Asian J. Math.
Vol. 22, No. 3, pp. 413–450, June 2018

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SZEGŐ KERNEL ASYMPTOTICS AND MORSE INEQUALITIES ON CR MANIFOLDS WITH S^1 ACTION*

CHIN-YU HSIAO† AND XIAOSHAN LI‡

Dedicated to Professor Ngaiming Mok for his 60th birthday

Abstract. Let X be a compact connected CR manifold of dimension $2n-1, n \geq 2$. We assume that there is a transversal CR locally free S^1 action on X. Let L^k be the k-th power of a rigid CR line bundle L over X. Without any assumption on the Levi-form of X, we obtain a scaling upper-bound for the partial Szegő kernel on $(0,q)$-forms with values in L^k. After integration, this gives the weak Morse inequalities. By a refined spectral analysis, we also obtain the strong Morse inequalities in CR setting. We apply the strong Morse inequalities to show that the Grauert-Riemenschneider criterion is also true in the CR setting.

Key words. CR manifolds, S^1-action, Szegő kernel asymptotics, Morse inequalities.

Mathematics Subject Classification. 32A25, 32J25, 32V20.

1. Introduction and statement of the main results. The problem of embedding CR manifolds is prominent in areas such as complex analysis, partial differential equations and differential geometry. Let X be a compact CR manifold of dimension $2n-1$, $n \geq 2$. When X is strongly pseudoconvex and dimension of X is greater than or equal to five, a classical theorem of L. Boutet de Monvel [6] asserts that X can be globally CR embedded into $\mathbb{C}^N$, for some $N \in \mathbb{N}$. For a strongly pseudoconvex CR manifold of dimension greater than five, the dimension of the kernel of the tangential Cauchy-Riemmann operator $\overline{\partial}_b$ is infinite and we can find many CR functions to embed X into complex space. Inspired by Kodaira, the first-named author and Marinescu introduced in [16] the idea of embedding CR manifolds by means of CR sections of tensor powers L^k of a CR line bundle $L \to X$. To study Kodaira type embedding theorems on CR manifolds, it is crucial to be able to know

Question 1.1. *When* $\dim H^0_b(X, L^k) \gtrsim k^n$, *for* k *large, where* $H^0_b(X, L^k)$ *denotes the space of global smooth CR sections of* L^k.

Inspired by Demailly [8, 9] (see also Getzler [11]), the first-named author and Marinescu established in [16] analogues of the holomorphic Morse inequalities of Demailly for CR manifolds.

Theorem 1.2 (Theorem 1.8, [16]). *We assume that* $Y(0)$ *and* $Y(1)$ *hold at each*

*Received June 30, 2016; accepted for publication June 1, 2017.
†Institute of Mathematics, Academia Sinica and National Center for Theoretical Sciences, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, Taiwan (chsiao@math.sinica.edu.tw or chinyu.hsiao@gmail.com). Chin-Yu Hsiao was partially supported by Taiwan Ministry of Science of Technology project 104-2628-M-001-003-MY2, the Golden-Jade fellowship of Kenda Foundation and Academia Sinica Career Development Award.
‡School of Mathematics and Statistics, Wuhan University, Hubei 430072, China (xiaoshanli@whu.edu.cn); & Institute of Mathematics, Academia Sinica, 6F, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, Taiwan (xiaoshanli@math.sinica.edu.tw). Xiaoshan Li was supported by National Natural Science Foundation of China (Grant No. 11501422).

