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ON HARNACK INEQUALITIES FOR WITTEN LAPLACIAN ON RIEMANNIAN MANIFOLDS WITH SUPER RICCI FLOWS*

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Dedicated to Prof. Ngaiming Mok for his 60th birthday

Abstract. In this paper, we prove the Li-Yau type Harnack inequality for the heat equation $\partial_t u = Lu$ associated with the time dependent Witten Laplacian on manifolds equipped with a variant of complete backward $(-K, m)$-super Perelman Ricci flows. Moreover, using a probabilistic approach we prove an improved Hamilton type Harnack inequality on manifolds equipped with complete $(-K)$-super Perelman Ricci flows.

Key words. Harnack inequality, super Perelman Ricci flows, Witten Laplacian.

Mathematics Subject Classification. Primary 58J35, 58J65; Secondary 60J60, 60H30.

1. Introduction.

1.1. Motivation. Differential Harnack inequality is an important topic in the study of heat equations and geometric flows on Riemannian manifolds. Let M be an n dimensional complete Riemannian manifold, u be a positive solution to the heat equation

$$\partial_t u = \Delta u. \tag{1}$$

In [9], Li and Yau proved that, if the Ricci curvature is bounded from below by a negative constant, i.e., $Ric \geq -K$, where $K \geq 0$ is a constant, then for all $\alpha > 1$, the following differential Harnack inequality holds

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{\sqrt{2}(\alpha - 1)}, \quad \forall t > 0. \tag{2}$$

In particular, if $Ric \geq 0$, then taking $\alpha \to 1$, the Li-Yau Harnack inequality holds

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}, \quad \forall t > 0. \tag{3}$$

In [8], on complete Riemannian manifolds with $Ric \geq -K$, Hamilton proved a variant of the Li-Yau type Harnack inequality for any positive solution to the heat equation (1)

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{n}{2t} e^{4Kt}, \quad \forall t > 0. \tag{4}$$

In particular, when $K = 0$, the above inequality reduces to the Li-Yau Harnack inequality (3) on complete Riemannian manifolds with non-negative Ricci curvature.

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In the same paper [8], Hamilton also proved a dimension free Harnack inequality on compact Riemannian manifolds with Ricci curvature bounded from below. More precisely, if $Ric \geq -K$, where $K \geq 0$ is a constant, then, for any positive and bounded solution u to the heat equation (1), it holds

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \log\left(\frac{A}{u}\right), \quad \forall t > 0, \tag{5}$$

where $A = \sup\{u(t,x) : x \in M, t \geq 0\}$. Indeed, the same result holds for positive solutions (with suitable growth condition) to the heat equation $\partial_t u = \Delta u$ on complete Riemannian manifolds with $Ric \geq -K$.

Let (M,g) be a complete Riemannian manifold, $\phi \in C^2(M)$ (called a potential function), and $d\mu = e^{-\phi}dv$, where v is the Riemannian volume measure on (M,g). The Witten Laplacian on (M,g,ϕ) is defined by

$$L = \Delta - \nabla\phi \cdot \nabla.$$

For all $u, v \in C_0^\infty(M)$, we have the integration by parts formula

$$\int_M \langle \nabla u, \nabla v \rangle d\mu = -\int_M Luv d\mu = -\int_M uLv d\mu.$$

By [2], for any $u \in C^\infty(M)$, the generalized Bochner formula holds

$$L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu \rangle = 2|\nabla^2 u|^2 + 2Ric(L)(\nabla u, \nabla u), \tag{6}$$

where $\nabla^2 u$ is the Hessian of u, $|\nabla^2 u|$ denotes its Hilbert-Schmidt norm, and

$$Ric(L) = Ric + \nabla^2 \phi.$$

In the literature, $Ric(L)$ is called the (infinite dimensional) Bakry-Emery Ricci curvature associated with the Witten Laplacian L on (M,g,ϕ). It plays as a good substitute of the Ricci curvature in many problems in comparison geometry and analysis on complete Riemannian manifolds with smooth weighted volume measures. See [2, 17, 10, 12] and references therein.

Following [2, 10], we introduce the m-dimensional Bakry-Emery Ricci curvature on (M,g,ϕ) by

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n},$$

where $m \geq n$ is a constant, and $m = n$ if and only if ϕ is a constant. When $m = \infty$, we have $Ric_{\infty,n}(L) = Ric(L)$. Following [2, 10], we say that the Witten Laplacian L satisfies the $CD(K,\infty)$ condition if $Ric(L) \geq K$, and L satisfies the $CD(K,m)$ condition if $Ric_{m,n}(L) \geq K$. Recall that, when $m \in \mathbb{N}$, the m-dimensional Bakry-Emery Ricci curvature $Ric_{m,n}(L)$ has a very natural geometric interpretation. Indeed, consider the warped product metric on $M^n \times S^{m-n}$ defined by

$$\widetilde{g} = g_M \bigoplus e^{-\frac{2\phi}{m-n}} g_{S^{m-n}}.$$

where S^{m-n} is the unit sphere in $\mathbb{R}^{m-n+1}$ with the standard metric $g_{S^{m-n}}$. By [17, 10], the quantity $Ric_{m,n}(L)$ is equal to the Ricci curvature of the above warped product metric $\widetilde{g}$ on $M^n \times S^{m-n}$ along the horizontal vector fields.

In [10, 11], the Li-Yau Harnack inequalities (2) and (3) were extended to positive solutions of the heat equation

$$\partial_t u = Lu, \tag{7}$$

associated with the Witten Laplacian on complete Riemannian manifolds with the $CD(K,m)$-condition for $K \in \mathbb{R}$ and $m \in [n,\infty)$. As application, two-sides Gaussian type heat kernel estimates and the Varadhan short time asymptotic behavior of the heat kernel for the Witten Laplacian were proved in [10, 11]. In [12], an improved version of Hamilton's Harnack inequality (5) was proved for any positive and bounded solution to the heat equation (7) of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K,\infty)$-condition. More precisely, letting (M,g) be a complete Riemannian manifold, $\phi \in C^2(M)$, and assuming that

$$Ric(L) \geq -K,$$

where $K \geq 0$ is a constant, then for any positive and bounded solution u to the heat equation (7), the following optimal dimension free differential Harnack inequality was proved in [11]

$$\frac{|\nabla u|^2}{u^2} \leq \frac{2K}{1-e^{-2Kt}} \log\left(\frac{A}{u}\right), \tag{8}$$

where $A = \sup\{u(t,x) : x \in M, t \geq 0\}$. As far as we know, the above estimate is sharp even for the heat equation $\partial_t u = \Delta u$ on complete Riemannian manifolds with Ricci curvature bounded from below by $-K$, i.e., $Ric \geq -K$. Using the inequality $\frac{2K}{1-e^{-2Kt}} \leq 2K + \frac{1}{t}$ for $K \geq 0$ and $t > 0$, Hamilton's Harnack inequality (5) for positive and bounded solution to the heat equation (7) can be derived from (8).

The aim of this paper is to extend the Li-Yau type and Hamilton type dimension free Harnack inequalities to positive solutions of the heat equation (7) for the time dependent Witten Laplacian on manifolds equipped with a family of complete time dependent Riemannian metrics and potentials. We would like to mention that the Li-Yau and the Hamilton type Harnack inequalities for heat equation $\partial_t u = \Delta u$ on compact or complete Ricci flow have been studied by many authors in the literature. See [4, 6, 19, 20] and references therein. In this paper we will prove the Li-Yau and Harmilton type Harnack inequalities for the heat equation of the time dependent Witten Laplacian on manifolds equipped with a variant of the complete backward $(-K,m)$-super Perelman Ricci flows and the complete $(-K)$-super Perelman Ricci flows which we will introduce in Section 1.2 below. Indeed, we can also extend the Li-Yau-Hamilton Harnack inequality (5) to positive solutions of the heat equation (7) for the time dependent Witten Laplacian on manifolds equipped with a variant of complete backward super Perelman Ricci flows. Due to the limit of space, we would like to do this in a separate paper.

