A_3A_4A_5 + 30\alpha^4 A_5^4 - 6\alpha^5 A_1A_2 - 6\alpha^5 A_1A_3 \\ &-6\alpha^5 A_1A_4 - 6\alpha^5 A_1A_5 - 6\alpha^5 A_2A_3 - 6\alpha^5 A_2A_4 - 6\alpha^5 A_2A_5 - 6\alpha^5 A_3A_4 \\ &-6\alpha^5 A_3A_5 - 6\alpha^5 A_4A_5 - 240\alpha^5 A_5^2 - 6\alpha^4 A_1A_2A_3 - 6\alpha^4 A_1A_2A_4 - 6\alpha^4 A_1A_2A_5 \\ &-6\alpha^4 A_1A_3A_4 - 6\alpha^4 A_1A_3A_5 - 6\alpha^4 A_1A_4A_5 - 6\alpha^4 A_2A_3A_4 - 6\alpha^4 A_2A_3A_5 \\ &-6\alpha^4 A_2A_4A_5 - 6\alpha^4 A_3A_4A_5 - 150\alpha^4 A_5^3 + 6\alpha^5 A_1 + 6\alpha^5 A_2 + 6\alpha^5 A_3 + 6\alpha^5 A_4 \\ &+120\alpha^5 A_5 + 6\alpha^4 A_1A_2 + 6\alpha^4 A_1A_3 + 6\alpha^4 A_1A_4 + 6\alpha^4 A_1A_5 + 6\alpha^4 A_2A_3 \\ &+6\alpha^4 A_2A_4 + 6\alpha^4 A_2A_5 + 6\alpha^4 A_3A_4 + 6\alpha^4 A_3A_5 + 6\alpha^4 A_4A_5 + 240\alpha^4 A_5^2 \\ &-5\alpha^4 A_1 - 5\alpha^4 A_2 - 5\alpha^4 A_3 - 5\alpha^4 A_4 - 119\alpha^4 A_5 - \alpha^3 A_1A_2 - \alpha^3 A_1A_3 \\ &-\alpha^3 A_1A_4 - \alpha^3 A_1A_5 - \alpha^3 A_2A_3 - \alpha^3 A_2A_4 - \alpha^3 A_2A_5 - \alpha^3 A_3A_4 - \alpha^3 A_3A_5 \\ &-\alpha^3 A_4A_5 + \alpha^2 A_1A_2A_3 + \alpha^2 A_1A_2A_4 + \alpha^2 A_1A_2A_5 + \alpha^2 A_1A_3A_4 + \alpha^2 A_1A_3A_5 \end{aligned}$$

$$
\begin{aligned}
&+\alpha^2A_1A_4A_5+\alpha^2A_2A_3A_4+\alpha^2A_2A_3A_5+\alpha^2A_2A_4A_5+\alpha^2A_3A_4A_5\\
&-\alpha A_1A_2A_3A_4-\alpha A_1A_2A_3A_5-\alpha A_1A_2A_4A_5-\alpha A_1A_3A_4A_5-\alpha A_2A_3A_4A_5\\
&+A_1A_2A_3A_4A_5-\alpha^4\Big)\\
=&\ \frac{1}{(1-\alpha)\alpha^4}\Delta_2.
\end{aligned}
$$

To apply the partial differentiation test, we must determine the domain of Δ_2. We note that $\frac{1}{A_5}<1$ and $\frac{2}{A_4}\le\frac{1}{A_4}+\frac{1}{A_5}\le 1$. Similarly, $\frac{3}{A_3}\le\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5}\le 1$, and a similar statement is true involving A_2 and A_1. Hence, we have

$$A_1\ge 5, A_2\ge 4, A_3\ge 3, A_4\ge 2, \text{ and } A_5>1.$$

Now that we have a domain established, we can begin applying the partial differentiation test to demonstrate that Δ_2 is positive.

We see that

$$\frac{\partial^5\Delta_2}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5}=6\alpha^5-6\alpha^4+1>0$$

for all $\alpha\in(0,1)$ (it is easy to check directly that it has minimal value 0.51136 or it can be proved using Lemma 2.2). Thus the partial derivative of Δ_2 with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5=1$.

$$\left.\frac{\partial^4\Delta_2}{\partial A_1\partial A_2\partial A_3\partial A_4}\right|_{A_5=1}=1-\alpha>0.$$

It follows that the partial of Δ_2 with respect to A_1, A_2, A_3, and A_5 is positive for all $A_4\ge 1, \alpha\in(0,1)$ because $\frac{\partial^3\Delta_2}{\partial A_1\partial A_2\partial A_3}$ is symmetric in A_4 and A_5. Hence, $\frac{\partial^3\Delta_2}{\partial A_1\partial A_2\partial A_3}$ is an increasing function of A_4, A_5 with a minimum at $A_4=A_5=1$.

$$\left.\frac{\partial^3\Delta_2}{\partial A_1\partial A_2\partial A_3}\right|_{A_4=A_5=1}=(1-\alpha)^2>0.$$

This is positive, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to A_3, A_4, and A_5, we know that $\frac{\partial^3\Delta_2}{\partial A_1\partial A_2\partial A_4}$ and $\frac{\partial^3\Delta_2}{\partial A_1\partial A_2\partial A_5}$ are positive over the given domain. Hence, $\frac{\partial^2\Delta_2}{\partial A_1\partial A_2}$ is an increasing function of A_3, A_4, and A_5 for all $A_3, A_4, A_5\ge 1$ and $\alpha\in(0,1)$. The minimum is at $A_3=A_4=A_5=1$.

$$\left.\frac{\partial^2\Delta_2}{\partial A_1\partial A_2}\right|_{A_3=A_4=A_5=1}=(1-\alpha)^3>0.$$

Because this is symmetric with respect to A_2, A_3, A_4, and A_5, we see that $\frac{\partial^2\Delta_2}{\partial A_1\partial A_3}$, $\frac{\partial^2\Delta_2}{\partial A_1\partial A_4}$, and $\frac{\partial^2\Delta_2}{\partial A_1\partial A_5}$ are positive over the given domain. Hence, $\frac{\partial\Delta_2}{\partial A_1}$ is an increasing function of A_2, A_3, A_4, and A_5 that is minimized at $A_2=A_3=A_4=A_5=1$.

$$\left.\frac{\partial\Delta_2}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=1}=(1-\alpha)^4>0.$$

Thus $\dfrac{\partial \Delta_2}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_2 in A_1, A_2, A_3, and A_4, we know that $\dfrac{\partial \Delta_2}{\partial A_2}$, $\dfrac{\partial \Delta_2}{\partial A_3}$, and $\dfrac{\partial \Delta_2}{\partial A_4}$ are positive over the given domain. Therefore, Δ_2 is an increasing function of A_1, A_2, A_3, and A_4. We can hence plug in the minimum values of A_1, A_2, A_3, and A_4 to determine a new polynomial in A_5 and α that we want to show is positive. We define

$$\begin{aligned}\Delta_3 &= \Delta_2|_{A_1=5,A_2=4,A_3=3,A_4=2}\\ &= -30\alpha^5 A_5^4 + 150\alpha^5 A_5^3 + 30\alpha^4 A_5^4 - 240\alpha^5 A_5^2 - 150\alpha^4 A_5^3 + 258\alpha^5 A_5 + 240\alpha^4 A_5^2\\ &\quad -138\alpha^5 - 257\alpha^4 A_5 + 151\alpha^4 - 14\alpha^3 A_5 - 71\alpha^3 + 71\alpha^2 A_5 + 154\alpha^2 - 154\alpha A_5\\ &\quad -120\alpha + 120A_5.\end{aligned}$$

We must show that Δ_3 is positive over the interval $\alpha \in \left(0, \frac{1}{2}\right]$ and $A_5 \geq 1$. We split this into two subcases:

Subcase I (a): $A_5 \geq 1.73$.

Subcase I (b): $A_5 < 1.73$.

We calculated the number 1.73 numerically, by noting that the partial differentiation test works normally for $A_5 \geq 1.73$ because all the partial derivatives remain positive. When $A_5 < 1.73$ some partial derivatives become negative.

3.1.1. Subcase I (a). In this subcase, we can apply the partial differentiation test normally. We begin by noting that

$$\frac{\partial^4 \Delta_3}{\partial A_5^4} = 720\alpha^4(1-\alpha) > 0.$$

We then consider

$$\left.\frac{\partial^3 \Delta_3}{\partial A_5^3}\right|_{A_5=1.73} = 345.6\alpha^4(1-\alpha) > 0.$$

Similarly,

$$\left.\frac{\partial^2 \Delta_3}{\partial A_5^2}\right|_{A_5=1.73} = 0.444\alpha^4(1-\alpha) > 0.$$

We continue by considering

$$\left.\frac{\partial \Delta_3}{\partial A_5}\right|_{A_5=1.73} = 153.07896\alpha^5 - 152.07896\alpha^4 - 14\alpha^3 + 71\alpha^2 - 154\alpha + 120 > 0$$

for $\alpha \in (0,1)$ (This can also be proven by Lemma 2.2). Finally, we evaluate Δ_3 at its minimum:

$$\Delta_3|_{A_5=1.73} = 97.9780377\alpha^5 - 83.2480377\alpha^4 - 95.22\alpha^3 + 276.83\alpha^2 - 386.42\alpha + 207.60 > 0.$$

This completes this subcase.

3.1.2. Subcase I (b). For Δ_3, the partial derivative test does not work normally for A_5 — some of the derivatives end up becoming negative. However, we can plot the polynomial Δ_3 over the interval $A_5 \in [1, 1.73)$ and $\alpha \in (0,1)$ using Mathematica and Maple and verify that it is non-negative in the region we consider. Its minimum value is 120, which occurs at $(A_5, \alpha) = (1, 0)$.

Hence, Subcase (b), and consequently Case I, is complete. □

3.2. Case II. In this case, $\lceil a_6 \rceil = 3$, so we can plug $v = 3$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

THEOREM 3.2. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $2 < a_6 \leq 3$, then*

$$6!\, P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 64.$$

Proof. In this case, $a_6 \in (2, 3]$, so we have to consider two levels — $x_6 = 1$ and $x_6 = 2$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our two subcases are:

Subcase II (a): $P_5(x_6 = 2) = 0$.
Subcase II (b): $P_5(x_6 = 2) > 0$.

3.2.1. Subcase II (a). We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.2. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$ because $a_6 \in (2, 3]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6!\, P_5(x_6 = 2) \leq 6[&(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 \\ &+ A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)]. \end{aligned}$$

As before, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.2, substituting in $a_i = \frac{A_i}{\alpha}$. We observe that this difference is equal to $\Delta_1 - 63$, where Δ_1 is from Case I above. Since we applied the partial differentiation test on $\Delta_2 = \Delta_1 \cdot \alpha^4(1 - \alpha)$ in Case I, here we need to show that

$$\Delta_4 := \Delta_2 - 63\alpha^4(1 - \alpha)$$

is positive. Since $\Delta_2 - \Delta_4$ is a function in α only, we must check that the value of Δ_4 at its minimum is positive (because all of the partial derivatives in the test are the same for Δ_2 and Δ_4). As in Case I, we have

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1,$$

so we must only check that

$$\Delta_4|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} = 63\alpha^5 - 49\alpha^4 - 85\alpha^3 + 225\alpha^2 - 274\alpha + 120 > 0.$$

Since this is true by Lemma 2.2, Δ_4 is always positive, and this subcase is complete.

3.2.2. Subcase II (b). In this subcase, we know that $P_5(x_6 = 2) > 0$, implying that $(1, 1, 1, 1, 1, 2)$ is a positive integral solution to the inequality in Theorem 3.2. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in \left(0, \frac{1}{3}\right]$ because $a_6 \in (2, 3]$. For simplicity, let $A_i = a_i \cdot \alpha_1$ for $i = 1, 2, 3, 4$, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6!\, P_5(x_6 = 2) \leq 6[&(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &-(A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)], \end{aligned}$$

and,

$$\begin{aligned} 6!\, P_5\,(x_6 = 1) \leq 6\Bigg[&\left(A_1 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_2 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_3 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right) \\ &\left(A_4 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right) - \left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)^5 \\ &+\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1}\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 2\right) \\ &\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 3\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 4\right)\Bigg]. \end{aligned}$$

Because $6!\, P_6 = 6!\,(P_5(x_6 = 1) + P_5(x_6 = 2))$, if we let Δ_5 be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 3.2, substituting in $a_i = \dfrac{A_i}{\alpha_1}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_5 = \frac{1}{(1-\alpha_1)\alpha_1^5}&(99A_1A_2A_3A_4A_5\alpha_1^6 - 84A_1A_2A_3A_4A_5\alpha_1^5 \\ &-102A_1A_2A_3A_4\alpha_1^6 - 102A_1A_2A_3A_5\alpha_1^6 - 102A_1A_2A_4A_5\alpha_1^6 - 102A_1A_3A_4A_5\alpha_1^6 \\ &-102A_2A_3A_4A_5\alpha_1^6 - 510A_5^4\alpha_1^6 + 15A_1A_2A_3A_4A_5\alpha_1^4 + 78A_1A_2A_3A_4\alpha_1^5 \\ &+78A_1A_2A_3A_5\alpha_1^5 + 108A_1A_2A_3\alpha_1^6 + 78A_1A_2A_4A_5\alpha_1^5 + 108A_1A_2A_4\alpha_1^6 \\ &+108A_1A_2A_5\alpha_1^6 + 78A_1A_3A_4A_5\alpha_1^5 + 108A_1A_3A_4\alpha_1^6 + 108A_1A_3A_5\alpha_1^6 \\ &+108A_1A_4A_5\alpha_1^6 + 78A_2A_3A_4A_5\alpha_1^5 + 108A_2A_3A_4\alpha_1^6 + 108A_2A_3A_5\alpha_1^6 \\ &+108A_2A_4A_5\alpha_1^6 + 108A_3A_4A_5\alpha_1^6 + 390A_5^4\alpha_1^5 + 2700A_5^3\alpha_1^6 - 12A_1A_2A_3A_4\alpha_1^4 \\ &-12A_1A_2A_3A_5\alpha_1^4 - 72A_1A_2A_3\alpha_1^5 - 12A_1A_2A_4A_5\alpha_1^4 - 72A_1A_2A_4\alpha_1^5 \\ &-72A_1A_2A_5\alpha_1^5 - 120A_1A_2\alpha_1^6 - 12A_1A_3A_4A_5\alpha_1^4 - 72A_1A_3A_4\alpha_1^5 - 72A_1A_3A_5\alpha_1^5 \\ &-120A_1A_3\alpha_1^6 - 72A_1A_4A_5\alpha_1^5 - 120A_1A_4\alpha_1^6 - 120A_1A_5\alpha_1^6 - 12A_2A_3A_4A_5\alpha_1^4 \\ &-72A_2A_3A_4\alpha_1^5 - 72A_2A_3A_5\alpha_1^5 - 120A_2A_3\alpha_1^6 - 72A_2A_4A_5\alpha_1^5 - 120A_2A_4\alpha_1^6 \\ &-120A_2A_5\alpha_1^6 - 72A_3A_4A_5\alpha_1^5 - 120A_3A_4\alpha_1^6 - 120A_3A_5\alpha_1^6 - 120A_4A_5\alpha_1^6 \\ &-60A_5^4\alpha_1^4 - 1800A_5^3\alpha_1^5 - 4800A_5^2\alpha_1^6 - 15A_1A_2A_3A_4A_5\alpha_1^2 + 12A_1A_2A_3A_4\alpha_1^3 \end{aligned}$$

$$
\begin{aligned}
&+12A_1A_2A_3A_5\alpha_1^3+12A_1A_2A_4A_5\alpha_1^3+72A_1A_2\alpha_1^5+12A_1A_3A_4A_5\alpha_1^3\\
&+72A_1A_3\alpha_1^5+72A_1A_4\alpha_1^5+72A_1A_5\alpha_1^5+144A_1\alpha_1^6+12A_2A_3A_4A_5\alpha_1^3\\
&+72A_2A_3\alpha_1^5+72A_2A_4\alpha_1^5+72A_2A_5\alpha_1^5+144A_2\alpha_1^6+72A_3A_4\alpha_1^5\\
&+72A_3A_5\alpha_1^5+144A_3\alpha_1^6+72A_4A_5\alpha_1^5+144A_4\alpha_1^6+60A_5^4\alpha_1^3+2880A_5^2\alpha_1^5\\
&+2880A_5\alpha_1^6+4A_1A_2A_3A_4A_5\alpha_1+2A_1A_2A_3A_4\alpha_1^2+2A_1A_2A_3A_5\alpha_1^2\\
&-8A_1A_2A_3\alpha_1^3+2A_1A_2A_4A_5\alpha_1^2-8A_1A_2A_4\alpha_1^3-8A_1A_2A_5\alpha_1^3+8A_1A_2\alpha_1^4\\
&+2A_1A_3A_4A_5\alpha_1^2-8A_1A_3A_4\alpha_1^3-8A_1A_3A_5\alpha_1^3+8A_1A_3\alpha_1^4-8A_1A_4A_5\alpha_1^3\\
&+8A_1A_4\alpha_1^4+8A_1A_5\alpha_1^4-80A_1\alpha_1^5+2A_2A_3A_4A_5\alpha_1^2-8A_2A_3A_4\alpha_1^3\\
&-8A_2A_3A_5\alpha_1^3+8A_2A_3\alpha_1^4-8A_2A_4A_5\alpha_1^3+8A_2A_4\alpha_1^4+8A_2A_5\alpha_1^4\\
&-80A_2\alpha_1^5-8A_3A_4A_5\alpha_1^3+8A_3A_4\alpha_1^4+8A_3A_5\alpha_1^4-80A_3\alpha_1^5+8A_4A_5\alpha_1^4\\
&-80A_4\alpha_1^5+90A_5^4\alpha_1^2-600A_5^3\alpha_1^3+960A_5^2\alpha_1^4-1904A_5\alpha_1^5+1008\alpha_1^6\\
&+13A_1A_2A_3A_4A_5-10A_1A_2A_3A_4\alpha_1-10A_1A_2A_3A_5\alpha_1+4A_1A_2A_3\alpha_1^2\\
&-10A_1A_2A_4A_5\alpha_1+4A_1A_2A_4\alpha_1^2+4A_1A_2A_5\alpha_1^2+8A_1A_2\alpha_1^3\\
&-10A_1A_3A_4A_5\alpha_1+4A_1A_3A_4\alpha_1^2+4A_1A_3A_5\alpha_1^2+8A_1A_3\alpha_1^3+4A_1A_4A_5\alpha_1^2\\
&+8A_1A_4\alpha_1^3+8A_1A_5\alpha_1^3-32A_1\alpha_1^4-10A_2A_3A_4A_5\alpha_1+4A_2A_3A_4\alpha_1^2\\
&+4A_2A_3A_5\alpha_1^2+8A_2A_3\alpha_1^3+4A_2A_4A_5\alpha_1^2+8A_2A_4\alpha_1^3+8A_2A_5\alpha_1^3\\
&-32A_2\alpha_1^4+4A_3A_4A_5\alpha_1^2+8A_3A_4\alpha_1^3+8A_3A_5\alpha_1^3-32A_3\alpha_1^4\\
&+8A_4A_5\alpha_1^3-32A_4\alpha_1^4+30A_5^4\alpha_1-300A_5^3\alpha_1^2+960A_5^2\alpha_1^3-944A_5\alpha_1^4-1040\alpha_1^5)\\
=\ &\frac{1}{(1-\alpha_1)\alpha_1^5}\Delta_6.
\end{aligned}
$$

Just like in Case I, we are trying to show that Δ_6 is positive for

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Although we only need to show this is true for $\alpha_1 \in (0, 1/3]$, we will demonstrate it true for the interval $\alpha_1 \in (0, 1/2]$ because it will aid us in a later case.

Thus, the first step in the partial differentiation test is determining that

$$\frac{\partial^5 \Delta_6}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 99\alpha_1^6 - 84\alpha_1^5 + 15\alpha_1^4 - 15\alpha_1^2 + 4\alpha_1 + 13 > 0$$

for all $\alpha_1 \in (0, \frac{1}{2}]$. Thus the partial derivative of Δ_6 with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = 1$.

$$\left.\frac{\partial^4 \Delta_6}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=1} = -3\alpha_1^6 - 6\alpha_1^5 + 3\alpha_1^4 + 12\alpha_1^3 - 13\alpha_1^2 - 6\alpha_1 + 13 > 0.$$

It follows that the partial of Δ_6 with respect to A_1, A_2, A_3, and A_5 is positive for all $A_4 \geq 1, \alpha_1 \in (0, \frac{1}{2}]$ because $\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_3}$ is symmetric in A_4 and A_5. Hence, $\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 and A_5 with a minimum at $A_4 = A_5 = 1$.

$$\left.\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=A_5=1} = 3\alpha_1^6 - 9\alpha_1^4 + 16\alpha_1^3 - 7\alpha_1^2 - 16\alpha_1 + 13 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to A_3, A_4, and A_5, we know that $\dfrac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_4}$ and $\dfrac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_5}$ are also positive over the given domain. Hence, $\dfrac{\partial^2 \Delta_2}{\partial A_1 \partial A_2}$ is an increasing function of A_3, A_4, and A_5 for all $A_3, A_4, A_5 \geq 1$ and $\alpha_1 \in (0, \frac{1}{2}]$. The minimum is at $A_3 = A_4 = A_5 = 1$.

$$\left.\frac{\partial^2 \Delta_6}{\partial A_1 \partial A_2}\right|_{A_3=A_4=A_5=1} = -3\alpha_1^6 + 6\alpha_1^5 - 13\alpha_1^4 + 20\alpha_1^3 + 3\alpha_1^2 - 26\alpha_1 + 13 > 0.$$

Because this is symmetric with respect to A_2, A_3, A_4, and A_5, we see that $\dfrac{\partial^2 \Delta_6}{\partial A_1 \partial A_3}$, $\dfrac{\partial^2 \Delta_6}{\partial A_1 \partial A_4}$, and $\dfrac{\partial^2 \Delta_6}{\partial A_1 \partial A_5}$ are positive over the given domain. Hence, $\dfrac{\partial \Delta_6}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 and A_5 minimized at $A_2 = A_3 = A_4 = A_5 = 1$.

$$\left.\frac{\partial \Delta_6}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=1} = 3\alpha_1^6 + 4\alpha_1^5 - 33\alpha_1^4 + 32\alpha_1^3 + 17\alpha_1^2 - 36\alpha_1 + 13 > 0.$$

Thus $\dfrac{\partial \Delta_6}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_6 in A_1, A_2, A_3, and A_4, we know that $\dfrac{\partial \Delta_6}{\partial A_2}$, $\dfrac{\partial \Delta_6}{\partial A_3}$, and $\dfrac{\partial \Delta_6}{\partial A_4}$ are positive over the given domain. Therefore, Δ_6 is an increasing function of A_1, A_2, A_3, and A_4. We can hence plug in the minimum values of A_1, A_2, A_3, and A_4 to determine a new polynomial in A_5 and α_1 that we want to show is positive. We define

$$\begin{aligned}
\Delta_7 &= \Delta_6|_{A_1=5, A_2=4, A_3=3, A_4=2} \\
&= -510A_5^4\alpha_1^6 + 390A_5^4\alpha_1^5 + 2700A_5^3\alpha_1^6 - 60A_5^4\alpha_1^4 - 1800A_5^3\alpha_1^5 - 4800A_5^2\alpha_1^6 \\
&\quad +60A_5^4\alpha_1^3 + 2880A_5^2\alpha_1^5 + 5040A_5\alpha_1^6 + 90A_5^4\alpha_1^2 - 600A_5^3\alpha_1^3 + 960A_5^2\alpha_1^4 \\
&\quad -4076A_5\alpha_1^5 - 1104\alpha_1^6 + 30A_5^4\alpha_1 - 300A_5^3\alpha_1^2 + 960A_5^2\alpha_1^3 - 880A_5\alpha_1^4 + 1224\alpha_1^5 \\
&\quad +1392A_5\alpha_1^3 - 1320\alpha_1^4 - 1208A_5\alpha_1^2 + 776\alpha_1^3 - 1060A_5\alpha_1 + 856\alpha_1^2 + 1560A_5 \\
&\quad -1200\alpha_1.
\end{aligned}$$

We must show that Δ_7 is positive over the interval $\alpha_1 \in (0, \frac{1}{2}]$ and $A_5 \geq 1$.

