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A GENERAL DEFECT RELATION AND HEIGHT INEQUALITY FOR DIVISORS IN SUBGENERAL POSITION*

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Dedicated to Professor Ngaiming Mok on the occasion of his sixtieth birthday

Abstract. In this paper, we establish a general defect relation for holomorphic curves into algebraic varieties intersecting divisors in subgeneral position, as well as a general height inequality for rational points approximating the given divisors in an algebraic variety.

Key words. Nevanlinna theory, Diophantine approximation, Second Main Theorem, Schmidt's subspace theorem.

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1. Introduction and statement of the main result. In this paper, we establish a general (new) defect relation for holomorphic curves into algebraic varieties, as well as establish a general (new) height inequality in Diophantine approximation for divisors in l-subgeneral position. The results obtained improve the previously known results of Ru ([Ru15], [Ru16]), Corvaja-Zannier ([CZ04Add]), Chen-Ru-Yan ([CRY]), Levin ([Lev14]), and Shi-Ru ([Shi-Ru15]) etc.. For the recent development in this direction, including the results for divisors in general position (i.e. the case when l is equal to the dimension of the variety), see [CZ04b], [EF02], [EF08], [Ru04], [Ru09], and [Lev09].

To state the precise results, we recall some notations. Let X be a complex projective variety. For an effective Cartier divisor D on X, the Weil function for D is given by

$$\lambda_D(x) = -\log \|s_D(x)\|, \tag{1}$$

where $\mathscr{O}(D)$ is the line bundle associated to D, s_D is the canonical section of $\mathscr{O}(D)$, i.e., a global section for which $(s_D) = D$, and $\|\cdot\|$ is any continuous metric on $\mathscr{O}(D)$. It is well defined, up to a bounded term, independent of the choices of the metric. In the case when $X = \mathbb{P}^n(\mathbb{C})$ and $D = \{Q = 0\} \subset \mathbb{P}^n(\mathbb{C})$ where Q is a homogeneous polynomial of degree d, λ_D can be chosen as, for $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C})\backslash \text{supp}D$,

$$\lambda_D(x) = \log \frac{(\max_{0\le i\le n} |x_i|^d) \cdot \|Q\|}{|Q(x)|},$$

where $\|Q\|$ is the maximum of the norm of the coefficients of Q. Let $f : \mathbb{C} \to X$ be a holomorphic map whose image is not contained in the support of divisor D on X. The *proximity function of f with respect to D* is defined by

$$m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi},$$

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where λ_D is the Weil function associated to D. The *counting function of f with respect to D* is defined by

$$N_f(r,D) = \int_1^r \frac{n_f(t,D)}{t} dt$$

where $n_f(t,D)$ is the number of zeros of $\rho \circ f$ inside $\{|z| < t\}$, counting multiplicities, where ρ is a local defining function of D. The *height* or *characteristic function of f with respect to D* is given by

$$T_{f,D}(r) := m_f(r,D) + N_f(r,D).$$

For any line sheaf $\mathscr{L}$ on X with $\dim H^0(X,\mathscr{L}) \geq 1$, we define $T_{f,\mathscr{L}}(r) := T_{f,D}(r)$ with $D = [s=0], s \in H^0(X,\mathscr{L})$. It is well defined up to a bounded term.

Let $D_1,\ldots,D_q$ be effective divisors on X. We say that $D_1,\ldots,D_q$ are in *l-subgeneral position* on X if for any subset $I \subseteq \{1,\ldots,q\}$ with $\#I \leq l+1$,

$$\dim \bigcap_{i\in I} \mathrm{Supp}\ D_i \leq l - \#I,$$

where $\dim \emptyset = -1$. In particular, the supports of any $l+1$ divisors in l-subgeneral position have empty intersection. If $l = \dim X$, then we say the divisors are in *general position* on X. Let $\mathscr{L}$ be a line sheaf on X, we use $h^0(\mathscr{L})$ to denote $\dim H^0(X,\mathscr{L})$, and $\mathscr{L}(-D)$ to denote the sheaf $\mathscr{L} \otimes \mathscr{O}(-D)$ for a given divisor D on X.

Definition 1.1. *Let $\mathscr{L}$ be a line sheaf and D be a nonzero effective Cartier divisor on a projective variety X. We define*

$$\gamma(\mathscr{L},D) := \limsup_{N\to+\infty} \frac{Nh^0(\mathscr{L}^N)}{\sum_{m\geq 1} h^0(\mathscr{L}^N(-mD))}, \tag{2}$$

where N passes over all positive integers such that $h^0(\mathscr{L}^N(-D)) \neq 0$. If no such N exists, then we define $\gamma(\mathscr{L},D) = +\infty$ (Note that $|\mathscr{L}^N|$ does not have to be base point free.)

Main Theorem (Analytic Part). *Let X be a complex projective variety of dimension n and let $D_1,\ldots,D_q$ be effective Cartier divisors, located in l-subgeneral position on X with $l+n-2>0$. Let $\mathscr{L}$ be a line sheaf on X with $h^0(\mathscr{L}^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^q m_f(r,D_j) \leq_{exc} \frac{l(l-1)}{(l+n-2)} \left(\max_{1\leq j\leq q} \gamma(\mathscr{L},D_j) + \epsilon \right) T_{f,\mathscr{L}}(r),$$

where $\leq_{exc}$ means the inequality holds for all $r \in (0,+\infty)$ except for a subset $E \subset (0,+\infty)$ with finite Lebesgue measure.

