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ON THE SECOND MAIN THEOREM OF NEVANLINNA THEORY FOR SINGULAR DIVISORS WITH (k, ℓ)-CONDITIONS*

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Dedicated to Professor Ngaiming Mok on his 60th Birthday

Abstract. In this paper, we introduce the (k, ℓ)-condition on divisors by using germ decompositions and a new ramification current as the curvature current of a singular metric. Then we prove Second Main Theorem type results of Nevanlinna theory for divisors satisfying our (k, ℓ)-condition with a new ramification term which produces an extra Characteristic Function term of a meromorphic map defined by Jacobian minors. Our Main theorem recovers Lang's result([La]) when $\ell = 1$, and covers the general position case when $k = 1$.

Key words. Ramification current, (k, ℓ)-condition, Second Main Theorem, differentiably non-degenerate, negative curvature method.

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0. Introduction. The most important result in Nevanlinna theory is the First and Second Main Theorem. H. Cartan extended Nevanlinna's First and Second Main Theorem to holomorphic curves in projective spaces intersecting hyperplanes, and Bloch considered holomorphic curves in Abelian varieties. Ahlfors([A]), following Weyl's work, used negative curvature method and associated curve to gave a geometric approach to the theory of holomorphic curves in projective spaces. Ru([R2]) derived a Second Main Theorem for algebraic non-degenerate holomorphic curves into complex projective algebraic variety intersecting hypersurfaces.

For any non-constant holomorphic map $f : \mathbb{C} \to \mathbb{CP}^1$ and any distinct points $a_1, \cdots, a_q$ in $\mathbb{CP}^1$, Nevanlinna's Second Main Theorem states that for any $\epsilon > 0$

$$qT_f(r) - \sum_{j=1}^{q} N_f(a_j, r) \leq (2 + \epsilon)T_f(r) \quad \|,$$

where $\|$ means that the inequality holds except on a Borel set of finite Lebesgue measure.

The counterpart in number theory of the Second Main Theorem is the Roth Theorem(see [R1] for the analogy between the Nevanlinna theory and Diophantine approximation), which states that for any algebraic number $\alpha \in \mathbb{R}$ of degree ≥ 2 and for any $\epsilon > 0$,

$$|a - \frac{p}{q}| \geq \frac{1}{q^{2+\epsilon}},$$

except for finitely many $\frac{p}{q} \in \mathbb{Q}$. Lang conjectured that the estimate in Roth Theorem can be sharpened to

$$|\alpha - \frac{p}{q}| \geq \frac{1}{q^2 \log^{1+\epsilon} q}.$$

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Motivated by Lang's conjecture, Wong([W]) proved a Second Main Theorem with better error term. Later, Wong's result was improved by Lang([La]).

The task of generalizing the Nevanlinna theory for (differentiably)non-degenerate holomorphic maps to higher dimensional case is very difficult in general. The main difficulty is to control the singularities of divisors. To handle the singularity issue, Carlson and Griffiths([CaGr]) established the Second Main Theorem under the simple normal crossings assumption about the singularity locus of the divisors. The method used in [CaGr] is now known as the negative curvature method which has its origin in the [A]. This method was successfully used by Siu and Yeung([SY1],[SY2],[SY3]) for geometric hyperbolicity problems. Chen-Ru-Yan([CRY]) also obtained a Second Main Theorem for algebraically degenerate holomorphic curve $f : \mathbb{C} \to \mathbb{CP}^n$ intersecting hypersurfaces in general position.

Based on the work of Griffiths et al and Hironaka's resolution of singularities, Sakai and Shiffman([Sa],[Sh]) obtained the Second Main Theorems and defect relations in more general setting. Inspired by Wong's method, we give a Second Main Theorem (Theorem 2.1) under the (k, ℓ)-condition (see Definition 2.1) of the divisor. This paper is arranged as follows. In Section 1, we fix related notations, and give a reformulation of the standard Calculus Lemma . In Section 2, we define the (k, ℓ)-condition (Definition 2.1) for complex hypersurface (in this paper, a complex hypersurface means a closed complex subspace defined by a nontrivial principal ideal sheaf) with smooth irreducible components according to its germ decompositions. Note that the germ decompositions in Definition 2.1 don't imply the global decomposition of the complex hypersurface itself(see examples given in Remark 2.1). When $\ell = 1$, the (k, ℓ)-condition equals to simple normal crossings of complexity k in [La]. When $k = 1$, the $(1, \ell)$-condition equals to ℓ-subgeneral position. We also introduce a new ramification current R_f (Definition 2.1) for any non-degenerate holomorphic map $f : \mathbb{C}^n \to M$ in terms of the curvature of a singular metric on the pulled-back anti-canonical bundle $f^*K_M^*$. It is different from the notion of ramification divisor introduced by Griffiths and King([GK]). This new ramification current not only helps us to get the truncated counting function in a natural way(see Lemma 2.2) but also produces an extra Characteristic Function term in Theorem 2.2. This extra Characteristic Function term comes from a meromorphic map $\mathbb{J}_f$ given by a linear system of $f^*K_M^*$(see (2.12)). It would be interesting if one can clarify potential relations between f and the associated meromorphic map $\mathbb{J}_f$. By using $\mathbb{J}_f$, we give a lower bound in (2.13) of the ramification current R_f. We prove a Second Main Theorem under our (k, ℓ)-condition of the hypersurface(see Theorem 2.1). Our Theorem 2.1 recovers Lang's SMT in [La]($m = n, \ell = 1$) and cover the case of general position condition($k = 1$).

1. Preliminaries. In this section, we give some basic notions and formulas that are frequently used in this paper. Let $z = (z_1, \cdots, z_n)$ be the natural holomorphic coordinate coordinates of $\mathbb{C}^n$, $n \geq 1$, we set

$$|z|^2 = \sum_{j=1}^{n} |z_j|^2,\ d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial),$$

$$\alpha = dd^c|z|^2 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}|z|^2 = \frac{\sqrt{-1}}{2\pi}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$

$$\beta = dd^c \log|z|^2\ (z \neq 0),$$

$$\alpha^k = \wedge^k \alpha,\ \beta^k = \wedge^k \beta,\ \alpha^0 = 1,\ \beta^0 = 1,$$
$$\gamma = d^c \log |z|^2 \wedge \beta^{n-1}.$$

Set $S_r = \{|z| = r\}$, $B_r = \{|z| \le r\}$. It is easy to see that

$$d\alpha = 0,\ d\beta = 0,\ \beta^n = 0,$$
$$\alpha^n = n! \left(\frac{\sqrt{-1}}{2\pi}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$
$$\int_{B_r} \alpha^n = r^{2n},\ \int_{S_r} \gamma = 1,\ \beta|_{S_r} = \frac{1}{r^2}\alpha|_{S_r}.$$

We need the following fact from [Ne], page 253.