We apply the partial differentiation test normally, beginning by noting that

$$\frac{\partial^4 \Delta_7}{\partial A_5^4} = -12240\alpha_1^6 + 9360\alpha_1^5 - 1440\alpha_1^4 + 1440\alpha_1^3 + 2160\alpha_1^2 + 720\alpha_1 > 0.$$

We then consider

$$\left.\frac{\partial^3 \Delta_7}{\partial A_5^3}\right|_{A_5=1} = 3960\alpha_1^6 - 1440\alpha_1^5 - 1440\alpha_1^4 - 2160\alpha_1^3 + 360\alpha_1^2 + 720\alpha_1 > 0.$$

Similarly,

$$\left.\frac{\partial^2 \Delta_7}{\partial A_5^2}\right|_{A_5=1} = 480\alpha_1^6 - 360\alpha_1^5 + 1200\alpha_1^4 - 960\alpha_1^3 - 720\alpha_1^2 + 360\alpha_1,$$

which is positive for $\alpha_1 \in (0, \frac{1}{3}]$. Although $\left.\dfrac{\partial^2 \Delta_7}{\partial A_5^2}\right|_{A_5=1}$ ends up becoming negative in the interval $\alpha_1 \in (\frac{1}{3}, \frac{1}{2}]$, this ends up being irrelevant because we can verify numerically that $\left.\frac{\partial \Delta_7}{\partial A_5}\right|_{A_5=1}$ is still positive. We do this by plotting $\left.\frac{\partial \Delta_7}{\partial A_5}\right|_{A_5=1}$ over that interval using Mathematica or Maple. Its minimum value is $\frac{16777}{32} = 524.281$, attained when $\alpha_1 = \frac{1}{2}$.

We continue by considering

$$\left.\frac{\partial \Delta_7}{\partial A_5}\right|_{A_5=1} = 1500\alpha_1^6 - 2156\alpha_1^5 + 800\alpha_1^4 + 1752\alpha_1^3 - 1748\alpha_1^2 - 940\alpha_1 + 1560 > 0$$

for $\alpha_1 \in \left(0, \frac{1}{2}\right)$. Finally, we evaluate Δ_7 at its minimum:

$$\Delta_7|_{A_5=1} = 1326\alpha_1^6 - 1382\alpha_1^5 - 1300\alpha_1^4 + 2588\alpha_1^3 - 562\alpha_1^2 - 2230\alpha_1 + 1560 > 0.$$

This completes this subcase, and hence Case II is complete. ∎

3.3. Case III. In this case, $\lceil a_6 \rceil = 4$, so we can plug $v = 4$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

THEOREM 3.3. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $3 < a_6 \leq 4$, then*

$$6!\, P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 729.$$

Proof. In this case, $a_6 \in (3, 4]$, so we have to consider three levels — $x_6 = 1$, $x_6 = 2$, and $x_6 = 3$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our three subcases are:

Subcase III (a): $P_5(x_6 = 3) = P_5(x_6 = 2) = 0$.
Subcase III (b): $P_5(x_6 = 3) = 0, P_5(x_6 = 2) > 0$.
Subcase III (c): $P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.

3.3.1. Subcase III (a). We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.3. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$ because $a_6 \in (3, 4]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6!\, P_5(x_6 = 1) \leq 6[&(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &-(A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)]. \end{aligned}$$

As before, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.3, substituting in $a_i = \frac{A_i}{\alpha}$. We observe that this difference is equal to $\Delta_1 - 728$, where Δ_1 is from Case I above. Since we applied the partial differentiation test on $\Delta_2 = \Delta_1 \cdot \alpha^4(1-\alpha)$ in Case I, here we need to show that

$$\Delta_8 := \Delta_2 - 728\alpha^4(1-\alpha)$$

is positive. To apply the partial differentiation test, we must determine the domain of Δ_8. By the same logic as in Cases I and II, we have

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, \text{ and } A_4 \geq A_5 \geq \frac{\alpha}{1-\alpha} > 2,$$

because $A_5 = a_5 \cdot \alpha \geq a_6 \cdot \alpha = \dfrac{\alpha}{1-\alpha}$ and $\alpha \in \left(\dfrac{2}{3}, \dfrac{3}{4}\right]$. Now that we have a domain established, we can begin applying the partial differentiation test to demonstrate that Δ_8 is positive.

We see that

$$\frac{\partial^5 \Delta_8}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 6\alpha^5 - 6\alpha^4 + 1 > 0$$

for all $\alpha \in (0,1)$. Thus the partial derivative of Δ_8 with respect to $A_1, A_2, A_3, A_4,$ and A_5 is positive and minimized at $A_5 = \dfrac{\alpha}{1-\alpha}$.

$$\left.\frac{\partial^4 \Delta_8}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{\alpha^2(12\alpha^4 - 18\alpha^3 + 6\alpha^2 + 1)}{(\alpha - 1)} > 0.$$

It follows that the partial of Δ_8 with respect to A_1, A_2, A_3, and A_5 is positive for all $A_4 \geq \dfrac{\alpha}{1-\alpha}, \alpha \in (0,1)$ because $\dfrac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_3}$ is symmetric in A_4 and A_5. Hence, $\dfrac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4, A_5 with a minimum at $A_4 = A_5 = \dfrac{\alpha}{1-\alpha}$.

$$\left.\frac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{\alpha^4(24\alpha^3 - 48\alpha^2 + 30\alpha - 5)}{(1-\alpha)^2} > 0.$$

This is positive, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to $A_3, A_4,$ and A_5, we know that $\dfrac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_4}$ and $\dfrac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_5}$ are positive over the given domain. Hence, $\dfrac{\partial^2 \Delta_8}{\partial A_1 \partial A_2}$ is an increasing function of A_3, A_4, and A_5 for all $A_3, A_4, A_5 \geq \dfrac{\alpha}{1-\alpha}$ and $\alpha \in \left(\dfrac{2}{3}, \dfrac{3}{4}\right]$. The minimum is at $A_3 = A_4 = A_5 = \dfrac{\alpha}{1-\alpha}$.

$$\left.\frac{\partial^2 \Delta_8}{\partial A_1 \partial A_2}\right|_{A_3 = A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{\alpha^4(48\alpha^4 - 120\alpha^3 + 109\alpha^2 - 42\alpha + 6)}{(1-\alpha)^3} > 0.$$

Because this is symmetric with respect to $A_2, A_3, A_4,$ and A_5, we see that $\dfrac{\partial^2 \Delta_8}{\partial A_1 \partial A_3}$, $\dfrac{\partial^2 \Delta_8}{\partial A_1 \partial A_4}$, and $\dfrac{\partial^2 \Delta_8}{\partial A_1 \partial A_5}$ are positive over the given domain. Hence, $\dfrac{\partial \Delta_8}{\partial A_1}$

is an increasing function of A_2, A_3, A_4 and A_5 minimized at $A_2 = A_3 = A_4 = A_5 = \dfrac{\alpha}{1-\alpha}$.

$$\left.\frac{\partial \Delta_8}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{\alpha^4(96\alpha^5 - 287\alpha^4 + 336\alpha^3 - 192\alpha^2 + 54\alpha - 6)}{(1-\alpha)^4} > 0.$$

Thus $\dfrac{\partial \Delta_8}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_8 in A_1, A_2, A_3, and A_4, we know that $\dfrac{\partial \Delta_8}{\partial A_2}$, $\dfrac{\partial \Delta_8}{\partial A_3}$, and $\dfrac{\partial \Delta_8}{\partial A_4}$ are positive over the given domain. Therefore, Δ_8 is an increasing function of A_1, A_2, A_3, and A_4. We can hence plug in the minimum values of A_1, A_2, A_3, and A_4 to determine a new polynomial in A_5 and α that we want to show is positive. We define

$$\begin{aligned}\Delta_9 &= \Delta_8|_{A_1=5,A_2=4,A_3=3,A_4=2}\\ &= -30\alpha^5A_5^4 + 150\alpha^5A_5^3 + 30\alpha^4A_5^4 - 240\alpha^5A_5^2 - 150\alpha^4A_5^3 + 258\alpha^5A_5 + 240\alpha^4A_5^2\\ &\quad +590\alpha^5 - 257\alpha^4A_5 - 577\alpha^4 - 14\alpha^3A_5 - 71\alpha^3 + 71\alpha^2A_5 + 154\alpha^2 - 154\alpha A_5\\ &\quad -120\alpha + 120A_5.\end{aligned}$$

We must show that Δ_9 is positive over the interval $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ and $A_5 \geq \dfrac{\alpha}{1-\alpha}$. We can apply the partial differentiation test normally. We begin by noting that

$$\frac{\partial^4 \Delta_9}{\partial A_5^4} = 720\alpha^4(1-\alpha) > 0.$$

We then consider

$$\left.\frac{\partial^3 \Delta_9}{\partial A_5^3}\right|_{A_5=\frac{\alpha}{1-\alpha}} = 180\alpha^4(9\alpha - 5) > 0.$$

Similarly,

$$\left.\frac{\partial^2 \Delta_9}{\partial A_5^2}\right|_{A_5=\frac{\alpha}{1-\alpha}} = \frac{60\alpha^4(29\alpha^2 - 31\alpha + 8)}{1-\alpha} > 0.$$

We continue by considering

$$\left.\frac{\partial \Delta_9}{\partial A_5}\right|_{A_5=\frac{\alpha}{1-\alpha}} = \frac{1308\alpha^7 - 2183\alpha^6 + 1238\alpha^5 - 158\alpha^4 - 310\alpha^3 + 499\alpha^2 - 394\alpha + 120}{(1-\alpha)^2} > 0,$$

for $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Finally, we evaluate Δ_9 at its minimum:

$$\Delta_9|_{A_5=\frac{\alpha}{1-\alpha}} = \frac{2\alpha^2(44\alpha^6 + 472\alpha^5 - 1216\alpha^4 + 898\alpha^3 - 46\alpha^2 - 197\alpha + 60)}{(1-\alpha)^3} > 0,$$

completing this subcase.

3.3.2. Subcase III (b). In this subcase, we know that $P_5(x_6 = 2) > 0$, implying that $(1, 1, 1, 1, 1, 2)$ is a positive integral solution to the inequality in Theorem 3.3. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in \left(\frac{1}{3}, \frac{1}{2}\right]$ because $a_6 \in (3, 4]$. For simplicity, let $A_i = a_i \cdot \alpha_1$ for $i = 1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \le 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned}6!\, P_5(x_6 = 2) \le 6[&(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)\\ &-(A_5-1)^5 + A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)],\end{aligned}$$

and,

$$\begin{aligned}6!\, P_5\,(x_6 = 1) \le 6\Bigg[&\left(A_1 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_2 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_3 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\\ &\left(A_4 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right) - \left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)^5\\ &+\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1}\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 2\right)\\ &\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 3\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 4\right)\Bigg].\end{aligned}$$

Because $6!\, P_6 = 6!\,(P_5(x_6 = 1) + P_5(x_6 = 2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 3.3, substituting in $a_i = \dfrac{A_i}{\alpha_1}$. We observe that this difference is equal to $\Delta_5 - 665$, where Δ_5 is from Case II above. Since we applied the partial differentiation test on $\Delta_6 = \Delta_5 \cdot 16\alpha^5(1-\alpha)$ in Case I, here we need to show that

$$\Delta_{10} := \Delta_6 - 665 \cdot 16\alpha_1^5(1-\alpha_1)$$

is positive. Since $\Delta_{10} - \Delta_6$ is a function of α_1 only, we only need to check that the value of Δ_{10} at its minimum is positive (because all of the partial derivatives in the test are the same for Δ_6 and Δ_{10}). Recall that we have

$$A_1 \ge 5, A_2 \ge 4, A_3 \ge 3, A_4 \ge 2, \text{ and } A_5 > \frac{2\alpha_1}{1-\alpha_1},$$

so we must only check that

$$\begin{aligned}&\Delta_{10}\big|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=\frac{2\alpha_1}{1-\alpha_1}}\\ &= \frac{1}{(1-\alpha_1)^3}(16\alpha_1(3094\alpha_1^8 - 2093\alpha_1^7 - 1272\alpha_1^6 + 1770\alpha_1^5 - 738\alpha_1^4\\ &\qquad +351\alpha_1^3 - 28\alpha_1^2 - 244\alpha_1 + 120))\\ &> 0.\end{aligned}$$

Since this is true, Δ_{10} is always positive, and this subcase is complete.

3.3.3. Subcase III (c). In this subcase, we know that $P_5(x_6 = 3) > 0$, implying that $(1, 1, 1, 1, 1, 1, 3)$ is a positive integral solution to the inequality in Theorem 3.3. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \le 1 - \frac{3}{a_6} := \alpha_2,$$

then $\alpha_2 \in \left(0, \frac{1}{4}\right]$ because $a_6 \in (3,4]$. For simplicity, let $A_i = a_i \cdot \alpha_2$ for $i = 1, 2, 3, 4,$ and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6!\, P_5(x_6 = 3) \leq 6[&(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &-(A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)], \end{aligned}$$

as well as

$$\begin{aligned} 6!\, P_5\,(x_6 = 2) \leq 6\Bigg[&\left(A_1 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_2 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_3 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right) \\ &\left(A_4 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right) - \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)^5 \\ &+\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2}\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 2\right) \\ &\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 3\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 4\right)\Bigg], \end{aligned}$$

and

$$\begin{aligned} 6!\, P_5\,(x_6 = 1) \leq 6\Bigg[&\left(A_1 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_2 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_3 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right) \\ &\left(A_4 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right) - \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)^5 \\ &+\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2}\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 2\right) \\ &\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 3\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 4\right)\Bigg]. \end{aligned}$$

Because $6!\, P_6 = 6!\,(P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3))$, if we let Δ_{11} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 3.4 and substituting in $a_i = \dfrac{A_i}{\alpha_2}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_{11} = \frac{1}{27\alpha_2^5(1-\alpha_2)}(&-980A_5^4\alpha_2^6 + 5400A_5^3\alpha_2^6 - 10080A_5^2\alpha_2^6 + 324A_1\alpha_2^6 \\ &-252A_1A_2\alpha_2^6 + 324A_2\alpha_2^6 - 252A_1A_3\alpha_2^6 + 216A_1A_2A_3\alpha_2^6 - 252A_2A_3\alpha_2^6 \\ &+324A_3\alpha_2^6 - 252A_1A_4\alpha_2^6 + 216A_1A_2A_4\alpha_2^6 - 252A_2A_4\alpha_2^6 + 216A_1A_3A_4\alpha_2^6 \\ &-196A_1A_2A_3A_4\alpha_2^6 + 216A_2A_3A_4\alpha_2^6 - 252A_3A_4\alpha_2^6 + 324A_4\alpha_2^6 - 252A_1A_5\alpha_2^6 \\ &+216A_1A_2A_5\alpha_2^6 - 252A_2A_5\alpha_2^6 + 216A_1A_3A_5\alpha_2^6 - 196A_1A_2A_3A_5\alpha_2^6 \\ &+216A_2A_3A_5\alpha_2^6 - 252A_3A_5\alpha_2^6 + 216A_1A_4A_5\alpha_2^6 - 196A_1A_2A_4A_5\alpha_2^6 \\ &+216A_2A_4A_5\alpha_2^6 - 196A_1A_3A_4A_5\alpha_2^6 + 184A_1A_2A_3A_4A_5\alpha_2^6 - 196A_2A_3A_4A_5\alpha_2^6 \end{aligned}$$

$$+216A_3A_4A_5\alpha_2^6 - 252A_4A_5\alpha_2^6 + 6480A_5\alpha_2^6 + 19656\alpha_2^6 + 580A_5^4\alpha_2^5 - 2700A_5^3\alpha_2^5$$
$$+4320A_5^2\alpha_2^5 - 135A_1\alpha_2^5 + 108A_1A_2\alpha_2^5 - 135A_2\alpha_2^5 + 108A_1A_3\alpha_2^5 - 108A_1A_2A_3\alpha_2^5$$
$$+108A_2A_3\alpha_2^5 - 135A_3\alpha_2^5 + 108A_1A_4\alpha_2^5 - 108A_1A_2A_4\alpha_2^5 + 108A_2A_4\alpha_2^5$$
$$-108A_1A_3A_4\alpha_2^5 + 116A_1A_2A_3A_4\alpha_2^5 - 108A_2A_3A_4\alpha_2^5 + 108A_3A_4\alpha_2^5 - 135A_4\alpha_2^5$$
$$+108A_1A_5\alpha_2^5 - 108A_1A_2A_5\alpha_2^5 + 108A_2A_5\alpha_2^5 - 108A_1A_3A_5\alpha_2^5 + 116A_1A_2A_3A_5\alpha_2^5$$
$$-108A_2A_3A_5\alpha_2^5 + 108A_3A_5\alpha_2^5 - 108A_1A_4A_5\alpha_2^5 + 116A_1A_2A_4A_5\alpha_2^5$$
$$-108A_2A_4A_5\alpha_2^5 + 116A_1A_3A_4A_5\alpha_2^5 - 124A_1A_2A_3A_4A_5\alpha_2^5 + 116A_2A_3A_4A_5\alpha_2^5$$
$$-108A_3A_4A_5\alpha_2^5 + 108A_4A_5\alpha_2^5 - 3213A_5\alpha_2^5 - 19737\alpha_2^5 - 80A_5^4\alpha_2^4 + 2160A_5^2\alpha_2^4$$
$$-108A_1\alpha_2^4 + 27A_1A_2\alpha_2^4 - 108A_2\alpha_2^4 + 27A_1A_3\alpha_2^4 + 27A_2A_3\alpha_2^4 - 108A_3\alpha_2^4$$
$$+27A_1A_4\alpha_2^4 + 27A_2A_4\alpha_2^4 - 16A_1A_2A_3A_4\alpha_2^4 + 27A_3A_4\alpha_2^4 - 108A_4\alpha_2^4 + 27A_1A_5\alpha_2^4$$
$$+27A_2A_5\alpha_2^4 - 16A_1A_2A_3A_5\alpha_2^4 + 27A_3A_5\alpha_2^4 - 16A_1A_2A_4A_5\alpha_2^4 - 16A_1A_3A_4A_5\alpha_2^4$$
$$+20A_1A_2A_3A_4A_5\alpha_2^4 - 16A_2A_3A_4A_5\alpha_2^4 + 27A_4A_5\alpha_2^4 - 3186A_5\alpha_2^4 + 80A_5^4\alpha_2^3$$
$$-1350A_5^3\alpha_2^3 + 3600A_5^2\alpha_2^3 + 36A_1A_2\alpha_2^3 + 36A_1A_3\alpha_2^3 - 27A_1A_2A_3\alpha_2^3$$
$$+36A_2A_3\alpha_2^3 + 36A_1A_4\alpha_2^3 - 27A_1A_2A_4\alpha_2^3 + 36A_2A_4\alpha_2^3 - 27A_1A_3A_4\alpha_2^3$$
$$+16A_1A_2A_3A_4\alpha_2^3 - 27A_2A_3A_4\alpha_2^3 + 36A_3A_4\alpha_2^3 + 36A_1A_5\alpha_2^3 - 27A_1A_2A_5\alpha_2^3$$
$$+36A_2A_5\alpha_2^3 - 27A_1A_3A_5\alpha_2^3 + 16A_1A_2A_3A_5\alpha_2^3 - 27A_2A_3A_5\alpha_2^3 + 36A_3A_5\alpha_2^3$$
$$-27A_1A_4A_5\alpha_2^3 + 16A_1A_2A_4A_5\alpha_2^3 - 27A_2A_4A_5\alpha_2^3 + 16A_1A_3A_4A_5\alpha_2^3$$
$$+16A_2A_3A_4A_5\alpha_2^3 - 27A_3A_4A_5\alpha_2^3 + 36A_4A_5\alpha_2^3 + 230A_5^4\alpha_2^2 - 1350A_5^3\alpha_2^2$$
$$+19A_1A_2A_3A_4\alpha_2^2 + 19A_1A_2A_3A_5\alpha_2^2 + 19A_1A_2A_4A_5\alpha_2^2 + 19A_1A_3A_4A_5\alpha_2^2$$
$$-20A_1A_2A_3A_4A_5\alpha_2^2 + 19A_2A_3A_4A_5\alpha_2^2 + 170A_5^4\alpha_2 - 20A_1A_2A_3A_4\alpha_2$$
$$-20A_1A_2A_3A_5\alpha_2 - 20A_1A_2A_4A_5\alpha_2 - 20A_1A_3A_4A_5\alpha_2 - 11A_1A_2A_3A_4A_5\alpha_2$$
$$-20A_2A_3A_4A_5\alpha_2 + 32A_1A_2A_3A_4A_5)$$
$$= \frac{1}{27\alpha_2^5(1-\alpha_2)}\Delta_{12}.$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_2 \in \left(0, \frac{1}{4}\right]$.