1.2. Statement of main results. Let $(M, g(t), \phi(t), t \in [0,T])$ be a manifold equipped with a family of time dependent complete Riemannian metrics $g(t)$ and potential functions $\phi(t) \in C^2(M)$, $t \in [0,T]$. In this paper, we call $(M, g(t), \phi(t), t \in [0,T])$ a (K,m)-super Perelman Ricci flow if the metric $g(t)$ and the potential function $\phi(t)$ satisfy the following inequality

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq Kg.$$

When $m = \infty$, i.e., if the metric $g(t)$ and the potential function $\phi(t)$ satisfy the following inequality

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) \geq Kg,$$

we call $(M, g(t), \phi(t), t \in [0, T])$ a (K, ∞)-super Perelman Ricci flow or a K-super Perelman Ricci flow. Note that, when $\phi(t) \equiv 0$, $t \in [0, T]$, we see that $(M, g(t), \phi(t) \equiv 0, t \in [0, T])$ is a (K, m)-super Perelman Ricci flow for any $m \in [n, \infty]$ if and only if $(M, g(t), t \in [0, T])$ is the K-super Ricci flow in the sense of Hamilton

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric \geq Kg.$$

On the other hand, we would like to mention that, the Perelman Ricci flow

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) = 0$$

has been introduced in [18] as the gradient flow of Perelman's $\mathcal{F}$-functional $\mathcal{F}(g, \phi) = \int_M (R + |\nabla\phi|^2) e^{-\phi} dv$ on $\mathcal{M} \times C^\infty(M)$ under the constraint condition that $d\mu = e^{-\phi} dv$ does not change in time, where $\mathcal{M}$ denotes the set of all Riemannian metrics on M.

Our first result, Theorem 1.1, extends the improved Hamilton type dimension free Harnack inequality (8) to positive and bounded solutions of the heat equation $\partial_t u = Lu$ for time dependent Witten Laplacian on manifolds equipped with a complete $(-K)$-super Perelman Ricci flow. As far as we know, our result is new even in the case of super Ricci flow without potential, i.e., $\phi(t) = 0$, $t \in [0, T]$. See Theorem 1.2.

THEOREM 1.1. *Let M be a manifold equipped with a family of time dependent complete Riemannian metrics and C^2-potentials $(g(t), \phi(t), t \in [0, T])$. Suppose that $(M, g(t), \phi(t), t \in [0, T])$ is a complete $(-K)$-super Perelman Ricci flow*

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) \geq -Kg, \tag{9}$$

where $K \geq 0$ is a constant independent of $t \in [0, T]$. Let u be a positive and bounded solution to the heat equation

$$\partial_t u = Lu,$$

where

$$L = \Delta_{g(t)} - \nabla_{g(t)}\phi(t) \cdot \nabla_{g(t)}$$

is the time dependent Witten Laplacian on $(M, g(t), \phi(t))$. Suppose that

$$\int_0^T \int_M \left(\left| \nabla \left(\frac{|\nabla u|^2}{u} \right)(y) \right|^2 + |\nabla(u \log u)|^2(y) \right) p_{0,t}(x, y) d\mu(y) dt < \infty, \tag{10}$$

where $p_{s,t}(x, y)$ denotes the fundamental solution to the heat equation $\partial_t u = Lu$ with respect to the weighted volume measure μ, $0 \leq s \leq t \leq T$. Then for all $x \in M$ and $t > 0$,

$$\frac{|\nabla u|^2}{u^2} \leq \frac{2K}{1 - e^{-2Kt}} \log\left(\frac{A}{u}\right), \tag{11}$$

where $A = \sup\{u(t,x) : x \in M, t \geq 0\}$*. Using the inequality* $\frac{2K}{1-e^{-2Kt}} \leq 2K + \frac{1}{t}$ *for* $K \geq 0$ *and* $t > 0$*, we have the Hamilton Harnack inequality*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \log\left(\frac{A}{u}\right). \tag{12}$$

In particular, when $K = 0$*, i.e.,* $(M, g(t), \phi(t), t \in [0,T])$ *is a manifold equipped with a complete super Perelman Ricci flow*

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) \geq 0,$$

we have

$$\frac{|\nabla u|^2}{u^2} \leq \frac{1}{t} \log\left(\frac{A}{u}\right).$$

In particular, when $\phi(t) \equiv 0$, $t \in [0,T]$, we have the following improved Hamilton type dimension free Harnack inequality for the heat equation (1) of the time dependent Laplace-Beltrami on manifolds with $(-K)$-super Ricci flows.

THEOREM 1.2. *Let* $(M, g(t), t \in [0,T])$ *be a manifold equipped with a complete* $(-K)$*-super Ricci flow*

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric \geq -Kg,$$

where $K \geq 0$ *is a constant independent of* $t \in [0,T]$*. Let* u *be a positive and bounded solution to the heat equation associated with the time dependent Laplace-Beltrami*

$$\partial_t u = \Delta u.$$

Suppose that

$$\int_0^T \int_M \left(\left|\nabla\left(\frac{|\nabla u|^2}{u}\right)(y)\right|^2 + |\nabla(u \log u)|^2(y)|\right) p_{0,t}(x,y) dvol(y) dt < \infty,$$

where $p_{s,t}(x,y)$ *denotes the fundamental solution to the heat equation* $\partial_t u = \Delta u$ *with respect to the volume measure* $dvol$*,* $0 \leq s \leq t \leq T$*. Then for all* $x \in M$ *and* $t > 0$*,*

$$\frac{|\nabla u|^2}{u^2} \leq \frac{2K}{1-e^{-2Kt}} \log\left(\frac{A}{u}\right),$$

where $A = \sup\{u(t,x) : x \in M, t \geq 0\}$*. Using the inequality* $\frac{2K}{1-e^{-2Kt}} \leq 2K + \frac{1}{t}$ *for* $K \geq 0$ *and* $t > 0$*, we have the Hamilton Harnack inequality*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \log\left(\frac{A}{u}\right).$$

In particular, when $K = 0$*, i.e.,* $(M, g(t), t \in [0,T])$ *is a manifold equipped with a complete super Ricci flow*

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric \geq 0,$$

we have

$$\frac{|\nabla u|^2}{u^2} \leq \frac{1}{t}\log\left(\frac{A}{u}\right). \tag{13}$$

Integrating the differential Harnack inequality (11) along geodesics on $(M, g(t))$, we have the following Harnack inequality for positive solutions of the heat equation of the time dependent Witten Laplacian on super Perelman Ricci flows.

COROLLARY 1.3. *Under the same condition and notation as in Theorem 1.1 and Theorem 1.2, for any $\delta > 0$, and for all $x, y \in M$, $0 < t < T$, we have*

$$u(x,t) \leq A^{\frac{\delta}{1+\delta}} u(y,t)^{\frac{1}{1+\delta}} \exp\left\{\frac{1+\delta^{-1}}{4(1+\delta)}\frac{2K}{1-e^{-2Kt}} d_t^2(x,y)\right\}.$$

where $d_t(x, y)$ denotes the distance between x and y in $(M, g(t))$. In particular, when $K = 0$, we have

$$u(x,t) \leq A^{\frac{\delta}{1+\delta}} u(y,t)^{\frac{1}{1+\delta}} \exp\left\{\frac{1+\delta^{-1}}{4(1+\delta)}\frac{d_t^2(x,y)}{t}\right\}.$$

The next result extends the Li-Yau type Harnack inequality to positive solutions of the heat equation $\partial_t u = Lu$ for time dependent Witten Laplacian on compact or complete Riemannian manifolds equipped with a variant of the backward $(-K, m)$-super Perelman Ricci flows.