Let $f(r)$ be an increasing function of $r \in [a, \infty)$ with piecewise continuous derivative. Let $g \in C^1[a, \infty)$ and $h > 0$ be Borel function defined on (b, ∞). Suppose that $f(r) > b$ holds on $r \in [a, \infty)$. Then

$$f'(r) \le g'(r) h(f(r)) \qquad \|_g, \tag{1.1}$$

where $\|_g$ means that the inequality holds except on a Borel set $I \subset [a, \infty)$ such that $\int_I dg \le \int_{f(a)}^{\infty} \frac{dr}{h(r)}$. It follows from

$$\int_I dg \le \int_I \frac{f'(r)}{h(f(r))} dr \le \int_a^{\infty} \frac{f'(r)}{h(f(r))} dr \le \int_{f(a)}^{\infty} \frac{dr}{h(r)}. \tag{1.2}$$

For our later use, we reformulate a variant of the above fact as follows.

Lemma 1.1 (Calculus Lemma). *Let $S(r)$ be a function on $(0, \infty)$ which tends to ∞ as $r \to \infty$. Assume that $S, r^{2n-1}\frac{dS}{dr} > e$ are both increasing functions with piecewise continuous derivatives, then for any $\epsilon, \delta \in (0, 1)$*

$$\frac{1}{r^{2n-1}} \frac{d}{dr}\left(r^{2n-1} \frac{dS}{dr}\right) \le \frac{S \log^2 S \log^{2+\delta} \log S}{r^2 \log^{\epsilon} r} \qquad \|_{\epsilon,\delta},$$

where $\|_{\epsilon,\delta}$ means that the inequality holds on $[a, \infty)$, a is a positive constant, with an exceptional Borel subset $I_{\epsilon,\delta} \subset [a, \infty)$ such that $\int_{I_{\epsilon,\delta}} \frac{dr}{r \log r} < \infty$.

Proof. Let $f(r) = S(r)$, $g(r) = \mu \log\log r$, $h(r) = r \log r \log^{\nu} \log r$ and $b = e$ with $\nu > 1 > \mu > 0$ to be specified, we obtain from (1.1) that

$$\frac{dS}{dr} \le \frac{\mu}{r \log r} \cdot S \cdot \log S \cdot \log^{\nu} \log S \qquad \|_g. \tag{1.3}$$

Keeping the same g, h as above and taking $f(r) = r^{2n-1}\frac{dS}{dr}$, we have

$$\frac{d}{dr}(r^{2n-1}\frac{dS}{dr}) \le \frac{\mu}{r \log r} \cdot (r^{2n-1}\frac{dS}{dr}) \cdot \log(r^{2n-1}\frac{dS}{dr}) \cdot \log^{\nu} \log(r^{2n-1}\frac{dS}{dr}) \qquad \|_g. \tag{1.4}$$

By (1.3) , we have

$$\log(r^{2n-1}\frac{dS}{dr}) \le \log(r^{2n-2} S^{2+\nu}) = (2n-2)\log r + (2+\nu)\log S \qquad \|_g, \tag{1.5}$$

and

$$\log^\nu \log(r^{2n-1}\frac{dS}{dr}) \le [(2n-2)\log r + \log(2+\nu) + \log\log S]^\nu \qquad \|_g$$
$$\le 3^{\nu-1}\left[(2n-2)^\nu \log^\nu r + \log^\nu(2+\nu) + \log^\nu \log S\right] \qquad \|_g. \quad (1.6)$$

Substituting (1.3), (1.5) and (1.6) into (1.4), we obtain the desired estimate when $\mu > 0$ is close to 0, $\nu > 1$ is close to 1 and a is large enough.

For the exceptional set, by (1.2) we have

$$\int_I \frac{dr}{r\log r} = \frac{1}{\mu}\int_I dg \le \frac{1}{\mu}\int_{\min(S(a),a^{2n-1}S'(a))}^\infty \frac{1}{h(r)}dr < +\infty.$$

The proof is thus complete. □

Let M be an m-dimensional complex manifold. By a complex hypersurface in M, we always mean a closed complex subspace of M defined by a (nontrivial)principal ideal sheaf. Every complex hypersurface D defines a divisor given by the defining ideal sheaf, we will denote the induced divisor by the same notation D.

Let $f : \mathbb{C}^n \to M$ be a holomorphic map and ω be a positive Hermitian $(1,1)$-form on M, the Characteristic Function of f with respect to ω is defined by

$$T_f(\omega, r) = \int_1^r \frac{dt}{t^{2n-1}}\int_{B_t} f^*\omega \wedge \alpha^{n-1}.$$

A holomorphic map $f : \mathbb{C}^n \to M$ is said to be **non-degenerate** if df has rank m at some point of $\mathbb{C}^n$(in this case, we have $n \ge m$). Let s be a holomorphic section of $\mathcal{O}(D)$ such that D is the zero divisor of s. We equip the line bundle $\mathcal{O}(D)$ with a Hermitian metric and introduce the Proximity Function $m_f(D,r)$ and the Counting Function $N_f(D,r)$ of f for D as follows.

$$m_f(D,r) = \int_{S_r} \log\frac{1}{\|f^*s\|}\gamma,$$
$$N_f(D,r) = \int_1^r \frac{dt}{t^{2n-1}}\int_{B_t} [f^*D] \wedge \alpha^{n-1},$$

where f^*D is the complex inverse image space of the complex hypersurface D. Let $\widetilde{f^*D}$ be the reduction of the complex hypersurface f^*D of $\mathbb{C}^n$, we define the Truncated Counting Function by

$$N_f^{(1)}(D,r) = \int_1^r \frac{dt}{t^{2n-1}}\int_{B_t} [\widetilde{f^*D}] \wedge \alpha^{n-1}.$$

When ω is closed and represents the first Chern class of a positive line bundle L over M, we denote $T_f(\omega,r) = T_f(L,r)$. By Green-Jensen's formula, we have the First Main Theorem

$$m_f(D,r) + N_f(D,r) = T_f(D,r) + O(1).$$

Note that the above definitions of non-degenracy, characteristic function, proximity function and counting function also make sense for meromorphic functions(e.g., see [NW]).