point of X. Then as $k \to \infty$,

$$
\begin{aligned}
&-\dim H^0_b(X,L^k)+\dim H^1_b(X,L^k)\\
&\leqslant \frac{k^n}{(2\pi)^n}\Big(-\int_X\int_{\mathbb{R}_{x,0}}\big|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)\big|\,ds\,dv_X(x)\\
&\quad+\int_X\int_{\mathbb{R}_{x,1}}\big|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)\big|\,ds\,dv_X(x)\Big)+o(k^n),
\end{aligned}
\tag{1.1}
$$

where $\mathcal{R}^L_x$ is the associated curvature of L at $x\in X$, $H^1_b(X,L^k)$ denotes the first $\overline{\partial}_b$ cohomology group with values in L^k, $\mathcal{L}_x$ denotes the Levi form of X at $x\in X$, and for $x\in X$, $q=0,1$,

$$
\begin{aligned}
\mathbb{R}_{x,q}=&\{s\in\mathbb{R};\ \mathcal{R}^L_x+2s\mathcal{L}_x \text{ has exactly } q \text{ negative eigenvalues}\\
&\text{and } n-1-q \text{ positive eigenvalues}\}.
\end{aligned}
\tag{1.2}
$$

When $Y(0)$ and $Y(1)$ hold, from Kohn's results we know that $\dim H^0_b(X,L^k)<\infty$ and $\dim H^1_b(X,L^k)<\infty$. From (1.1), we see that if

$$
\int_X\int_{\mathbb{R}_{x,0}}\big|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)\big|\,dsdv_X(x)>\int_X\int_{\mathbb{R}_{x,1}}\big|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)\big|\,dsdv_X(x)
\tag{1.3}
$$

then L is big, that is $\dim H^0_b(X,L^k)\gtrsim k^n$. This is a very general criterion and it is desirable to refine it in some cases where (1.3) is not easy to verify. In general, it is very difficult to see when (1.3) holds even L is positive. The problem comes from the presence of positive eigenvalues of $\mathcal{R}^L_x$ and negative eigenvalues of $\mathcal{L}_x$. By using Theorem 1.2 to approach Question 1.1, we always have to impose extra conditions linking the Levi form and the curvature of the line bundle L. Similar problems also appear in the works of Marinescu [19, 20], Berman [4] where they studied the $\overline{\partial}$-Neumann cohomology groups associated to a high power of a given holomorphic line bundle on a compact complex manifold with boundary. In order to get many holomorphic sections, they also have to assume that, close to the boundary, the curvature of the line bundle is adapted to the Levi form of the boundary. In [13], by carefully studying semi-classical behaviour of microlocal Fourier transforms of the extreme functions for the spaces of lower energy forms of the associated Kohn Laplacian, the first-named author prove that L is big when L is positive, $Y(0)$ and $Y(1)$ hold on X under certain Sasakian conditions on X and L without any extra condition linking the Levi form of X and the curvature of L. All these developments need the assumptions that the Levi form satisfies condition $Y(0)$ and $Y(1)$.

However, in some important problems in CR geometry, we need to know when L is big without any assumption of the Levi form. For example, Ohsawa and Sibony [23] studied Kodaira type embedding theorems on Levi-flat CR manifolds. In their work, it is important to understand the space $H^0_b(X,L^k)$ for k large. Adachi [1] constructed a positive CR line bundle L over a Levi-flat compact CR manifold X of dimension $2n-1$ such that $\dim H^0_b(X,L^k)\lesssim k^{n-1}<k^n$ for k large. We are lead to ask

Question 1.3. *Can we establish some kind of Morse inequalities and Grauert-Riemenschneider criterion on some class of CR manifolds without any Levi-curvature assumption?*

The purpose of this work is to answer Question 1.3.

1.1. Our main results. Let us now formulate our main results. We refer to section 1.2 for some standard notations and terminology used here. Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n-1$, $n \geqslant 2$. Let L be a rigid CR line bundle over X. For every $u \in \Omega^{0,q}(X, L^k)$, we can define $Tu \in \Omega^{0,q}(X, L^k)$ and we have

$$T\overline{\partial}_b = \overline{\partial}_b T \quad \text{on } \Omega^{0,q}(X, L^k), \tag{1.4}$$

where $\overline{\partial}_b : \Omega^{0,q}(X, L^k) \to \Omega^{0,q+1}(X, L^k)$ denotes the tangential Cauchy-Riemann operator. For every $m \in \mathbb{Z}$, put

$$\Omega^{0,q}_m(X, L^k) := \left\{u \in \Omega^{0,q}(X, L^k);\, Tu = imu\right\}. \tag{1.5}$$