To start the partial differentiation test, we see that

$$\frac{\partial^5 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 184\alpha_2^6 - 124\alpha_2^5 + 20\alpha_2^4 - 20\alpha_2^2 - 11\alpha_2 + 32 > 0$$

for all $\alpha_2 \in (0, \frac{1}{4}]$. Thus the partial derivative of Δ_{12} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = 1$.

$$\left.\frac{\partial^4 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=1} = -12\alpha_2^6 - 8\alpha_2^5 + 4\alpha_2^4 + 16\alpha_2^3 - \alpha_2^2 - 31\alpha_2 + 32 > 0.$$

It follows that the partial of Δ_{12} with respect to A_1, A_2, A_3, and A_5 is positive for all $A_4 \geq 1, \alpha_2 \in (0, \frac{1}{4}]$ because $\frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3}$ is symmetric in A_4 and A_5. Hence, $\frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 and A_5 with a minimum at $A_4 = A_5 = 1$.

$$\left.\frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=A_5=1} = 8\alpha_2^6 - 12\alpha_2^4 + 5\alpha_2^3 + 18\alpha_2^2 - 51\alpha_2 + 32 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to A_3, A_4, and A_5, we know that $\frac{\partial^3\Delta_{12}}{\partial A_1\partial A_2\partial A_4}$ and $\frac{\partial^3\Delta_{12}}{\partial A_1\partial A_2\partial A_5}$ are also positive over the given domain. Hence, $\frac{\partial^2\Delta_{12}}{\partial A_1\partial A_2}$ is an increasing function of A_3, A_4, and A_5 for all $A_3, A_4, A_5 \geq 1$ and $\alpha_2 \in (0, \frac{1}{4}]$. The minimum is at $A_3 = A_4 = A_5 = 1$.

$$\left.\frac{\partial^2\Delta_{12}}{\partial A_1\partial A_2}\right|_{A_3=A_4=A_5=1} = -8\alpha_2^6 + 8\alpha_2^5 - \alpha_2^4 + 3\alpha_2^3 + 37\alpha_2^2 - 71\alpha_2 + 32 > 0.$$

Because this is symmetric with respect to A_2, A_3, A_4, and A_5, we see that $\frac{\partial^2\Delta_{12}}{\partial A_1\partial A_3}$, $\frac{\partial^2\Delta_{12}}{\partial A_1\partial A_4}$, and $\frac{\partial^2\Delta_{12}}{\partial A_1\partial A_5}$ are positive over the given domain. Hence, $\frac{\partial\Delta_{12}}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 and A_5 minimized at $A_2 = A_3 = A_4 = A_5 = 1$.

$$\left.\frac{\partial\Delta_{12}}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=1} = 12\alpha_2^6 - 11\alpha_2^5 - 44\alpha_2^4 + 46\alpha_2^3 + 56\alpha_2^2 - 91\alpha_2 + 32 > 0.$$

Thus $\frac{\partial\Delta_{12}}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_{12} in A_1, A_2, A_3, and A_4, we know that $\frac{\partial\Delta_{12}}{\partial A_2}$, $\frac{\partial\Delta_{12}}{\partial A_3}$, and $\frac{\partial\Delta_{12}}{\partial A_4}$ are positive over the given domain. Therefore, Δ_{12} is an increasing function of A_1, A_2, A_3, and A_4. We can hence plug in the minimum values of A_1, A_2, A_3, and A_4 to determine a new polynomial in A_5 and α_2 that we want to show is positive. We define

$$\begin{aligned}
\Delta_{13} &= \Delta_{12}|_{A_1=5,A_2=4,A_3=3,A_4=2} \\
&= -980A_5^4\alpha_2^6 + 580A_5^4\alpha_2^5 + 5400A_5^3\alpha_2^6 - 80A_5^4\alpha_2^4 - 2700A_5^3\alpha_2^5 - 10080A_5^2\alpha_2^6 \\
&\quad +80A_5^4\alpha_2^3 + 4320A_5^2\alpha_2^5 + 10184A_5\alpha_2^6 + 230A_5^4\alpha_2^2 - 1350A_5^3\alpha_2^3 + 2160A_5^2\alpha_2^4 \\
&\quad -6385A_5\alpha_2^5 + 16044\alpha_2^6 + 170A_5^4\alpha_2 - 1350A_5^3\alpha_2^2 + 3600A_5^2\alpha_2^3 - 2872A_5\alpha_2^4 \\
&\quad -16671\alpha_2^5 + 1051A_5\alpha_2^3 - 1515\alpha_2^4 + 526A_5\alpha_2^2 + 318\alpha_2^3 - 4400A_5\alpha_2 + 2280\alpha_2^2 \\
&\quad +3840A_5 - 2400\alpha_2.
\end{aligned}$$

We must show that Δ_{13} is positive over the interval $\alpha_2 \in (0, \frac{1}{4}]$ and $A_5 \geq 1$.

We apply the partial differentiation test normally, beginning by noting that

$$\frac{\partial^4\Delta_{13}}{\partial A_5^4} = -23520\alpha_2^6 + 13920\alpha_2^5 - 1920\alpha_2^4 + 1920\alpha_2^3 + 5520\alpha_2^2 + 4080\alpha_2 > 0.$$

We then consider

$$\left.\frac{\partial^3\Delta_{13}}{\partial A_5^3}\right|_{A_5=1} = 8880\alpha_2^6 - 2280\alpha_2^5 - 1920\alpha_2^4 - 6180\alpha_2^3 - 2580\alpha_2^2 + 4080\alpha_2 > 0.$$

Similarly,

$$\left.\frac{\partial^2\Delta_{13}}{\partial A_5^2}\right|_{A_5=1} = 480\alpha_2^6 - 600\alpha_2^5 + 3360\alpha_2^4 + 60\alpha_2^3 - 5340\alpha_2^2 + 2040\alpha_2,$$

which is positive for $\alpha_2 \in (0, \frac{1}{4}]$. We continue by considering

$$\left.\frac{\partial \Delta_{13}}{\partial A_5}\right|_{A_5=1} = 2304\alpha_2^6 - 3525\alpha_2^5 + 1128\alpha_2^4 + 4521\alpha_2^3 - 2604\alpha_2^2 - 3720\alpha_2 + 3840 > 0.$$

Finally, we evaluate Δ_{13} at its minimum:

$$\Delta_{13}|_{A_5=1} = 20568\alpha_2^6 - 20856\alpha_2^5 - 2307\alpha_2^4 + 3699\alpha_2^3 + 1686\alpha_2^2 - 6630\alpha_2 + 3840 > 0.$$

This completes this subcase, and hence Case III is complete. □

3.4. Case IV. In this case, $\lceil a_6 \rceil = 5$, so we can plug $v = 5$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

THEOREM 3.4. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $4 < a_6 \leq 5$, then*

$$6!\, P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 4096.$$

Proof. In this case, $a_6 \in (4, 5]$, so we have to consider four levels — $x_6 = 1$, $x_6 = 2$, $x_6 = 3$ and $x_6 = 4$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our four subcases are:
Subcase IV (a): $P_5(x_6 = 4) = P_5(x_6 = 3) = P_5(x_6 = 2) = 0$.
Subcase IV (b): $P_5(x_6 = 4) = P_5(x_6 = 3) = 0, P_5(x_6 = 2) > 0$.
Subcase IV (c): $P_5(x_6 = 4) = 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.
Subcase IV (d): $P_5(x_6 = 4) > 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.

3.4.1. Subcase IV (a). We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.4. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right]$ because $a_6 \in (4, 5]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Whereas we previously bounded the number of positive integral solutions to this inequality using the Yau Number Theoretic Conjecture for $n = 5$, we will now use the Yau Geometric Conjecture for $n = 5$, proven in [1]. This gives us the bound

$$6!\, P_5(x_6 = 1) \leq 6\left[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - p_5(\lceil A_5 \rceil)\right], \quad (3)$$

where p_5 is the function defined in (1). Note that since $p_5(v)$ is increasing for $v \geq 4$. Also, since $A_5 = a_5 \cdot \alpha > \frac{3}{4} \cdot 4 = 3$, we note that $\lceil A_5 \rceil \geq 4$. Thus, we maximize the RHS of (3) by substituting $p_5(4) = 243$ in for $p_5(\lceil A_5 \rceil)$. Hence, we have

$$6!\, P_5(x_6 = 1) \leq 6\left[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - 243\right]. \quad (4)$$

As before, we take the difference obtained by subtracting the RHS of (4) from the RHS of Theorem 3.4, substituting in $a_i = \dfrac{A_i}{\alpha}$ and $a_6 = \dfrac{1}{1-\alpha}$, yielding

$$
\begin{aligned}
\Delta_{14} = \frac{1}{2\alpha^4(1-\alpha)}&(5262\alpha^5 - 5264\alpha^4 + 12\alpha^5 A_1 - 10\alpha^4 A_1 - 12\alpha^5 A_1A_2 + 12\alpha^4 A_1A_2\\
&-2\alpha^3 A_1A_2 + 12\alpha^5 A_1A_2A_3 - 12\alpha^4 A_1A_2A_3 + 2\alpha^2 A_1A_2A_3 - 12\alpha^5 A_1A_2A_3A_4\\
&+12\alpha^4 A_1A_2A_3A_4 - 2\alpha A_1A_2A_3A_4 + 12\alpha^5 A_1A_2A_3A_4A_5 - 12\alpha^4 A_1A_2A_3A_4A_5\\
&+2A_1A_2A_3A_4A_5 - 12\alpha^5 A_1A_2A_3A_5 + 12\alpha^4 A_1A_2A_3A_5 - 2\alpha A_1A_2A_3A_5\\
&+12\alpha^5 A_1A_2A_4 - 12\alpha^4 A_1A_2A_4 + 2\alpha^2 A_1A_2A_4 - 12\alpha^5 A_1A_2A_4A_5 + 12\alpha^4 A_1A_2A_4A_5\\
&-2\alpha A_1A_2A_4A_5 + 12\alpha^5 A_1A_2A_5 - 12\alpha^4 A_1A_2A_5 + 2\alpha^2 A_1A_2A_5 - 12\alpha^5 A_1A_3\\
&+12\alpha^4 A_1A_3 - 2\alpha^3 A_1A_3 + 12\alpha^5 A_1A_3A_4 - 12\alpha^4 A_1A_3A_4 + 2\alpha^2 A_1A_3A_4\\
&-12\alpha^5 A_1A_3A_4A_5 + 12\alpha^4 A_1A_3A_4A_5 - 2\alpha A_1A_3A_4A_5 + 12\alpha^5 A_1A_3A_5 - 12\alpha^4 A_1A_3A_5\\
&+2\alpha^2 A_1A_3A_5 - 12\alpha^5 A_1A_4 + 12\alpha^4 A_1A_4 - 2\alpha^3 A_1A_4 + 12\alpha^5 A_1A_4A_5 - 12\alpha^4 A_1A_4A_5\\
&+2\alpha^2 A_1A_4A_5 - 12\alpha^5 A_1A_5 + 12\alpha^4 A_1A_5 - 2\alpha^3 A_1A_5 + 12\alpha^5 A_2 - 10\alpha^4 A_2 - 12\alpha^5 A_2A_3\\
&+12\alpha^4 A_2A_3 - 2\alpha^3 A_2A_3 + 12\alpha^5 A_2A_3A_4 - 12\alpha^4 A_2A_3A_4 + 2\alpha^2 A_2A_3A_4\\
&-12\alpha^5 A_2A_3A_4A_5 + 12\alpha^4 A_2A_3A_4A_5 - 2\alpha A_2A_3A_4A_5 + 12\alpha^5 A_2A_3A_5 - 12\alpha^4 A_2A_3A_5\\
&+2\alpha^2 A_2A_3A_5 - 12\alpha^5 A_2A_4 + 12\alpha^4 A_2A_4 - 2\alpha^3 A_2A_4 + 12\alpha^5 A_2A_4A_5 - 12\alpha^4 A_2A_4A_5\\
&+2\alpha^2 A_2A_4A_5 - 12\alpha^5 A_2A_5 + 12\alpha^4 A_2A_5 - 2\alpha^3 A_2A_5 + 12\alpha^5 A_3 - 10\alpha^4 A_3\\
&-12\alpha^5 A_3A_4 + 12\alpha^4 A_3A_4 - 2\alpha^3 A_3A_4 + 12\alpha^5 A_3A_4A_5 - 12\alpha^4 A_3A_4A_5 + 2\alpha^2 A_3A_4A_5\\
&-12\alpha^5 A_3A_5 + 12\alpha^4 A_3A_5 - 2\alpha^3 A_3A_5 + 12\alpha^5 A_4 - 10\alpha^4 A_4 - 12\alpha^5 A_4A_5 + 12\alpha^4 A_4A_5\\
&-2\alpha^3 A_4A_5 + 12\alpha^5 A_5 - 10\alpha^4 A_5)\\
= \frac{1}{2\alpha^4(1-\alpha)}&\Delta_{15}.
\end{aligned}
$$

We now proceed with the partial derivative test on Δ_{15} with $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right]$ and

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{\alpha}{1-\alpha},$$

like in previous cases.

$$\frac{\partial^5 \Delta_{15}}{\partial A_1\, \partial A_2\, \partial A_3\, \partial A_4\, \partial A_5} = 12\alpha^5 - 12\alpha^4 + 2 > 0, \quad \alpha \in \left(\tfrac{3}{4}, \tfrac{4}{5}\right].$$

Thus the partial derivative of Δ_{15} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = \dfrac{\alpha}{1-\alpha}$.

$$\left.\frac{\partial^4 \Delta_{15}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\left(12\alpha^6 - 18\alpha^5 + 6\alpha^4 + \alpha^2\right)}{1-\alpha} > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{15}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\alpha^4\left(24\alpha^3 - 48\alpha^2 + 30\alpha - 5\right)}{(1-\alpha)^2} > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\left.\frac{\partial^2 \Delta_{15}}{\partial A_1 \partial A_2}\right|_{A_3 = A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\alpha^4\left(48\alpha^4 - 120\alpha^3 + 109\alpha^2 - 42\alpha + 6\right)}{(1-\alpha)^3} > 0, \text{ and}$$

$$\left.\frac{\partial\Delta_{15}}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{2\alpha^4\left(96\alpha^5-287\alpha^4+336\alpha^3-192\alpha^2+54\alpha-6\right)}{(1-\alpha)^4} > 0.$$

We also observe that $A_5 = a_5\alpha \geq 5\cdot\frac{3}{4} = \frac{15}{4}$, and since $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5$, our minimum for Δ_{15} is

$$\Delta_{15}|_{A_1=5,A_2=4,A_3=A_4=A_5=\frac{15}{4}}$$
$$= \frac{264600\alpha^5-263368\alpha^4-10460\alpha^3+42075\alpha^2-84375\alpha+67500}{32},$$

which is positive over our desired interval, completing this subcase.

3.4.2. Subcase IV (b). In this subcase, we know that $P_5(x_6=2) > 0$, implying that $(1,1,1,1,1,2)$ is a positive integral solution to the inequality in Theorem 3.4. Thus, if

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5} \leq 1-\frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in \left(\frac{1}{2},\frac{3}{4}\right]$ because $a_6 \in (4,5]$. For simplicity, let $A_i = a_i\cdot\alpha_1$ for $i = 1, 2, 3, 4,$ and 5. This yields the new inequality

$$\frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned}6!\,P_5(x_6=2) \leq 6[&(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)\\ &-(A_5-1)^5+A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)],\end{aligned}$$

and,

$$\begin{aligned}6!\,P_5\left(x_6=1\right) \leq 6\Bigg[&\left(A_1\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_2\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_3\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\\ &\left(A_4\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)-\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)^5\\ &+\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-2\right)\\ &\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-3\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-4\right)\Bigg].\end{aligned}$$

Because $6!\,P_6 = 6!\,(P_5(x_6=1)+P_5(x_6=2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 3.4, substituting in $a_i = \frac{A_i}{\alpha_1}$, yielding

$$\begin{aligned}\Delta_{16} = \frac{1}{16\alpha_1^5(1-\alpha_1)}(&-510A_5^4\alpha_1^6+2700A_5^3\alpha_1^6-4800A_5^2\alpha_1^6+144A_1\alpha_1^6-120A_1A_2\alpha_1^6\\ &+144A_2\alpha_1^6-120A_1A_3\alpha_1^6+108A_1A_2A_3\alpha_1^6-120A_2A_3\alpha_1^6+144A_3\alpha_1^6-120A_1A_4\alpha_1^6\\ &+108A_1A_2A_4\alpha_1^6-120A_2A_4\alpha_1^6+108A_1A_3A_4\alpha_1^6-102A_1A_2A_3A_4\alpha_1^6+108A_2A_3A_4\alpha_1^6\\ &-120A_3A_4\alpha_1^6+144A_4\alpha_1^6-120A_1A_5\alpha_1^6+108A_1A_2A_5\alpha_1^6-120A_2A_5\alpha_1^6\end{aligned}$$

$$
\begin{aligned}
&+108A_1A_3A_5\alpha_1^6 - 102A_1A_2A_3A_5\alpha_1^6 + 108A_2A_3A_5\alpha_1^6 - 120A_3A_5\alpha_1^6 + 108A_1A_4A_5\alpha_1^6 \\
&-102A_1A_2A_4A_5\alpha_1^6 + 108A_2A_4A_5\alpha_1^6 - 102A_1A_3A_4A_5\alpha_1^6 + 99A_1A_2A_3A_4A_5\alpha_1^6 \\
&-102A_2A_3A_4A_5\alpha_1^6 + 108A_3A_4A_5\alpha_1^6 - 120A_4A_5\alpha_1^6 + 2880A_5\alpha_1^6 + 65520\alpha_1^6 + 390A_5^4\alpha_1^5 \\
&-1800A_5^3\alpha_1^5 + 2880A_5^2\alpha_1^5 - 80A_1\alpha_1^5 + 72A_1A_2\alpha_1^5 - 80A_2\alpha_1^5 + 72A_1A_3\alpha_1^5 \\
&-72A_1A_2A_3\alpha_1^5 + 72A_2A_3\alpha_1^5 - 80A_3\alpha_1^5 + 72A_1A_4\alpha_1^5 - 72A_1A_2A_4\alpha_1^5 + 72A_2A_4\alpha_1^5 \\
&-72A_1A_3A_4\alpha_1^5 + 78A_1A_2A_3A_4\alpha_1^5 - 72A_2A_3A_4\alpha_1^5 + 72A_3A_4\alpha_1^5 - 80A_4\alpha_1^5 \\
&+72A_1A_5\alpha_1^5 - 72A_1A_2A_5\alpha_1^5 + 72A_2A_5\alpha_1^5 - 72A_1A_3A_5\alpha_1^5 + 78A_1A_2A_3A_5\alpha_1^5 \\
&-72A_2A_3A_5\alpha_1^5 + 72A_3A_5\alpha_1^5 - 72A_1A_4A_5\alpha_1^5 + 78A_1A_2A_4A_5\alpha_1^5 - 72A_2A_4A_5\alpha_1^5 \\
&+78A_1A_3A_4A_5\alpha_1^5 - 84A_1A_2A_3A_4A_5\alpha_1^5 + 78A_2A_3A_4A_5\alpha_1^5 - 72A_3A_4A_5\alpha_1^5 \\
&+72A_4A_5\alpha_1^5 - 1904A_5\alpha_1^5 - 65552\alpha_1^5 - 60A_5^4\alpha_1^4 + 960A_5^2\alpha_1^4 - 32A_1\alpha_1^4 + 8A_1A_2\alpha_1^4 \\
&-32A_2\alpha_1^4 + 8A_1A_3\alpha_1^4 + 8A_2A_3\alpha_1^4 - 32A_3\alpha_1^4 + 8A_1A_4\alpha_1^4 + 8A_2A_4\alpha_1^4 - 12A_1A_2A_3A_4\alpha_1^4 \\
&+8A_3A_4\alpha_1^4 - 32A_4\alpha_1^4 + 8A_1A_5\alpha_1^4 + 8A_2A_5\alpha_1^4 - 12A_1A_2A_3A_5\alpha_1^4 + 8A_3A_5\alpha_1^4 \\
&-12A_1A_2A_4A_5\alpha_1^4 - 12A_1A_3A_4A_5\alpha_1^4 + 15A_1A_2A_3A_4A_5\alpha_1^4 - 12A_2A_3A_4A_5\alpha_1^4 \\
&+8A_4A_5\alpha_1^4 - 944A_5\alpha_1^4 + 60A_5^4\alpha_1^3 - 600A_5^3\alpha_1^3 + 960A_5^2\alpha_1^3 + 8A_1A_2\alpha_1^3 \\
&+8A_1A_3\alpha_1^3 - 8A_1A_2A_3\alpha_1^3 + 8A_2A_3\alpha_1^3 + 8A_1A_4\alpha_1^3 - 8A_1A_2A_4\alpha_1^3 + 8A_2A_4\alpha_1^3 \\
&-8A_1A_3A_4\alpha_1^3 + 12A_1A_2A_3A_4\alpha_1^3 - 8A_2A_3A_4\alpha_1^3 + 8A_3A_4\alpha_1^3 + 8A_1A_5\alpha_1^3 - 8A_1A_2A_5\alpha_1^3 \\
&+8A_2A_5\alpha_1^3 - 8A_1A_3A_5\alpha_1^3 + 12A_1A_2A_3A_5\alpha_1^3 - 8A_2A_3A_5\alpha_1^3 + 8A_3A_5\alpha_1^3 - 8A_1A_4A_5\alpha_1^3 \\
&+12A_1A_2A_4A_5\alpha_1^3 - 8A_2A_4A_5\alpha_1^3 + 12A_1A_3A_4A_5\alpha_1^3 + 12A_2A_3A_4A_5\alpha_1^3 - 8A_3A_4A_5\alpha_1^3 \\
&+8A_4A_5\alpha_1^3 + 90A_5^4\alpha_1^2 - 300A_5^3\alpha_1^2 + 4A_1A_2A_3\alpha_1^2 + 4A_1A_2A_4\alpha_1^2 + 4A_1A_3A_4\alpha_1^2 \\
&+2A_1A_2A_3A_4\alpha_1^2 + 4A_2A_3A_4\alpha_1^2 + 4A_1A_2A_5\alpha_1^2 + 4A_1A_3A_5\alpha_1^2 + 2A_1A_2A_3A_5\alpha_1^2 \\
&+4A_2A_3A_5\alpha_1^2 + 4A_1A_4A_5\alpha_1^2 + 2A_1A_2A_4A_5\alpha_1^2 + 4A_2A_4A_5\alpha_1^2 + 2A_1A_3A_4A_5\alpha_1^2 \\
&-15A_1A_2A_3A_4A_5\alpha_1^2 + 2A_2A_3A_4A_5\alpha_1^2 + 4A_3A_4A_5\alpha_1^2 + 30A_5^4\alpha_1 - 10A_1A_2A_3A_4\alpha_1 \\
&-10A_1A_2A_3A_5\alpha_1 - 10A_1A_2A_4A_5\alpha_1 - 10A_1A_3A_4A_5\alpha_1 + 4A_1A_2A_3A_4A_5\alpha_1 \\
&-10A_2A_3A_4A_5\alpha_1 + 13A_1A_2A_3A_4A_5) \\
=& \frac{1}{16\alpha_1^5(1-\alpha_1)}\Delta_{17}.
\end{aligned}
$$