2. The Second Main Theorem.

2.1. The (k, ℓ)-condition and ramification current. The main difficulty in the higher dimensional case lies in the singularities of divisors. To handle the singularity issue, Carlson and Griffiths([CaGr]) established the Second Main Theorem under the simple normal crossing assumption about the singularity locus of the divisors. The method used in [CaGr] is known as the negative curvature method. Later, by using different singular metrics, the main result in [CaGr] was improved by many authors(e.g. [La], [W]).

Based on the work of Griffiths et al and Hironaka's resolution of singularities, Sakai and Shiffman([Sa], [Sh]) obtained the Second Main Theorems and defect relations in more general setting. We will give Second Main Theorem type results under the following assumption about singularity locus.

Definition 2.1. *Let D be a complex hypersurface with smooth irreducible components. We say that D satisfies the (k, ℓ)-condition for positive integers k, ℓ if at each $x \in D$ we have (i) there are reduced germs of divisors $(D_1, x), \cdots, (D_{\ell_x}, x)$ on M with $\ell_x \leq \ell$ such that*

$$(D, x) = \sum_{\nu=1}^{\ell_x} (D_\nu, x);$$

(ii) for each $1 \leq \nu \leq \ell_x$, (D_ν, x) has a complete intersection at x and $\operatorname{codim}_x D_\nu \leq k$, *where* codim *denotes the codimension in M.*

Remark 2.1.

1. *When $\ell = 1$, the (k, ℓ)-condition is equivalent to the condition that D has simple normal crossings of complexity k as defined in [La]. When $k = 1$, the (k, ℓ)-condition is equivalent to the ℓ-subgeneral position condition, i.e., there are at most ℓ irreducible components of D passing through a common point. If $k =$ dimension of M, then the second condition in Definition 2.1 reduces to the property that (D_ν, x) is a complete intersection at $x, 1 \leq \nu \leq \ell_x$.*
2. *Since D has smooth (global)irreducible components, every irreducible component of the reduced germ (D_ν, x) in Definition 2.1 is represented by a global irreducible component of D which certainly depends on x. For example, it is easy to find smooth curves D_1, D_2, D_3 in $\mathbb{P}^2$ such that $D_i, D_j (i \neq j)$ are tangent somewhere and $D_1 \cap D_2 \cap D_3 = \emptyset$. Then the divisor $D := D_1 + D_2 + D_3$ is located in 2-subgeneral position whereas each pair of the curves D_1, D_2, D_3 does not have normal crossings. Since there are singular points which are not ordinary double points, according to Remark 5 in [Sa], D does not have quasi-negligible singularities in the sense of Sakai.*
3. *Now we give another example where $k, \ell \neq 1$. Let's consider the following smooth curves D_1, D_2, D_3, D_4 in $\mathbb{P}^2$ defined by polynomials $P_1(z) = z_1, P_2(z) = z_2, P_3(z) = z_0 z_1 - \sqrt{\frac{2}{5}} z_2^2, P_4(z) = z_0^2 z_2 + \frac{3}{5} z_1^3 - z_1 z_2^2$. Since $D_1 \cap D_2 \cap D_3 \cap D_4 = \{[1, 0, 0]\}$, the divisor $D := D_1 + D_2 + D_3 + D_4$ is located in 4-subgeneral position. At the point $[1, 0, 0]$, D_1 is tangent to D_3 and transversal to D_4, while D_2 is tangent to D_4 and transversal to D_3. According to Definition 2.1, the divisor $D := D_1 + D_2 + D_3 + D_4$ satisfies our $(2, 2)$-condition. Let $a_{13} = [1, 0, 0], a_{14} = [0, 0, 1], a_{34} = [\sqrt{\frac{2}{5}}, 1, 1]$ and $a_{23} = [0, 1, 0]$, then it is easy to see that D_i and D_j are tangent at the point*

a_{ij}. Hence, $D_i + D_j$ does not have simple normal crossings except the case $i = 1, j = 2$. In above examples, the divisor D can not be written as a sum of two divisors both of which have normal crossings. Moreover, D does not have quasi-negligible singularities in the sense of Sakai.

Let $f : \mathbb{C}^n \to M$ $(n \geq m)$ be a non-degenerate holomorphic map, we define a relevant $(1,1)$-current R_f as follows. Take an arbitrary point $z \in \mathbb{C}^n$ and a local coordinate chart $(U; w_1, \cdots, w_m)$ of $f(z) \in M$. For any $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, we denote by $J_{i_1 \cdots i_m}$ the determinant of the m by m matrix which is formed by selecting the i_a-th$(1 \leq a \leq m)$ columns in the Jacobian $J(f) := (\frac{\partial w_a \circ f}{\partial z_i})_{1 \leq a \leq m, 1 \leq i \leq n}$. By definition, these local data $\sum\limits_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} |J_{i_1 \cdots i_m}|^2$ patch together to give a singular metric on the pulled-back line bundle $f^* K_M^*$ over $\mathbb{C}^n$ where K_M is the canonical bundle of M. This observation allows us to introduce the notion of ramification current of f in terms of the curvature of this singular metric on $f^* K_M^*$.

Definition 2.2. *Let M be a complex manifold of dimension m, $f : \mathbb{C}^n \to M$ $(n \geq m)$ be a non-degenerate holomorphic map, the ramification current R_f of f is defined as*

$$R_f := dd^c \log \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} |J_{i_1 \cdots i_m}|^2 \right). \tag{2.1}$$

It is easy to see that R_f is a globally defined positive $(1,1)$-current on $\mathbb{C}^n$. The Counting Function of R_f is defined by

$$N(R_f, r) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} R_f \wedge \alpha^{n-1}.$$

Remark 2.2. *Since* $\mathrm{Supp} R_f$ *possibly has interior points, in general, R_f is no longer an integration current along a complex hypersurface unless $n = m$. So our definition of the ramification current R_f is different from the notion of ramification divisor introduced in [GK]. This difference results in a new ramification term and an extra Characteristic Function term in Theorems 2.1 and Theorem 2.2 respectively. Fixing some $1 \leq i_1 < \cdots < i_m \leq n$ with $J_{i_1 \cdots i_m} \not\equiv 0$, Griffiths and King defined the ramification divisor as the zero divisor of $J_{i_1 \cdots i_m}$ ([GK]). The advantage of our ramification current lies in the geometric inequality (2.13) below.*

2.2. A Second Main Theorem. The following Lemma is needed in our proof of Theorem 2.1.

Lemma 2.1. *Let $L_1, \cdots, L_p$ be Hermitian holomorphic line bundles over a complex manifold M and $s_j \in H^0(M, L_j), 1 \leq j \leq p$. Assume that at some point $x \in M$ the zero divisors of $s_1, \cdots, s_q$ are all smooth and have complete intersection. Then there exists a positive constant $C > 0$ such that*

$$\frac{\left(\sum\limits_{j=1}^{p} \psi_j \sqrt{-1} \partial s_j \wedge \overline{\partial s_j} + \phi \omega \right)^m}{\omega^m} \geq C \phi^{m-p} \prod_{j=1}^{p} \psi_j,$$

holds in a neighbourhood of x*, where* ϕ, ψ_j, $1 \le j \le p$ *are arbitrary non-negative functions, and* ω *is a positive Hermitian* $(1,1)$*-form on* M.