From (1.4), we have the $\overline{\partial}_b$-complex for every $m \in \mathbb{Z}$:

$$\overline{\partial}_b : \cdots \to \Omega^{0,q-1}_m(X, L^k) \to \Omega^{0,q}_m(X, L^k) \to \Omega^{0,q+1}_m(X, L^k) \to \cdots. \tag{1.6}$$

For every $m \in \mathbb{Z}$, the m-th Fourier component of $\overline{\partial}_b$ cohomology is given by

$$H^q_{b,m}(X, L^k) := \frac{\operatorname{Ker} \overline{\partial}_b : \Omega^{0,q}_m(X, L^k) \to \Omega^{0,q+1}_m(X, L^k)}{\operatorname{Im} \overline{\partial}_b : \Omega^{0,q-1}_m(X, L^k) \to \Omega^{0,q}_m(X, L^k)}. \tag{1.7}$$

The starting point of this paper is that without any Levi curvature assumption, for every $m \in \mathbb{Z}$ and every $q = 0, 1, 2, \ldots, n-1$, we have

$$\dim H^q_{b,m}(X, L^k) < \infty. \tag{1.8}$$

Fix $\lambda \geq 0$ and set $H^q_{b,\leq\lambda}(X, L^k) := \bigoplus_{m\in\mathbb{Z}, |m|\leq\lambda} H^q_{b,m}(X, L^k)$. In this work, we study the asymptotic behavior of the space $H^q_{b,\leq k\delta}(X, L^k)$ and its partial Szegő kernel. Our main results are the following

Theorem 1.4 (weak Morse inequalities). *For k large and for every $q = 0, 1, 2, \ldots, n-1$, we have*

$$\begin{aligned} &\dim H^q_{b,\leq k\delta}(X, L^k) \\ &\leq (2\pi)^{-n} \frac{(-1)^q}{(n-1)!} k^n \int_X \int_{\mathbb{R}_{x,q} \bigcap [-\delta,\delta]} (i\mathcal{R}^L_x + i2s\mathcal{L}_x)^{n-1} \wedge (-\omega_0(x)) ds + o(k^n), \end{aligned} \tag{1.9}$$

where $\mathcal{R}^L_x$ denotes the curvature of L, $\mathcal{L}_x$ denotes the Levi form of X, ω_0 is the unique global non-vanishing real one form determined by $\langle \omega_0 , U \rangle = 0$, $\forall U \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0 , T \rangle = -1$ and

$$\begin{aligned} \mathbb{R}_{x,q} :=\{&s \in \mathbb{R} : \mathcal{R}^L_x + 2s\mathcal{L}_x \text{ has exactly } q \text{ negative} \\ &\text{and } n-1-q \text{ positive eigenvalues}\}. \end{aligned} \tag{1.10}$$

Although the eigenvalues of the Hermitian quadratic form $\mathcal{R}^L_x + 2s\mathcal{L}_x$, $s \in \mathbb{R}$ are calculated with respect to the rigid Hermitian metric $\langle \cdot | \cdot \rangle$, the sign does not depend on the metric. Note that $\mathcal{R}^L_x, \mathcal{L}_x \in T^{*1,0}_x X \wedge T^{*0,1}_x X$ (see Definition 1.13). Hence, $(\mathcal{R}^L_x + 2s\mathcal{L}_x)^{n-1} \wedge (-\omega_0(x))$ is a global $2n-1$ form on X. Any Hermitian fiber metric h^L on L induces a curvature $\mathcal{R}^L$. It is easy to see that the integral in (1.9) does not depend on the choice of Hermitian fiber metric of L.