THEOREM 1.4. *Let $(M, g(t), t \in [0, T])$ be a Riemannian manifold with a family of time dependent complete Riemannian metrics $g(t)$ and potentials $\phi(t) \in C^2(M)$, $t \in [0, T]$. Let $L = \Delta_{g(t)} - \nabla_{g(t)}\phi(t) \cdot \nabla_{g(t)}$, and u be a positive solution to the heat equation $\partial_t u = Lu$. Let $\partial_t g = 2h$ and $\alpha > 1$. Suppose that there exist two constants $K \geq 0$ and $m > n$ independent of $t \in [0, T]$ such that*[1]

$$\frac{1}{2}(1-\alpha)\partial_t g + Ric_{m,n}(L) \geq -Kg, \tag{14}$$

and assume that $A^2 = \max\left[|h|^2 + \frac{(\mathrm{Tr}h)^2}{m-n}\right] < \infty$ and $B = \max|S| < \infty$, where

$$S(\cdot) = 2h(\nabla\phi, \cdot) - \langle 2\mathrm{div}h - \nabla\mathrm{Tr}_g h + \nabla\partial_t\phi, \cdot\rangle + \frac{2\mathrm{Tr}h}{m-n}\langle\nabla\phi, \cdot\rangle.$$

If M is compact, then for any $\gamma > 0$ and $t \in (0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1 + \sqrt{1 + \frac{t^2}{m}\left(4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{2B^2}{\gamma}\right)}\right].$$

If $(M, g(t), t \in [0, T])$ are complete, then for any $\gamma > 0$ and $t \in (0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1 + C_4(K_2 + \sqrt{K_1})t + \sqrt{(1 + C_4(K_2 + \sqrt{K_1})t)^2 + \frac{Dt^2}{m}}\right] \tag{15}$$

[1]Indeed, since $\alpha > 1$, (14) is a variant of the backward $(-K, m)$-super Perelman Ricci flows, for example, when $\alpha = 2$, (14) is the backward $(-K, m)$-super Perelman Ricci flow, i.e., $-\frac{1}{2}\partial_t g + Ric_{m,n}(L) \geq -Kg$.

where C_4 is a constant depending only on m, K_1 and K_2 are two positive constants such that $Ric_{m,n}(L) \geq -K_1$ and $h \geq -K_2$, and $D = 4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{B^2}{2\gamma}$.

When $\phi(t) \equiv 0$, $t \in [0,T]$, we have the following Li-Yau Harnack inequality for positive solutions of the heat equation $\partial_t u = \Delta_{g(t)} u$ on complete Riemannian manifolds equipped with a variant of the backward $(-K)$-super Ricci flows.

THEOREM 1.5. *Let $(M, g(t), t \in [0,T])$ be a manifold equipped with a family of time dependent complete Riemannian metrics $g(t)$. Let u be a positive solution to the heat equation*

$$\partial_t u = \Delta_{g(t)} u.$$

Let $\partial_t g = 2h$ and $\alpha > 1$. Suppose that there exist two constants $K \geq 0$ and $m > n$ independent of $t \in [0,T]$ such that[2]

$$\frac{1}{2}(1-\alpha)\partial_t g + Ric \geq -Kg, \tag{16}$$

and assume that $A^2 = \max\left[|h|^2 + \frac{(\mathrm{Tr}h)^2}{m-n}\right] < \infty$ and $B = \max|S| < \infty$, where

$$S(\cdot) = -\langle 2\mathrm{div}h - \nabla \mathrm{Tr}_g h, \cdot\rangle.$$

If M is compact, then for any $\gamma > 0$ and $t \in (0,T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1 + \sqrt{1 + \frac{t^2}{m}\left(4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{2B^2}{\gamma}\right)}\right]. \tag{17}$$

If $(M, g(t), t \in [0,T])$ are complete, then for any $\gamma > 0$ and $t \in (0,T]$, we have

$$\begin{aligned}&\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u}\\ &\leq \frac{m\alpha^2}{4t}\left[1 + C_4(m)(K_2 + \sqrt{K_1})t + \sqrt{(1 + C_4(m)(K_2+\sqrt{K_1})t)^2 + \frac{Dt^2}{m}}\right].\end{aligned} \tag{18}$$

where C_4 is a constant depending only on m, K_1 and K_2 are two positive constants such that $Ric \geq -K_1$ and $h \geq -K_2$, and $D = 4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{B^2}{2\gamma}$.

By standard method as in Li-Yau [9] and Chow et al [6], integrating the above Li-Yau differential Harnack quantity along paths on the space-time, we can derive the following parabolic Harnack inequality for the solution of the heat equation on different points in space-time.

COROLLARY 1.6. *Let $(M, \widetilde{g})$ be a complete Riemannian manifold, $(g(t), \phi(t), t \in [0,T])$ be a family of complete Riemannian metrics and C^2-potentials on M. Assuming that for each $t \in [0,T]$, there exists a constant $C > 0$ such that*

$$C^{-1}\widetilde{g} \leq g(t) \leq C\widetilde{g}.$$

[2]Indeed, since $\alpha > 1$, (16) is a variant of the backward $(-K)$-super Ricci flows, for example, for $\alpha = 2$, (16) is the backward $(-K)$-super Ricci flow, i.e., $-\frac{1}{2}\partial_t g + Ric \geq -Kg$.

Let u be a positive solution to the heat equation $\partial_t u = Lu$. Then, under the same condition and notation as in Theorem 1.4 or Theorem 1.5, for any $\alpha > 1$, $x_1, x_2 \in M$ and $0 < t_1 < t_2 \leq T$, we have

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \leq e^{-C_7(t_2-t_1)} \left(\frac{t_1}{t_2}\right)^{\frac{m\alpha}{2}} \exp\left(-\frac{C\alpha}{4}\frac{d_{\tilde{g}}^2(x_1,x_)}{t_2-t_1}\right),$$

where $C_7 = C_4(K_2 + \sqrt{K_1}) + \sqrt{\frac{D}{m}}$ *and* $D = \frac{(2K+\gamma)^2}{(\alpha-1)^2} + \frac{2B^2}{m\gamma} + \frac{4A^2}{m}$.

To end this Section, let us give some remarks and compare our results with known results in the literature.

Remark 1.7.

- In [20], Q. Zhang proved (13) for positive and bounded solutions to the heat equation $\partial_t u = \Delta_{g(t)} u$ on complete Riemmanian manifolds equipped with the Ricci flow $\partial_t g = -2Ric$. By a probabilistic approach, Guo, Philliposki and Thalmaier [7] proved the Hamilton Harnack inequality (13) for the backward heat equation $\partial_t u = -\Delta_{g(t)} u$ on Riemannian manifolds equipped with complete backward super Ricci flow $\partial_t g \leq 2Ric$. See also [4] for Hamilton type Harnack inequality on Riemannian manifolds equipped with complete Ricci flow with $|Ric| \leq K$. In [12], the second author give two proofs of the optimal Hamilton dimension free Harnack inequality (8) for positive and bounded solution to the heat equation $\partial_t u = Lu$ on complete Riemannian manifolds $Ric(L) \geq -K$. The proof of Theorem 1.1 is similar to the probabilistic proof of (8) given in [12]. For another proof of Theorem 1.1 derived from the reversal logarithmic Sobolev inequality on complete $(-K)$-super Perelman Ricci flows, see [14].
- A local version of the Li-Yau Harnack inequality in Theorem 1.4 and Theorem 1.5 is proved in Section 4.2, see Theorem 4.2.
- In [19], J. Sun proved the Li-Yau Harnack inequality for positive solutions of the heat equation $\partial_t u = \Delta_{g(t)} u$ on manifold M with a family of complete Riemannian metrics $(g(t), t \in [0,T])$. The assumptions in [19] are given by: $\partial_t g = 2h$, $Ric \geq -K_1 g$, $-K_2 g \leq h \leq K_3 g$, and $|\nabla h| \leq K_4$. Under these conditions, for any $\alpha > 1$, Sun proved that

  $$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{n\alpha^2}{t} + C(K_1 + K_2 + K_3 + K_4 + \sqrt{2K_4}),$$

  where C depends only on n and α. Indeed, under these conditions, we have

  $$\frac{1}{2}(1-\alpha)\partial_t g + Ric \geq -(K_1 + (\alpha-1)K_3)g,$$

  i.e., (16) holds with $K = K_1 + (\alpha-1)K_3$. Moreover,

  $$|h|^2 \leq n(K_2+K_3)^2.$$

  Using the inequality $|\mathrm{Tr}h|^2 \leq n|h|^2$, we have

  $$A^2 = \max\left[|h|^2 + \frac{(\mathrm{Tr}h)^2}{m-n}\right] \leq \frac{mn}{m-n}(K_2+K_3)^2.$$