Proof. By assumption, there is a local coordinate chart $(U; w_1, \cdots, w_m)$ of M with $x = (0, \cdots, 0)$ such that there are smooth functions $h_j > 0$ such that

$$\|s_j\|^2 = h_j |w_j|^2,$$

which implies the following

$$\sqrt{-1}\partial s_j \wedge \overline{\partial s_j} = \sqrt{-1} h_j dw_j \wedge d\bar{w}_j + O(|w|).$$

Since $dw_1 \wedge \cdots \wedge dw_p \neq 0$ in an open neighborhood of x and $\partial s_j \wedge \overline{\partial s_j}, 1 \le j \le p$, are positive $(1,1)$-forms, we have

$$\begin{aligned}\left(\sqrt{-1}\sum_{j=1}^{p} \psi_j \partial s_j \wedge \overline{\partial s_j} + \phi\omega\right)^m &\ge \sqrt{-1} m \phi^{m-p} (\prod_{j=1}^{p} \psi_j \partial s_j \wedge \overline{\partial s_j}) \wedge \omega^{m-p} \\ &\ge C\phi^{m-p}(\prod_{j=1}^{p}\psi_j)\omega^m\end{aligned}$$

where C is a positive constant independent of $\phi, \psi_j, 1 \le j \le p$. □

Now we prove a Second Main Theorem under the (k,ℓ)-condition.

Theorem 2.1 (SMT). *Let* M *be a compact complex manifold of dimension* m*,and* $f : \mathbb{C}^n \to M$ *be a non-degenerate holomorphic map. Let* D *be a complex hypersurface in* M *satisfying the* (k,ℓ)*-condition. Then for any* $\epsilon, \delta \in (0,1)$*, we have*

$$\begin{aligned}T_f(\mathcal{O}(D) + \ell K_M, r) \le N_f(D,r) - \ell N(R_f, r) + \frac{\ell}{2}(m+k)\log T_f(\omega, r) \\ + \ell m \log\log T_f(\omega,r) + (\ell m + \delta)\log\log\log T_f(\omega, r) \\ - \ell m \log r - \frac{\ell m \epsilon}{2}\log\log r \quad \|_{\epsilon,\delta}\end{aligned}$$

where K_M *is the canonical bundle of* M*,* ω *is a positive Hermitian form on* M.

Proof. Let $D = D_1 + \cdots + D_q$ be the irreducible decomposition. For $1 \le j \le q$, let $s_j \in H^0(M, \mathcal{O}(D_j))$ be a section such that D_j is the zero divisor of s_j. By choosing an appropriate metric on $\mathcal{O}(D_j)$, we may assume that $\|s_j\|^2 < \frac{1}{e}$. Let $0 < \lambda < 1$ be a constant to be specified. Then we know by definition

$$\begin{aligned}2\sum_{j=1}^{q} m_f(D_j, r) &= \sum_{j=1}^{q}\int_{S_r}\log\frac{1}{\|s_j\circ f\|^2}\gamma = \int_{S_r}\log(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|^2})\gamma \\ &= \int_{S_r}\log(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|^{2\lambda}})\gamma + \int_{S_r}\log(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|^{2-2\lambda}})\gamma \\ &:= I + II.\end{aligned}$$

Since M is compact, we may assume $c_1(\mathcal{O}(D)) \le C\omega$ on M for some constant $C > 0$. For the term I, using the First Main Theorem, we derive

$$I = 2\lambda\int_{S_r}\log(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|})\gamma = 2\lambda\sum_{j=1}^{q} m_f(D_j, r) \le 2C\lambda T_f(\omega, r) + O(1).$$

We give an upper bound for II as follows.

For every $x \in D$, by the (k,ℓ)-condition, we may assume without loss of generality that $D_{j_1}, \cdots, D_{j_{k_x}}$ are the components which pass through x and that there is a decomposition $\{j_1, \cdots, j_{k_x}\} = I_1 \cup \cdots \cup I_{\ell_x}$ with $\ell_x \le \ell, |I_\nu| \le k$ for $1 \le \nu \le \ell_x$ such that $D_i,\ i \in I_\nu$, have complete intersection at x. By the Arithmetic-Geometric Mean Inequality, we have the following estimate in a small neighbourhood U_x of x minus D that

$$\prod_{j=1}^{q} \frac{1}{\|s_j\|^{\frac{2-2\lambda}{m}}} \le C_x \left(\sum_{\nu=1}^{\ell_x} \frac{1}{\prod\limits_{j \in I_\nu} \|s_j\|^{\frac{2-2\lambda}{m}}} \right)^{\ell}, \tag{2.2}$$

where C_x is a positive constant independent of λ.