We now proceed with the partial derivative test on Δ_{17} with $\alpha_1 \in \left(\frac{1}{2}, \frac{3}{5}\right]$ and

$$
A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{2\alpha_1}{1-\alpha_1},
$$

like in previous cases.

$$
\frac{\partial^5 \Delta_{17}}{\partial A_1\, \partial A_2\, \partial A_3\, \partial A_4\, \partial A_5} = 99\alpha_1^6 - 84\alpha_1^5 + 15\alpha_1^4 - 15\alpha_1^2 + 4\alpha_1 + 13 > 0, \quad \alpha_1 \in \left(\tfrac{1}{2}, \tfrac{3}{5}\right].
$$

Thus the partial derivative of Δ_{17} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = \dfrac{2\alpha_1}{1-\alpha_1}$.

$$
\left.\frac{\partial^4 \Delta_{17}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5 = \frac{2\alpha_1}{1-\alpha_1}} = \frac{4\alpha_1\left(75\alpha_1^6 - 87\alpha_1^5 + 30\alpha_1^4 - 6\alpha_1^3 - 5\alpha_1^2 + 5\alpha_1 + 4\right)}{1-\alpha_1} > 0.
$$

We continue with,

$$
\left.\frac{\partial^3 \Delta_{17}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4 = A_5 = \frac{2\alpha_1}{1-\alpha_1}} = \frac{16\alpha_1^2\left(57\alpha_1^6 - 84\alpha_1^5 + 42\alpha_1^4 - 11\alpha_1^3 + 3\alpha_1 + 1\right)}{(1-\alpha_1)^2} > 0.
$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\begin{aligned}&\frac{\partial^2\Delta_{17}}{\partial A_1\partial A_2}\bigg|_{A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}}\\ &=\frac{16\alpha_1^3\left(174\alpha_1^6-312\alpha_1^5+205\alpha_1^4-68\alpha_1^3+12\alpha_1^2+4\alpha_1+1\right)}{(1-\alpha_1)^3}>0,\ \text{and}\\ &\frac{\partial\Delta_{17}}{\partial A_1}\bigg|_{A_2=A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}}\\ &=\frac{16\alpha_1^4\left(534\alpha_1^6-1133\alpha_1^5+929\alpha_1^4-386\alpha_1^3+88\alpha_1^2-\alpha_1+1\right)}{(1-\alpha_1)^4}>0.\end{aligned}$$

Finally, our minimum for Δ_{17} is

$$\begin{aligned}&\Delta_{17}|_{A_1=5,A_2=4,A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}}\\ &=\frac{16\alpha_1^2}{(1-\alpha_1)^3}\left(-2019\alpha_1^7+16182\alpha_1^6-26485\alpha_1^5+17652\alpha_1^4-4241\alpha_1^3-262\alpha_1^2+73\alpha_1+60\right),\end{aligned}$$

which is positive over our desired interval, completing this subcase.

3.4.3. Subcase IV (c). In this subcase, we know that $P_5(x_6=3)>0$, implying that $(1,1,1,1,1,3)$ is a positive integral solution to the inequality in Theorem 3.4. Thus, if

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5}\le 1-\frac{3}{a_6}:=\alpha_2,$$

then $\alpha_2\in\left(\frac{1}{4},\frac{2}{5}\right]$ because $a_6\in(4,5]$. For simplicity, let $A_i=a_i\cdot\alpha_2$ for $i=1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5}\le 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned}6!\,P_5(x_6=3)\le 6[&(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)\\ &-(A_5-1)^5+A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)],\end{aligned}$$

as well as

$$\begin{aligned}6!\,P_5\left(x_6=2\right)\le 6\Bigg[&\left(A_1\cdot\frac{1+2\alpha_2}{3\alpha_2}-1\right)\left(A_2\cdot\frac{1+2\alpha_2}{3\alpha_2}-1\right)\left(A_3\cdot\frac{1+2\alpha_2}{3\alpha_2}-1\right)\\ &\left(A_4\cdot\frac{1+2\alpha_2}{3\alpha_2}-1\right)\left(A_5\cdot\frac{1+2\alpha_2}{3\alpha_2}-1\right)-\left(A_5\cdot\frac{1+2\alpha_2}{3\alpha_2}-1\right)^5\\ &+\left(A_5\cdot\frac{1+2\alpha_2}{3\alpha_2}\right)\left(A_5\cdot\frac{1+2\alpha_2}{3\alpha_2}-1\right)\left(A_5\cdot\frac{1+2\alpha_2}{3\alpha_2}-2\right)\\ &\left(A_5\cdot\frac{1+2\alpha_2}{3\alpha_2}-3\right)\left(A_5\cdot\frac{1+2\alpha_2}{3\alpha_2}-4\right)\Bigg],\end{aligned}$$

and

$$6!\,P_5\left(x_6=1\right)\le 6\Bigg[\left(A_1\cdot\frac{2+\alpha_2}{3\alpha_2}-1\right)\left(A_2\cdot\frac{2+\alpha_2}{3\alpha_2}-1\right)\left(A_3\cdot\frac{2+\alpha_2}{3\alpha_2}-1\right)$$

$$\left(A_4\cdot\frac{2+\alpha_2}{3\alpha_2}-1\right)\left(A_5\cdot\frac{2+\alpha_2}{3\alpha_2}-1\right)-\left(A_5\cdot\frac{2+\alpha_2}{3\alpha_2}-1\right)^5$$
$$+\left(A_5\cdot\frac{2+\alpha_2}{3\alpha_2}\right)\left(A_5\cdot\frac{2+\alpha_2}{3\alpha_2}-1\right)\left(A_5\cdot\frac{2+\alpha_2}{3\alpha_2}-2\right)$$
$$\left(A_5\cdot\frac{2+\alpha_2}{3\alpha_2}-3\right)\left(A_5\cdot\frac{2+\alpha_2}{3\alpha_2}-4\right)\Bigg].$$

Because $6!\,P_6 = 6!\,(P_5(x_6=1)+P_5(x_6=2)+P_5(x_6=3))$, if we let Δ_{11} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 3.4 and substituting in $a_i = \dfrac{A_i}{\alpha_2}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned}
\Delta_{18} = {} & \frac{1}{27\alpha_2^5(1-\alpha_2)}(-980A_5^4\alpha_2^6+5400A_5^3\alpha_2^6-10080A_5^2\alpha_2^6+324A_1\alpha_2^6-252A_1A_2\alpha_2^6\\
&+324A_2\alpha_2^6-252A_1A_3\alpha_2^6+216A_1A_2A_3\alpha_2^6-252A_2A_3\alpha_2^6+324A_3\alpha_2^6-252A_1A_4\alpha_2^6\\
&+216A_1A_2A_4\alpha_2^6-252A_2A_4\alpha_2^6+216A_1A_3A_4\alpha_2^6-196A_1A_2A_3A_4\alpha_2^6+216A_2A_3A_4\alpha_2^6\\
&-252A_3A_4\alpha_2^6+324A_4\alpha_2^6-252A_1A_5\alpha_2^6+216A_1A_2A_5\alpha_2^6-252A_2A_5\alpha_2^6\\
&+216A_1A_3A_5\alpha_2^6-196A_1A_2A_3A_5\alpha_2^6+216A_2A_3A_5\alpha_2^6-252A_3A_5\alpha_2^6+216A_1A_4A_5\alpha_2^6\\
&-196A_1A_2A_4A_5\alpha_2^6+216A_2A_4A_5\alpha_2^6-196A_1A_3A_4A_5\alpha_2^6+184A_1A_2A_3A_4A_5\alpha_2^6\\
&-196A_2A_3A_4A_5\alpha_2^6+216A_3A_4A_5\alpha_2^6-252A_4A_5\alpha_2^6+6480A_5\alpha_2^6+110565\alpha_2^6\\
&+580A_5^4\alpha_2^5-2700A_5^3\alpha_2^5+4320A_5^2\alpha_2^5-135A_1\alpha_2^5+108A_1A_2\alpha_2^5-135A_2\alpha_2^5\\
&+108A_1A_3\alpha_2^5-108A_1A_2A_3\alpha_2^5+108A_2A_3\alpha_2^5-135A_3\alpha_2^5+108A_1A_4\alpha_2^5\\
&-108A_1A_2A_4\alpha_2^5+108A_2A_4\alpha_2^5-108A_1A_3A_4\alpha_2^5+116A_1A_2A_3A_4\alpha_2^5-108A_2A_3A_4\alpha_2^5\\
&+108A_3A_4\alpha_2^5-135A_4\alpha_2^5+108A_1A_5\alpha_2^5-108A_1A_2A_5\alpha_2^5+108A_2A_5\alpha_2^5\\
&-108A_1A_3A_5\alpha_2^5+116A_1A_2A_3A_5\alpha_2^5-108A_2A_3A_5\alpha_2^5+108A_3A_5\alpha_2^5-108A_1A_4A_5\alpha_2^5\\
&+116A_1A_2A_4A_5\alpha_2^5-108A_2A_4A_5\alpha_2^5+116A_1A_3A_4A_5\alpha_2^5-124A_1A_2A_3A_4A_5\alpha_2^5\\
&+116A_2A_3A_4A_5\alpha_2^5-108A_3A_4A_5\alpha_2^5+108A_4A_5\alpha_2^5-3213A_5\alpha_2^5-110646\alpha_2^5-80A_5^4\alpha_2^4\\
&+2160A_5^2\alpha_2^4-108A_1\alpha_2^4+27A_1A_2\alpha_2^4-108A_2\alpha_2^4+27A_1A_3\alpha_2^4+27A_2A_3\alpha_2^4\\
&-108A_3\alpha_2^4+27A_1A_4\alpha_2^4+27A_2A_4\alpha_2^4-16A_1A_2A_3A_4\alpha_2^4+27A_3A_4\alpha_2^4-108A_4\alpha_2^4\\
&+27A_1A_5\alpha_2^4+27A_2A_5\alpha_2^4-16A_1A_2A_3A_5\alpha_2^4+27A_3A_5\alpha_2^4-16A_1A_2A_4A_5\alpha_2^4\\
&-16A_1A_3A_4A_5\alpha_2^4+20A_1A_2A_3A_4A_5\alpha_2^4-16A_2A_3A_4A_5\alpha_2^4+27A_4A_5\alpha_2^4-3186A_5\alpha_2^4\\
&+80A_5^4\alpha_2^3-1350A_5^3\alpha_2^3+3600A_5^2\alpha_2^3+36A_1A_2\alpha_2^3+36A_1A_3\alpha_2^3-27A_1A_2A_3\alpha_2^3\\
&+36A_2A_3\alpha_2^3+36A_1A_4\alpha_2^3-27A_1A_2A_4\alpha_2^3+36A_2A_4\alpha_2^3-27A_1A_3A_4\alpha_2^3\\
&+16A_1A_2A_3A_4\alpha_2^3-27A_2A_3A_4\alpha_2^3+36A_3A_4\alpha_2^3+36A_1A_5\alpha_2^3-27A_1A_2A_5\alpha_2^3\\
&+36A_2A_5\alpha_2^3-27A_1A_3A_5\alpha_2^3+16A_1A_2A_3A_5\alpha_2^3-27A_2A_3A_5\alpha_2^3+36A_3A_5\alpha_2^3\\
&-27A_1A_4A_5\alpha_2^3+16A_1A_2A_4A_5\alpha_2^3-27A_2A_4A_5\alpha_2^3+16A_1A_3A_4A_5\alpha_2^3\\
&+16A_2A_3A_4A_5\alpha_2^3-27A_3A_4A_5\alpha_2^3+36A_4A_5\alpha_2^3+230A_5^4\alpha_2^2-1350A_5^3\alpha_2^2\\
&+19A_1A_2A_3A_4\alpha_2^2+19A_1A_2A_3A_5\alpha_2^2+19A_1A_2A_4A_5\alpha_2^2+19A_1A_3A_4A_5\alpha_2^2\\
&-20A_1A_2A_3A_4A_5\alpha_2^2+19A_2A_3A_4A_5\alpha_2^2+170A_5^4\alpha_2-20A_1A_2A_3A_4\alpha_2\\
&-20A_1A_2A_3A_5\alpha_2-20A_1A_2A_4A_5\alpha_2-20A_1A_3A_4A_5\alpha_2-11A_1A_2A_3A_4A_5\alpha_2\\
&-20A_2A_3A_4A_5\alpha_2+32A_1A_2A_3A_4A_5)\\
= {} & \frac{1}{27\alpha_2^5(1-\alpha_2)}\Delta_{19}.
\end{aligned}$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_2 \in \left(\frac{1}{4}, \frac{2}{5}\right]$. We begin with

$$\frac{\partial^5 \Delta_{19}}{\partial A_1\, \partial A_2\, \partial A_3\, \partial A_4\, \partial A_5} = 184\alpha_2^6 - 124\alpha_2^5 + 20\alpha_2^4 - 20\alpha_2^2 - 11\alpha_2 + 32 > 0, \quad \alpha_2 \in \left(\frac{1}{4}, \frac{2}{5}\right].$$

Thus the partial derivative of Δ_{19} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = 1$.

$$\left.\frac{\partial^4 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=1} = -12\alpha_2^6 - 8\alpha_2^5 + 4\alpha_2^4 + 16\alpha_2^3 - \alpha_2^2 - 31\alpha_2 + 32 > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=2, A_5=1} = -4\alpha_2^6 - 8\alpha_2^5 - 8\alpha_2^4 + 21\alpha_2^3 + 17\alpha_2^2 - 82\alpha_2 + 64 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\left.\frac{\partial^2 \Delta_{19}}{\partial A_1 \partial A_2}\right|_{A_3=3, A_4=2, A_5=1} = -8\alpha_2^6 - 8\alpha_2^5 - 29\alpha_2^4 + 50\alpha_2^3 + 89\alpha_2^2 - 286\alpha_2 + 192 > 0, \text{ and}$$

$$\left.\frac{\partial \Delta_{19}}{\partial A_1}\right|_{A_2=4, A_3=3, A_4=2, A_5=1} = -20\alpha_2^6 - 11\alpha_2^5 - 158\alpha_2^4 + 215\alpha_2^3 + 470\alpha_2^2 - 1264\alpha_2 + 768 > 0.$$

Finally, our minimum for Δ_{19} is

$$\begin{aligned} &\Delta_{19}|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} \\ &= 3\left(37159\alpha_2^6 - 37255\alpha_2^5 - 769\alpha_2^4 + 1233\alpha_2^3 + 562\alpha_2^2 - 2210\alpha_2 + 1280\right), \end{aligned}$$

which is positive over our desired interval, completing this subcase.

3.4.4. Subcase IV (d). In this subcase, we know that $P_5(x_6 = 4) > 0$, implying that $(1, 1, 1, 1, 1, 4)$ is a positive integral solution to the inequality in Theorem 3.4. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{4}{a_6} := \alpha_3,$$

then $\alpha_3 \in \left(0, \frac{1}{5}\right]$ because $a_6 \in (4, 5]$. For simplicity, let $A_i = a_i \cdot \alpha_3$ for $i = 1, 2, 3, 4$, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6!\, P_5(x_6 = 4) \leq 6[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)$$

$$-(A_5-1)^5 + A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)],$$

as well as

$$\begin{aligned}6!\,P_5\,(x_6=3) \le 6\Bigg[&\left(A_1\cdot\frac{1+3\alpha_3}{4\alpha_3}-1\right)\left(A_2\cdot\frac{1+3\alpha_3}{4\alpha_3}-1\right)\left(A_3\cdot\frac{1+3\alpha_3}{4\alpha_3}-1\right)\\ &\left(A_4\cdot\frac{1+3\alpha_3}{4\alpha_3}-1\right)\left(A_5\cdot\frac{1+3\alpha_3}{4\alpha_3}-1\right)-\left(A_5\cdot\frac{1+3\alpha_3}{4\alpha_3}-1\right)^5\\ &+\left(A_5\cdot\frac{1+3\alpha_3}{4\alpha_3}\right)\left(A_5\cdot\frac{1+3\alpha_3}{4\alpha_3}-1\right)\left(A_5\cdot\frac{1+3\alpha_3}{4\alpha_3}-2\right)\\ &\left(A_5\cdot\frac{1+3\alpha_3}{4\alpha_3}-3\right)\left(A_5\cdot\frac{1+3\alpha_3}{4\alpha_3}-4\right)\Bigg],\end{aligned}$$

and

$$\begin{aligned}6!\,P_5\,(x_6=2) \le 6\Bigg[&\left(A_1\cdot\frac{2+2\alpha_3}{4\alpha_3}-1\right)\left(A_2\cdot\frac{2+2\alpha_3}{4\alpha_3}-1\right)\left(A_3\cdot\frac{2+2\alpha_3}{4\alpha_3}-1\right)\\ &\left(A_4\cdot\frac{2+2\alpha_3}{4\alpha_3}-1\right)\left(A_5\cdot\frac{2+2\alpha_3}{4\alpha_3}-1\right)-\left(A_5\cdot\frac{2+2\alpha_3}{4\alpha_3}-1\right)^5\\ &+\left(A_5\cdot\frac{2+2\alpha_3}{4\alpha_3}\right)\left(A_5\cdot\frac{2+2\alpha_3}{4\alpha_3}-1\right)\left(A_5\cdot\frac{2+2\alpha_3}{4\alpha_3}-2\right)\\ &\left(A_5\cdot\frac{2+2\alpha_3}{4\alpha_3}-3\right)\left(A_5\cdot\frac{2+2\alpha_3}{4\alpha_3}-4\right)\Bigg],\end{aligned}$$

and

$$\begin{aligned}6!\,P_5\,(x_6=1) \le 6\Bigg[&\left(A_1\cdot\frac{3+\alpha_3}{4\alpha_3}-1\right)\left(A_2\cdot\frac{3+\alpha_3}{4\alpha_3}-1\right)\left(A_3\cdot\frac{3+\alpha_3}{4\alpha_3}-1\right)\\ &\left(A_4\cdot\frac{3+\alpha_3}{4\alpha_3}-1\right)\left(A_5\cdot\frac{3+\alpha_3}{4\alpha_3}-1\right)-\left(A_5\cdot\frac{3+\alpha_3}{4\alpha_3}-1\right)^5\\ &+\left(A_5\cdot\frac{3+\alpha_3}{4\alpha_3}\right)\left(A_5\cdot\frac{3+\alpha_3}{4\alpha_3}-1\right)\left(A_5\cdot\frac{3+\alpha_3}{4\alpha_3}-2\right)\\ &\left(A_5\cdot\frac{3+\alpha_3}{4\alpha_3}-3\right)\left(A_5\cdot\frac{3+\alpha_3}{4\alpha_3}-4\right)\Bigg].\end{aligned}$$