Theorem 1.5 (strong Morse inequalities). *For k large and for every $q = 0, 1, 2, \ldots, n-2$, we have*

$$\begin{aligned}&\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{b,\leq k\delta}(X,L^k)\\&\leq (2\pi)^{-n}\frac{k^n}{(n-1)!}(-1)^q\sum_{j=0}^{q}\int_X\int_{\mathbb{R}_{x,j}\bigcap[-\delta,\delta]}(i\mathcal{R}^L_x+i2s\mathcal{L}_x)^{n-1}\wedge(-\omega_0(x))ds+o(k^n),\end{aligned}\tag{1.11}$$

and when $q = n-1$, we have asymptotic Hirzebruch-Riemann-Roch theorem

$$\begin{aligned}&\sum_{j=0}^{n-1}(-1)^{j}\dim H^j_{b,\leq k\delta}(X,L^k)\\&= (2\pi)^{-n}\frac{k^n}{(n-1)!}\sum_{j=0}^{n-1}\int_X\int_{\mathbb{R}_{x,j}\bigcap[-\delta,\delta]}(i\mathcal{R}^L_x+i2s\mathcal{L}_x)^{n-1}\wedge(-\omega_0(x))ds+o(k^n).\end{aligned}\tag{1.12}$$

Demailly [8, 9] proved remarkable asymptotic Morse inequalities for the $\overline{\partial}$ complex constructed over the line bundle L^k on compact complex manifold as $k \to \infty$, where L is a holomorphic Hermitian line bundle. He solved with their help a generalized version of the Grauert-Riemenschneider. The original version of the conjecture had been solved previously by Siu [21, 22]. Shortly after, Bismut[5] gave a heat equation proof of Demailly's inequalities which involves probability theory.

Definition 1.6. *We say that (L, h^L) is a positive rigid CR line bundle over X if for any point $p \in X$, $\mathcal{R}^L_p$ is a positive Hermitian quadratic over $T^{1,0}_pX$.*

Assume that $\mathcal{R}^L$ is positive. The point of this paper is that if $\delta > 0$ is small enough then $\mathbb{R}_{x,j} \cap [\delta, \delta] = \emptyset$, $\forall x \in X$ and for every $j = 1, 2, \ldots, n-1$. From this observation, (1.9) and (1.12), we conclude that

$$\begin{aligned}&\dim H^0_{b,\leq k\delta}(X,L^k)\\&= (2\pi)^{-n}\frac{1}{(n-1)!}k^n\int_X\int_{\mathbb{R}_{x,0}\bigcap[-\delta,\delta]}(i\mathcal{R}^L_x+i2s\mathcal{L}_x)^{n-1}\wedge(-\omega_0(x))ds+o(k^n).\end{aligned}$$

Hence, $\dim H^0_{b,\leq k\delta}(X, L^k) \approx k^n$. We conclude that

Theorem 1.7. *If L is a positive rigid CR line bundle, then L is big, that is $\dim H^0_b(X, L^k) \gtrsim k^n$ when $k \gg 1$.*

We notice that from Theorem 1.4 and Theorem 1.5 and some simple argument, we can easily deduce Demailly's weak and strong Morse inequalities (see the proof of Corollary 1.27).

Definition 1.8. *We say that condition $X(q)$ holds on X if there is a $\delta > 0$ such that $\mathbb{R}_{x,q} \bigcap [-\delta, \delta] = \emptyset$, $\forall\ x \in X$.*

In this work, we generalize Grauert-Riemenschneider criterion to CR manifolds with S^1 action and to general $(0, q)$-forms.

Theorem 1.9 (Grauert-Riemenschneider criterion). *Given $q \in \{0, 1, \ldots, n-1\}$, assume that $X(q-1)$ and $X(q+1)$ hold on X. Then, for some $\delta > 0$,*

$$\begin{aligned}&\dim H^q_{b,\leq k\delta}(X,L^k)\\&= (2\pi)^{-n}\frac{(-1)^q}{(n-1)!}k^n\int_X\int_{\mathbb{R}_{x,q}\bigcap[-\delta,\delta]}(i\mathcal{R}^L_x+i2s\mathcal{L}_x)^{n-1}\wedge(-\omega_0(x))ds+o(k^n).\end{aligned}\tag{1.13}$$

DEFINITION 1.10. *We say that L is a semi-positive rigid CR line bundle over X if there exists a constant $\delta > 0$ such that $\mathcal{R}^L_x + 2s\mathcal{L}_x$ is a semi-positive Hermitian quadratic over $T^{1,0}_x X$ for any $x \in X$, $|s| < \delta$.*

When L is semi-positive, it is easy to see that condition $X(1)$ holds on X. From this observation and Theorem 1.9, we obtain the Grauert-Riemenschneider criterion in the CR setting.