A direct computation using the definition of the first Chern form $c_1(\mathcal{O}(D_j))$ implies that

$$dd^c \|s_j\|^{2\lambda} + \lambda \|s_j\|^{2\lambda} c_1(\mathcal{O}(D_j)) = \frac{\sqrt{-1}\lambda^2 \partial s_j \wedge \overline{\partial s_j}}{2\pi \|s_j\|^{2-2\lambda}} \tag{2.3}$$

holds outside D_j. Set

$$\eta_\nu := \frac{1}{\lambda^2} \sum_{j \in I_\nu} \left(dd^c \|s_j\|^{2\lambda} + \lambda \|s_j\|^{2\lambda} c_1(\mathcal{O}(D_j)) \right) + \frac{1}{\lambda}\omega = \sum_{j \in I_\nu} \frac{\sqrt{-1}\partial s_j \wedge \overline{\partial s_j}}{2\pi \|s_j\|^{2-2\lambda}} + \frac{1}{\lambda}\omega,$$

$$\eta := \frac{1}{\lambda^2} \sum_{j=1}^{q} \left(dd^c \|s_j\|^{2\lambda} + \lambda \|s_j\|^{2\lambda} c_1(\mathcal{O}(D_j)) \right) + \frac{\ell}{\lambda}\omega.$$

Then by Lemma 2.1, the above equality yields

$$\frac{A_{x,\nu}}{\prod\limits_{j \in I_\nu} \|s_j\|^{2-2\lambda}} \omega^m \le \frac{1}{\lambda^{k-m}} \eta_\nu^m,\ 1 \le \nu \le \ell(x), \tag{2.4}$$

in $U_x \backslash D$ where $A_{x,\nu}$ is a positive constant independent of λ and U_x is a neighbourhood of x . Pulling back to $\mathbb{C}^n$, we have

$$\frac{A_{x,\nu}\Omega}{\prod\limits_{j \in I_\nu} \|s_j \circ f\|^{2-2\lambda}} \alpha^n \le \frac{1}{\lambda^{k-m}} f^* \eta_\nu^m \wedge \alpha^{n-m}, \tag{2.5}$$

where Ω is defined by

$$f^* \omega^m \wedge \alpha^{n-m} = \Omega \cdot \alpha^n. \tag{2.6}$$

Since $\eta_\nu \le \eta, 1 \le \nu \le \ell_x$, by using the pointwise singular value decomposition of the tangent map df, we have

$$\begin{aligned} \frac{1}{\lambda^{k-m}} \sum_{\nu=1}^{\ell_x} f^* \eta_\nu^m \wedge \alpha^{n-m} &\le \frac{1}{\lambda^{k-m}} f^* \eta^m \wedge \alpha^{n-m} \\ &= \frac{1}{\binom{n}{m} \lambda^{k-m}} \det(\eta \circ df \circ df^*) \alpha^n \\ &\le \frac{1}{\binom{n}{m} \lambda^{k-m}} \left(\frac{\mathrm{tr}(f^*\eta)}{m} \right)^m \alpha^n \end{aligned} \tag{2.7}$$

where df^* is the Hermitian adjoint of df and we have identified η as an endomorphism of TM via the metric ω. Combining the inequalities (2.2), (2.5) and (2.7), we have the following estimate on $f^{-1}(U_x \setminus D)$

$$\begin{aligned}\log\left(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|^{\frac{2-2\lambda}{m}}}\right) &\leq \ell\log\left(\sum_{\nu=1}^{\ell_x}\frac{1}{\prod\limits_{j\in I_\nu}\|s_j\circ f\|^{\frac{2-2\lambda}{m}}}\right)+\log C_x\\ &= \ell\log\left(\sum_{\nu=1}^{\ell_x}\frac{\Omega^{\frac{1}{m}}}{\prod\limits_{j\in I_\nu}\|s_j\circ f\|^{\frac{2-2\lambda}{m}}}\right)-\frac{\ell}{m}\log\Omega+\log C_x\\ &\leq \ell\log\left(\frac{1}{\lambda^{\frac{k-m}{m}}}\mathrm{tr}(f^*\eta)\right)-\frac{\ell}{m}\log\Omega+\log C'_x,\end{aligned}$$

where C_x, C'_x are positive constants both independent of λ. By the compactness of D, we can cover D by finitely many open sets on which the inequality (2.4) holds. Since M is compact, the above inequality trivially holds on $\mathbb{C}^n \setminus f^{-1}(V)$ where V is an open neighbourhood of D. Thus, we obtain the following the inequality

$$\log\left(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|^{\frac{2-2\lambda}{m}}}\right) \leq \ell\log\left(\frac{1}{\lambda^{\frac{k-m}{m}}}\mathrm{tr}(f^*\eta)\right)-\frac{\ell}{m}\log\Omega+C \tag{2.8}$$

on $\mathbb{C}^n \setminus f^{-1}(D)$ where C is a positive constant independent of λ. From the estimate (2.8) and the assumption that f is non-degenerate, it is easy to see

$$\begin{aligned}II &= \int_{S_r}\log\left(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|^{2-2\lambda}}\right)\gamma = m\int_{S_r}\log\left(\prod_{j=1}^{q}\frac{1}{\|s_j\circ f\|^{\frac{2-2\lambda}{m}}}\right)\gamma\\ &\leq \ell m\log\int_{S_r}\left(\frac{1}{\lambda^{\frac{k-m}{m}}}\mathrm{tr}(f^*\eta)\right)\gamma-\ell\int_{S_r}\log\Omega\gamma+C.\end{aligned}$$

Set

$$S(r)=\int_1^r\frac{dt}{t^{2n-1}}\int_{B_t}\mathrm{tr}(f^*\eta)\alpha^n,$$

then we know by definition

$$2n\int_{S_r}\mathrm{tr}(f^*\eta)\gamma=\frac{1}{r^{2n-1}}\frac{d}{dr}\left(r^{2n-1}\frac{d}{dr}S(r)\right). \tag{2.9}$$

Since M is compact, we may assume $c_1(\mathcal{O}(D)) \leq C\omega$ on M. By the definition of η

and Green-Jensen's formula, we have

$$\begin{aligned}
S(r) &= \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} f^*\eta \wedge \alpha^{n-1} \\
&\leq \frac{\ell + C}{\lambda} T_f(\omega, r) + \frac{1}{\lambda^2} \left(\sum_{j=1}^q \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} dd^c \|s_j\|^{2\lambda} \wedge \alpha^{n-1} \right) \\
&= \frac{\ell + C}{\lambda} T_f(\omega, r) + \frac{1}{\lambda^2} \left(\sum_{j=1}^q \frac{1}{2} \int_{S_r} \|s_j\|^{2\lambda} \gamma + O(1) \right) \\
&\leq \frac{\ell + C}{\lambda} T_f(\omega, r) + \frac{1}{\lambda^2} \left(\frac{\ell}{2e} + O(1) \right),
\end{aligned}$$

where C is a positive constant independent of λ.