Because $6!\,P_6 = 6!\,(P_5(x_6=1)+P_5(x_6=2)+P_5(x_6=3)+P_5(x_6=4))$, if we let Δ_{20} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 3.4 and substituting in $a_i = \dfrac{A_i}{\alpha_3}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned}\Delta_{20} = \frac{1}{128\alpha_3^5(1-\alpha_3)}&(-5310A_5^4\alpha_3^6+30000A_5^3\alpha_3^6-57600A_5^2\alpha_3^6+1920A_1\alpha_3^6 1440A_1A_2\alpha_3^6\\ &+1920A_2\alpha_3^6-1440A_1A_3\alpha_3^6+1200A_1A_2A_3\alpha_3^6-1440A_2A_3\alpha_3^6+1920A_3\alpha_3^6\\ &-1440A_1A_4\alpha_3^6+1200A_1A_2A_4\alpha_3^6-1440A_2A_4\alpha_3^6+1200A_1A_3A_4\alpha_3^6-1062A_1A_2A_3A_4\alpha_3^6\\ &+1200A_2A_3A_4\alpha_3^6-1440A_3A_4\alpha_3^6+1920A_4\alpha_3^6-1440A_1A_5\alpha_3^6+1200A_1A_2A_5\alpha_3^6\\ &-1440A_2A_5\alpha_3^6+1200A_1A_3A_5\alpha_3^6-1062A_1A_2A_3A_5\alpha_3^6+1200A_2A_3A_5\alpha_3^6-1440A_3A_5\alpha_3^6\\ &+1200A_1A_4A_5\alpha_3^6-1062A_1A_2A_4A_5\alpha_3^6+1200A_2A_4A_5\alpha_3^6-1062A_1A_3A_4A_5\alpha_3^6\end{aligned}$$

$$
\begin{aligned}
&+975A_1A_2A_3A_4A_5\alpha_3^6 - 1062A_2A_3A_4A_5\alpha_3^6 + 1200A_3A_4A_5\alpha_3^6 - 1440A_4A_5\alpha_3^6\\
&+38400A_5\alpha_3^6 + 524160\alpha_3^6 + 2550A_5^4\alpha_3^5 - 12000A_5^3\alpha_3^5 + 19200A_5^2\alpha_3^5 - 640A_1\alpha_3^5\\
&+480A_1A_2\alpha_3^5 - 640A_2\alpha_3^5 + 480A_1A_3\alpha_3^5 - 480A_1A_2A_3\alpha_3^5 + 480A_2A_3\alpha_3^5 - 640A_3\alpha_3^5\\
&+480A_1A_4\alpha_3^5 - 480A_1A_2A_4\alpha_3^5 + 480A_2A_4\alpha_3^5 - 480A_1A_3A_4\alpha_3^5 + 510A_1A_2A_3A_4\alpha_3^5\\
&-480A_2A_3A_4\alpha_3^5 + 480A_3A_4\alpha_3^5 - 640A_4\alpha_3^5 + 480A_1A_5\alpha_3^5 - 480A_1A_2A_5\alpha_3^5\\
&+480A_2A_5\alpha_3^5 - 480A_1A_3A_5\alpha_3^5 + 510A_1A_2A_3A_5\alpha_3^5 - 480A_2A_3A_5\alpha_3^5 + 480A_3A_5\alpha_3^5\\
&-480A_1A_4A_5\alpha_3^5 + 510A_1A_2A_4A_5\alpha_3^5 - 480A_2A_4A_5\alpha_3^5 + 510A_1A_3A_4A_5\alpha_3^5\\
&-540A_1A_2A_3A_4A_5\alpha_3^5 + 510A_2A_3A_4A_5\alpha_3^5 - 480A_3A_4A_5\alpha_3^5 + 480A_4A_5\alpha_3^5 - 15232A_5\alpha_3^5\\
&-524672\alpha_3^5 - 300A_5^4\alpha_3^4 + 11520A_5^2\alpha_3^4 - 768A_1\alpha_3^4 + 160A_1A_2\alpha_3^4 - 768A_2\alpha_3^4\\
&+160A_1A_3\alpha_3^4 + 160A_2A_3\alpha_3^4 - 768A_3\alpha_3^4 + 160A_1A_4\alpha_3^4 + 160A_2A_4\alpha_3^4\\
&-60A_1A_2A_3A_4\alpha_3^4 + 160A_3A_4\alpha_3^4 - 768A_4\alpha_3^4 + 160A_1A_5\alpha_3^4 + 160A_2A_5\alpha_3^4\\
&-60A_1A_2A_3A_5\alpha_3^4 + 160A_3A_5\alpha_3^4 - 60A_1A_2A_4A_5\alpha_3^4 - 60A_1A_3A_4A_5\alpha_3^4\\
&+75A_1A_2A_3A_4A_5\alpha_3^4 - 60A_2A_3A_4A_5\alpha_3^4 + 160A_4A_5\alpha_3^4 - 22656A_5\alpha_3^4 + 300A_5^4\alpha_3^3\\
&-7200A_5^3\alpha_3^3 + 26880A_5^2\alpha_3^3 + 288A_1A_2\alpha_3^3 + 288A_1A_3\alpha_3^3 - 160A_1A_2A_3\alpha_3^3\\
&+288A_2A_3\alpha_3^3 + 288A_1A_4\alpha_3^3 - 160A_1A_2A_4\alpha_3^3 + 288A_2A_4\alpha_3^3 - 160A_1A_3A_4\alpha_3^3\\
&+60A_1A_2A_3A_4\alpha_3^3 - 160A_2A_3A_4\alpha_3^3 + 288A_3A_4\alpha_3^3 + 288A_1A_5\alpha_3^3 - 160A_1A_2A_5\alpha_3^3\\
&+288A_2A_5\alpha_3^3 - 160A_1A_3A_5\alpha_3^3 + 60A_1A_2A_3A_5\alpha_3^3 - 160A_2A_3A_5\alpha_3^3 + 288A_3A_5\alpha_3^3\\
&-160A_1A_4A_5\alpha_3^3 + 60A_1A_2A_4A_5\alpha_3^3 - 160A_2A_4A_5\alpha_3^3 + 60A_1A_3A_4A_5\alpha_3^3\\
&+60A_2A_3A_4A_5\alpha_3^3 - 160A_3A_4A_5\alpha_3^3 + 288A_4A_5\alpha_3^3 + 1290A_5^4\alpha_3^2 - 10800A_5^3\alpha_3^2\\
&-48A_1A_2A_3\alpha_3^2 - 48A_1A_2A_4\alpha_3^2 - 48A_1A_3A_4\alpha_3^2 + 130A_1A_2A_3A_4\alpha_3^2 - 48A_2A_3A_4\alpha_3^2\\
&-48A_1A_2A_5\alpha_3^2 - 48A_1A_3A_5\alpha_3^2 + 130A_1A_2A_3A_5\alpha_3^2 - 48A_2A_3A_5\alpha_3^2 - 48A_1A_4A_5\alpha_3^2\\
&+130A_1A_2A_4A_5\alpha_3^2 - 48A_2A_4A_5\alpha_3^2 + 130A_1A_3A_4A_5\alpha_3^2 - 75A_1A_2A_3A_4A_5\alpha_3^2\\
&+130A_2A_3A_4A_5\alpha_3^2 - 48A_3A_4A_5\alpha_3^2 + 1470A_5^4\alpha_3 - 90A_1A_2A_3A_4\alpha_3 - 90A_1A_2A_3A_5\alpha_3\\
&-90A_1A_2A_4A_5\alpha_3 - 90A_1A_3A_4A_5\alpha_3 - 100A_1A_2A_3A_4A_5\alpha_3 - 90A_2A_3A_4A_5\alpha_3\\
&+177A_1A_2A_3A_4A_5)\\
=\ & \frac{1}{128\alpha_3^5(1-\alpha_3)}\Delta_{21}.
\end{aligned}
$$

Once again, our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_3 \in \left(0, \frac{1}{5}\right]$. We begin with

$$\frac{\partial^5 \Delta_{21}}{\partial A_1\,\partial A_2\,\partial A_3\,\partial A_4\,\partial A_5} = 975\alpha_3^6 - 540\alpha_3^5 + 75\alpha_3^4 - 75\alpha_3^2 - 100\alpha_3 + 177 > 0, \ \ \alpha_3 \in \left(0, \frac{1}{5}\right].$$

Thus the partial derivative of Δ_{21} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = 1$.

$$\left.\frac{\partial^4 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=1} = -87\alpha_3^6 - 30\alpha_3^5 + 15\alpha_3^4 + 60\alpha_3^3 + 55\alpha_3^2 - 190\alpha_3 + 177 > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=2, A_5=1} = -36\alpha_3^6 - 30\alpha_3^5 - 30\alpha_3^4 + 20\alpha_3^3 + 192\alpha_3^2 - 470\alpha_3 + 354 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\begin{aligned}&\left.\frac{\partial^2\Delta_{21}}{\partial A_1\partial A_2}\right|_{A_3=3,A_4=2,A_5=1}\\ &= -2\left(36\alpha_3^6+15\alpha_3^5+25\alpha_3^4+6\alpha_3^3-346\alpha_3^2+795\alpha_3-531\right)>0, \text{ and}\\ &\left.\frac{\partial\Delta_{21}}{\partial A_1}\right|_{A_2=4,A_3=3,A_4=2,A_5=1}\\ &= -4\left(45\alpha_3^6+25\alpha_3^5+92\alpha_3^4-70\alpha_3^3-755\alpha_3^2+1725\alpha_3-1062\right)>0.\end{aligned}$$

Finally, our minimum for Δ_{21} is

$$\begin{aligned}&\Delta_{21}|_{A_1=5,A_2=4,A_3=3,A_4=2,A_5=1}\\ &= 530142\alpha_3^6-531374\alpha_3^5-16028\alpha_3^4+24900\alpha_3^3+6310\alpha_3^2-35190\alpha_3+21240,\end{aligned}$$

which is positive over our desired interval, completing this subcase, and thus completing Case IV. □

3.5. Case V. In this case, $\lceil a_6\rceil = 6$, so we can plug $v = 6$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

THEOREM 3.5. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3}+\frac{x_4}{a_4}+\frac{x_5}{a_5}+\frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $5 < a_6 \leq 6$, then*

$$6!\,P_6 \leq (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1) - 14905.$$

Proof. In this case, $a_6 \in (5, 6]$, so we have to consider five levels — $x_6 = 1$, $x_6 = 2$, $x_6 = 3$, $x_6 = 4$, and $x_6 = 5$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our five subcases are:

Subcase V (a): $P_5(x_6 = 5) = P_5(x_6 = 4) = P_5(x_6 = 3) = P_5(x_6 = 2) = 0$.
Subcase V (b): $P_5(x_6 = 5) = P_5(x_6 = 4) = P_5(x_6 = 3) = 0, P_5(x_6 = 2) > 0$.
Subcase V (c): $P_5(x_6 = 5) = P_5(x_6 = 4) = 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.
Subcase V (d): $P_5(x_6 = 5) = 0, P_5(x_6 = 4) > 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.
Subcase V (e): $P_5(x_6 = 5) > 0, P_5(x_6 = 4) > 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.

3.5.1. Subcase V (a). We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.5. Thus, if

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5} \leq 1-\frac{1}{a_6} := \alpha,$$

then $\alpha \in \left(\frac{4}{5}, \frac{5}{6}\right]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5} \leq 1.$$

Just like in Subcase IV (a), we will now use the Yau Geometric Conjecture for $n = 5$, proven in [1]. This gives us the bound

$$6!\,P_5(x_6 = 1) \leq 6\left[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1) - p_5(\lceil A_5\rceil)\right], \tag{5}$$

where p_5 is the function defined in (1). Note that since $p_5(v)$ is increasing for $v \geq 4$. Also, since $A_5 = a_5 \cdot \alpha > \frac{4}{5} \cdot 5 = 4$, we note that $\lceil A_5 \rceil \geq 5$. Thus, we maximize the RHS of (5) by substituting $p_5(5) = 904$ in for $p_5(\lceil A_5 \rceil)$. Hence, we have

$$6!\, P_5(x_6 = 1) \leq 6\left[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - 904\right]. \quad (6)$$

As before, we take the difference obtained by subtracting the RHS of (6) from the RHS of Theorem 3.4, substituting in $a_i = \dfrac{A_i}{\alpha}$ and $a_6 = \dfrac{1}{1-\alpha}$, yielding

$$\begin{aligned}
\Delta_{22} = \frac{1}{2\alpha^4(1-\alpha)}&(18948\alpha^5 - 18950\alpha^4 + 12\alpha^5 A_1 - 10\alpha^4 A_1 - 12\alpha^5 A_1A_2 + 12\alpha^4 A_1A_2 \\
&-2\alpha^3 A_1A_2 + 12\alpha^5 A_1A_2A_3 - 12\alpha^4 A_1A_2A_3 + 2\alpha^2 A_1A_2A_3 - 12\alpha^5 A_1A_2A_3A_4 \\
&+12\alpha^4 A_1A_2A_3A_4 - 2\alpha A_1A_2A_3A_4 + 12\alpha^5 A_1A_2A_3A_4A_5 - 12\alpha^4 A_1A_2A_3A_4A_5 \\
&+2A_1A_2A_3A_4A_5 - 12\alpha^5 A_1A_2A_3A_5 + 12\alpha^4 A_1A_2A_3A_5 - 2\alpha A_1A_2A_3A_5 + 12\alpha^5 A_1A_2A_4 \\
&-12\alpha^4 A_1A_2A_4 + 2\alpha^2 A_1A_2A_4 - 12\alpha^5 A_1A_2A_4A_5 + 12\alpha^4 A_1A_2A_4A_5 - 2\alpha A_1A_2A_4A_5 \\
&+12\alpha^5 A_1A_2A_5 - 12\alpha^4 A_1A_2A_5 + 2\alpha^2 A_1A_2A_5 - 12\alpha^5 A_1A_3 + 12\alpha^4 A_1A_3 - 2\alpha^3 A_1A_3 \\
&+12\alpha^5 A_1A_3A_4 - 12\alpha^4 A_1A_3A_4 + 2\alpha^2 A_1A_3A_4 - 12\alpha^5 A_1A_3A_4A_5 + 12\alpha^4 A_1A_3A_4A_5 \\
&-2\alpha A_1A_3A_4A_5 + 12\alpha^5 A_1A_3A_5 - 12\alpha^4 A_1A_3A_5 + 2\alpha^2 A_1A_3A_5 - 12\alpha^5 A_1A_4 \\
&+12\alpha^4 A_1A_4 - 2\alpha^3 A_1A_4 + 12\alpha^5 A_1A_4A_5 - 12\alpha^4 A_1A_4A_5 + 2\alpha^2 A_1A_4A_5 - 12\alpha^5 A_1A_5 \\
&+12\alpha^4 A_1A_5 - 2\alpha^3 A_1A_5 + 12\alpha^5 A_2 - 10\alpha^4 A_2 - 12\alpha^5 A_2A_3 + 12\alpha^4 A_2A_3 - 2\alpha^3 A_2A_3 \\
&+12\alpha^5 A_2A_3A_4 - 12\alpha^4 A_2A_3A_4 + 2\alpha^2 A_2A_3A_4 - 12\alpha^5 A_2A_3A_4A_5 + 12\alpha^4 A_2A_3A_4A_5 \\
&-2\alpha A_2A_3A_4A_5 + 12\alpha^5 A_2A_3A_5 - 12\alpha^4 A_2A_3A_5 + 2\alpha^2 A_2A_3A_5 - 12\alpha^5 A_2A_4 \\
&+12\alpha^4 A_2A_4 - 2\alpha^3 A_2A_4 + 12\alpha^5 A_2A_4A_5 - 12\alpha^4 A_2A_4A_5 + 2\alpha^2 A_2A_4A_5 - 12\alpha^5 A_2A_5 \\
&+12\alpha^4 A_2A_5 - 2\alpha^3 A_2A_5 + 12\alpha^5 A_3 - 10\alpha^4 A_3 - 12\alpha^5 A_3A_4 + 12\alpha^4 A_3A_4 - 2\alpha^3 A_3A_4 \\
&+12\alpha^5 A_3A_4A_5 - 12\alpha^4 A_3A_4A_5 + 2\alpha^2 A_3A_4A_5 - 12\alpha^5 A_3A_5 + 12\alpha^4 A_3A_5 - 2\alpha^3 A_3A_5 \\
&+12\alpha^5 A_4 - 10\alpha^4 A_4 - 12\alpha^5 A_4A_5 + 12\alpha^4 A_4A_5 - 2\alpha^3 A_4A_5 + 12\alpha^5 A_5 - 10\alpha^4 A_5) \\
= \frac{1}{2\alpha^4(1-\alpha)}&\Delta_{23}.
\end{aligned}$$

We now proceed with the partial derivative test on Δ_{23} with $\alpha \in \left(\frac{4}{5}, \frac{5}{6}\right]$ and

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{\alpha}{1-\alpha},$$

like in previous cases.

$$\frac{\partial^5 \Delta_{23}}{\partial A_1\, \partial A_2\, \partial A_3\, \partial A_4\, \partial A_5} = 12\alpha^5 - 12\alpha^4 + 2 > 0, \quad \alpha \in \left(\tfrac{4}{5}, \tfrac{5}{6}\right].$$

Thus the partial derivative of Δ_{23} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = \dfrac{\alpha}{1-\alpha}$.

$$\left.\frac{\partial^4 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\left(12\alpha^6 - 18\alpha^5 + 6\alpha^4 + \alpha^2\right)}{1-\alpha} > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\alpha^4\left(24\alpha^3 - 48\alpha^2 + 30\alpha - 5\right)}{(1-\alpha)^2} > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\left.\frac{\partial^2 \Delta_{23}}{\partial A_1 \partial A_2}\right|_{A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{2\alpha^4\left(48\alpha^4-120\alpha^3+109\alpha^2-42\alpha+6\right)}{(1-\alpha)^3} > 0, \text{ and}$$

$$\left.\frac{\partial \Delta_{23}}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{2\alpha^4\left(96\alpha^5-287\alpha^4+336\alpha^3-192\alpha^2+54\alpha-6\right)}{(1-\alpha)^4} > 0.$$

We must now only test Δ_{23} at its minimum. We observe that if $A_5 >= 5$, then we have

$$\Delta_{23}|_{A_1=5,A_2=A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = 2(5-6\alpha)^2\left(434\alpha^3+290\alpha^2+175\alpha+125\right),$$

which is positive over our desired interval. We must now only consider the minimum of Δ_{23} when we have $4 \leq \frac{\alpha}{1-\alpha} \leq A_5 < 5$. We observe that since the partial differentiation test told us that all the partial derivatives of Δ_{23} are positive, indicating that to minimize Δ_{23}, we set $A_2 = A_3 = A_4 = A_5 = x$ for some $x \in (4, 5]$. Thus, we have $\dfrac{1}{A_1} \leq 1 - \dfrac{4}{x}$, or that

$$A_1 \geq \frac{x}{4-x}.$$

Hence, we consider

$$\begin{aligned}\Delta_{24} &= \Delta_{23}|_{A_1=\frac{x}{4-x},A_2=A_3=A_4=A_5=x}\\ &= \frac{1}{4-x}(-75792\alpha^5+75800\alpha^4-2\alpha x^5+2x^5+48\alpha^5x^4-48\alpha^4x^4+8\alpha^2x^4\\ &\quad -192\alpha^5x^3+192\alpha^4x^3-12\alpha^3x^3-20\alpha^2x^3+288\alpha^5x^2-280\alpha^4x^2+40\alpha^3x^2\\ &\quad +18768\alpha^5x-18800\alpha^4x).\end{aligned}$$

It can easily be numerically verified that $\Delta_{24} \geq 0$ over the interval $x \in (4, 5), \alpha \in \left(\dfrac{4}{5}, \dfrac{5}{6}\right)$, completing this subcase.

3.5.2. Subcase V (b). In this subcase, we know that $P_5(x_6 = 2) > 0$, implying that $(1, 1, 1, 1, 1, 2)$ is a positive integral solution to the inequality in Theorem 3.5. Thus, if

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5} \leq 1-\frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in \left(\frac{3}{5}, \frac{2}{3}\right]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_1$ for $i = 1, 2, 3, 4$, and 5. This yields the new inequality

$$\frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6!\, P_5(x_6 = 2) \leq 6[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)$$

$$-(A_5-1)^5+A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)],$$

and,

$$\begin{aligned}6!\,P_5\,(x_6=1)\le 6\Bigg[&\left(A_1\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_2\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_3\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\\&\left(A_4\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)-\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)^5\\&+\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-1\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-2\right)\\&\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-3\right)\left(A_5\cdot\frac{1+\alpha_1}{2\alpha_1}-4\right)\Bigg].\end{aligned}$$

Because $6!\,P_6=6!\,(P_5(x_6=1)+P_5(x_6=2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 3.5, substituting in $a_i=\dfrac{A_i}{\alpha_1}$, yielding

$$\begin{aligned}\Delta_{25}=\frac{1}{16\alpha_1^5(1-\alpha_1)}&(-510A_5^4a_1^6+2700A_5^3a_1^6-4800A_5^2a_1^6+144A_1a_1^6-120A_1A_2a_1^6\\
&+144A_2a_1^6-120A_1A_3a_1^6+108A_1A_2A_3a_1^6-120A_2A_3a_1^6+144A_3a_1^6-120A_1A_4a_1^6\\
&+108A_1A_2A_4a_1^6-120A_2A_4a_1^6+108A_1A_3A_4a_1^6-102A_1A_2A_3A_4a_1^6+108A_2A_3A_4a_1^6\\
&-120A_3A_4a_1^6+144A_4a_1^6-120A_1A_5a_1^6+108A_1A_2A_5a_1^6-120A_2A_5a_1^6+108A_1A_3A_5a_1^6\\
&-102A_1A_2A_3A_5a_1^6+108A_2A_3A_5a_1^6-120A_3A_5a_1^6+108A_1A_4A_5a_1^6-102A_1A_2A_4A_5a_1^6\\
&+108A_2A_4A_5a_1^6-102A_1A_3A_4A_5a_1^6+99A_1A_2A_3A_4A_5a_1^6-102A_2A_3A_4A_5a_1^6\\
&+108A_3A_4A_5a_1^6-120A_4A_5a_1^6+2880A_5a_1^6+238464a_1^6+390A_5^4a_1^5-1800A_5^3a_1^5\\
&+2880A_5^2a_1^5-80A_1a_1^5+72A_1A_2a_1^5-80A_2a_1^5+72A_1A_3a_1^5-72A_1A_2A_3a_1^5+72A_2A_3a_1^5\\
&-80A_3a_1^5+72A_1A_4a_1^5-72A_1A_2A_4a_1^5+72A_2A_4a_1^5-72A_1A_3A_4a_1^5+78A_1A_2A_3A_4a_1^5\\
&-72A_2A_3A_4a_1^5+72A_3A_4a_1^5-80A_4a_1^5+72A_1A_5a_1^5-72A_1A_2A_5a_1^5+72A_2A_5a_1^5\\
&-72A_1A_3A_5a_1^5+78A_1A_2A_3A_5a_1^5-72A_2A_3A_5a_1^5+72A_3A_5a_1^5-72A_1A_4A_5a_1^5\\
&+78A_1A_2A_4A_5a_1^5-72A_2A_4A_5a_1^5+78A_1A_3A_4A_5a_1^5-84A_1A_2A_3A_4A_5a_1^5\\
&+78A_2A_3A_4A_5a_1^5-72A_3A_4A_5a_1^5+72A_4A_5a_1^5-1904A_5a_1^5-238496a_1^5-60A_5^4a_1^4\\
&+960A_5^2a_1^4-32A_1a_1^4+8A_1A_2a_1^4-32A_2a_1^4+8A_1A_3a_1^4+8A_2A_3a_1^4-32A_3a_1^4\\
&+8A_1A_4a_1^4+8A_2A_4a_1^4-12A_1A_2A_3A_4a_1^4+8A_3A_4a_1^4-32A_4a_1^4+8A_1A_5a_1^4\\
&+8A_2A_5a_1^4-12A_1A_2A_3A_5a_1^4+8A_3A_5a_1^4-12A_1A_2A_4A_5a_1^4-12A_1A_3A_4A_5a_1^4\\
&+15A_1A_2A_3A_4A_5a_1^4-12A_2A_3A_4A_5a_1^4+8A_4A_5a_1^4-944A_5a_1^4+60A_5^4a_1^3-600A_5^3a_1^3\\
&+960A_5^2a_1^3+8A_1A_2a_1^3+8A_1A_3a_1^3-8A_1A_2A_3a_1^3+8A_2A_3a_1^3+8A_1A_4a_1^3-8A_1A_2A_4a_1^3\\
&+8A_2A_4a_1^3-8A_1A_3A_4a_1^3+12A_1A_2A_3A_4a_1^3-8A_2A_3A_4a_1^3+8A_3A_4a_1^3+8A_1A_5a_1^3\\
&-8A_1A_2A_5a_1^3+8A_2A_5a_1^3-8A_1A_3A_5a_1^3+12A_1A_2A_3A_5a_1^3-8A_2A_3A_5a_1^3+8A_3A_5a_1^3\\
&-8A_1A_4A_5a_1^3+12A_1A_2A_4A_5a_1^3-8A_2A_4A_5a_1^3+12A_1A_3A_4A_5a_1^3+12A_2A_3A_4A_5a_1^3\\
&-8A_3A_4A_5a_1^3+8A_4A_5a_1^3+90A_5^4a_1^2-300A_5^3a_1^2+4A_1A_2A_3a_1^2+4A_1A_2A_4a_1^2\\
&+4A_1A_3A_4a_1^2+2A_1A_2A_3A_4a_1^2+4A_2A_3A_4a_1^2+4A_1A_2A_5a_1^2+4A_1A_3A_5a_1^2\\
&+2A_1A_2A_3A_5a_1^2+4A_2A_3A_5a_1^2+4A_1A_4A_5a_1^2+2A_1A_2A_4A_5a_1^2+4A_2A_4A_5a_1^2\\
&+2A_1A_3A_4A_5a_1^2-15A_1A_2A_3A_4A_5a_1^2+2A_2A_3A_4A_5a_1^2+4A_3A_4A_5a_1^2+30A_5^4a_1\\
&-10A_1A_2A_3A_4a_1-10A_1A_2A_3A_5a_1-10A_1A_2A_4A_5a_1-10A_1A_3A_4A_5a_1\\
&+4A_1A_2A_3A_4A_5a_1-10A_2A_3A_4A_5a_1+13A_1A_2A_3A_4A_5)\end{aligned}$$

$$= \frac{1}{16\alpha_1^5(1-\alpha_1)}\Delta_{26}.$$

We now proceed with the partial derivative test on Δ_{26} with $\alpha_1 \in \left(\frac{3}{5}, \frac{2}{3}\right]$ and