The above theorem has a counterpart in Diophantine approximation. We use the standard notations in Diophantine approximation (see for example, [Lan87], [Voj87], or [Vojcm]). For a number field k, recall that M_k denotes the set of places of k, and that k_v denotes the completion of k at a place $v \in M_k$. Norms $\|\cdot\|_v$ on k are normalized so that

$$\|x\|_v = |\sigma(x)|^{[k_v:\mathbb{R}]} \qquad \text{or} \qquad \|p\|_v = p^{-[k_v:\mathbb{Q}_p]}$$

if $v \in M_k$ is an archimedean place corresponding to an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$ or a non-archimedean place lying over a rational prime p, respectively. For each $v \in M_k$, we denote by $\lambda_{D,v}$ as the Weil function of D with respect to v (for the definition, see [Lan87], [Ru16] or [Vojcm]). Let $\mathscr{L}$ be a line sheaf, we denote by $h_{\mathscr{L}}(x)$ the height of $x \in X(k)$ with respect to the line sheaf $\mathscr{L}$.

Main Theorem (Arithmetic Part). *Let X be a projective variety of dimension n, and let $D_1, \dots, D_q$ be effective Cartier divisors, located in l-subgeneral position on X with $l+n-2>0$, both defined over a number field k. Let $\mathscr{L}$ be a line sheaf on X with $h^0(\mathscr{L}^N) \geq 1$ for N big enough. Let $S \subset M_k$ be a finite set of places. Let $\lambda_{D_j,v}, 1 \leq j \leq q$, be the Weil function associated to D_j for $v \in S$. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q}\sum_{v\in S}\lambda_{D_j,v}(x) \leq \frac{l(l-1)}{(l+n-2)}\left(\max_{1\leq j\leq q}\gamma(\mathscr{L}, D_j)+\epsilon\right)h_{\mathscr{L}}(x),$$

holds for all k-rational points outside a proper Zariski closed subset of X.

The proof of the result on the arithmetic case is similar to the complex case by replacing H. Cartan's theorem with Schmidt's subspace theorem (see, for example, [Ru16]), so the rest of the paper will only focus on the complex part.

2. The consequences of the Main Theorem. In this section, we derive some consequences from the Main Theorem. Let $D := D_1 + \cdots + D_q$ where $D_1, \dots, D_q$ are effective Cartier divisors on X. Write

$$\gamma(D_j) = \gamma(\mathscr{O}(D), D_j). \tag{3}$$

To compute $\gamma(D_j)$, we consider the following two cases.

2.1. The divisors are ample and linearly equivalent. Assume that each D_j, $1 \leq j \leq q$, is linearly equivalent to a fixed ample divisor A on X. We write $h^0(D) := h^0(\mathscr{O}(D))$. By the Riemann-Roch theorem, with $n = \dim X$, we have

$$h^0(ND) = h^0(qNA) = \frac{(qN)^n A^n}{n!} + o(N^n)$$

and

$$h^0(ND - mD_j) = h^0((qN-m)A) = \frac{(qN-m)^n A^n}{n!} + o(N^n).$$

Thus

$$\sum_{m\geq 1} h^0(ND - mD_j) = \frac{A^n}{n!}\sum_{l=0}^{qN-1} l^n + o(N^{n+1}) = \frac{A^n(qN-1)^{n+1}}{(n+1)!} + o(N^{n+1}).$$

Hence

$$\gamma(D_j) = \lim_{N\to\infty}\frac{N\frac{(qN)^n A^n}{n!} + o(N^{n+1})}{\frac{A^n(qN-1)^{n+1}}{(n+1)!} + o(N^{n+1})} = \frac{n+1}{q}.$$

Therefore, the Main Theorems imply the following results due to Shi-Ru ([Shi-Ru15]).

Theorem 2.1 (Theorem 1 in [Shi-Ru15]). *Let X be a complex projective variety of dimension n and let $D_1, \ldots, D_q$ be effective Cartier divisors, located in l-subgeneral position on X with $l+n-2>0$. Assume that each D_j, $1 \le j \le q$, is linearly equivalent to a fixed ample divisor A. Let $f: \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} m_f(r, D_j) \le_{exc} \frac{l(l-1)(n+1+\epsilon)}{(l+n-2)} T_{f,A}(r).$$

Theorem 2.2 (Arithmetic Part). *Let X be a projective variety of dimension n, and let $D_1, \ldots, D_q$ be effective Cartier divisors, located in l-subgeneral position on X with $l+n-2>0$, both defined over a number field k. Assume that each D_j, $1 \le j \le q$, is linearly equivalent to a fixed ample divisor A. Let $S \subset M_k$ be a finite set of places. Let $\lambda_{D_j,v}, 1 \le j \le q$, be the Weil function associated to D_j for $v \in S$. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \sum_{v \in S} \lambda_{D_j,v}(x) \le \frac{l(l-1)(n+1+\epsilon)}{(l+n-2)} h_A(x),$$

holds for all k-rational points outside a proper Zariski closed subset of X.