On the other hand

$$B = |2\mathrm{div}h - \nabla \mathrm{Tr}_g h| = |2g^{ij}\nabla_i h_{jl} - g^{ij}\nabla_l h_{ij}| \le 3|g||\nabla h| \le 3\sqrt{n}K_4.$$

Therefore, Theorem 1.5 applies and yields the following estimate

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u}\\
&\le \frac{m\alpha^2}{4t}\left[1 + C_4(K_2 + \sqrt{K_1})t\right]\\
&\quad + \frac{m\alpha^2}{4t}\sqrt{(1 + C_4(K_2 + \sqrt{K_1})t)^2 + \frac{t^2}{m}\left(4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{B^2}{2\gamma}\right)}\\
&\le \frac{m\alpha^2}{2t} + C_{m,n,\alpha,\gamma}(1 + \sqrt{K_1} + K_1 + K_2 + K_3 + K_4),
\end{aligned}$$

where $C_{m,n,\alpha,\gamma}$ depends only on m, n, α and γ. Taking $m = 2n$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \le \frac{n\alpha^2}{t} + C_{n,\alpha,\gamma}(1 + \sqrt{K_1} + K_1 + K_2 + K_3 + K_4).$$

- In the case of the Ricci flow, i.e., $\partial_t g = -2Ric$, we have $h = -Ric$, $\mathrm{Tr}h = -R$, where R is the scalar curvature. In this case, $\frac{1}{2}(1-\alpha)\partial_t g + Ric = \alpha Ric$. Thus (16) reads as

$$\alpha Ric \ge -Kg.$$

Note that the second Bianchi identity says that

$$\mathrm{div}Ric - \frac{1}{2}\nabla R = 0.$$

Thus $B \equiv 0$ for all $t \in [0, T]$. Let u be a positive solution to the heat equation $\partial_t u = \Delta_{g(t)} u$ on a Ricci flow $(M, g(t), t \in [0, T])$ with $Ric \ge -\alpha^{-1}Kg$. In the case M is compact, then for any $t \in (0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \le \frac{m\alpha^2}{4t}\left[1 + \sqrt{1 + \frac{4t^2}{m}\left(A^2 + \frac{mK^2}{(\alpha-1)^2}\right)}\right]. \tag{19}$$

In the case $(M, g(t), t \in [0, T])$ are complete, then for any $t \in (0, T]$, we have

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u}\\
&\le \frac{m\alpha^2}{4t}\left[1 + C_4(K_2 + \sqrt{K_1})t\right]\\
&\quad + \frac{m\alpha^2}{4t}\sqrt{(1 + C_4(K_2 + \sqrt{K_1})t)^2 + \frac{4t^2}{m}\left(A^2 + \frac{mK^2}{(\alpha-1)^2}\right)},
\end{aligned} \tag{20}$$

where C_4 is a constant depending only on m, $K_1 \ge 0$ and $K_2 \ge 0$ are two constants such that $Ric \ge -K_1 g$ and $h \ge -K_2 g$, i.e., $-K_1 g \le Ric \le \frac{K_2}{2}g$. See also Bailesteanu-Cao-Pulemotov [4] for the Li-Yau type Harnack estimate on complete Riemannian manifolds with Ricci flow such that $|Ric| \le K$.

- In the case of the backward Ricci flow, i.e., $\partial_t g = 2Ric$, we have $h = Ric$, $\mathrm{Tr}h = R$, where R is the scalar curvature. In this case, for any $\alpha > 1$, $\frac{1}{2}(1-\alpha)\partial_t g + Ric = (2-\alpha)Ric$, and (16) reads as

$$(2-\alpha)Ric \geq -Kg.$$

By the second Bianchi identity, $B \equiv 0$ for all $t \in [0,T]$. Let u be a positive solution to the heat equation $\partial_t u = \Delta_{g(t)}u$ on a backward Ricci flow $(M, g(t), t \in [0,T])$ with $(2-\alpha)Ric \geq -Kg$. Then, if M is compact, for any $t \in (0,T]$, (19) holds, and if $(M, g(t), t \in [0,T])$ are complete, (20) holds. In particular, if $(M, g(t), t \in [0,T])$ is a backward Ricci flow and $\alpha = 2$, we have $K = 0$. In this case, if M is compact, then for any $t \in (0,T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1 + \sqrt{1 + \frac{4A^2t^2}{m}}\right],$$

and if $(M, g(t), t \in [0,T])$ are complete, then for any $t \in (0,T]$, we have

$$\begin{aligned}&\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u}\\ &\leq \frac{m\alpha^2}{4t}\left[1 + C_4(K_2 + \sqrt{K_1})t + \sqrt{(1 + C_4(K_2 + \sqrt{K_1})t)^2 + \frac{4A^2t^2}{m}}\right], \quad (21)\end{aligned}$$

where C_4 is a constant depending only on m, $K_1 \geq 0$ and $K_2 \geq 0$ are two constants such that $Ric \geq -K_1 g$ and $h \geq -K_2 g$, i.e., $Ric \geq -\frac{K_2}{2}g$. Thus $K_2 = 2K_1$ and (21) reads as follows

$$\begin{aligned}&\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u}\\ &\leq \frac{m\alpha^2}{4t}\left[1 + C_4(K_1 + \sqrt{K_1})t + \sqrt{(1 + C_4(K_1 + \sqrt{K_1})t)^2 + \frac{4A^2t^2}{m}}\right].\end{aligned}$$

- In the case $g(t)$ and $\phi(t)$ are independent of $t \in [0,T]$, we have $A = B = 0$, and $K_2 = 0$. Thus, on any compact or complete Riemannian manifold with $Ric_{m,n}(L) \geq -Kg$, for all $t \in (0,T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1 + C_4\sqrt{K}t + \sqrt{(1 + C_4\sqrt{K}t)^2 + \frac{4K^2t^2}{(\alpha-1)^2}}\right].$$

Hence

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{2t}\left[1 + C_4\sqrt{K}t + \frac{Kt}{\alpha-1}\right]. \quad (22)$$

From the proof of Theorem 1.5, we see that $C_4 = C(m-1)$ for some constant $C > 0$. In particular, on any complete Riemannian manifold with $Ric_{m,n}(L) \geq 0$, we recapture the generalized Li-Yau Harnack inequality for any positive solution to the heat equation $\partial_t u = Lu$ (see [10, 11])

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{m}{2t}.$$

Taking $m = n$, $\phi \equiv 0$, and $L = \Delta$, then, for any positive solution to the heat equation $\partial_t u = \Delta u$ on a complete Riemannian manifold with $Ric \geq -K$, and for any $\alpha > 1$, we have (compare with (2))

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n\alpha^2}{2t} \left[1 + C_4 \sqrt{K} t + \frac{Kt}{\alpha - 1} \right],$$

Here $C_4 = C(n-1)$ for some constant $C > 0$. In particular, we recapture the Li-Yau Harnack inequality (3) for any positive solution to the heat equation $\partial_t u = \Delta u$ on any complete Riemannian manifold with non-negative Ricci curvature.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 and Corollary 1.3. In Section 3, we prove the Li-Yau Harnack inequality for the heat equation of time dependent Witten Laplacian on compact Riemannian manifolds equipped with a variant of the backward $(-K, m)$-super Perelman Ricci flows, i.e., Theorem 1.5. In Section 4, we prove Theorem 1.5 on complete Riemannian manifolds equipped with a variant of the backward $(-K, m)$-super Perelman Ricci flows. This paper is an improved version of a part of the authors' preprint [14], which will be divided into three papers due to the limit of the space. See also [15, 16].