THEOREM 1.11. *If L is a semi-positive rigid CR line bundle and positive at a point, then L is big.*

1.2. Set up and terminology. Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n-1, n \geq 2$, where $T^{1,0}X$ is a CR structure of X. That is, $T^{1,0}X$ is a subbundle of rank $n-1$ of the complexified tangent bundle $\mathbb{C}TX$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. We assume that X admits a S^1 action: $S^1 \times X \to X$. We use $e^{i\theta}$ to denote the S^1 action. For $x \in X$, we say that the period of x is $\frac{2\pi}{\ell}$, $\ell \in \mathbb{N}$, if $e^{i\theta} \circ x \neq x$, for every $0 < \theta < \frac{2\pi}{\ell}$ and $e^{i\frac{2\pi}{\ell}} \circ x = x$. For each $\ell \in \mathbb{N}$, put

$$X_\ell = \left\{x \in X; \text{the period of } x \text{ is } \tfrac{2\pi}{\ell}\right\} \tag{1.14}$$

and let $p = \min\{\ell \in \mathbb{N};\ X_\ell \neq \emptyset\}$. It is well-known that if X is connected, then X_p is an open and dense subset of X (see Duistermaat-Heckman [10] and Appendix in [15]) and the Lebesgue measure $m(X \setminus X_p) = 0$. For simplicity, in this work, we assume that $p = 1$ and we denote $X_{\rm reg} := X_1$.

Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the S^1 action given as follows

$$(Tu)(x) = \frac{\partial}{\partial\theta}\left(u(e^{i\theta}x)\right)\Big|_{\theta=0}, u \in C^\infty(X). \tag{1.15}$$

DEFINITION 1.12. *We say that the S^1 action $e^{i\theta}, 0 \leq \theta < 2\pi$, is CR if*

$$[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X).$$

Furthermore, we say that the S^1 action is transversal if for each $x \in X$,

$$T(x) \oplus T^{1,0}_x(X) \oplus T^{0,1}_x X = \mathbb{C}T_x X.$$

We assume throughout that $(X, T^{1,0}X)$ is a CR manifold with a transversal CR S^1 action $e^{i\theta}, 0 \leq \theta < 2\pi$ and we let T be the global vector field induced by the S^1 action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle\, \omega_0\,,\, U\,\rangle = 0$, for every $U \in T^{1,0}X \oplus T^{0,1}X$ and $\langle\, \omega_0\,,\, T\,\rangle = -1$.

DEFINITION 1.13. *For $x \in X$, the Levi-form $\mathcal{L}_x$ is the Hermitian quadratic form on $T^{1,0}_x X$ defined as follows. For any $U, V \in T^{1,0}_x X$, pick $\mathcal{U}, \mathcal{V} \in C^\infty(X, T^{1,0}X)$ such that $\mathcal{U}(x) = U, \mathcal{V}(x) = V$. Set*

$$\mathcal{L}_x(U, \overline{V}) = \frac{1}{2i}\langle[\mathcal{U}, \overline{\mathcal{V}}](x), \omega_0(x)\rangle \tag{1.16}$$

where $[\,,\,]$ denotes the Lie bracket. Note that $\mathcal{L}_x$ does not depend on the choice of $\mathcal{U}$ and $\mathcal{V}$.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. Define the vector bundle of $(0,q)$-forms by $T^{*0,q}X := \Lambda^q T^{*0,1}X$. Let $D \subset X$ be an open subset. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega_0^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D. Similarly, if E is a vector bundle, then we let $\Omega^{0,q}(D,E)$ denote the space of smooth sections of $T^{*0,q}X \otimes E$ over D and let $\Omega_0^{0,q}(D,E)$ be the subspace of $\Omega^{0,q}(D,E)$ whose elements have compact support in D.