2.2. Big and nef case. We use the standard notion of the intersection theory (see [Ful98] for a thorough modern account, or [Laz04]). We will use the notation D^n to denote the intersection number of the n-fold intersection of D with itself. A Cartier divisor D (or invertible sheaf $\mathscr{O}(D)$) on X is said to be **numerically effective**, or **nef**, if $D.C \ge 0$ for any closed integral curve C on X. We will use the following lemma for nef divisors (see [Kle66]).

Lemma 2.3. *Let $n = \dim X$, and if $D_1, \ldots, D_n$ are nef divisors on X, then*

$$D_1.D_2....D_n \ge 0.$$

An effective divisor D is said to be **big** if $D^n > 0$.

We now assume that $D_1, \ldots, D_q$ are big and nef.

Definition 2.4. *Suppose that X is a complete variety of dimension n. Let $D_1, \ldots, D_q$ be effective Cartier divisors on X, and let $D = D_1 + D_2 + \cdots + D_q$. We say that D* ***has equi-degree with respect to*** *$D_1, D_2, \ldots, D_q$ if $D_i.D^{n-1} = \frac{1}{q}D^n$ for all $i = 1, \ldots, q$.*

The important result associated to the concept of equi-degree is the following lemma regarding $D := D_1 + \cdots + D_q$ where each D_j is only assumed to be big and nef for $1 \le j \le q$.

Lemma 2.5 (Lemma 9.7 in [Lev09]). *Let X be a projective variety of dimension n. If $D_j, 1 \le j \le q$, are big and nef Cartier divisors, then there exist positive real numbers r_j such that $D = \sum_{j=1}^{q} r_j D_j$ has equi-degree.*

We first compute $\gamma(D_j)$ under an additional assumption that $D_1, \ldots, D_q$ are of equi-degree, i.e.

$$D_j.D^{n-1} = \frac{1}{q} D^n \quad \text{for} \quad j = 1, \ldots, q. \tag{4}$$

where $D := D_1 + \cdots + D_q$. We use the following lemma from Autissier [Aut1].

LEMMA 2.6 (Lemma 4.2 in [Aut1]). *Suppose E is a big and base-point free Cartier divisor on a projective variety X of dimension n, and F is a nef Cartier divisor on X such that $F - E$ is also nef. Let $\beta > 0$ be a positive real number. Then, for any positive integers N and m with $1 \le m \le \beta N$, we have*

$$h^0(NF - mE) \ge \frac{F^n}{n!}N^n - \frac{F^{n-1}\,.\,E}{(n-1)!}N^{n-1}m + \frac{(n-1)F^{n-2}\,.\,E^2}{n!}N^{n-2}\min\{m^2, N^2\} + O(N^{n-1})$$

where O depends on β.

Let $n = \dim X$, and assume that $n \ge 2$. Fix $1 \le i \le q$ and apply Lemma 2.6 by taking $\beta = \frac{D^n}{nD^{n-1}.D_i}$, we get

$$\begin{aligned}
&\sum_{m=1}^{\infty} h^0(ND - mD_i)\\
&\ge \sum_{m=1}^{[\beta N]}\left(\frac{D^n}{n!}N^n - \frac{D^{n-1}\,.\,D_i}{(n-1)!}N^{n-1}m + \frac{D^{n-2}.D_i^2}{n!}N^{n-2}\min\{m^2, N^2\}\right) + O(N^n)\\
&\ge \left(\frac{D^n}{n!}\beta - \frac{D^{n-1}\,.\,D_i}{(n-1)!}\frac{\beta^2}{2} + \frac{D^{n-2}.D_i^n}{n!}g(\beta)\right)N^{n+1} + O(N^n)\\
&= \left(\frac{\beta}{2} + \frac{D^{n-2}.D_i^2}{D^n}g(\beta)\right)D^n\frac{N^{n+1}}{n!} + O(N^n)\\
&\ge \left(\frac{\beta}{2} + \hat{\alpha}\right)Nh^0(ND) + O(N^n)
\end{aligned} \tag{5}$$

where $\hat{\alpha} := \frac{\min_{1\le j\le q} D_j^n}{D^n}g(\beta)$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by $g(x) = \frac{x^3}{3}$ if $x \le 1$ and $g(x) = x - \frac{2}{3}$ for $x \ge 1$. Note that, from (4), $\beta = \frac{D^n}{nD^{n-1}.D_i} = \frac{q}{n}$, so $g(\beta) \ge \frac{1}{3n^3}$. This, together with (5) and the definition of $\gamma(D_i)$, implies that

$$\gamma(D_i) = \limsup_{N\to+\infty}\frac{Nh^0(ND)}{\sum_{m\ge1}h^0(ND - mD_i)} \le \frac{1}{\frac{\beta}{2} + \hat{\alpha}} = \frac{2n}{q + 2n\hat{\alpha}}. \tag{6}$$

Notice that

$$\hat{\alpha} = \frac{\min_{1\le j\le q} D_j^n}{D^n}g(\beta) \ge \frac{\min_{1\le j\le q} D_j^n}{3n^3D^n}.$$

By applying the Main Theorem (in both complex and arithmetic cases) with $\epsilon = \frac{2n^2\hat{\alpha}}{(q+n\hat{\alpha})(q+2n\hat{\alpha})}$, it implies the following two theorems.