2. Hamilton type Harnack inequality on super Ricci flows. To prove the Hamilton type Harnack inequality on $(-K)$-super Perelman Ricci flows, we extend the probabilistic approach which was used in time independent case in [12]. First we introduce the L-diffusion process on $(M, g(t), t \in [0, T])$. Following [1], let $(U_t, t \in [0, T])$ be the solution of the Stratonovich SDE on the orthonormal frame bundle $(O(M), \widetilde{g}(t), t \in [0, T])$, where $\widetilde{g}(t)$ is the Sasaki Riemannian metric on $O(M)$ defined by the Riemnnian metric on $(M, g(t))$

$$dU_t = \sum_{i=1}^{n} H_i(U_t) \circ dW_t^i - \left[(\nabla\phi)^H (U_t) + \sum_{\alpha,\beta=1}^{n} \frac{\partial g}{\partial t}(U_t e_\alpha, U_t e_\beta) V_{\alpha,\beta}(U_t) \right] dt,$$
$$U_0 = u \in (O(M), \widetilde{g}(0)),$$

where $\{H_i\}_{i=1}^n$ denote the canonical vector fields on $(O(M), \widetilde{g}(t))$, $\{V_{\alpha,\beta}\}_{\alpha,\beta=1}^n$ denote the canonical vertical vector fields on $(O(M), \widetilde{g}(t))$, and $(\nabla\phi)^H$ denotes the horizontal lift of the vector field $\nabla\phi$ from $(M, g(t))$ to $(O(M), \widetilde{g}(t))$. The L-diffusion process on $(M, g(t))$ is defined by

$$X_t = \pi(U_t)$$

By [1], for smooth $f \in [0, T] \times M \to \mathbb{R}$, Ito's formula holds

$$df(t, X_t) = (\partial_t + L) f(t, X_t) dt + \sum_{i=1}^{n} \nabla_{e_i} f(t, X_t) dW_t^i.$$

Proof of Theorem 1.1. By direct calculation and the generalized Bochner formula, we have

$$(\partial_t - L) \frac{|\nabla u|^2}{u} = -\frac{2}{u} \left| \nabla^2 u - \frac{\nabla u \otimes \nabla u}{u} \right|^2 - \frac{2}{u} \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla u, \nabla u). \quad (23)$$

Thus, on manifold with a $(-K)$-super Perelman Ricci flow, we have

$$(\partial_t - L)\frac{|\nabla u|^2}{u} \leq 2K\frac{|\nabla u|^2}{u}.$$

Note that

$$(\partial_t - L)(u\log(A/u)) = \frac{|\nabla u|^2}{u}.$$

Let $\psi(t) = \frac{1-e^{-2Kt}}{2K}$. Then $\psi'(t) + 2K\psi(t) = 1$. Define

$$h(x,t) := \psi(t)\frac{|\nabla u|^2}{u} - u\log(A/u).$$

Then at $t = 0$, $h \leq 0$, and for $t > 0$, it holds

$$(\partial_t - L)h \leq [\psi'(t) + 2K\psi(t) - 1]\,\frac{|\nabla u|^2}{u} = 0.$$

In the case M is compact, the maximum principle yields that $h(x,t) \leq 0$ for all time $t > 0$ and $x \in M$. In the case $(M, g(t), t \in [0,T])$ is a complete non-compact Riemannian manifold with a $(-K)$-super Perelman Ricci flow, we can give a probabilistic proof to (11) as follows. Let X_t be the L-diffusion process on $(M, g(t))$ starting from $X_0 = x$. Applying Itô's formula to $h(X_t, T-t)$, $t \in [0,T]$, we have

$$h(X_t, T-t) = h(X_0, T) + \int_0^t \nabla h(X_s, T-s)dW_s + \int_0^t \left(L - \frac{\partial}{\partial t}\right) h(X_s, T-s)ds,$$

where the second term in the right hand side is the Itô's stochastic integral with respect to the Brownian motion $\{W_s, s \in [0,t]\}$. In particular, taking $t = T$, we obtain

$$h(X_T, 0) = h(X_0, T) + \int_0^T \nabla h(X_s, T-s)\cdot dW_s + \int_0^T \left(L - \frac{\partial}{\partial t}\right) h(X_s, T-s)ds.$$

Note that, under the condition (10), we have

$$\mathbb{E}\left[\int_0^T |\nabla h(X_s, T-s)|^2 ds\right] < \infty.$$

Hence $M_t = \int_0^t \nabla h(X_s, T-s)dW_s$ is a martingale with respect to $\mathcal{F}_t = \sigma(W_s, s \leq t)$. Taking the expectation on both sides, the martingale property of Itô's integral implies that

$$E\left[h(X_T, 0)\right] = h(x,T) + E\left[\int_0^T \left(L - \frac{\partial}{\partial t}\right) h(X_s, T-s)ds\right] \geq h(x,T).$$

As $h(y, 0) \leq 0$ for all $y \in M$, we derive that $h(x,T) \leq 0$ for all $T > 0$ and $x \in M$. □

Proof of Corollary 1.3. The proof is similar to the one of Theorem 1.1 in [8]. Let $l(x,t) = \log A/u(x,t)$. Then the differential Harnack inequality (11) in Theorem 1.1 implies

$$|\nabla\sqrt{l(x,t)}| = \frac{1}{2}\frac{|\nabla l(x,t)|}{\sqrt{l(x,t)}} \leq \frac{1}{2}\sqrt{\frac{2K}{1-e^{-2Kt}}}.$$

Fix $x, y \in M$ and integrate along a geodesic on $(M, g(t))$ linking x and y, the above inequality yields

$$\sqrt{\log A/u(x,t)} \leq \sqrt{\log A/u(y,t)} + \frac{1}{2}\sqrt{\frac{2K}{1-e^{-2Kt}}}d_t(x,y).$$

where $d_t(x, y)$ denotes the distance between x and y in $(M, g(t))$. Combining this with the elementary inequality

$$(a+b)^2 \leq (1+\delta)a^2 + (1+\delta^{-1})b^2,$$

we can derive the desired Harnack inequality for u in Corollary 1.3. □

3. Li-Yau Harnack inequality on compact super Perelman Ricci flows. Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let $f = \log u$. Then

$$(L - \partial_t)f = -|\nabla f|^2.$$

Let

$$F = t(|\nabla f|^2 - \alpha f_t).$$

3.1. The commutator $[\partial_t, L]f$. Let M be a manifold with a family of time dependent metrics $(g(t), t \in [0, T])$ and potentials $\phi(t) \in C^2(M)$, $t \in [0, T]$. Let $\partial_t g = 2h$.

LEMMA 3.1. *For any $f \in C^\infty(M)$, we have*

$$\partial_t|\nabla f|^2 = -2h(\nabla f, \nabla f) + 2\langle\nabla f, \nabla f_t\rangle,$$

and

$$[\partial_t, L]f = -2\langle h, \nabla^2 f\rangle + 2h(\nabla\phi, \nabla f) - \langle 2\mathrm{div}h - \nabla \mathrm{Tr}_g h + \nabla\partial_t\phi, \nabla f\rangle. \qquad (24)$$

Proof. By direct calculation, we have

$$\partial_t|\nabla f|^2 = \partial_t(g^{ij}(t)\nabla_i f\nabla_j f) = \partial_t g^{ij}(t)\nabla_i f\nabla_j f + 2g^{ij}(t)\nabla_i f\nabla_j f_t.$$

Note that

$$\partial_t g^{ij}(t) = -\partial_t g_{ij}(t) = -2h_{ij}.$$

The first equality follows. On the other hand, by [5, 19], we have

$$\partial_t\Delta_{g(t)}f = \Delta_{g(t)}\partial_t f - 2\langle h, \nabla^2 f\rangle - 2\langle \mathrm{div}h - \frac{1}{2}\nabla\mathrm{Tr}_g h, \nabla f\rangle.$$

Combining this with

$$\partial_t\langle\nabla\phi, \nabla f\rangle = -\partial_t g(\nabla\phi, \nabla f) + \langle\nabla\phi_t, \nabla f\rangle + \langle\nabla\phi, \nabla f_t\rangle,$$

we obtain (24) in Lemma 3.1. □

LEMMA 3.2.