Let $\lambda = \frac{1}{1+T_f(\omega,r)}$, we can derive from the above inequality that

$$\log S(r) \leq 2 \log T_f(\omega, r) + O(1).$$

From the Calculus Lemma 1.1 and (2.9), it follows that

$$\begin{aligned}
II &\leq \ell m \log \left(\frac{1}{2n\lambda^{\frac{k-m}{m}}} \frac{1}{r^{2n-1}} \frac{d}{dr} \left(r^{2n-1} \frac{d}{dr} S \right) \right) - \ell \int_{S_r} \log \Omega \gamma + O(1) \\
&\leq \ell m \log \frac{S \log^2 S \log^{2+\delta} \log S}{r^2 \log^\epsilon r} + \ell(k-m) \log T_f(\omega, r) - \ell \int_{S_r} \log \Omega\gamma + O(1) \quad \|_{\epsilon,\delta} \\
&\leq \ell(m+k) \log T_f(\omega, r) + 2\ell m \log\log T_f(\omega, r) + (2+\delta)\ell m \log\log\log T_f(\omega, r) \\
&\quad - \ell \int_{S_r} \log \Omega\gamma - 2\ell m \log r - \ell m\epsilon \log\log r + O(1) \quad \|_{\epsilon,\delta}. \qquad (2.10)
\end{aligned}$$

According to definition (2.6), Ω is locally given by

$$\Omega = \frac{1}{\binom{n}{m}} \det(g_{a\bar{b}}) \cdot \det(J(f)J(f)^*)$$

where $J(f)^*$ denotes the complex conjugate transpose of the Jacobian matrix $J(f)$ and $g_{a\bar{b}} (1 \leq a, b \leq m)$ are the coefficients of ω in a local coordinate chart, i.e. $\omega = \sqrt{-1} \sum_{1\leq a,b\leq m} g_{a\bar{b}} dw_a \wedge d\bar{w}_b$. By Cauchy-Binet formula, Ω could be rewritten as

$$\Omega = \frac{\det(g_{a\bar{b}})}{\binom{n}{m}} \left(\sum_{1\leq i_1<i_2<\cdots<i_m\leq n} |J_{i_1\cdots i_m}|^2 \right),$$

from which we know, by using Green-Jensen's formula and the definition (2.1), that

$$\int_{S_r} \log \Omega\gamma = 2T_f(K_M, r) + 2N(R_f, r) + O(1). \qquad (2.11)$$

Substituting (2.11) into (2.10), we combine the upper bounds of I, II as follows

$$\begin{aligned}
\sum_{j=1}^q m_f(D_j, r) &\leq \frac{\ell(m+k)}{2} \log T_f(\omega, r) + \ell m \log\log T_f(\omega, r) + \frac{(2+\delta)\ell m}{2} \log\log\log T_f(\omega, r) \\
&\quad - \ell T_f(K_M, r) - \ell N(R_f, r) - km \log r - \frac{\ell m\epsilon}{2} \log\log r \quad \|_{\epsilon,\delta}
\end{aligned}$$

which gives the desired estimate. The proof of Theorem 2.1 is complete. □

REMARK 2.3. *The special case of Theorem 2.1 where $\ell = 1, m = n$ is exactly Lang's Second Main Theorem([La]). We can always arrange a given complex hypersurface D(whose irreducible components are assumed to be smooth) as $D = \sum_j D_j$ where all D_j have simple normal crossings, e.g., each D_j is an irreducible component of D. Then we obtain a Second Main Theorem by adding up Lang's Second Main Theorem for D_j. However, the example in item 3 of Remark 2.1 shows that our (k,ℓ)-condition helps to obtain improved Second Main Theorem type results.*

2.3. The Associated Meromorphic Map $\mathbb{J}_f$. Let $f : \mathbb{C}^n \to M$ $(n \geq m)$ be a non-degenerate holomorphic map. We return to the minors $J_{i_1\cdots i_m}(1 \leq i_1 < i_2 < \cdots < i_m \leq n)$ in the definition of R_f. These local holomorphic data $\{J_{i_1\cdots i_m}\}_{1\leq i_1<i_2<\cdots<i_m\leq n}$, as a system of global sections of $f^*K_M^*$, define a meromorphic map

$$\begin{aligned} \mathbb{J}_f : \mathbb{C}^n &\to \mathbb{P}^{\binom{n}{m}-1} \\ z &\mapsto [J_{i_1\cdots i_m}(z)]_{1\leq i_1<i_2<\cdots<i_m\leq n} \end{aligned} \tag{2.12}$$

whose base locus is the subvariety locally defined by functions $J_{i_1\cdots i_m}(1 \leq i_1 < i_2 < \cdots < i_m \leq n)$. We will use the local representation of $\mathbb{J}_f$ to give a global lower bound of the ramification divisor R_f in terms of an arbitrary complex hypersurface.

Let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^{\binom{n}{m}-1}$. We have a positive $(1,1)$-current $\mathbb{J}_f^*\omega_{\mathrm{FS}}$ on $\mathbb{C}^n$ which is defined as the pull-back of ω_{FS} by using a reduced representation of the meromorphic map $\mathbb{J}_f$. Note that, for any $z \in \mathbb{C}^n$, canceling the greatest common divisor in $\mathcal{O}_{\mathbb{C}^n,z}$ of the germs of $J_{i_1\cdots i_m}(1 \leq i_1 < i_2 < \cdots < i_m \leq n)$ yields a reduced representation of $\mathbb{J}_f$ in a neighbourhood of z.

On the other hand, when the germs of $J_{i_1\cdots i_m}(1 \leq i_1 < i_2 < \cdots < i_m \leq n)$ are simultaneously divisible by the germ of a holomorphic function h, by Poincaré-Lelong formula and Definition 2.1, the $(1,1)$-current R_f is locally bounded from below by the sum of $\mathbb{J}_f^*\omega_{\mathrm{FS}}$ and the integration current along the zero divisor of h. Consequently, it follows that for any complex hypersurface D satisfying the (k,ℓ)-condition, we have the following current inequality

$$\ell R_f \geq \mathbb{J}_f^*\omega_{\mathrm{FS}} + f^*D - \widetilde{f^*D}, \tag{2.13}$$

where $\widetilde{f^*D}$ is the reduction of the complex inverse image space f^*D. Integrating the inequality (2.13) gives

LEMMA 2.2. *Let M be a complex manifold of dimension m, $f : \mathbb{C}^n \to M$ be a non-degenerate holomorphic map. For any complex hypersurface D in M satisfying the (k,ℓ)-condition, we have*

$$T_{\mathbb{J}_f}(\mathcal{O}_{\mathbb{P}^N}(1), r) + N_f(D,r) - \ell N(R_f, r) \leq N_f^{(1)}(D,r)$$

where $N := \binom{n}{m} - 1$ and $T_{\mathbb{J}_f}(\mathcal{O}_{\mathbb{P}^N}(1), r) := \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} \mathbb{J}_f^*\omega_{\mathrm{FS}} \wedge \alpha^{n-1}$ is the Characteristic Function of the meromorphic map $\mathbb{J}_f$ defined by (2.12).