THEOREM 2.7 (Analytic Part). *Let X be a complex projective variety of dimension $n \ge 2$, and let $D_1, \ldots, D_q$ be big and nef Cartier divisors on X, located in l-subgeneral position on X. Assume that $D := \sum_{j=1}^q D_j$ has equi-degree with respect to $D_1, \ldots, D_q$. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then*

$$m_f(r, D) \le_{exc} \frac{l(l-1)}{(l+n-2)}\frac{2n}{q + C'}T_{f,D}(r),$$

where

$$C' = \frac{\min_{1\le j\le q} D_j^n}{3n^2 D^n}.$$

THEOREM 2.8 (Arithmetic Part). *Let k be a number field and let $S \subseteq M_k$ be a finite set containing all archimedean places. Let X be a projective variety of dimension $n \ge 2$, and let $D_1, \dots, D_q$ be big and nef Cartier divisors on X located in l-subgeneral position, both defined over k. Assume that $D := \sum_{j=1}^q D_j$ has equi-degree respect to $D_1, \dots, D_q$. Let $\lambda_{D,\upsilon}$ be the Weil function associated to D for $\upsilon \in S$. Then the inequality*

$$\sum_{\upsilon\in S} \lambda_{D,\upsilon}(x) \le \frac{l(l-1)}{(l+n-2)} \frac{2n}{q+C'} h_D(x),$$

holds for all k-rational points $x \in X(k)$ outside of a proper Zariski-closed subset of X, where

$$C' = \frac{\min_{1\le j\le q} D_j^n}{3n^2 D^n}.$$

In the general case that $D_1, \dots, D_q$ are only assumed to be big and nef, from Lemma 2.5, there are positive real numbers $r_i > 0$ such that $D := \sum_{i=1}^q r_i D_i$ has equi-degree. Note that the divisors $r_j D_j$ and D_j have the same support. In this case, we have the following results.

THEOREM 2.9 (Analytic Part). *Let X be a complex projective variety of dimension $n \ge 2$, and let $D_1, \dots, D_q$ be effective, big, and nef Cartier divisors on X, located in l-subgeneral position. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ has equi-degree (such numbers exist due to Lemma 2.5). Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then*

$$\sum_{j=1}^q r_j m_f(r, D_j) \le_{exc} \frac{l(l-1)}{(l+n-2)} \frac{2n}{q+C} \left(\sum_{j=1}^q r_j T_{f,D_j}(r) \right)$$

where

$$C = \frac{\min_{1\le j\le q}(r_j^n D_j^n)}{6n^2 4^n D^n}.$$

THEOREM 2.10 (Arithmetic Part). *Let k be a number field and let $S \subseteq M_k$ be a finite set containing all archimedean places. Let X be a projective variety of dimension $n \ge 2$, and let $D_1, \dots, D_q$ be effective, big, and nef Cartier divisors on X, located in l-subgeneral position, both defined over k. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ has equi-degree (such numbers exist due to Lemma 2.5). Let $\lambda_{D_j,\upsilon}, 1 \le j \le q$, be the Weil function associated to D_j for $\upsilon \in S$. Then the inequality*

$$\sum_{j=1}^q r_j \sum_{\upsilon\in S} \lambda_{D_j,\upsilon}(x) \le \frac{l(l-1)}{(l+n-2)} \frac{2n}{q+C} \left(\sum_{j=1}^q r_j h_{D_j}(x) \right)$$

holds for all $x \in X(k)$ outside a proper Zariski-closed subset of X, where

$$C = \frac{\min_{1\le j\le q}(r_j^n D_j^n)}{6n^2 4^n D^n}.$$

We note that Theorems 2.9 and 2.10 improve the earlier results in [Ru15] (see Theorem 5.6 in [Ru15]) and [Ru16] (see Theorem 4.1 in [Ru16]). Again, we only prove Theorem 2.9.

Proof of Theorem 2.9. Let $D := \sum_{j=1}^q r_j D_j$. Denote by

$$\alpha := \frac{\min_{1\le j\le q}(r_j^n D_j^n)}{3n^3 4^n D^n}. \tag{7}$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can choose (positive) rational numbers $a_1, \dots, a_q$ such that

$$|a_j - r_j| \le \frac{\delta_1 \min_{1\le j\le q} r_j}{2} \min\left\{1, \frac{1}{\frac{l(l-1)}{(l+n-2)}\frac{2n}{q}\frac{1}{(1+n\alpha/q)}}\right\}, \quad 1 \le j \le q, \tag{8}$$

and

$$\left|\frac{D'^n}{a_j D_j . D'^{n-1}} - q\right| < \delta_2, \quad 1 \le j \le q, \tag{9}$$

where $D' := a_1 D_1 + \cdots + a_q D_q$, and δ_1, δ_2 will be chosen later (see (18) and (14)). Note that, with the above choices, we have

$$D'^n \ge \frac{1}{2^n} D^n \quad \text{and} \quad D'^n \le 2^n D^n. \tag{10}$$

To clear out the denominators, we let d be the product of the denominators of $a_1, \dots, a_q$. Similar to (5), by applying Lemma 2.6 with $F = dD'$ and $E = da_i D_i$, we have

$$\sum_{m=1}^{\infty} h^0(NdD' - mda_i D_i) \ge \left(\frac{\beta}{2} + \hat{\alpha}\right) N h^0(NdD') + O(N^n) \tag{11}$$

where

$$\beta = \frac{d^n D'^n}{n(dD')^{n-1} . (da_i D_i)} = \frac{D'^n}{nD'^{n-1}.(a_i D_i)},$$

$$\hat{\alpha} = \frac{\min_{1\le j\le q}(da_j D_j)^n}{(dD')^n} g(\beta) = \frac{\min_{1\le j\le q}(a_j D_j)^n}{D'^n} g(\beta),$$

and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function defined by

$$g(x) = \begin{cases} \frac{x^3}{3}, & 0 < x \le 1 \\ x - \frac{2}{3}, & x \ge 1. \end{cases}$$