$$(L-\partial_t)F = 2t\left(|\nabla^2 f|^2 + (Ric(L) + (1-\alpha)h)(\nabla f, \nabla f)\right) - 2\langle\nabla f, \nabla F\rangle - t^{-1}F + \alpha t[\partial_t, L]f. \qquad (25)$$

Proof. By the Bochner formula and using

$$\partial_t|\nabla f|^2_{g(t)} = -\partial_t g(t)(\nabla f, \nabla f) + 2\langle \nabla f, \nabla f_t\rangle_{g(t)},$$

we have

$$\begin{aligned} LF &= tL|\nabla f|^2 - \alpha t L f_t \\ &= 2t\left(|\nabla^2 f|^2 + Ric(L)(\nabla f, \nabla f) + \langle \nabla f, \nabla L f\rangle\right) - \alpha t L\partial_t f \\ &= 2t\left(|\nabla^2 f|^2 + Ric(L)(\nabla f, \nabla f) + \langle \nabla f, \nabla (f_t - |\nabla f|^2)\rangle\right) - \alpha t L\partial_t f \\ &= 2t\left(|\nabla^2 f|^2 + Ric(L)(\nabla f, \nabla f)\right) - 2\langle \nabla f, \nabla F\rangle + 2(1-\alpha)t\langle \nabla f, \nabla f_t\rangle - \alpha t L\partial_t f \\ &= 2t\left(|\nabla^2 f|^2 + Ric(L)(\nabla f, \nabla f)\right) - 2\langle \nabla f, \nabla F\rangle \\ &\qquad +2t(1-\alpha)h(\nabla f, \nabla f) + (1-\alpha)t\partial_t|\nabla f|^2 - \alpha t L\partial_t f. \end{aligned}$$

On the other hand

$$\begin{aligned} \partial_t F &= (|\nabla f|^2 - \alpha f_t) + t\partial_t|\nabla f|^2 - \alpha t f_{tt} \\ &= (|\nabla f|^2 - \alpha f_t) + t\partial_t|\nabla f|^2 - \alpha t\partial_t(Lf + |\nabla f|^2) \\ &= (|\nabla f|^2 - \alpha f_t) + (1-\alpha)t\partial_t|\nabla f|^2 - \alpha t\partial_t Lf. \end{aligned}$$

Combing above formulas, we derive (25). □

LEMMA 3.3. *For any $\alpha > 1$, we have*

$$\begin{aligned} &(L-\partial_t)F \\ &\geq \frac{2F^2}{\alpha^2 mt} + \frac{4(\alpha-1)}{m\alpha^2}|\nabla f|^2 F + \frac{2t(\alpha-1)^2}{m\alpha^2}|\nabla f|^4 - \frac{t\alpha^2}{2}\left[\frac{(\mathrm{Tr}h)^2}{m-n} + |h|^2\right] \\ &\quad +2t(Ric_{m,n}(L) + (1-\alpha)h)(\nabla f, \nabla f) - 2\langle \nabla f, \nabla F\rangle - t^{-1}F + \alpha t S(\nabla f). \end{aligned} \tag{26}$$

Proof. Substituting $[\partial_t, L]f$ into (25), we have

$$\begin{aligned} (L-\partial_t)F = 2t&\left|\nabla^2 f - \frac{\alpha h}{2}\right|^2 - \frac{t\alpha^2|h|^2}{2} + 2t(Ric(L) + (1-\alpha)h)(\nabla f, \nabla f) \\ &-2\langle \nabla f, \nabla F\rangle - t^{-1}F + \alpha t S_1(\nabla f), \end{aligned}$$

where

$$S_1(\nabla f) = 2h(\nabla\phi, \nabla f) - \langle 2\mathrm{div}h - \nabla \mathrm{Tr}_g h + \nabla\phi_t, \nabla f\rangle.$$

Using the inequality $|S|^2 \geq \frac{1}{n}|\mathrm{Tr}S|^2$ for $n\times n$ symmetric matrices S and the Cauchy-Schwartz inequality $(a+b)^2 \geq \frac{a^2}{1+\varepsilon} - \frac{b^2}{\varepsilon}$ for all $\varepsilon > 0$, we can obtain

$$\begin{aligned} \left|\nabla^2 f - \frac{\alpha h}{2}\right|^2 &\geq \frac{1}{n}\left|\Delta f - \frac{\alpha \mathrm{Tr}h}{2}\right|^2 \\ &\geq \frac{|Lf|^2}{n(1+\varepsilon)} - \frac{\left|\nabla\phi\cdot\nabla f - \frac{\alpha\mathrm{Tr}h}{2}\right|^2}{n\varepsilon}. \end{aligned}$$

Let $m := n(1+\varepsilon)$. Then

$$\begin{aligned}
&(L-\partial_t)F \\
&\geq \frac{2t}{m}|Lf|^2 - \frac{2t}{m-n}\left|\nabla\phi\cdot\nabla f - \frac{\alpha \mathrm{Tr}h}{2}\right|^2 - \frac{t\alpha^2|h|^2}{2} + 2t(Ric(L)+(1-\alpha)h)(\nabla f,\nabla f) \\
&\quad -2\langle\nabla f,\nabla F\rangle - t^{-1}F + \alpha t S_1(\nabla f) \\
&= \frac{2t}{m}|Lf|^2 - \frac{t\alpha^2(\mathrm{Tr}h)^2}{2(m-n)} - \frac{t\alpha^2|h|^2}{2} + 2t(Ric_{m,n}(L)+(1-\alpha)h)(\nabla f,\nabla f) \\
&\quad -2\langle\nabla f,\nabla F\rangle - t^{-1}F + \alpha t S_1(\nabla f) + \frac{2\alpha t \mathrm{Tr}h}{m-n}\langle\nabla\phi,\nabla f\rangle. \qquad (27)
\end{aligned}$$

Let

$$S(\cdot) = S_1(\cdot) + \frac{2\mathrm{Tr}h}{m-n}\langle\nabla\phi,\cdot\rangle.$$

Substituting $-LF = |\nabla f|^2 - f_t = \frac{F}{\alpha t} + \frac{\alpha-1}{\alpha}|\nabla f|^2$ into (27), we have

$$\begin{aligned}
(L-\partial_t)F \geq{}& \frac{2t}{m}\left[\frac{F}{\alpha t} + \frac{\alpha-1}{\alpha}|\nabla f|^2\right]^2 - \frac{t\alpha^2}{2}\left[\frac{(\mathrm{Tr}h)^2}{m-n} + |h|^2\right] \\
&+2t(Ric_{m,n}(L)+(1-\alpha)h)(\nabla f,\nabla f) - 2\langle\nabla f,\nabla F\rangle - t^{-1}F + \alpha t S(\nabla f).
\end{aligned}$$

This completes the proof of Lemma 3.3. □

Note that $A^2 = \max\left[|h|^2 + \frac{(\mathrm{Tr}h)^2}{m-n}\right]$, $B = \max|S|$. Under the assumption (14), i.e., $Ric_{m,n}(L) + (1-\alpha)h \geq -K$, when $F \geq 0$ we have

$$\begin{aligned}
(L-\partial_t)F \geq{}& \frac{2F^2}{\alpha^2 mt} + \frac{2t(\alpha-1)^2}{m\alpha^2}|\nabla f|^4 - \frac{t\alpha^2A^2}{2} \\
&-2Kt|\nabla f|^2 - 2\langle\nabla f,\nabla F\rangle - t^{-1}F - \alpha Bt|\nabla f|.
\end{aligned}$$

Using the inequality

$$ax^4 + bx^2 + cx \geq -\frac{(b-\gamma)^2}{4a} - \frac{c^2}{4\gamma},$$

where $\gamma > 0$ is any positive constant, we can derive that

$$\frac{2t(\alpha-1)^2}{m\alpha^2}|\nabla f|^4 - 2Kt|\nabla f|^2 - \alpha Bt|\nabla f| \geq -\frac{m\alpha^2 t(2K+\gamma)^2}{8(\alpha-1)^2} - \frac{\alpha^2B^2t}{4\gamma}.$$