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{2\alpha_1}{1-\alpha_1},$$

like in previous cases.

$$\frac{\partial^5 \Delta_{26}}{\partial A_1\,\partial A_2\,\partial A_3\,\partial A_4\,\partial A_5} = 99\alpha_1^6 - 84\alpha_1^5 + 15\alpha_1^4 - 15\alpha_1^2 + 4\alpha_1 + 13 > 0, \quad \alpha_1 \in \left(\tfrac{3}{5}, \tfrac{2}{3}\right].$$

Thus the partial derivative of Δ_{26} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = \dfrac{2\alpha_1}{1-\alpha_1}$.

$$\left.\frac{\partial^4 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=\frac{2\alpha_1}{1-\alpha_1}} = \frac{4\alpha_1\left(75\alpha_1^6 - 87\alpha_1^5 + 30\alpha_1^4 - 6\alpha_1^3 - 5\alpha_1^2 + 5\alpha_1 + 4\right)}{1-\alpha_1} > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}} = \frac{16\alpha_1^2\left(57\alpha_1^6 - 84\alpha_1^5 + 42\alpha_1^4 - 11\alpha_1^3 + 3\alpha_1 + 1\right)}{(1-\alpha_1)^2} > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\left.\frac{\partial^2 \Delta_{26}}{\partial A_1 \partial A_2}\right|_{A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}}$$
$$= \frac{16\alpha_1^3\left(174\alpha_1^6 - 312\alpha_1^5 + 205\alpha_1^4 - 68\alpha_1^3 + 12\alpha_1^2 + 4\alpha_1 + 1\right)}{(1-\alpha_1)^3} > 0, \text{ and}$$
$$\left.\frac{\partial \Delta_{26}}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}}$$
$$= \frac{16\alpha_1^4\left(534\alpha_1^6 - 1133\alpha_1^5 + 929\alpha_1^4 - 386\alpha_1^3 + 88\alpha_1^2 - \alpha_1 + 1\right)}{(1-\alpha_1)^4} > 0.$$

We must now only test Δ_{26} at its minimum. We observe that if $A_5 >= 3.9$, then we have

$$\Delta_{26}|_{A_1=5, A_2=4, A_3=A_4=A_5=3.9}$$
$$= 246714a_1^6 - 245222a_1^5 - 1692.76a_1^4 + 5984.03a_1^3 - 9061.53a_1^2 - 2778.87a_1 + 15422.9,$$

which is positive over our desired interval. We must now only consider the minimum of Δ_{26} when we have $3 \leq \frac{2\alpha_1}{1-\alpha_1} \leq A_5 < 3.9$.

If $A_5 < 3.9$, we have

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} \leq \frac{29}{39},$$

which must have positive integral solutions. Thus, $A_1 \geq 4 \cdot \dfrac{39}{29}$ and $A_2 \geq 3 \cdot \dfrac{39}{29}$, with $A_3 \geq A_4 \geq A_5 \geq \frac{2\alpha_1}{1-\alpha_1} \geq 3$ still true from before, and $A_5 < 3.9$ by assumption.

With these new bounds on A_1, A_2, A_3, A_4, and A_5, we can use the Yau Geometric Conjecture for $n = 5$ to bound $P_6 = P_5(x_6 = 1) + P_5(x_6 = 2)$, like in Subcase V (a). Thus, we get

$$6!\, P_5(x_6 = 2) \le 6\,[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - p_5(\lceil A_5 \rceil)]$$

and,

$$6!\, P_5\,(x_6 = 1) \le 6\Bigg[\left(A_1 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_2 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_3 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right) \left(A_4 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right) - p_5\left(\left\lceil A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} \right\rceil\right)\Bigg].$$

Note that $\lceil A_5 \rceil = \left\lceil \frac{2\alpha_1}{1-\alpha_1} \right\rceil = 4$, so $p_5(\lceil A_5 \rceil) = p_5(4) = 243$. Similarly, $\left\lceil A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} \right\rceil = \left\lceil \frac{2\alpha_1}{1-\alpha_1} \cdot \frac{1+\alpha_1}{2\alpha_1} \right\rceil = \left\lceil \frac{1+\alpha_1}{1-\alpha_1} \right\rceil = 5$, so $p_5\left(\left\lceil A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} \right\rceil\right) = p_5(5) = 904$. Hence, our bounds are actually

$$6!\, P_5(x_6 = 2) \le 6\,[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - 243]\,,$$

and,

$$6!\, P_5\,(x_6 = 1) \le 6\Bigg[\left(A_1 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_2 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_3 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right) \left(A_4 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right)\left(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1\right) - 904\Bigg].$$

Because $6!\, P_6 = 6!\,(P_5(x_6 = 1) + P_5(x_6 = 2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 3.5, substituting in $a_i = \dfrac{A_i}{\alpha_1}$, yielding

$$\begin{aligned}
\Delta_{27} = \frac{1}{16\alpha_1^5(1-\alpha_1)}(&-510A_5^4\alpha_1^6 + 2700A_5^3\alpha_1^6 - 4800A_5^2\alpha_1^6 + 144A_1\alpha_1^6 - 120A_1A_2\alpha_1^6 \\
&+144A_2\alpha_1^6 - 120A_1A_3\alpha_1^6 + 108A_1A_2A_3\alpha_1^6 - 120A_2A_3\alpha_1^6 + 144A_3\alpha_1^6 - 120A_1A_4\alpha_1^6 \\
&+108A_1A_2A_4\alpha_1^6 - 120A_2A_4\alpha_1^6 + 108A_1A_3A_4\alpha_1^6 - 102A_1A_2A_3A_4\alpha_1^6 + 108A_2A_3A_4\alpha_1^6 \\
&-120A_3A_4\alpha_1^6 + 144A_4\alpha_1^6 - 120A_1A_5\alpha_1^6 + 108A_1A_2A_5\alpha_1^6 - 120A_2A_5\alpha_1^6 \\
&+108A_1A_3A_5\alpha_1^6 - 102A_1A_2A_3A_5\alpha_1^6 + 108A_2A_3A_5\alpha_1^6 - 120A_3A_5\alpha_1^6 + 108A_1A_4A_5\alpha_1^6 \\
&-102A_1A_2A_4A_5\alpha_1^6 + 108A_2A_4A_5\alpha_1^6 - 102A_1A_3A_4A_5\alpha_1^6 + 99A_1A_2A_3A_4A_5\alpha_1^6 \\
&-102A_2A_3A_4A_5\alpha_1^6 + 108A_3A_4A_5\alpha_1^6 - 120A_4A_5\alpha_1^6 + 2880A_5\alpha_1^6 + 238464\alpha_1^6 \\
&+390A_5^4\alpha_1^5 - 1800A_5^3\alpha_1^5 + 2880A_5^2\alpha_1^5 - 80A_1\alpha_1^5 + 72A_1A_2\alpha_1^5 - 80A_2\alpha_1^5 + 72A_1A_3\alpha_1^5 \\
&-72A_1A_2A_3\alpha_1^5 + 72A_2A_3\alpha_1^5 - 80A_3\alpha_1^5 + 72A_1A_4\alpha_1^5 - 72A_1A_2A_4\alpha_1^5 + 72A_2A_4\alpha_1^5 \\
&-72A_1A_3A_4\alpha_1^5 + 78A_1A_2A_3A_4\alpha_1^5 - 72A_2A_3A_4\alpha_1^5 + 72A_3A_4\alpha_1^5 - 80A_4\alpha_1^5 \\
&+72A_1A_5\alpha_1^5 - 72A_1A_2A_5\alpha_1^5 + 72A_2A_5\alpha_1^5 - 72A_1A_3A_5\alpha_1^5 + 78A_1A_2A_3A_5\alpha_1^5 \\
&-72A_2A_3A_5\alpha_1^5 + 72A_3A_5\alpha_1^5 - 72A_1A_4A_5\alpha_1^5 + 78A_1A_2A_4A_5\alpha_1^5 - 72A_2A_4A_5\alpha_1^5 \\
&+78A_1A_3A_4A_5\alpha_1^5 - 84A_1A_2A_3A_4A_5\alpha_1^5 + 78A_2A_3A_4A_5\alpha_1^5 - 72A_3A_4A_5\alpha_1^5 \\
&+72A_4A_5\alpha_1^5 - 1904A_5\alpha_1^5 - 238496\alpha_1^5 - 60A_5^4\alpha_1^4 + 960A_5^2\alpha_1^4 - 32A_1\alpha_1^4 \\
&+8A_1A_2\alpha_1^4 - 32A_2\alpha_1^4 + 8A_1A_3\alpha_1^4 + 8A_2A_3\alpha_1^4 - 32A_3\alpha_1^4 + 8A_1A_4\alpha_1^4
\end{aligned}$$

$$+8A_2A_4\alpha_1^4 - 12A_1A_2A_3A_4\alpha_1^4 + 8A_3A_4\alpha_1^4 - 32A_4\alpha_1^4 + 8A_1A_5\alpha_1^4 + 8A_2A_5\alpha_1^4$$
$$-12A_1A_2A_3A_5\alpha_1^4 + 8A_3A_5\alpha_1^4 - 12A_1A_2A_4A_5\alpha_1^4 - 12A_1A_3A_4A_5\alpha_1^4$$
$$+15A_1A_2A_3A_4A_5\alpha_1^4 - 12A_2A_3A_4A_5\alpha_1^4 + 8A_4A_5\alpha_1^4 - 944A_5\alpha_1^4 + 60A_5^4\alpha_1^3$$
$$-600A_5^3\alpha_1^3 + 960A_5^2\alpha_1^3 + 8A_1A_2\alpha_1^3 + 8A_1A_3\alpha_1^3 - 8A_1A_2A_3\alpha_1^3 + 8A_2A_3\alpha_1^3 + 8A_1A_4\alpha_1^3$$
$$-8A_1A_2A_4\alpha_1^3 + 8A_2A_4\alpha_1^3 - 8A_1A_3A_4\alpha_1^3 + 12A_1A_2A_3A_4\alpha_1^3 - 8A_2A_3A_4\alpha_1^3 + 8A_3A_4\alpha_1^3$$
$$+8A_1A_5\alpha_1^3 - 8A_1A_2A_5\alpha_1^3 + 8A_2A_5\alpha_1^3 - 8A_1A_3A_5\alpha_1^3 + 12A_1A_2A_3A_5\alpha_1^3 - 8A_2A_3A_5\alpha_1^3$$
$$+8A_3A_5\alpha_1^3 - 8A_1A_4A_5\alpha_1^3 + 12A_1A_2A_4A_5\alpha_1^3 - 8A_2A_4A_5\alpha_1^3 + 12A_1A_3A_4A_5\alpha_1^3$$
$$+12A_2A_3A_4A_5\alpha_1^3 - 8A_3A_4A_5\alpha_1^3 + 8A_4A_5\alpha_1^3 + 90A_5^4\alpha_1^2 - 300A_5^3\alpha_1^2 + 4A_1A_2A_3\alpha_1^2$$
$$+4A_1A_2A_4\alpha_1^2 + 4A_1A_3A_4\alpha_1^2 + 2A_1A_2A_3A_4\alpha_1^2 + 4A_2A_3A_4\alpha_1^2 + 4A_1A_2A_5\alpha_1^2$$
$$+4A_1A_3A_5\alpha_1^2 + 2A_1A_2A_3A_5\alpha_1^2 + 4A_2A_3A_5\alpha_1^2 + 4A_1A_4A_5\alpha_1^2 + 2A_1A_2A_4A_5\alpha_1^2$$
$$+4A_2A_4A_5\alpha_1^2 + 2A_1A_3A_4A_5\alpha_1^2 - 15A_1A_2A_3A_4A_5\alpha_1^2 + 2A_2A_3A_4A_5\alpha_1^2 + 4A_3A_4A_5\alpha_1^2$$
$$+30A_5^4\alpha_1 - 10A_1A_2A_3A_4\alpha_1 - 10A_1A_2A_3A_5\alpha_1 - 10A_1A_2A_4A_5\alpha_1 - 10A_1A_3A_4A_5\alpha_1$$
$$+4A_1A_2A_3A_4A_5\alpha_1 - 10A_2A_3A_4A_5\alpha_1 + 13A_1A_2A_3A_4A_5)$$
$$= \frac{1}{16\alpha_1^5(1-\alpha_1)}\Delta_{28}.$$

We note that Δ_{28} and Δ_{26} are defined similarly, with their only difference being an α_1 polynomial that is subtracted. Since $\Delta_{28} - \Delta_{26}$ is a function of α_1 only, all the partial derivatives have already been shown positive. Hence, we only need to deal with showing that the minimum value of Δ_{28} is non-negative. We note that

$$\Delta_{28}|_{A_1=4\cdot\frac{39}{29}, A_2=3\cdot\frac{39}{29}, A_3=A_4=\frac{7}{2}, A_5=\frac{2\alpha_1}{1-\alpha_1}}$$
$$= \frac{2\alpha_1}{841(\alpha_1-1)}\left(43835559\alpha_1^6 - 95014566\alpha_1^5 + 48838016\alpha_1^4\right.$$
$$\left.-130694\alpha_1^3 + 2463073\alpha_1^2 - 437892\alpha_1 - 1788696\right),$$

which is non-negative for $\alpha_1 \in \left(\frac{3}{5}, \frac{2}{3}\right]$. Thus, we need only consider when $A_4 < \frac{7}{2}$. This leaves us with,

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} \leq 1 - \frac{10}{39} - \frac{2}{7} = \frac{125}{273},$$

which must have positive integral solutions, implying that $A_1 \geq 3 \cdot \frac{273}{125}$ and $A_2 \geq 2 \cdot \frac{273}{125}$. Thus, Δ_{28} is minimized at

$$\Delta_{28}|_{A_1=3\cdot\frac{273}{125}, A_2=2\cdot\frac{273}{125}, A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}}$$
$$= \frac{16\alpha_1^3}{15625(\alpha_1-1)^3}\left(75990474\alpha_1^6 - 411281087\alpha_1^5 + 690459330\alpha_1^4 - 479368418\alpha_1^3\right.$$
$$\left.+121590162\alpha_1^2 - 1618071\alpha_1 - 447174\right),$$

which is zero over our desired interval, completing this subcase.

3.5.3. Subcase V (c). In this subcase, we know that $P_5(x_6 = 3) > 0$, implying that $(1,1,1,1,1,3)$ is a positive integral solution to the inequality in Theorem 3.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{3}{a_6} := \alpha_2,$$

then $\alpha_2 \in \left(\frac{2}{5}, \frac{1}{2}\right]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_2$ for $i = 1$, 2, 3, 4, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6!\, P_5(x_6 = 3) \leq 6[&(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &-(A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)], \end{aligned}$$

as well as

$$\begin{aligned} 6!\, P_5\,(x_6 = 2) \leq 6\Bigg[&\left(A_1 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_2 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_3 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right) \\ &\left(A_4 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right) - \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)^5 \\ &+\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2}\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 2\right) \\ &\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 3\right)\left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 4\right)\Bigg], \end{aligned}$$

and

$$\begin{aligned} 6!\, P_5\,(x_6 = 1) \leq 6\Bigg[&\left(A_1 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_2 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_3 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right) \\ &\left(A_4 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right) - \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)^5 \\ &+\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2}\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 2\right) \\ &\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 3\right)\left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 4\right)\Bigg]. \end{aligned}$$

Because $6!\, P_6 = 6!\,(P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3))$, if we let Δ_{29} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 3.5 and substituting in $a_i = \dfrac{A_i}{\alpha_2}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_{29} = \frac{1}{27\alpha_2^5(1-\alpha_2)}(&-980A_5^4\alpha_2^6 + 5400A_5^3\alpha_2^6 - 10080A_5^2\alpha_2^6 + 324A_1\alpha_2^6 - 252A_1A_2\alpha_2^6 \\ &+324A_2\alpha_2^6 - 252A_1A_3\alpha_2^6 + 216A_1A_2A_3\alpha_2^6 - 252A_2A_3\alpha_2^6 + 324A_3\alpha_2^6 - 252A_1A_4\alpha_2^6 \\ &+216A_1A_2A_4\alpha_2^6 - 252A_2A_4\alpha_2^6 + 216A_1A_3A_4\alpha_2^6 - 196A_1A_2A_3A_4\alpha_2^6 + 216A_2A_3A_4\alpha_2^6 \\ &-252A_3A_4\alpha_2^6 + 324A_4\alpha_2^6 - 252A_1A_5\alpha_2^6 + 216A_1A_2A_5\alpha_2^6 - 252A_2A_5\alpha_2^6 \\ &+216A_1A_3A_5\alpha_2^6 - 196A_1A_2A_3A_5\alpha_2^6 + 216A_2A_3A_5\alpha_2^6 - 252A_3A_5\alpha_2^6 + 216A_1A_4A_5\alpha_2^6 \\ &-196A_1A_2A_4A_5\alpha_2^6 + 216A_2A_4A_5\alpha_2^6 - 196A_1A_3A_4A_5\alpha_2^6 + 184A_1A_2A_3A_4A_5\alpha_2^6 \\ &-196A_2A_3A_4A_5\alpha_2^6 + 216A_3A_4A_5\alpha_2^6 - 252A_4A_5\alpha_2^6 + 6480A_5\alpha_2^6 + 402408\alpha_2^6 \\ &+580A_5^4\alpha_2^5 - 2700A_5^3\alpha_2^5 + 4320A_5^2\alpha_2^5 - 135A_1\alpha_2^5 + 108A_1A_2\alpha_2^5 - 135A_2\alpha_2^5 \\ &+108A_1A_3\alpha_2^5 - 108A_1A_2A_3\alpha_2^5 + 108A_2A_3\alpha_2^5 - 135A_3\alpha_2^5 + 108A_1A_4\alpha_2^5 \end{aligned}$$

$$
\begin{aligned}
&-108A_1A_2A_4\alpha_2^5+108A_2A_4\alpha_2^5-108A_1A_3A_4\alpha_2^5+116A_1A_2A_3A_4\alpha_2^5-108A_2A_3A_4\alpha_2^5\\
&+108A_3A_4\alpha_2^5-135A_4\alpha_2^5+108A_1A_5\alpha_2^5-108A_1A_2A_5\alpha_2^5+108A_2A_5\alpha_2^5\\
&-108A_1A_3A_5\alpha_2^5+116A_1A_2A_3A_5\alpha_2^5-108A_2A_3A_5\alpha_2^5+108A_3A_5\alpha_2^5-108A_1A_4A_5\alpha_2^5\\
&+116A_1A_2A_4A_5\alpha_2^5-108A_2A_4A_5\alpha_2^5+116A_1A_3A_4A_5\alpha_2^5-124A_1A_2A_3A_4A_5\alpha_2^5\\
&+116A_2A_3A_4A_5\alpha_2^5-108A_3A_4A_5\alpha_2^5+108A_4A_5\alpha_2^5-3213A_5\alpha_2^5-402489\alpha_2^5-80A_5^4\alpha_2^4\\
&+2160A_5^2\alpha_2^4-108A_1\alpha_2^4+27A_1A_2\alpha_2^4-108A_2\alpha_2^4+27A_1A_3\alpha_2^4+27A_2A_3\alpha_2^4-108A_3\alpha_2^4\\
&+27A_1A_4\alpha_2^4+27A_2A_4\alpha_2^4-16A_1A_2A_3A_4\alpha_2^4+27A_3A_4\alpha_2^4-108A_4\alpha_2^4+27A_1A_5\alpha_2^4\\
&+27A_2A_5\alpha_2^4-16A_1A_2A_3A_5\alpha_2^4+27A_3A_5\alpha_2^4-16A_1A_2A_4A_5\alpha_2^4-16A_1A_3A_4A_5\alpha_2^4\\
&+20A_1A_2A_3A_4A_5\alpha_2^4-16A_2A_3A_4A_5\alpha_2^4+27A_4A_5\alpha_2^4-3186A_5\alpha_2^4+80A_5^4\alpha_2^3\\
&-1350A_5^3\alpha_2^3+3600A_5^2\alpha_2^3+36A_1A_2\alpha_2^3+36A_1A_3\alpha_2^3-27A_1A_2A_3\alpha_2^3+36A_2A_3\alpha_2^3\\
&+36A_1A_4\alpha_2^3-27A_1A_2A_4\alpha_2^3+36A_2A_4\alpha_2^3-27A_1A_3A_4\alpha_2^3+16A_1A_2A_3A_4\alpha_2^3\\
&-27A_2A_3A_4\alpha_2^3+36A_3A_4\alpha_2^3+36A_1A_5\alpha_2^3-27A_1A_2A_5\alpha_2^3+36A_2A_5\alpha_2^3\\
&-27A_1A_3A_5\alpha_2^3+16A_1A_2A_3A_5\alpha_2^3-27A_2A_3A_5\alpha_2^3+36A_3A_5\alpha_2^3-27A_1A_4A_5\alpha_2^3\\
&+16A_1A_2A_4A_5\alpha_2^3-27A_2A_4A_5\alpha_2^3+16A_1A_3A_4A_5\alpha_2^3+16A_2A_3A_4A_5\alpha_2^3\\
&-27A_3A_4A_5\alpha_2^3+36A_4A_5\alpha_2^3+230A_5^4\alpha_2^2-1350A_5^3\alpha_2^2+19A_1A_2A_3A_4\alpha_2^2\\
&+19A_1A_2A_3A_5\alpha_2^2+19A_1A_2A_4A_5\alpha_2^2+19A_1A_3A_4A_5\alpha_2^2-20A_1A_2A_3A_4A_5\alpha_2^2\\
&+19A_2A_3A_4A_5\alpha_2^2+170A_5^4\alpha_2-20A_1A_2A_3A_4\alpha_2-20A_1A_2A_3A_5\alpha_2-20A_1A_2A_4A_5\alpha_2\\
&-20A_1A_3A_4A_5\alpha_2-11A_1A_2A_3A_4A_5\alpha_2-20A_2A_3A_4A_5\alpha_2+32A_1A_2A_3A_4A_5)\\
=&\ \frac{1}{27\alpha_2^5(1-\alpha_2)}\Delta_{30}.
\end{aligned}
$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq A_5 \geq \frac{3\alpha_2}{1-\alpha_2}.$$