Fix $\theta_0 \in [0, 2\pi)$. Let

$$de^{i\theta_0} : \mathbb{C}T_xX \to \mathbb{C}T_{e^{i\theta_0}x}X$$

denote the differential map of $e^{i\theta_0} : X \to X$. By the property of transversal CR S^1 action, we can check that

$$\begin{aligned} de^{i\theta_0} &: T_x^{1,0}X \to T^{1,0}_{e^{i\theta_0}x}X, \\ de^{i\theta_0} &: T_x^{0,1}X \to T^{0,1}_{e^{i\theta_0}x}X, \\ de^{i\theta_0}&(T(x)) = T(e^{i\theta_0}x). \end{aligned} \tag{1.17}$$

Let $(de^{i\theta_0})^* : \Lambda^q(\mathbb{C}T^*X) \to \Lambda^q(\mathbb{C}T^*X)$ be the pull back of $de^{i\theta_0}, q = 0, 1 \cdots, n-1$. From (1.17), we can check that for $q = 0, 1, \cdots, n-1$,

$$(de^{i\theta_0})^* : T^{*0,q}_{e^{i\theta_0}x}X \to T_x^{*0,q}X. \tag{1.18}$$

Let $u \in \Omega^{0,q}(X)$ and define Tu as follows. For any $X_1, \cdots, X_q \in T_x^{1,0}X$,

$$Tu(X_1, \cdots, X_q) := \frac{\partial}{\partial \theta}\left((de^{i\theta})^* u(X_1, \cdots, X_q)\right)\Big|_{\theta=0}. \tag{1.19}$$

From the definition of Tu it is easy to check that $Tu = L_T u$ for $u \in \Omega^{0,q}(X)$, where $L_T u$ is the Lie derivative of u along the direction T. It is straightforward to see that (see also the discussion after Theorem 1.31)

$$T\overline{\partial}_b = \overline{\partial}_b T \text{ on } \Omega^{0,q}(X). \tag{1.20}$$

Definition 1.14. *Let $D \subset X$ be an open set. We say that a function $u \in C^\infty(D)$ is rigid if $Tu = 0$. We say that a function $u \in C^\infty(X)$ is Cauchy-Riemann (CR for short) if $\overline{\partial}_b u = 0$. We say that $u \in C^\infty(X)$ is rigid CR if $\overline{\partial}_b u = 0$ and $Tu = 0$.*

Definition 1.15. *Let E be a complex vector bundle over X. We say that E is rigid (resp. CR, resp. rigid CR) if there exists an open cover $(U_j)_j$ of X and trivializing frames $\left\{f_j^1, f_j^2, \ldots, f_j^r\right\}$ on U_j, such that the corresponding transition matrices are rigid (resp. CR, resp. rigid CR). The frames $\left\{f_j^1, f_j^2, \ldots, f_j^r\right\}$ are called rigid (resp. CR, resp. rigid CR) frames.*

Example 1.16. *Let X be a compact CR manifold with a locally free transversal CR S^1 action. Let $\{Z_j\}_j$ be a trivializing frame of $T^{1,0}X$ defined in (1.59). It is easy to check that the transition functions of such frames are rigid CR and thus $T^{1,0}X$ is a rigid CR vector bundle. Moreover,* $\det T^{1,0}X$ *the determinant bundle of $T^{1,0}X$ is a rigid CR line bundle.*

Example 1.17. *Let* $(L,h) \xrightarrow{\pi} M$ *be a Hermitian line bundle over a complex manifold* M*. Consider the circle bundle* $X = \{v \in L : h(v) = 1\}$ *over* M*. Then* X *is a compact CR manifold with a globally free transversal CR* S^1 *action. Let* E *be a holomorphic vector bundle over* M*. Then the restriction of the pull back* $\pi^* E|_X$ *on* X *is a rigid CR vector bundle over* X*.*