Using (9), we get

$$\beta = \frac{D'^n}{nD'^{n-1}.(a_i D_i)} \ge \frac{q - \delta_2}{n} \tag{12}$$

and, also it is easy to see that $\beta \geq \frac{1}{n}$ since $D'^{n-1}.(a_iD_i) \leq D'^n$, and so $g(\beta) \geq \frac{1}{3n^3}$. Hence, by also noticing (10) and (7),

$$\hat{\alpha} = \frac{\min_{1\leq j\leq q}(a_j^n D_j^n)}{D'^n} g(\beta) \geq \frac{\min_{1\leq j\leq q}(r_j^n D_j^n)}{3n^3 4^n D^n} = \alpha. \tag{13}$$

Hence, by combining (11), (12) and (13), we get

$$\sum_{m=1}^{\infty} h^0(NdD' - mda_iD_i) > \left(\frac{q-\delta_2}{2n} + \alpha\right) Nh^0(NdD') + O(N^n),$$

so by the definition of γ (see (3)),

$$\gamma(da_iD_i) \leq \frac{2n}{q - \delta_2 + 2n\alpha} \leq \frac{2n}{q + \frac{3}{2}n\alpha}$$

by taking

$$\delta_2 = \min\{1, \frac{n\alpha}{2}\}. \tag{14}$$

Applying the Main Theorem with $\epsilon = \frac{n^2\alpha}{(q+n\alpha)(q+\frac{3}{2}n\alpha)}$, we get

$$\sum_{j=1}^{q} a_j m_f(r, D_j) \leq_{exc} \frac{l(l-1)}{(l+n-2)} \frac{2n}{(q+n\alpha)} \left(\sum_{j=1}^{q} a_j T_{f,D_j}(r)\right). \tag{15}$$

To use the above result to derive our conclusion, notice (8) gives us

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \sum_{j=1}^{q} a_j m_f(r, D_j) + \frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right) \sum_{j=1}^{q} m_f(r, D_j) \tag{16}$$

and

$$\sum_{j=1}^{q} a_j T_{f,D_j}(r) \leq \sum_{j=1}^{q} r_j T_{f,D_j}(r) + \frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right) \frac{1}{\frac{l(l-1)}{(l+n-2)}\frac{2n}{q+n\alpha}} \sum_{j=1}^{q} T_{f,D_j}(r). \tag{17}$$

Using (15), (16) and the First Main Theorem,

$$\begin{aligned}
\sum_{j=1}^{q} r_j m_f(r, D_j) &\leq \sum_{j=1}^{q} a_j m_f(r, D_j) + \frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right) \sum_{j=1}^{q} m_f(r, D_j) \\
&\leq_{exc} \frac{l(l-1)}{(l+n-2)} \frac{2n}{(q+n\alpha)} \left(\sum_{j=1}^{q} a_j T_{f,D_j}(r)\right) + \frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right) \sum_{j=1}^{q} T_{f,D_j}(r) \\
&\leq_{exc} \frac{l(l-1)}{(l+n-2)} \frac{2n}{(q+n\alpha)} \left(\sum_{j=1}^{q} a_j T_{f,D_j}(r)\right) + \frac{\delta_1}{2} \sum_{j=1}^{q} r_j T_{f,D_j}(r).
\end{aligned}$$

This, together with (17) gives

$$\begin{aligned}
&\sum_{j=1}^{q} r_j m_f(r, D_j)\\
&\leq_{exc} \frac{l(l-1)}{(l+n-2)} \frac{2n}{(q+n\alpha)} \left(\sum_{j=1}^{q} r_j T_{f,D_j}(r)\right) + \frac{\delta_1}{2}\left(\min_{1\leq i\leq q} r_i\right) \sum_{j=1}^{q} T_{f,D_j}(r)\\
&+ \frac{\delta_1}{2} \sum_{j=1}^{q} r_j T_{f,D_j}(r)\\
&\leq_{exc} \left(\frac{l(l-1)}{(l+n-2)} \frac{2n}{q+n\alpha} + \delta_1\right) \sum_{j=1}^{q} r_j T_{f,D_j}(r)\\
&= \left(\frac{l(l-1)}{(l+n-2)} \frac{2n}{q+\frac{n\alpha}{2}}\right) \sum_{j=1}^{q} r_j T_{f,D_j}(r)
\end{aligned}$$

by choosing

$$\delta_1 = \frac{l(l-1)n^2\alpha}{(l+n-2)(q+n\alpha)(q+\frac{n\alpha}{2})}. \tag{18}$$

Thus

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq_{exc} \left(\frac{l(l-1)}{(l+n-2)} \frac{2n}{q+C}\right) \sum_{j=1}^{q} r_j T_{f,D_j}(r)$$

where,

$$C = \frac{n\alpha}{2} = \frac{\min_{1\leq j\leq q}(r_j^n D_j^n)}{6n^2 4^n D^n}. \tag{19}$$

This finishes the proof. □

3. Proof of the Main Theorem. To prove the Main Theorem, we need the following version of H. Cartan's theorem (see [Ru97], or [Voj97]). Note that, for the arithmetic part, we use Schmidt's subspace theorem instead.