Note also that we have $\alpha_2 \in \left(\frac{2}{5}, \frac{1}{2}\right]$. We begin with

$$\frac{\partial^5 \Delta_{30}}{\partial A_1\, \partial A_2\, \partial A_3\, \partial A_4\, \partial A_5} = 184\alpha_2^6 - 124\alpha_2^5 + 20\alpha_2^4 - 20\alpha_2^2 - 11\alpha_2 + 32 > 0, \quad \alpha_2 \in \left(\frac{2}{5}, \frac{1}{2}\right].$$

Thus the partial derivative of Δ_{30} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = 1$.

$$\left.\frac{\partial^4 \Delta_{30}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=1} = -12\alpha_2^6 - 8\alpha_2^5 + 4\alpha_2^4 + 16\alpha_2^3 - \alpha_2^2 - 31\alpha_2 + 32 > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{30}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=2, A_5=1} = -4\alpha_2^6 - 8\alpha_2^5 - 8\alpha_2^4 + 21\alpha_2^3 + 17\alpha_2^2 - 82\alpha_2 + 64 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\left.\frac{\partial^2 \Delta_{30}}{\partial A_1 \partial A_2}\right|_{A_3=3, A_4=2, A_5=1} = -8\alpha_2^6 - 8\alpha_2^5 - 29\alpha_2^4 + 50\alpha_2^3 + 89\alpha_2^2 - 286\alpha_2 + 192 > 0, \text{ and}$$

$$\left.\frac{\partial \Delta}{\partial A_1}\right|_{A_2=4, A_3=3, A_4=2, A_5=1} = -20\alpha_2^6 - 11\alpha_2^5 - 158\alpha_2^4 + 215\alpha_2^3 + 470\alpha_2^2 - 1264\alpha_2 + 768 > 0.$$

Finally, our minimum for Δ_{30} is

$$\begin{aligned}
&\Delta_{30}\big|_{A_1=5,A_2=4,A_3=3,A_4=A_5=\frac{3\alpha_2}{1-\alpha_2}} \\
&= \frac{9\alpha_2^2}{(\alpha_2-1)^3}\left(15156\alpha_2^7 - 171949\alpha_2^6 + 272493\alpha_2^5 - 182673\alpha_2^4\right. \\
&\left. +44099\alpha_2^3 + 972\alpha_2^2 + 1152\alpha_2 - 1120\right),
\end{aligned}$$

which is positive over our desired interval, completing this subcase.

3.5.4. Subcase V (d). In this subcase, we know that $P_5(x_6 = 4) > 0$, implying that $(1, 1, 1, 1, 1, 4)$ is a positive integral solution to the inequality in Theorem 3.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \le 1 - \frac{4}{a_6} := \alpha_3,$$

then $\alpha_3 \in \left(\frac{1}{5}, \frac{1}{3}\right]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_3$ for $i = 1, 2, 3, 4,$ and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \le 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned}
6!\, P_5(x_6 = 4) \le 6[&(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1) \\
&-(A_5-1)^5 + A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)],
\end{aligned}$$

as well as

$$\begin{aligned}
6!\, P_5\left(x_6 = 3\right) \le 6\Bigg[&\left(A_1 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1\right)\left(A_2 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1\right)\left(A_3 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1\right) \\
&\left(A_4 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1\right)\left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1\right) - \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1\right)^5 \\
&+\left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3}\right)\left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1\right)\left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 2\right) \\
&\left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 3\right)\left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 4\right)\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
6!\, P_5\left(x_6 = 2\right) \le 6\Bigg[&\left(A_1 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1\right)\left(A_2 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1\right)\left(A_3 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1\right) \\
&\left(A_4 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1\right)\left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1\right) - \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1\right)^5 \\
&+\left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3}\right)\left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1\right)\left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 2\right) \\
&\left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 3\right)\left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 4\right)\Bigg],
\end{aligned}$$

and

$$6!\, P_5\left(x_6 = 1\right) \le 6\Bigg[\left(A_1 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1\right)\left(A_2 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1\right)\left(A_3 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1\right)$$

$$\left(A_4 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1\right)\left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1\right) - \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1\right)^5$$
$$+\left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3}\right)\left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1\right)\left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 2\right)$$
$$\left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 3\right)\left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 4\right)\Bigg].$$

Because $6!\,P_6 = 6!\,(P_5(x_6=1)+P_5(x_6=2)+P_5(x_6=3)+P_5(x_6=4))$, if we let Δ_{20} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 3.4 and substituting in $a_i = \dfrac{A_i}{\alpha_3}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned}
\Delta_{31} = \frac{1}{128\alpha_3^5(1-\alpha_3)}&(-5310A_5^4\alpha_3^6 + 30000A_5^3\alpha_3^6 - 57600A_5^2\alpha_3^6 + 1920A_1\alpha_3^6 - 1440A_1A_2\alpha_3^6\\
&+1920A_2\alpha_3^6 - 1440A_1A_3\alpha_3^6 + 1200A_1A_2A_3\alpha_3^6 - 1440A_2A_3\alpha_3^6 + 1920A_3\alpha_3^6\\
&-1440A_1A_4\alpha_3^6 + 1200A_1A_2A_4\alpha_3^6 - 1440A_2A_4\alpha_3^6 + 1200A_1A_3A_4\alpha_3^6\\
&-1062A_1A_2A_3A_4\alpha_3^6 + 1200A_2A_3A_4\alpha_3^6 - 1440A_3A_4\alpha_3^6 + 1920A_4\alpha_3^6 - 1440A_1A_5\alpha_3^6\\
&+1200A_1A_2A_5\alpha_3^6 - 1440A_2A_5\alpha_3^6 + 1200A_1A_3A_5\alpha_3^6 - 1062A_1A_2A_3A_5\alpha_3^6\\
&+1200A_2A_3A_5\alpha_3^6 - 1440A_3A_5\alpha_3^6 + 1200A_1A_4A_5\alpha_3^6 - 1062A_1A_2A_4A_5\alpha_3^6\\
&+1200A_2A_4A_5\alpha_3^6 - 1062A_1A_3A_4A_5\alpha_3^6 + 975A_1A_2A_3A_4A_5\alpha_3^6 - 1062A_2A_3A_4A_5\alpha_3^6\\
&+1200A_3A_4A_5\alpha_3^6 - 1440A_4A_5\alpha_3^6 + 38400A_5\alpha_3^6 + 1907712\alpha_3^6 + 2550A_5^4\alpha_3^5 - 12000A_5^3\alpha_3^5\\
&+19200A_5^2\alpha_3^5 - 640A_1\alpha_3^5 + 480A_1A_2\alpha_3^5 - 640A_2\alpha_3^5 + 480A_1A_3\alpha_3^5 - 480A_1A_2A_3\alpha_3^5\\
&+480A_2A_3\alpha_3^5 - 640A_3\alpha_3^5 + 480A_1A_4\alpha_3^5 - 480A_1A_2A_4\alpha_3^5 + 480A_2A_4\alpha_3^5\\
&-480A_1A_3A_4\alpha_3^5 + 510A_1A_2A_3A_4\alpha_3^5 - 480A_2A_3A_4\alpha_3^5 + 480A_3A_4\alpha_3^5 - 640A_4\alpha_3^5\\
&+480A_1A_5\alpha_3^5 - 480A_1A_2A_5\alpha_3^5 + 480A_2A_5\alpha_3^5 - 480A_1A_3A_5\alpha_3^5 + 510A_1A_2A_3A_5\alpha_3^5\\
&-480A_2A_3A_5\alpha_3^5 + 480A_3A_5\alpha_3^5 - 480A_1A_4A_5\alpha_3^5 + 510A_1A_2A_4A_5\alpha_3^5 - 480A_2A_4A_5\alpha_3^5\\
&+510A_1A_3A_4A_5\alpha_3^5 - 540A_1A_2A_3A_4A_5\alpha_3^5 + 510A_2A_3A_4A_5\alpha_3^5 - 480A_3A_4A_5\alpha_3^5\\
&+480A_4A_5\alpha_3^5 - 15232A_5\alpha_3^5 - 1908224\alpha_3^5 - 300A_5^4\alpha_3^4 + 11520A_5^2\alpha_3^4 - 768A_1\alpha_3^4\\
&+160A_1A_2\alpha_3^4 - 768A_2\alpha_3^4 + 160A_1A_3\alpha_3^4 + 160A_2A_3\alpha_3^4 - 768A_3\alpha_3^4 + 160A_1A_4\alpha_3^4\\
&+160A_2A_4\alpha_3^4 - 60A_1A_2A_3A_4\alpha_3^4 + 160A_3A_4\alpha_3^4 - 768A_4\alpha_3^4 + 160A_1A_5\alpha_3^4\\
&+160A_2A_5\alpha_3^4 - 60A_1A_2A_3A_5\alpha_3^4 + 160A_3A_5\alpha_3^4 - 60A_1A_2A_4A_5\alpha_3^4 - 60A_1A_3A_4A_5\alpha_3^4\\
&+75A_1A_2A_3A_4A_5\alpha_3^4 - 60A_2A_3A_4A_5\alpha_3^4 + 160A_4A_5\alpha_3^4 - 22656A_5\alpha_3^4 + 300A_5^4\alpha_3^3\\
&-7200A_5^3\alpha_3^3 + 26880A_5^2\alpha_3^3 + 288A_1A_2\alpha_3^3 + 288A_1A_3\alpha_3^3 - 160A_1A_2A_3\alpha_3^3 + 288A_2A_3\alpha_3^3\\
&+288A_1A_4\alpha_3^3 - 160A_1A_2A_4\alpha_3^3 + 288A_2A_4\alpha_3^3 - 160A_1A_3A_4\alpha_3^3 + 60A_1A_2A_3A_4\alpha_3^3\\
&-160A_2A_3A_4\alpha_3^3 + 288A_3A_4\alpha_3^3 + 288A_1A_5\alpha_3^3 - 160A_1A_2A_5\alpha_3^3 + 288A_2A_5\alpha_3^3\\
&-160A_1A_3A_5\alpha_3^3 + 60A_1A_2A_3A_5\alpha_3^3 - 160A_2A_3A_5\alpha_3^3 + 288A_3A_5\alpha_3^3 - 160A_1A_4A_5\alpha_3^3\\
&+60A_1A_2A_4A_5\alpha_3^3 - 160A_2A_4A_5\alpha_3^3 + 60A_1A_3A_4A_5\alpha_3^3 + 60A_2A_3A_4A_5\alpha_3^3\\
&-160A_3A_4A_5\alpha_3^3 + 288A_4A_5\alpha_3^3 + 1290A_5^4\alpha_3^2 - 10800A_5^3\alpha_3^2 - 48A_1A_2A_3\alpha_3^2\\
&-48A_1A_2A_4\alpha_3^2 - 48A_1A_3A_4\alpha_3^2 + 130A_1A_2A_3A_4\alpha_3^2 - 48A_2A_3A_4\alpha_3^2 - 48A_1A_2A_5\alpha_3^2\\
&-48A_1A_3A_5\alpha_3^2 + 130A_1A_2A_3A_5\alpha_3^2 - 48A_2A_3A_5\alpha_3^2 - 48A_1A_4A_5\alpha_3^2\\
&+130A_1A_2A_4A_5\alpha_3^2 - 48A_2A_4A_5\alpha_3^2 + 130A_1A_3A_4A_5\alpha_3^2 - 75A_1A_2A_3A_4A_5\alpha_3^2\\
&+130A_2A_3A_4A_5\alpha_3^2 - 48A_3A_4A_5\alpha_3^2 + 1470A_5^4\alpha_3 - 90A_1A_2A_3A_4\alpha_3 - 90A_1A_2A_3A_5\alpha_3\\
&-90A_1A_2A_4A_5\alpha_3 - 90A_1A_3A_4A_5\alpha_3 - 100A_1A_2A_3A_4A_5\alpha_3 - 90A_2A_3A_4A_5\alpha_3\\
&+177A_1A_2A_3A_4A_5)
\end{aligned}$$

$$= \frac{1}{128\alpha_3^5(1-\alpha_3)}\Delta_{32}.$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_3 \in \left(\frac{1}{5}, \frac{1}{3}\right]$. We begin with

$$\frac{\partial^5 \Delta_{32}}{\partial A_1\, \partial A_2\, \partial A_3\, \partial A_4\, \partial A_5} = 975\alpha_3^6 - 540\alpha_3^5 + 75\alpha_3^4 - 75\alpha_3^2 - 100\alpha_3 + 177 > 0, \;\; \alpha_3 \in \left(\frac{1}{5}, \frac{1}{3}\right].$$

Thus the partial derivative of Δ_{32} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = 1$.

$$\left.\frac{\partial^4 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=1} = -87\alpha_3^6 - 30\alpha_3^5 + 15\alpha_3^4 + 60\alpha_3^3 + 55\alpha_3^2 - 190\alpha_3 + 177 > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=2, A_5=1} = -36\alpha_3^6 - 30\alpha_3^5 - 30\alpha_3^4 + 20\alpha_3^3 + 192\alpha_3^2 - 470\alpha_3 + 354 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\begin{aligned} &\left.\frac{\partial^2 \Delta_{32}}{\partial A_1 \partial A_2}\right|_{A_3=3, A_4=2, A_5=1} \\ &= -2\left(36\alpha_3^6 + 15\alpha_3^5 + 25\alpha_3^4 + 6\alpha_3^3 - 346\alpha_3^2 + 795\alpha_3 - 531\right) > 0, \text{ and} \\ &\left.\frac{\partial \Delta_{32}}{\partial A_1}\right|_{A_2=4, A_3=3, A_4=2, A_5=1} \\ &= -4\left(45\alpha_3^6 + 25\alpha_3^5 + 92\alpha_3^4 - 70\alpha_3^3 - 755\alpha_3^2 + 1725\alpha_3 - 1062\right) > 0. \end{aligned}$$

Finally, our minimum for Δ_{32} is

$$\begin{aligned} &\Delta_{32}|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} \\ &= 2(956847\alpha_3^6 - 957463\alpha_3^5 - 8014\alpha_3^4 + 12450\alpha_3^3 + 3155\alpha_3^2 - 17595\alpha_3 + 10620), \end{aligned}$$

which is positive over our desired interval, completing this subcase.

3.5.5. Subcase V (e). In this subcase, we know that $P_5(x_6 = 5) > 0$, implying that $(1, 1, 1, 1, 1, 5)$ is a positive integral solution to the inequality in Theorem 3.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{5}{a_6} := \alpha_4,$$

then $\alpha_4 \in \left(0, \frac{1}{6}\right]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_4$ for $i = 1, 2, 3, 4$, and 5. This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6!\,P_5(x_6=5) \le 6[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1) \\ -(A_5-1)^5 + A_5(A_5-1)(A_5-2)(A_5-3)(A_5-4)],$$

as well as

$$6!\,P_5\,(x_6=4) \le 6\bigg[\left(A_1\cdot\frac{1+4\alpha_4}{5\alpha_4}-1\right)\left(A_2\cdot\frac{1+4\alpha_4}{5\alpha_4}-1\right)\left(A_3\cdot\frac{1+4\alpha_4}{5\alpha_4}-1\right) \\ \left(A_4\cdot\frac{1+4\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{1+4\alpha_4}{5\alpha_4}-1\right)-\left(A_5\cdot\frac{1+4\alpha_4}{5\alpha_4}-1\right)^5 \\ +\left(A_5\cdot\frac{1+4\alpha_4}{5\alpha_4}\right)\left(A_5\cdot\frac{1+4\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{1+4\alpha_4}{5\alpha_4}-2\right) \\ \left(A_5\cdot\frac{1+4\alpha_4}{5\alpha_4}-3\right)\left(A_5\cdot\frac{1+4\alpha_4}{5\alpha_4}-4\right)\bigg],$$

and

$$6!\,P_5\,(x_6=3) \le 6\bigg[\left(A_1\cdot\frac{2+3\alpha_4}{5\alpha_4}-1\right)\left(A_2\cdot\frac{2+3\alpha_4}{5\alpha_4}-1\right)\left(A_3\cdot\frac{2+3\alpha_4}{5\alpha_4}-1\right) \\ \left(A_4\cdot\frac{2+3\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{2+3\alpha_4}{5\alpha_4}-1\right)-\left(A_5\cdot\frac{2+3\alpha_4}{5\alpha_4}-1\right)^5 \\ +\left(A_5\cdot\frac{2+3\alpha_4}{5\alpha_4}\right)\left(A_5\cdot\frac{2+3\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{2+3\alpha_4}{5\alpha_4}-2\right) \\ \left(A_5\cdot\frac{2+3\alpha_4}{5\alpha_4}-3\right)\left(A_5\cdot\frac{2+3\alpha_4}{5\alpha_4}-4\right)\bigg],$$

and

$$6!\,P_5\,(x_6=2) \le 6\bigg[\left(A_1\cdot\frac{3+2\alpha_4}{5\alpha_4}-1\right)\left(A_2\cdot\frac{3+2\alpha_4}{5\alpha_4}-1\right)\left(A_3\cdot\frac{3+2\alpha_4}{5\alpha_4}-1\right) \\ \left(A_4\cdot\frac{3+2\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{3+2\alpha_4}{5\alpha_4}-1\right)-\left(A_5\cdot\frac{3+2\alpha_4}{5\alpha_4}-1\right)^5 \\ +\left(A_5\cdot\frac{3+2\alpha_4}{5\alpha_4}\right)\left(A_5\cdot\frac{3+2\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{3+2\alpha_4}{5\alpha_4}-2\right) \\ \left(A_5\cdot\frac{3+2\alpha_4}{5\alpha_4}-3\right)\left(A_5\cdot\frac{3+2\alpha_4}{5\alpha_4}-4\right)\bigg],$$

and finally,

$$6!\,P_5\,(x_6=1) \le 6\bigg[\left(A_1\cdot\frac{4+\alpha_4}{5\alpha_4}-1\right)\left(A_2\cdot\frac{4+\alpha_4}{5\alpha_4}-1\right)\left(A_3\cdot\frac{4+\alpha_4}{5\alpha_4}-1\right) \\ \left(A_4\cdot\frac{4+\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{4+\alpha_4}{5\alpha_4}-1\right)-\left(A_5\cdot\frac{4+\alpha_4}{5\alpha_4}-1\right)^5 \\ +\left(A_5\cdot\frac{4+\alpha_4}{5\alpha_4}\right)\left(A_5\cdot\frac{4+\alpha_4}{5\alpha_4}-1\right)\left(A_5\cdot\frac{4+\alpha_4}{5\alpha_4}-2\right) \\ \left(A_5\cdot\frac{4+\alpha_4}{5\alpha_4}-3\right)\left(A_5\cdot\frac{4+\alpha_4}{5\alpha_4}-4\right)\bigg].$$