Hence

$$(L-\partial_t)F \geq \frac{2F^2}{\alpha^2 mt} - \frac{F}{t} - 2\langle\nabla f,\nabla F\rangle - \frac{t\alpha^2A^2}{2} - \frac{m\alpha^2t(2K+\gamma)^2}{8(\alpha-1)^2} - \frac{\alpha^2B^2t}{4\gamma}. \qquad (28)$$

3.2. Proof of Theorem 1.4 in compact case. Assume that $F \geq 0$, otherwise Theorem 1.4 is obviously true. When M is compact, let (x_0,t_0) be the point where F achieves the maximum on $M\times[0,T]$. Then $\nabla F(x_0,t_0) = 0$, $\Delta F(x_0,t_0) \leq 0$ and $\partial_t F(x_0,t_0) \geq 0$. Therefore, at (x_0,t_0),

$$(L-\partial_t)F \leq 0.$$

By (28), we have

$$0 \geq \frac{2F^2}{\alpha^2 m t_0} - \frac{F}{t_0} - \frac{t_0 \alpha^2 A^2}{2} - \frac{m\alpha^2 t_0 (2K+\gamma)^2}{8(\alpha-1)^2} - \frac{\alpha^2 B^2 t_0}{4\gamma}.$$

This yields, for any $t \in (0, T]$, we have

$$\begin{aligned} F &\leq \frac{m\alpha^2}{4}\left[1+\sqrt{1+\frac{t_0^2}{m}\left(4A^2+\frac{m(2K+\gamma)^2}{(\alpha-1)^2}+\frac{2B^2}{\gamma}\right)}\right] \\ &\leq \frac{m\alpha^2}{4}\left[1+\sqrt{1+\frac{T^2}{m}\left(4A^2+\frac{m(2K+\gamma)^2}{(\alpha-1)^2}+\frac{2B^2}{\gamma}\right)}\right]. \end{aligned}$$

In particular, at time $t = T$, we derive the Li-Yau Harnack inequality in Theorem 1.5

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1+\sqrt{1+\frac{t^2}{m}\left(4A^2+\frac{m(2K+\gamma)^2}{(\alpha-1)^2}+\frac{2B^2}{\gamma}\right)}\right].$$

□

4. Proof of Theorem 1.4 in complete case.

4.1. A lemma. Fix $o \in M$. Let $Q_{2R,T} = \{(x,t) \in M \times [0,T] : d(x,o,t) \leq 2R, t \in [0,T]\}$. Let $\eta \in C^2([0,\infty),[0,1])$ be such that $\eta(r) = 1$ on $[0,1]$, $\eta = 0$ on $[2,\infty)$, $0 \leq \eta \leq 1$ on $[1,2]$, $\eta'(r) \leq 0$, $|\eta'(r)|^2 \leq C_1\eta(r)$ and $\eta''(r) \geq -C_2$, where C_1 and C_2 are two positive constants. Define

$$\psi(x,t) = \psi(d(x,o,t)) = \eta\left(\frac{d(x,o,t)}{R}\right) = \eta\left(\frac{\rho(x,t)}{R}\right),$$

where $\rho(x,t) = d(x,o,t)$ denotes the geodesic distance between x and o on $(M, g(t))$. We need the Laplacian comparison theorem on manifolds with time dependent metrics and potentials.

Lemma 4.1. *Let M be a complete Riemannian manifold equipped with a family of time dependent complete Riemannian metrics $g(t)$ and potentials $\phi(t)$, $t \in [0,T]$. Let*

$$\partial_t g = 2h.$$

Suppose that $Ric_{m,n}(L) \geq -K_1$, $h \geq -K_2$, where K_1, K_2 are two positive constants. Then

$$(L-\partial_t)\psi \geq -C_1 K_2 \psi^{1/2} - \frac{C_1}{R}(m-1)\sqrt{K_1}\coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}.$$

Proof. By [10], as $Ric_{m,n}(L) \geq -K_1$, the following Laplacian comparison theorem holds

$$Ld(x_0,x,t) \leq (m-1)\sqrt{K_1}\rho\coth(\sqrt{K_1}\rho),$$

and

$$\begin{aligned} L\psi &= \eta'(d(x_0,x,t)/R)\frac{Ld(x_0,x,t)}{R} + \eta''(d(x_0,x,t)/R)\frac{|\nabla d(x_0,x,t)|^2}{R^2} \\ &\geq -\frac{C_1}{R}(m-1)\sqrt{K_1}\coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}. \end{aligned}$$

On the other hand, let $\gamma : [a, b] \to M$ be a fixed path such that $\gamma(a) = x$ and $\gamma(b) = y$. Let $S = \dot{\gamma}(s)$. Given a time $t_0 \in [0, T]$, assuming that γ is parameterize by the arc length with respect to metric $g(t_0)$ on M, then $|S| = 1$ at time $t = t_0$. Moreover, the evolution of the length of γ with respect to $g(t)$ is given by

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=t_0} L_{g(t)}(\gamma) &= \int_a^b \left.\frac{d}{dt}\right|_{t=t_0} \sqrt{g(t)(S,S)}ds \\ &= \frac{1}{2}\int_a^b \left.\frac{\partial_t g(t)(S,S)}{\sqrt{g(t)(S,S)}}\right|_{t=t_0} ds \\ &= \frac{1}{2}\int_a^b \left.\frac{\partial g(t)}{\partial t}(S,S)\right|_{t=t_0} ds. \end{aligned}$$

This yields, under the assumption $h \geq -K_2$, where $K_2 \geq 0$,

$$\partial_t d(x,y,t) = \int_a^b h(S,S)ds \geq -K_2 d(x,y,t).$$

Since $-C_1\eta^{1/2}(r) \leq \eta'(r) \leq 0$, and $K_2 \geq 0$, it holds

$$\begin{aligned} -\partial_t\psi &= -\frac{\eta'(\rho/R)\partial_t d(x_0,x,t)}{R} \\ &\geq \frac{\eta'(\rho/R)K_2 d(x_0,x,t)}{R} \\ &\geq -\frac{C_1K_2}{R}\psi^{1/2}d(x_0,x,t). \end{aligned}$$

Combining this with the lower bound of $L\psi$, we have

$$(L-\partial_t)\psi \geq -C_1K_2\psi^{1/2} - \frac{C_1}{R}(m-1)\sqrt{K_1}\coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}.$$

The proof of Lemma 4.1 is completed. □

4.2. The local version of the Li-Yau Harnack inequality. In this subsection we prove a local version of the Li-Yau type Harnack inequality for positive solutions to the heat equation $\partial_t u = Lu$ on complete Riemannian manicolds with a variant of the backward $(-K, m)$-super Perelman Ricci flows. More precisely, we have the following

THEOREM 4.2. *Let $(M, g(t), \phi(t), t \in [0, T])$ be a manifold equipped with a family of time dependent complete Riemannian metrics and C^2-potentials. Under the same condition as in Theorem 1.5, for any $\alpha > 1$ and $R > 0$, we have the following local Li-Yau type differential Harnack inequality on $Q_{2R,T} = \{(x,t) \in M \times [0,T] : d(x,o,t) \leq 2R, t \in [0,T]\}$*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1 + Et + \sqrt{(1+Et)^2 + \frac{Dt^2}{m}}\right], \tag{29}$$

where $E = C_4(K_2 + \sqrt{K_1}) + \frac{C_5}{R} + \frac{C_6}{R^2}$, $D = 4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{2B^2}{\gamma}$, C_4, C_5 are constants depending only on m, and C_6 is a constant depending only on m and α.