Theorem 3.1. *Let n be a positive integer, let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, and let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained in a hyperplane. Then, for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{H_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \leq_{exc} (n+1+\epsilon)T_f(r),$$

where the maximum is taken over all $J \subset \{1, \ldots, q\}$ such that the hyperplanes $\{H_j, j \in J\}$ are in general position.

For the convenient use of Cartan's theorem, we prove the following slightly improved result.

Theorem 3.2. *Let X be a complex projective variety and let D be a Cartier divisor on X, let V be a nonzero linear subspace of $H^0(X, \mathscr{O}(D))$, and let $s_1, \ldots, s_q$*

be nonzero elements of V. For each $i = 1, \dots, q$, let D_j be the Cartier divisor (s_j). Let $f\colon \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for any $\epsilon > 0$,

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \leq_{exc} (\dim V + \epsilon)T_{f,D}(r).$$

Here the set J ranges over all subsets of $\{1, \dots, q\}$ such that the sections $(s_j)_{j\in J}$ are linearly independent.

Proof. Let $d = \dim V$. We may assume that $d > 1$ (otherwise, all D_j are the same divisor, and the sets J have at most one element each, so the theorem follows immediately from the First Main Theorem).

Let $\Phi\colon X \dashrightarrow \mathbb{P}^{d-1}$ be the rational map associated to the linear system V. Let X' be the closure of the graph of Φ, and let $p\colon X' \to X$ and $\phi\colon X' \to \mathbb{P}^{d-1}$ be the projection morphisms. Let $\tilde{f} :: \mathbb{C} \to X'$ be the lifting of f.

Note that, even though Φ extends to the morphism $\phi\colon X' \to \mathbb{P}^{d-1}$, the linear system of $H^0(X', p^*\mathscr{O}(D))$ corresponding to V may still have base points. What is true, however, is that there is an effective Cartier divisor B on X' such that, for each nonzero $s \in V$, there is a hyperplane H in $\mathbb{P}^{d-1}$ such that $p^*(s) - B = \phi^*H$. (More precisely, $\phi^*\mathscr{O}(1) \cong \mathscr{O}(p^*D - B)$. The map

$$\alpha\colon H^0(X', \mathscr{O}(p^*D - B)) \to H^0(X, \mathscr{O}(p^*D))$$

defined by tensoring with the canonical global section s_B of $\mathscr{O}(B)$ is injective, and its image contains $p^*(V)$. The preimage $\alpha^{-1}(p^*(V))$ corresponds to a base-point-free linear system for the divisor $p^*D - B$.)

For each $j = 1, \dots, q$, let H_j be the hyperplane in $\mathbb{P}^{d-1}$ for which $p^*(s_j) - B = \phi^*H_j$. Then,

$$\lambda_{p^*D_j} = \lambda_{\phi^*H_j} + \lambda_B + O(1)\ . \tag{20}$$

By functoriality of Weil functions, $\lambda_{p^*D_j}(\tilde{f}(z)) = \lambda_{D_j}(f(z))$. Therefore it will suffice to prove the inequality

$$\int_0^{2\pi} \left(\max_J \sum_{j\in J} \lambda_{H_j}(\phi(\tilde{f})(re^{i\theta})) + \lambda_B(\tilde{f}(re^{i\theta}))\right) \frac{d\theta}{2\pi}$$
$$\leq_{exc} (\dim V + \epsilon)T_{f,D}(r).$$

For any subset J of $\{1, \dots, q\}$, the sections s_j, $j \in J$, are linearly independent elements of V if and only if the hyperplanes H_j, $j \in J$, lie in general position in $\mathbb{P}^{d-1}$. Thus we may apply the above H. Cartan's Theorem to obtain that

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{H_j}(\phi(\tilde{f})(re^{i\theta}))\frac{d\theta}{2\pi} \leq_{exc} (\dim V + \epsilon)T_{\phi(\tilde{f})}(r). \tag{21}$$

From (20), we get $T_{\phi(\tilde{f})}(r) = T_{f,D}(r) - T_{\tilde{f},B}(r) + O(1)$. On the other hand, since each set J as above has at most $\dim V$ elements and B is effective, we get

$$(\#J)\lambda_B(x) \leq (\dim V)\lambda_B(x) + O(1)$$

for all $x \in X'$. Hence

$$\int_0^{2\pi}\left(\max_J \sum_{j\in J}\lambda_{H_j}(\phi(\tilde{f})(re^{i\theta}))+\lambda_B(\tilde{f}(re^{i\theta}))\right)\frac{d\theta}{2\pi}$$
$$\leq_{exc} (\dim V+\epsilon)T_{f,D}(r)-(\dim V+\epsilon)T_{\tilde{f},B}(r)+(\dim V)m_{\tilde{f}}(r,B)$$
$$\leq_{exc} (\dim V+\epsilon)T_{f,D}(r),$$

where, in the last inequality, we used the first main theorem that $m_{\tilde{f}}(r,B)\leq T_{\tilde{f},B}(r)+O(1)$. This finishes the proof. ∎

We also need the following joint-filtration lemma.