Combining Theorem 2.1 and Lemma 2.2 gives

Theorem 2.2 (SMT with an extra Characteristic Function Term). *Under the assumptions of Theorem 2.1, the following inequality holds for any* $\epsilon, \delta \in (0,1)$

$$T_{\mathbb{J}_f}(\mathcal{O}_{\mathbb{P}^N}(1), r) + T_f(\mathcal{O}(D) + \ell K_M, r) \leq N_f^{(1)}(D, r) + \frac{\ell}{2}(m+k)\log T_f(\omega, r)$$
$$+ \ell m \log\log T_f(\omega, r) + (\ell m + \delta)\log\log\log T_f(\omega, r) - \ell m \log r - \frac{\ell m \epsilon}{2}\log\log r \quad \|_{\epsilon,\delta}$$

where $N := \binom{n}{m} - 1$ *and* $\mathbb{J}_f$ *is defined by (2.12).*

We remark that the above Theorem 2.1 and Theorem 2.2 can be extended to meromorphic mappings.

3. Some Applications. The first application is about the growth order of non-degenerate holomorphic maps which is an immediate consequence of Theorem 2.2.

Corollary 3.1. *Let* M *be a compact complex manifold of dimension* m, $D_1, \cdots, D_q$ *be smooth hypersurfaces in* ℓ*-subgeneral position(i.e.,* $k = 1$*). Let* $f : \mathbb{C}^n \to M \setminus \bigcup_{j=1}^{q} D_j$ *be a non-degenerate holomorphic map. (i)If* $\sum_{j=1}^{q} \mathcal{O}(D_j) + \ell K_M \geq 0$, *then for every* $\sigma \in (0, \frac{2m}{m+1})$, *we have*

$$T_f(\omega, r) \geq r^{\sigma} \quad \|_{\sigma},$$

where $\|_{\sigma}$ *means that the inequality holds outside* I_{σ} *for some Borel set* I_{σ} *with* $\int_{I_{\sigma}} \frac{1}{r log r} dr < \infty$, ω *is a positive Hermitian form on* M*. In particular, when* M *is a projective manifold* f *is a transcendental map of order* $\rho_f \geq \frac{2m}{m+1}$*. (ii) If we also assume the meromorphic map* $\mathbb{J}_f$ *is not constant, then*

$$T_f(\omega, r) \geq r^{\frac{2m}{m+1}+\sigma} \quad \|_{\sigma}$$

holds for any sufficiently small $\sigma > 0$.

Now we give some applications in the case $k =$ the dimension of the ambient complex manifold(e.g., D is located in ℓ-subgeneral position).

We have the following degeneracy result for holomorphic maps.

Corollary 3.2. *Let* M *be a projective manifold of dimension* m, $D_1, \cdots, D_q$ *be smooth hypersurfaces satisfying the* (m, ℓ)*-condition. If* $\sum_{j=1}^{q} \mathcal{O}(D_j) + \ell K_M$ *is big, then any holomorphic* $f : \mathbb{C}^n \to M \setminus \bigcup_{j=1}^{q} D_j$ $(n \geq m)$ *is degenerate.*

Proof. Since $\sum_{j=1}^{q} \mathcal{O}(D_j) + \ell K_M$ is a big line bundle, there exists an integer $p > 0$ such that $p\left(\sum_{j=1}^{q} \mathcal{O}(D_j) + \ell K_M\right) = L + E$ where L is ample and E is effective. From First Main Theorem we know that $T_f(E, r)$ is bounded from below by a constant. It suffices to choose the Hermitian form ω in Theorem 2.2 to be a positive $(1,1)$-form representing the first Chern class of the positive line bundle L. By assumption, f omits $D_j's(1 \leq j \leq q)$, we also have $\sum_{j=1}^{q} N_f^{(1)}(D_j, r) \equiv 0$. If f is non-degenerate, we would get the contradiction $T_f(\omega, r) \leq (p\ell m + \delta)\log T_f(\omega, r) \quad \|_{\epsilon,\delta}$. □

Remark 3.1.

1. *The example given in Remark 5.6 of [Sa] also shows that there are irreducible curves in $\mathbb{P}^2$ with a single singular point such that the degeneracy result in Corollary 3.2 fails. So, the smoothness assumption of the hypersurfaces can not be simply dropped.*
2. *For the example in Remark 2.1, 3, we have* $\sum_{j=1}^{q} \mathcal{O}(D_j) + \ell K_M = \mathcal{O}_{\mathbb{P}^2}(1)$.

The next result is a criterion for rationality without normal crossing assumption.

Corollary 3.3. *Let M be a projective manifold of dimension m and $D_1, \cdots, D_q$ be smooth hypersurfaces satisfying the (m, ℓ)-condition such that $\sum_{j=1}^{q} \mathcal{O}(D_j) + \ell K_M$ is big. Then for any non-degenerate holomorphic map $f : \mathbb{C}^n \to M$, f is rational if and only if $f^*D_1, \cdots, f^*D_q$ are all algebraic hypersurfaces in $\mathbb{C}^n$. Moreover, when f is a non-degenerate rational map, we also have $\sum_{j=1}^{q} \deg f^*D_j > \ell m$.*

Proof. By Chow's Theorem, all $D_1, \cdots, D_q$ are algebraic. Hence, if f is a rational map then $f^*D_1, \cdots, f^*D_q$ must be algebraic hypersurfaces in $\mathbb{C}^n$. Conversely, we assume that $f^*D_1, \cdots, f^*D_q$ are algebraic. By using Poincaré-Lelong formula and Green-Jensen formula, it is easy to see that

$$N_f(D_j, r) = O(\log r), \ 1 \le j \le q.$$

By the same arguement in the proof of Corollary 3.2, there exists an interger $p > 0$ such that $T_f(\sum_{j=1}^{q} \mathcal{O}(D_j) + \ell K_M) + O(1) \ge \frac{1}{p} T_f(\omega, r)$, wher ω is a positive Hermitian $(1,1)$-form. Then we know that for any $\epsilon, \delta \in (0,1)$

$$T_f(\omega, r) \le 2p \sum_{j=1}^{q} N_f^{(1)}(D_j, r) \quad \|_{\epsilon,\delta} \quad \le O(\log r) \quad \|_{\epsilon,\delta},$$