Because $6!\,P_6 = 6!\,(P_5(x_6=1)+P_5(x_6=2)+P_5(x_6=3)+P_5(x_6=4)+P_5(x_6=5))$, if we let Δ_{33} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 3.5 and substituting in $a_i = \dfrac{A_i}{\alpha_4}$, then we merely have to apply the partial differentiation test for the expression

$$
\begin{aligned}
\Delta_{33} = {} & \frac{1}{3125\alpha_4^5(1-\alpha_4)}(-146850A_5^4\alpha_4^6 + 843750A_5^3\alpha_4^6 - 1650000A_5^2\alpha_4^6 + 56250A_1\alpha_4^6 \\
& -41250A_1A_2\alpha_4^6 + 56250A_2\alpha_4^6 - 41250A_1A_3\alpha_4^6 + 33750A_1A_2A_3\alpha_4^6 - 41250A_2A_3\alpha_4^6 \\
& +56250A_3\alpha_4^6 - 41250A_1A_4\alpha_4^6 + 33750A_1A_2A_4\alpha_4^6 - 41250A_2A_4\alpha_4^6 + 33750A_1A_3A_4\alpha_4^6 \\
& -29370A_1A_2A_3A_4\alpha_4^6 + 33750A_2A_3A_4\alpha_4^6 - 41250A_3A_4\alpha_4^6 + 56250A_4\alpha_4^6 \\
& -41250A_1A_5\alpha_4^6 + 33750A_1A_2A_5\alpha_4^6 - 41250A_2A_5\alpha_4^6 + 33750A_1A_3A_5\alpha_4^6 \\
& -29370A_1A_2A_3A_5\alpha_4^6 + 33750A_2A_3A_5\alpha_4^6 - 41250A_3A_5\alpha_4^6 + 33750A_1A_4A_5\alpha_4^6 \\
& -29370A_1A_2A_4A_5\alpha_4^6 + 33750A_2A_4A_5\alpha_4^6 - 29370A_1A_3A_4A_5\alpha_4^6 \\
& +26550A_1A_2A_3A_4A_5\alpha_4^6 - 29370A_2A_3A_4A_5\alpha_4^6 + 33750A_3A_4A_5\alpha_4^6 - 41250A_4A_5\alpha_4^6 \\
& +1125000A_5\alpha_4^6 + 46575000\alpha_4^6 + 59250A_5^4\alpha_4^5 - 281250A_5^3\alpha_4^5 + 450000A_5^2\alpha_4^5 \\
& -15625A_1\alpha_4^5 + 11250A_1A_2\alpha_4^5 - 15625A_2\alpha_4^5 + 11250A_1A_3\alpha_4^5 - 11250A_1A_2A_3\alpha_4^5 \\
& +11250A_2A_3\alpha_4^5 - 15625A_3\alpha_4^5 + 11250A_1A_4\alpha_4^5 - 11250A_1A_2A_4\alpha_4^5 + 11250A_2A_4\alpha_4^5 \\
& -11250A_1A_3A_4\alpha_4^5 + 11850A_1A_2A_3A_4\alpha_4^5 - 11250A_2A_3A_4\alpha_4^5 + 11250A_3A_4\alpha_4^5 \\
& -15625A_4\alpha_4^5 + 11250A_1A_5\alpha_4^5 - 11250A_1A_2A_5\alpha_4^5 + 11250A_2A_5\alpha_4^5 - 11250A_1A_3A_5\alpha_4^5 \\
& +11850A_1A_2A_3A_5\alpha_4^5 - 11250A_2A_3A_5\alpha_4^5 + 11250A_3A_5\alpha_4^5 - 11250A_1A_4A_5\alpha_4^5 \\
& +11850A_1A_2A_4A_5\alpha_4^5 - 11250A_2A_4A_5\alpha_4^5 + 11850A_1A_3A_4A_5\alpha_4^5 \\
& -12450A_1A_2A_3A_4A_5\alpha_4^5 + 11850A_2A_3A_4A_5\alpha_4^5 - 11250A_3A_4A_5\alpha_4^5 + 11250A_4A_5\alpha_4^5 \\
& -371875A_5\alpha_4^5 - 46590625\alpha_4^5 - 6000A_5^4\alpha_4^4 + 300000A_5^2\alpha_4^4 - 25000A_1\alpha_4^4 + 4375A_1A_2\alpha_4^4 \\
& -25000A_2\alpha_4^4 + 4375A_1A_3\alpha_4^4 + 4375A_2A_3\alpha_4^4 - 25000A_3\alpha_4^4 + 4375A_1A_4\alpha_4^4 \\
& +4375A_2A_4\alpha_4^4 - 1200A_1A_2A_3A_4\alpha_4^4 + 4375A_3A_4\alpha_4^4 - 25000A_4\alpha_4^4 + 4375A_1A_5\alpha_4^4 \\
& +4375A_2A_5\alpha_4^4 - 1200A_1A_2A_3A_5\alpha_4^4 + 4375A_3A_5\alpha_4^4 - 1200A_1A_2A_4A_5\alpha_4^4 \\
& -1200A_1A_3A_4A_5\alpha_4^4 + 1500A_1A_2A_3A_4A_5\alpha_4^4 - 1200A_2A_3A_4A_5\alpha_4^4 + 4375A_4A_5\alpha_4^4 \\
& -737500A_5\alpha_4^4 + 6000A_5^4\alpha_4^3 - 187500A_5^3\alpha_4^3 + 900000A_5^2\alpha_4^3 + 10000A_1A_2\alpha_4^3 \\
& +10000A_1A_3\alpha_4^3 - 4375A_1A_2A_3\alpha_4^3 + 10000A_2A_3\alpha_4^3 + 10000A_1A_4\alpha_4^3 - 4375A_1A_2A_4\alpha_4^3 \\
& +10000A_2A_4\alpha_4^3 - 4375A_1A_3A_4\alpha_4^3 + 1200A_1A_2A_3A_4\alpha_4^3 - 4375A_2A_3A_4\alpha_4^3 \\
& +10000A_3A_4\alpha_4^3 + 10000A_1A_5\alpha_4^3 - 4375A_1A_2A_5\alpha_4^3 + 10000A_2A_5\alpha_4^3 - 4375A_1A_3A_5\alpha_4^3 \\
& +1200A_1A_2A_3A_5\alpha_4^3 - 4375A_2A_3A_5\alpha_4^3 + 10000A_3A_5\alpha_4^3 - 4375A_1A_4A_5\alpha_4^3 \\
& +1200A_1A_2A_4A_5\alpha_4^3 - 4375A_2A_4A_5\alpha_4^3 + 1200A_1A_3A_4A_5\alpha_4^3 + 1200A_2A_3A_4A_5\alpha_4^3 \\
& -4375A_3A_4A_5\alpha_4^3 + 10000A_4A_5\alpha_4^3 + 34500A_5^4\alpha_4^2 - 375000A_5^3\alpha_4^2 - 2500A_1A_2A_3\alpha_4^2 \\
& -2500A_1A_2A_4\alpha_4^2 - 2500A_1A_3A_4\alpha_4^2 + 3775A_1A_2A_3A_4\alpha_4^2 - 2500A_2A_3A_4\alpha_4^2 \\
& -2500A_1A_2A_5\alpha_4^2 - 2500A_1A_3A_5\alpha_4^2 + 3775A_1A_2A_3A_5\alpha_4^2 - 2500A_2A_3A_5\alpha_4^2 \\
& -2500A_1A_4A_5\alpha_4^2 + 3775A_1A_2A_4A_5\alpha_4^2 - 2500A_2A_4A_5\alpha_4^2 + 3775A_1A_3A_4A_5\alpha_4^2 \\
& -1500A_1A_2A_3A_4A_5\alpha_4^2 + 3775A_2A_3A_4A_5\alpha_4^2 - 2500A_3A_4A_5\alpha_4^2 + 53100A_5^4\alpha_4 \\
& -1880A_1A_2A_3A_4\alpha_4 - 1880A_1A_2A_3A_5\alpha_4 - 1880A_1A_2A_4A_5\alpha_4 - 1880A_1A_3A_4A_5\alpha_4 \\
& -3175A_1A_2A_3A_4A_5\alpha_4 - 1880A_2A_3A_4A_5\alpha_4 + 4700A_1A_2A_3A_4A_5) \\
= {} & \frac{1}{3125\alpha_4^5(1-\alpha_4)}\Delta_{34}.
\end{aligned}
$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_4 \in \left(0, \frac{1}{6}\right]$. We begin with

$$\frac{\partial^5 \Delta_{34}}{\partial A_1\, \partial A_2\, \partial A_3\, \partial A_4\, \partial A_5} = 25\left(1062\alpha_4^6 - 498\alpha_4^5 + 60\alpha_4^4 - 60\alpha_4^2 - 127\alpha_4 + 188\right) > 0, \quad \alpha_4 \in \left(0, \frac{1}{6}\right].$$

Thus the partial derivative of Δ_{34} with respect to A_1, A_2, A_3, A_4, and A_5 is positive and minimized at $A_5 = 1$.

$$\left.\frac{\partial^4 \Delta_{34}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=1} = -5\left(564\alpha_4^6 + 120\alpha_4^5 - 60\alpha_4^4 - 240\alpha_4^3 - 455\alpha_4^2 + 1011\alpha_4 - 940\right) > 0.$$

We continue with,

$$\left.\frac{\partial^3 \Delta_{34}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=2, A_5=1} = -5\left(252\alpha_4^6 + 120\alpha_4^5 + 120\alpha_4^4 + 155\alpha_4^3 - 1165\alpha_4^2 + 2398\alpha_4 - 1880\right) > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2, and A_3 is positive. Furthermore,

$$\left.\frac{\partial^2 \Delta_{34}}{\partial A_1 \partial A_2}\right|_{A_3=3, A_4=2, A_5=1} = -5\left(504\alpha_4^6 + 120\alpha_4^5 - 35\alpha_4^4 + 610\alpha_4^3 - 3505\alpha_4^2 + 7946\alpha_4 - 5640\right) > 0, \text{ and}$$

$$\left.\frac{\partial \Delta_{34}}{\partial A_1}\right|_{A_2=4, A_3=3, A_4=2, A_5=1} = -25\left(252\alpha_4^6 + 127\alpha_4^5 + 210\alpha_4^4 - 275\alpha_4^3 - 2610\alpha_4^2 + 6808\alpha_4 - 4512\right) > 0.$$

Finally, our minimum for Δ_{34} is

$$\Delta_{34}|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} = 5\left(9352104\alpha_4^6 - 9355070\alpha_4^5 - 114085\alpha_4^4 + 182585\alpha_4^3 - 9730\alpha_4^2 - 168604\alpha_4 + 112800\right),$$

which is positive over our desired interval, completing this subcase, and concluding Case V. □

3.6. Case VI.
In this case, we are trying to prove:

THEOREM 3.6. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 6$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. Define $\mu = (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)$. If $P_6 > 0$, then*

$$6!\, P_6 \leq \mu - (1 + 114v - 259v^2 + 205v^3 - 70v^4 + 9v^5)|_{v=a_6-\beta+1},$$

where $v = \lceil a_6 \rceil$ *and* β *is the fractional part of* a_6. *Note that the fractional part* β *of* a_6 *is one of* $\frac{a_6}{a_1}, \frac{a_6}{a_2}, \frac{a_6}{a_3}, \frac{a_6}{a_4}$, *or* $\frac{a_6}{a_5}$. *Equality holds if and only if* $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \in \mathbb{Z}$.

Proof. Before we begin the proof, it will be beneficial to explicitly write out the sharp GLY Conjecture for $n = 6$, which was proven in [17].

Theorem 3.7 (Sharp GLY Conjecture for $n = 6$). *Let* $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 >= 5$ *be real numbers. Then,*

$$\begin{aligned}720P_6 \leq{}& a_1a_2a_3a_4a_5a_6 - \frac{5}{2}(a_1a_2a_3a_4a_5 + a_1a_2a_3a_4a_6 + a_1a_2a_3a_5a_6 + a_1a_2a_4a_5a_6\\&+a_1a_3a_4a_5a_6 + a_2a_3a_4a_5a_6) - 24(a_1 + a_2 + a_3 + a_4 + a_5) + \frac{137}{5}(a_1a_2\\&+a_1a_3 + a_1a_4 + a_1a_5 + a_2a_3 + a_2a_4 + a_2a_5 + a_3a_4 + a_3a_5 + a_4a_5)\\&-\frac{45}{2}(a_1a_2a_3 + a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_4 + a_1a_3a_5 + a_1a_4a_5 + a_2a_3a_4\\&+a_2a_3a_5 + a_2a_4a_5 + a_3a_4a_5) + 17(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5\\&+a_1a_3a_4a_5 + a_2a_3a_4a_5).\end{aligned}$$

Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \in \mathbb{Z}_+$.

To prove Theorem 3.6, we want to show that the RHS of its inequality is greater than the RHS of the inequality in Theorem 3.7. Taking their difference and substituting in $A_i = \frac{a_i}{a_6}$ for $i = 1, 2, 3, 4$, and 5, we get

$$\begin{aligned}\Delta_{35} ={}& \frac{1}{10}(15A_1A_2A_3A_4A_5a_6^5 + 15A_1A_2A_3A_4a_6^5 - 160A_1A_2A_3A_4a_6^4 + 15A_1A_2A_3A_5a_6^5\\&-160A_1A_2A_3A_5a_6^4 + 10A_1A_2A_3a_6^4 + 215A_1A_2A_3a_6^3 + 15A_1A_2A_4A_5a_6^5\\&-160A_1A_2A_4A_5a_6^4 + 10A_1A_2A_4a_6^4 + 215A_1A_2A_4a_6^3 + 10A_1A_2A_5a_6^4 + 215A_1A_2A_5a_6^3\\&-10A_1A_2a_6^3 - 264A_1A_2a_6^2 + 15A_1A_3A_4A_5a_6^5 - 160A_1A_3A_4A_5a_6^4 + 10A_1A_3A_4a_6^4\\&+215A_1A_3A_4a_6^3 + 10A_1A_3A_5a_6^4 + 215A_1A_3A_5a_6^3 - 10A_1A_3a_6^3 - 264A_1A_3a_6^2\\&+10A_1A_4A_5a_6^4 + 215A_1A_4A_5a_6^3 - 10A_1A_4a_6^3 - 264A_1A_4a_6^2 - 10A_1A_5a_6^3 - 264A_1A_5a_6^2\\&+10A_1a_6^2 + 230A_1a_6 + 15A_2A_3A_4A_5a_6^5 - 160A_2A_3A_4A_5a_6^4 + 10A_2A_3A_4a_6^4\\&+215A_2A_3A_4a_6^3 + 10A_2A_3A_5a_6^4 + 215A_2A_3A_5a_6^3 - 10A_2A_3a_6^3 - 264A_2A_3a_6^2\\&+10A_2A_4A_5a_6^4 + 215A_2A_4A_5a_6^3 - 10A_2A_4a_6^3 - 264A_2A_4a_6^2 - 10A_2A_5a_6^3 - 264A_2A_5a_6^2\\&+10A_2a_6^2 + 230A_2a_6 + 10A_3A_4A_5a_6^4 + 215A_3A_4A_5a_6^3 - 10A_3A_4a_6^3 - 264A_3A_4a_6^2\\&-10A_3A_5a_6^3 - 264A_3A_5a_6^2 + 10A_3a_6^2 + 230A_3a_6 - 10A_4A_5a_6^3 - 264A_4A_5a_6^2 + 10A_4a_6^2\\&+230A_4a_6 + 10A_5a_6^2 + 230A_5a_6 - 90a_6^5 + 450a_6^4\beta + 250a_6^4 - 900a_6^3\beta^2 - 1000a_6^3\beta\\&-150a_6^3 + 900a_6^2\beta^3 + 1500a_6^2\beta^2 + 450a_6^2\beta - 260a_6^2 - 450a_6\beta^4 - 1000a_6\beta^3 - 450a_6\beta^2\\&+520a_6\beta + 230a_6 + 90\beta^5 + 250\beta^4 + 150\beta^3 - 260\beta^2 - 240\beta + 10),\end{aligned}$$

which is symmetric in A_1, A_2, A_3, A_4, andA_5. Thus, without loss of generality, let $\beta = \frac{a_6}{a_5}$ and substitute it into Δ_{35}, yielding

$$\begin{aligned}\Delta_{36} ={}& \frac{1}{10A_5^5}(15A_1A_2A_3a_6^5A_5^6 + 15A_1A_2A_4a_6^5A_5^6 + 15A_1A_3A_4a_6^5A_5^6 + 15A_1A_2A_3A_4a_6^5A_5^6\\&+15A_2A_3A_4a_6^5A_5^6 + 10A_1A_2a_6^4A_5^6 + 10A_1A_3a_6^4A_5^6 - 160A_1A_2A_3a_6^4A_5^6 + 10A_2A_3a_6^4A_5^6\\&+10A_1A_4a_6^4A_5^6 - 160A_1A_2A_4a_6^4A_5^6 + 10A_2A_4a_6^4A_5^6 - 160A_1A_3A_4a_6^4A_5^6\end{aligned}$$

$$-160A_2A_3A_4a_6^4A_5^6+10A_3A_4a_6^4A_5^6-10A_1a_6^3A_5^6+215A_1A_2a_6^3A_5^6-10A_2a_6^3A_5^6$$
$$+215A_1A_3a_6^3A_5^6+215A_2A_3a_6^3A_5^6-10A_3a_6^3A_5^6+215A_1A_4a_6^3A_5^6+215A_2A_4a_6^3A_5^6$$
$$+215A_3A_4a_6^3A_5^6-10A_4a_6^3A_5^6-264A_1a_6^2A_5^6-264A_2a_6^2A_5^6-264A_3a_6^2A_5^6-264A_4a_6^2A_5^6$$
$$+10a_6^2A_5^6+230a_6A_5^6+15A_1A_2A_3A_4a_6^5A_5^5-90a_6^5A_5^5+10A_1A_2A_3a_6^4A_5^5$$
$$+10A_1A_2A_4a_6^4A_5^5+10A_1A_3A_4a_6^4A_5^5-160A_1A_2A_3A_4a_6^4A_5^5+10A_2A_3A_4a_6^4A_5^5$$
$$+250a_6^4A_5^5-10A_1A_2a_6^3A_5^5-10A_1A_3a_6^3A_5^5+215A_1A_2A_3a_6^3A_5^5-10A_2A_3a_6^3A_5^5$$
$$-10A_1A_4a_6^3A_5^5+215A_1A_2A_4a_6^3A_5^5-10A_2A_4a_6^3A_5^5+215A_1A_3A_4a_6^3A_5^5$$
$$+215A_2A_3A_4a_6^3A_5^5-10A_3A_4a_6^3A_5^5-150a_6^3A_5^5+10A_1a_6^2A_5^5-264A_1A_2a_6^2A_5^5$$
$$+10A_2a_6^2A_5^5-264A_1A_3a_6^2A_5^5-264A_2A_3a_6^2A_5^5+10A_3a_6^2A_5^5-264A_1A_4a_6^2A_5^5$$
$$-264A_2A_4a_6^2A_5^5-264A_3A_4a_6^2A_5^5+10A_4a_6^2A_5^5-260a_6^2A_5^5+230A_1a_6A_5^5+230A_2a_6A_5^5$$
$$+230A_3a_6A_5^5+230A_4a_6A_5^5+230a_6A_5^5+10A_5^5+450a_6^4A_5^4-1000a_6^3A_5^4+450a_6^2A_5^4$$
$$+520a_6A_5^4-240A_5^4-900a_6^3A_5^3+1500a_6^2A_5^3-450a_6A_5^3-260A_5^3+900a_6^2A_5^2$$
$$-1000a_6A_5^2+150A_5^2-450a_6A_5+250A_5+90)$$
$$=\frac{1}{10A_5^5}\Delta_{37}.$$

We apply the partial differentiation test to Δ_{37} over the interval $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5 \geq 1$ and $a_6 > 6$. We begin with,

$$\frac{\partial^{10}\Delta_{37}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5^6} = 10800a_6^5 > 0.$$

We continue:

$$\left.\frac{\partial^{9}\Delta_{37}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5^5}\right|_{A_5=1} = 600a_6^4(21a_6-32) > 0,$$

$$\left.\frac{\partial^{8}\Delta_{37}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5^4}\right|_{A_5=1} = 2400a_6^4(3a_6-8) > 0,$$

$$\left.\frac{\partial^{7}\Delta_{37}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5^3}\right|_{A_5=1} = 300a_6^4(9a_6-32) > 0,$$

$$\left.\frac{\partial^{6}\Delta_{37}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5^2}\right|_{A_5=1} = 50a_6^4(15a_6-64) > 0,$$

$$\left.\frac{\partial^{5}\Delta_{37}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5^1}\right|_{A_5=1} = 5a_6^4(33a_6-160) > 0,$$

$$\left.\frac{\partial^{4}\Delta_{37}}{\partial A_1\partial A_2\partial A_3\partial A_4}\right|_{A_4=A_5=1} = 10a_6^4(3a_6-16) > 0,$$

$$\left.\frac{\partial^{3}\Delta_{37}}{\partial A_1\partial A_2\partial A_3}\right|_{A_3=A_4=A_5=1} = 5a_6^3\left(9a_6^2-62a_6+43\right),$$

which briefly dips under zero (for $a_6 < 6.15$), but this is a non-issue because:

$$\left.\frac{\partial^2 \Delta_{37}}{\partial A_1 \partial A_2}\right|_{A_3=A_4=A_5=1} = a_6^2 \left(60a_6^3 - 450a_6^2 + 635a_6 - 264\right),$$

is always positive. We continue, noting that,

$$\left.\frac{\partial \Delta_{37}}{\partial A_1}\right|_{A_2=A_3=A_4=A_5=1} = a_6 \left(75a_6^4 - 580a_6^3 + 1250a_6^2 - 1046a_6 + 230\right) > 0.$$

Thus, we must only test the minimum value of Δ_{37}, which is:

$$\Delta_{37}|_{A_1=A_2=A_3=A_4=A_5=1} = 0,$$

indicating that Δ_{37} is non-negative, and that this case is complete. □

4. Conclusion. We have thus proven the six-dimensional case of the Yau Geometric Conjecture. The statement of the Conjecture contains a condition that $P_6 > 0$ for it to hold. However, Yau and Zuo [22, 23] proved a similar statement in the case that $p_g = 0$ (the geometric genus of the singularity is zero). Thus, this paper completely solves the problem of finding a sharp upper bound in the six-dimensional case.

This research project raises a number of interesting questions regarding further study of P_n and Q_n. For instance, can the Yau Geometric Conjecture be proven in general using the methods outlined in this paper (splitting the simplex into smaller levels and summing upper estimates)? Can the general case of the Yau Number-Theoretic Conjecture be approached similarly?

It is also important to consider lower bounds of P_n and Q_n. Currently, upper bounds have been extensively studied (as evidenced by the number of conjectures and theorems on the topic), but relatively little is known about lower bounds. A lower bound for P_n and Q_n is still interesting for the applications outlined in the Introduction of this paper. In fact a sharp lower bound can be used to estimate the geometric genus of weighted homogeneous isolated complete intersection singularities.

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