Lemma 3.3 ([CZ04a], Lemma 3.2). *Let $V=W_1\supset W_2\supset\cdots\supset W_h$ and $V=W_1^*\supset W_2^*\supset\cdots\supset W_{h^*}^*$ be two filtrations of V Then there exists a basis $v_1,\dots,v_d$ of V which contains a basis of each W_j and W_j^*.*

Lemma 3.4 ([Vojcm], Lemma 20.7). *Let X be a smooth complex projective variety and let D be an effective divisor on X. Let σ_0 be the set of all prime divisors occurring in D. Let*

$$\Sigma := \left\{\sigma\subseteq\sigma_0 \mid \bigcap_{E\in\sigma} E\neq\emptyset\right\}.$$

For $D=\sum_{E\in\sigma_0}(\operatorname{ord}_E D)E$ and $\sigma\in\Sigma$, write

$$D=D_{\sigma,1}+D_{\sigma,2}$$

where $D_{\sigma,1}:=\sum_{E\in\sigma}(\operatorname{ord}_E D)E$ and $D_{\sigma,2}:=\sum_{E\notin\sigma}(\operatorname{ord}_E D)E$. Then there exists a constant C, depending only on X and D, such that

$$\min_{\sigma\in\Sigma}\lambda_{D_{\sigma,2}}(x)\leq C$$

for all $x\in X$.

Proof of the Main Theorem. Let $x=f(z)\in X\backslash\operatorname{Supp} D$. By Lemma 3.4, there exists a $\sigma\in\Sigma$ (which depends on z) for which

$$\lambda_{D_{\sigma,2}}(f(z))\leq C.$$

From the condition that $D_1,\dots,D_q$ are in l-subgeneral position, there are $D_{1,z},\dots,D_{l,z}\in\{D_1,\dots,D_q\}$ such that the prime divisors $E\in\sigma$ with $f(z)\in E$ only occur in $D_{1,z},\dots,D_{l,z}$. Hence

$$\sum_{j=1}^{q}\lambda_{D_j}(f(z))\leq\sum_{\alpha=1}^{l}\lambda_{D_{\alpha,z}}(f(z))+O(1). \tag{22}$$

Fix $D_{\mu,z}$ and $D_{\nu,z}$ in $\{D_{1,z},\dots,D_{l,z}\}$ with $\mu\neq\nu$. For any $\epsilon>0$, we choose $N\geq N_0$ big enough such that

$$\frac{Nh^0(\mathscr{L}^N)}{\sum_{m=1}^{\infty}h^0(\mathscr{L}^N(-mD_j))}\leq\gamma(\mathscr{L},D_j)+\epsilon/2 \tag{23}$$

for $1 \leq j \leq q$. Fix such N. We consider the following two filtrations for the vector space $V := H^0(X, \mathscr{L}^N)$:

$$H^0(X, \mathscr{L}^N) = W_{\mu,0} \supset W_{\mu,1} \supset W_{\mu,2} \supset \cdots \supset \{0\},$$

and

$$H^0(X, \mathscr{L}^N) = W_{\nu,0} \supset W_{\nu,1} \supset W_{\nu,2} \supset \cdots \supset \{0\},$$

where

$$W_{\mu,k} = H^0(X, \mathscr{L}^N(-kD_{\mu,z})) \text{ and } W_{\nu,k} = H^0(X, \mathscr{L}^N(-kD_{\nu,z})).$$

Note here we regard $H^0(X, \mathscr{L}^N(-kD_{\mu,z}))$ as a subspace of $H^0(X, \mathscr{L}^N)$ by sending $s \mapsto s \otimes s^k_{D_{\mu,z}} \in H^0(X, \mathscr{L}^N)$ for any $s \in H^0(X, \mathscr{L}^N(-kD_{\mu,z}))$ where $s_{D_{\mu,z}}$ is the canonical section of $\mathscr{O}(D_{\mu,z})$. Hence, for any section $s \in H^0(X, \mathscr{L}^N(-kD_{\mu,z})) \subset H^0(X, \mathscr{L}^N)$, we have

$$(s) \geq kD_{\mu,z} \tag{24}$$

where (s) is the zero divisor of s on X. Similar result also holds for $s \in H^0(X, \mathscr{L}^N(-kD_{\nu,z})) \subset H^0(X, \mathscr{L}^N)$. By Lemma 3.3, there exists a basis $B(\mu, \nu, z)$ of $H^0(X, \mathscr{L}^N)$ which contains a basis of each $W_{\mu,k}$ and $W_{\nu,k}$. Note that $B(\mu, \nu, z)$ depends on $D_{\mu,z}$. Hence, by noticing (24),

$$\begin{aligned}\sum_{s \in B(\mu,\nu,z)} (s) &\geq \sum_{k=0}^{\infty} k \left(h^0(\mathscr{L}^N(-kD_{\mu,z})) - h^0(\mathscr{L}^N(-(k+1)D_{\mu,z}))\right) D_{\mu,z} \\ &= \sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_{\mu,z}))D_{\mu,z} \\ &\geq \left(\min_{1\leq j\leq q} \sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))\right) D_{\mu,z}.\end{aligned}$$

Similarly, we also have

$$\sum_{s \in B(\mu,\nu,z)} (s) \geq \left(\min_{1\leq j\leq q} \sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))\right) D_{\nu,z}.$$

It follows that

$$\sum_{s \in B(\mu,\nu,z)} (s) \geq \left(\min_{1\leq j\leq q} \sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))\right) \operatorname{lcm}(D_{\mu,z}, D_{\nu,z}), \tag{25}$$

where, for any two divisors D_1 and D_2 on X, we denote $\operatorname{lcm}(D_1, D_2) = \sum_E \max\{\operatorname{ord}_E D_1, \operatorname{ord}_E D_2\}E$ where the sum runs over all prime divisors E on X.