Proof. Let $F = t(|\nabla \log u|^2 - \alpha \partial_t \log u)$. Assume that $F \geq 0$, otherwise (29) is obviously true. Since ρ is Lipschitz on the complement of the cut locus of o, ψ is a Lipschitz function with support in $Q_{2R,T}$. As explained in Li and Yau [43], an argument of Calabi [3] allows us to apply the maximum principle to ψF. Let $(x_0, t_0) \in M \times [0, T]$ be a point where ψF achieves the maximum. Then, at (x_0, t_0),

$$\partial_t(\psi F) \geq 0,\ \Delta(\psi F) \leq 0,\ \nabla(\psi F) = 0,$$

which yields

$$(L - \partial_t)(\psi F) = \Delta(\psi F) - \nabla\phi \cdot \nabla(\psi F) - \partial_t(\psi F) \leq 0.$$

Note that

$$(L - \partial_t)(\psi F) = \psi(L - \partial_t)F + (L - \partial_t)\psi F + 2\nabla\psi \cdot \nabla F.$$

By Lemma 4.1, we have

$$(L - \partial_t)\psi \geq -C_1 K_2 \psi^{1/2} - \frac{C_1}{R}(m-1)\sqrt{K_1}\coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}.$$

Therefore, at (x_0, t_0), we have

$$0 \geq \psi(L - \partial_t)F + 2\nabla\psi \cdot \nabla F - A(R, T)F, \tag{30}$$

where

$$A(R, T) := C_1 K_2 \psi^{1/2} + \frac{C_1}{R}(m-1)\sqrt{K_1}\coth(\sqrt{K_1}\rho) + \frac{C_2}{R^2}.$$

Denote

$$C_3 = A(R, T) + 2|\nabla\psi|^2\psi^{-1}.$$

Using $\sqrt{K_1}\rho\coth(\sqrt{K_1}\rho) \leq 1 + \sqrt{K_1}\rho$, we have

$$\begin{aligned} C_3 &\leq C_1 K_2 + \frac{C_1(m-1)(1 + \sqrt{K_1}R)}{R} + \frac{2C_1 + C_2}{R^2} \\ &\leq C_1(K_2 + (m-1)\sqrt{K_1}) + \frac{C_1(m-1)}{R} + \frac{2C_1 + C_2}{R^2}. \end{aligned}$$

To simplify the notation, write

$$C_3 \leq C_4(K_2 + \sqrt{K_1}) + \frac{C_5}{R} + \frac{C_6}{R^2}.$$

Note that, at (x_0, t_0), $\nabla\psi \cdot \nabla F = -\psi|\nabla\psi|^2 F$. Substituting (26) into (30), at (x_0, t_0), we have

$$\begin{aligned} 0 &\geq \psi(L - \partial_t)F - A(R, T)F + 2\nabla\psi \cdot \nabla F \\ &\geq \psi(L - \partial_t)F - (A(R, T) + 2|\nabla\psi|^2\psi^{-1})F \\ &\geq \frac{2\psi F^2}{m\alpha^2 t} - \left(\frac{\psi}{t} + C_3\right)F + \frac{4(\alpha-1)\psi|\nabla f|^2 F}{m\alpha^2} - \frac{2C_2}{R}\psi^{1/2}|\nabla f|F \\ &\quad + \psi t\left[\frac{2(\alpha-1)^2}{m\alpha^2}|\nabla f|^4 - 2K|\nabla f|^2 - \alpha B|\nabla f| - \frac{\alpha^2 A^2}{2}\right]. \end{aligned}$$

By the inequality $ax^2 - bx \geq \frac{4b^2}{a}$ and (28), and multiplying the both sides by ψt_0, we have

$$0 \geq \frac{2(\psi F)^2}{m\alpha^2} - \left(\psi + C_3 t + \frac{m\alpha^2 C_2^2 t}{4(\alpha-1)R^2}\right)\psi F$$
$$-\frac{\alpha^2\psi^2 t^2}{8}\left(4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{2B^2}{\gamma}\right).$$

Let $D = 4A^2 + \frac{m(2K+\gamma)^2}{(\alpha-1)^2} + \frac{2B^2}{\gamma}$. We see that, at any $(x,t) \in Q_{R,T}$, we have

$$F(x,t) \leq (\psi F)(x_0,t_0)$$
$$\leq \frac{m\alpha^2}{4}\left[1 + C_3 t_0 + \frac{m\alpha^2 C_2^2 t_0}{4(\alpha-1)R^2} + \sqrt{\left(1 + C_3 t_0 + \frac{m\alpha^2 C_2^2 t_0}{4(\alpha-1)R^2}\right)^2 + \frac{D\psi^2 t_0^2}{m}}\right]$$
$$\leq \frac{m\alpha^2}{4}\left[1 + C_3 T + \frac{m\alpha^2 C_2^2 T}{4(\alpha-1)R^2} + \sqrt{\left(1 + C_3 T + \frac{m\alpha^2 C_2^2 T}{4(\alpha-1)R^2}\right)^2 + \frac{D\psi^2 T^2}{m}}\right].$$

In particular, taking $t = T$, we have

$$F(x,t) \leq \frac{m\alpha^2}{4}\left[1 + \left(C_4(K_2 + \sqrt{K_1}) + \frac{C_5}{R} + \frac{C_6}{R^2} + \frac{m\alpha^2 C_2^2}{4(\alpha-1)R^2}\right)t\right]$$
$$+\frac{m\alpha^2}{4}\sqrt{\left(1 + \left(C_4(K_2 + \sqrt{K_1}) + \frac{C_5}{R} + \frac{C_6}{R^2} + \frac{m\alpha^2 C_2^2}{4(\alpha-1)R^2}\right)t\right)^2 + \frac{Dt^2}{m}}.$$

This completes the proof of the Li-Yau Harnack inequality on $Q_{2R,T}$. □

4.3. Proof of Theorem 1.4 and Corollary 1.6.

Proof of Theorem 1.4. When $(M, g(t), t \in [0,T])$ are complete non-compact, taking $R \to \infty$ in (29), we obtain the Li-Yau differential Harnack inequality on $M \times (0,T]$

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t}\left[1 + C_4(K_2 + \sqrt{K_1})t + \sqrt{(1 + C_4(K_2 + \sqrt{K_1})t)^2 + \frac{Dt^2}{m}}\right].$$

This completes the proof of Theorem 1.4. □

Proof of Corollary 1.6. Let $\gamma : [t_1,t_2] \to M$ be a smooth path with $\gamma(t_i) = x_i$, $i = 1,2$. Then

$$\log\frac{u(x_2,t_2)}{u(x_1,t_1)} = \int_{t_1}^{t_2} \frac{d}{dt}\log u(\gamma(t),t)dt$$
$$= \int_{t_1}^{t_2} \left(\partial_t \log u + \langle\nabla \log u, \dot\gamma(t)\rangle\right)dt.$$

By the Li-Yau Harnack differential inequality in Theorem 1.4, we have

$$\log\frac{u(x_2,t_2)}{u(x_1,t_1)}$$
$$\geq \int_{t_1}^{t_2}\left(\frac{1}{\alpha}|\nabla \log u|^2 - \frac{m\alpha}{4t}\left(2(1 + (C_4(K_2 + \sqrt{K_1}) + \sqrt{Dm^{-1}})t\right) + \langle\nabla \log u, \dot\gamma(t)\rangle\right)dt$$
$$\geq -\frac{\alpha}{4}\int_{t_1}^{t_2}|\dot\gamma(t)|^2 dt - \frac{m\alpha}{2}\log\left(\frac{t_2}{t_1}\right) - C_7(t_2 - t_1).$$

Therefore, for any path γ on M with $\gamma(t_i) = x_i$, $i = 1, 2$, we have

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq e^{-C_7(t_2-t_1)} \left(\frac{t_1}{t_2}\right)^{\frac{m\alpha}{2}} \exp\left(-\frac{\alpha}{4}\int_{t_1}^{t_2} |\dot{\gamma}(t)|^2_{g(t)} dt\right).$$

Let $\gamma(t)$ be a constant speed minimal geodesic linking x_1 and x_2 on $(M, \widetilde{g})$. By assumption, we have

$$|\dot{\gamma}(t)|^2_{g(t)} \leq C\, |\dot{\gamma}(t)|^2_{\widetilde{g}} = \frac{C d^2_{\widetilde{g}}(x_1,x_2)}{(t_2-t_1)^2}.$$

This yields

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq e^{-C_7(t_2-t_1)} \left(\frac{t_1}{t_2}\right)^{\frac{m\alpha}{2}} \exp\left(-\frac{C\alpha}{4}\frac{d^2_{\widetilde{g}}(x_1,x_2)}{t_2-t_1}\right).$$

□

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