i.e., there exists a constant $A > 0$ such that

$$T_f(\omega, r) \le A \log r, \quad r \notin I_{\epsilon,\delta}. \tag{3.1}$$

Fix a positive number B satisfying $\log B > \int_{I_{\epsilon,\delta}} \frac{dr}{r \log r}$. Then we can find a sequence of numbers $r_k \in [e^{B^{k-1}}, e^{B^k}) \setminus I_{\epsilon,\delta}, k = 1, 2, \cdots$. For any $r \ge e$, there is a unique $k \ge 1$ such that $r \in [e^{B^{k-1}}, e^{B^k})$. From the inequality (3.1), it follows that

$$T_f(\omega, r) \le T_f(\omega, r_{k+1}) \le A \log r_{k+1} \le AB^2 \log r$$

i.e., $T_f(\omega, r) = O(\log r)$ which implies that f is a rational map. The last statement is due to $N_f(D_j, r) \le \deg f^*D_j \cdot \log r + O(1)$ and the term $-\ell m \log r$ in Theorem2.2. The proof is thus complete. □

By using the Gauss equation for holomorphic bisectional curvatures, we have the following consequence of Theorem 2.2.

Corollary 3.4. *Let $D_1, \cdots, D_q$ be smooth hypersurfaces of an m-dimensional projective manifold M satisfying the (m, ℓ)-condition and let $f : \mathbb{C}^n \to M$ be a non-degenerate holomorphic map. If there are positive integers $d_1, \cdots, d_q$ and a very ample divisor A such that $D_j \sim d_j A, 1 \leq j \leq q$, then for any $\epsilon \in (0, 1)$ and $\delta > 0$, we have*

$$\begin{aligned}&\big(d_1 + \cdots + d_q - \ell(m+1)\big)T_f(\mathcal{O}(A), r)\\ \leq &\sum_{j=1}^{q} N_f^{(1)}(D_j, r) + \frac{\ell(m+1)}{2}\log T_f(\mathcal{O}(A), r)\\ &+(\ell m + \delta)\log\log T_f(\mathcal{O}(A), r) - \ell m \log r - \frac{\ell m \epsilon}{2}\log\log r \quad \|_{\epsilon,\delta}.\end{aligned}$$

Proof. We first use the very ample divisor A to embed M into some projective space $\mathbb{P}^N$ and endow M with the pull-backed metric induced by the Fubini-Study metric $\omega_{\rm FS}$ on $\mathbb{P}^N$. Let R^M, R be the curvature tensors of $(M, \omega_{\rm FS}|_M)$ and $(\mathbb{P}^N, \omega_{\rm FS})$ respectively. By the Gauss equation for holomorphic bisectional curvatures and the fact that $\omega_{\rm FS}$ has constant holomorphic bisectional curvature, we know that

$$\begin{aligned}R^M_{u\bar{u}v\bar{v}} &= R_{u\bar{u}v\bar{v}} - \|\sigma(u, v)\|^2\\ &= 2\pi(\|u\|^2\|v\|^2 + |\langle u, v\rangle|^2) - \|\sigma(u, v)\|^2\end{aligned}$$

holds for any $u, v \in T_x^{1,0}M (x \in M)$ where σ is the second fundamental form of $(M, \omega_{\rm FS}|_M)$ in $(\mathbb{P}^N, \omega_{\rm FS})$. Summing with respect to v over a unitary basis of $T_x^{1,0}M$ shows that the Ricci form of $\omega_{\rm FS}|_M$ satisfies

$$\mathrm{Ric}_{\omega_{\rm FS}|_{\rm M}} \leq 2\pi(m+1)\omega_{\rm FS}|_M,$$

and consequently

$$c_1(K_M) \geq -(m+1)\omega_{\rm FS}|_M.$$

By definition, we also have $\mathcal{O}(D_1)+\cdots+\mathcal{O}(D_q) = (d_1+\cdots+d_q)\mathcal{O}(A)$ and $c_1(\mathcal{O}(A)) = \omega_{\rm FS}|_M$. Now the desired estimate follows from Theorem 2.2. □

REFERENCES

[A] L. Ahlfors, *The theory of meromorphic curves*, Acta Soc. Sci. Fenn. Nova Ser. A, 3 (1941), pp. 171–183.

[CaGr] J. Carlson and P. Griffiths, *A defect relation for equidimensional holomorphic mappings between algebraic varieties*, Ann. of Math.(2), 95 (1972), pp. 557–584.

[CRY] Z. H. Chen, M. Ru, and Q. M. Yan, *The degenerated second main theorem and schmidt's subspace theorem*, Science China Mathematics, 55:7 (2012), pp. 1367–1380.

[GK] P. Griffiths and J. King, *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta Mathematica, 130:1 (1973), pp. 145–220.

[La] S. Lang, *The error term in Nevanlinna theory. II*, Bull. Amer. Math. Soc. (N.S.), 22:1 (1990), pp. 115–125.

[Ne] R. Nevanlinna, *Analytic functions*, Springer-Verlag. (1970).

[NW] J. Noguchi and J. Winkelmann, *Nevanlinna theory in several complex variables and Diophantine approximation*, Springer Science & Business Media. (2013).

[R1] M. Ru, *Nevanlinna theory and its relation to Diophantine approximation*, Singapore: World Scientific, (2001).

[R2] M. Ru, *Holomorphic curves into algebraic varieties*, Ann. Math. (2), 169 (2009), pp. 255–267.

[Sa] F. Sakai, *Degeneracy of holomorphic maps with ramification*, Inventiones Mathematicae, 26 (1974), pp. 213–229.

[Sh] B. Shiffman, *Nevanlinna defect relations for singular divisors*, Inventiones Mathematicae, 31:2 (1975), pp. 155–182.

[SY1] Y.-T. Siu and S.-K. Yeung, *Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane*, Invent. Math., 124:1 (1996), pp. 573–618.

[SY2] Y.-T. Siu and S.-K. Yeung, *A generalized Bloch's theorem and the hyperbolicity of the complement of an ample divisor in an abelian variety*, Math. Ann., 306:4 (1996), pp. 743–758.

[SY3] Y.-T. Siu and S.-K. Yeung, *Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees*, Amer. J. Math., 119:5 (1997), pp. 1139–1172.

[W] P.-M. Wong, *On the second main theorem of Nevanlinna theory*, American Journal of Mathematics, 111:4 (1989), pp. 549–583.