Next, we claim that

$$\sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{l} \operatorname{lcm}(D_{\mu,z}, D_{\nu,z}) \geq (l+n-2) \sum_{\alpha=1}^{l} D_{\alpha,z}. \tag{26}$$

Indeed, fix $\mu \in \{1, \dots, l\}$. We will first show that every irreducible component E of $D_{\mu,z}$ can belong to at most $l-n$ divisors $D_{\nu,z}, \nu \neq \mu$. For sake of contradiction, assume there exists an irreducible element E of $D_{\mu,z}$ belonging to at least $l-n+2$ divisors $D_{\alpha,z}$. Then

$$E \subseteq \bigcap_{\alpha} \text{Supp } D_{\alpha,z},$$

with α indexing the divisors E belongs to, so

$$\dim \bigcap_{\alpha} \text{Supp } D_{\alpha,z} \geq \dim E = n-1 > n-2 = l-(l-n+2).$$

This contradicts the fact that $D_1, \dots, D_q$ are in l-subgeneral position. So any irreducible component E of $D_{\mu,z}$ can belong to at most $l-n$ divisors $D_{\nu,z}, \nu \neq \mu$, and so

$$\begin{aligned}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{l} \text{lcm}(D_{\mu,z}, D_{\nu,z}) &\geq (l-1-(l-n))D_{\mu,z} + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{l} D_{\nu,z} \\ &= (n-1)D_{\mu,z} + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{l} D_{\nu,z}.\end{aligned}$$

Summing over all μ proves the claim.

Combining (25) and (26), it gives

$$\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{l} \sum_{s\in B(\mu,\nu,z)} (s) \geq (l+n-2)\left(\min_{1\leq j\leq q} \sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))\right) \sum_{\alpha=1}^{l} D_{\alpha,z}.$$

Denote by $\mathcal{B}$ the collection of all $B(\mu,\nu,z), 1 \leq \mu,\nu \leq l, z \in \mathbb{C}$. Note that $\mathcal{B}$ is a finite set since there are only finite many choices of $\{D_{\mu,z}, D_{\nu,z}\} \subset \{D_1, \dots, D_q\}$. By the property of Weil functions, the above inequality gives

$$\max_{B\in\mathcal{B}} \sum_{s\in B} \lambda_s(f(z)) \geq \frac{(l+n-2)}{l(l-1)} \left(\min_{1\leq j\leq q} \sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))\right) \sum_{\alpha=1}^{l} \lambda_{D_{\alpha,z}}(f(z)).$$

Combining this with (22) yields

$$\begin{aligned}&\sum_{j=1}^{q} \lambda_{D_j}(f(z)) \\ &\leq \frac{l(l-1)}{l+n-2} \max_{1\leq j\leq q} \frac{1}{\sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))} \max_{B\in\mathcal{B}} \sum_{s\in B} \lambda_s(f(z)) + O(1).\end{aligned} \tag{27}$$

Therefore

$$\begin{aligned}&\sum_{j=1}^{q} m_f(r, D_j) = \sum_{j=1}^{q} \int_0^{2\pi} \lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \\ &\leq \frac{l(l-1)}{l+n-2} \max_{1\leq j\leq q} \frac{1}{\sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))} \int_0^{2\pi} \max_{B\in\mathcal{B}} \sum_{s\in B} \lambda_s(f(re^{i\theta}))\frac{d\theta}{2\pi} \\ &+O(1).\end{aligned}$$

Applying Theorem 3.2 with $V := H^0(X, \mathscr{L}^N)$, we get

$$\int_0^{2\pi} \max_{B\in\mathcal{B}} \sum_{s\in B} \lambda_s(f(re^{i\theta}))\frac{d\theta}{2\pi} \leq_{exc} \left(h^0(\mathscr{L}^N) + \frac{\epsilon}{2N}\right) T_{f,\mathscr{L}^N}(r)$$
$$\leq_{exc} \left(Nh^0(\mathscr{L}^N) + \frac{\epsilon}{2}\right) T_{f,\mathscr{L}}(r) + O(1).$$

Thus, we get, by noticing (23),

$$\sum_{j=1}^{q} m_f(r, D_j)$$
$$\leq_{exc} \frac{l(l-1)}{l+n-2}\left(\max_{1\leq j\leq q} \frac{Nh^0(\mathscr{L}^N)}{\sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_j))} + \frac{\epsilon}{2}\right) T_{f,\mathscr{L}}(r)$$
$$\leq_{exc} \frac{l(l-1)}{l+n-2}\left(\max_{1\leq j\leq q} \gamma(\mathscr{L}, D_j) + \epsilon\right) T_{f,\mathscr{L}}(r)$$

which finishes the proof. □

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