From now on, let L be a rigid CR line bundle over X. We fix an open covering $(U_j)_j$ and a family $(s_j)_j$ of rigid CR frames s_j on U_j. Let L^k be the k-th tensor power of L. Then $(s_j^{\otimes k})_j$ are rigid CR frames for L^k. Let s be a rigid CR frame of L on an open subset $D \subset X$ and locally for any $u \in \Omega^{0,q}(X,L)$, write $u = \tilde{u} \otimes s$, $\tilde{u} \in \Omega^{0,q}(D)$, we define $Tu = T\tilde{u} \otimes s$. Since the transition functions are rigid CR, Tu is well defined. Moreover, we have

$$T\overline{\partial}_b = \overline{\partial}_b T \text{ on } \Omega^{0,q}(X,L). \tag{1.21}$$

Fix a Hermitian fiber metric h^L on L. If s is a local rigid CR frame of L on an open subset $D \subset X$, then the local weight of h^L with respect to s is the function $\Phi \in C^\infty(D,\mathbb{R})$ for which

$$|s(x)|^2_{h^L} = e^{-\Phi(x)}, x \in D. \tag{1.22}$$

Definition 1.18. *Let* L *be a rigid CR line bundle and let* h^L *be a Hermitian metric on* L*. The curvature of* (L,h^L) *is the the Hermitian quadratic form* $R^L = R^{(L,h)}$ *on* $T^{1,0}X$ *defined by*

$$R^L_p(U,\overline{V}) = \frac{1}{2}\left\langle d(\overline{\partial}_b\Phi - \partial_b\Phi)(p), U \wedge \overline{V}\,\right\rangle,\ U,V \in T^{1,0}_pX,\ p \in D. \tag{1.23}$$

Due to [16, Proposition 4.2], R^L is a well-defined global Hermitian form, since the transition functions between different rigid CR frames are annihilated by T.

1.3. Hermitian CR geometry. Fix a smooth Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$, T is orthogonal to $T^{1,0}X \oplus T^{0,1}X$ and $\langle T|T\rangle = 1$. The Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ induces by duality a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0,q)$-forms $T^{*0,q}X, q = 0,1\cdots,n-1$. We shall also denote all these induced metrics by $\langle\cdot|\cdot\rangle$. For every $v \in T^{*0,q}X$, we write $|v|^2 := \langle v|v\rangle$. We have the pointwise orthogonal decompositions

$$\begin{aligned}\mathbb{C}T^*X &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0 : \lambda \in \mathbb{C}\},\\ \mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T : \lambda \in \mathbb{C}\}.\end{aligned} \tag{1.24}$$

Definition 1.19. *Let* D *be an open set and let* $V \in C^\infty(D,\mathbb{C}TX)$ *be a vector on* D*. We say that* V *is rigid if*

$$de^{i\theta}(V(x)) = V(e^{i\theta}x) \tag{1.25}$$

for any $x, \theta \in [0,2\pi)$ *satisfying* $x \in D, e^{i\theta}x \in D$*.*

Definition 1.20. *Let* $\langle\cdot|\cdot\rangle$ *be a Hermitian metric on* $\mathbb{C}TX$*. We say that* $\langle\cdot|\cdot\rangle$ *is rigid if for rigid vector fields* V, W *on* D*, where* D *is any on open set, we have*

$$\langle V(x)|W(x)\rangle = \langle (de^{i\theta}V)(e^{i\theta}x)|(de^{i\theta}W)(e^{i\theta}x)\rangle, \forall x \in D, \theta \in [0,2\pi). \tag{1.26}$$

From theorem 9.2 in [14], there is always a rigid Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X\perp T^{0,1}X, T\perp(T^{1,0}X\oplus T^{0,1}X), \langle T|T\rangle=1$ and $\langle u|v\rangle$ is real if u,v are real tangent vectors. Until further notice, we fix a rigid Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X\perp T^{0,1}X, T\perp(T^{1,0}X\oplus T^{0,1}X)$ and $\langle T|T\rangle=1$.

Definition 1.21. *Let L be a rigid CR line bundle. A Hermitian fiber metric h^L on L is said to be rigid if $T\Phi=0$ for local weight Φ with respect to any rigid CR frame.*

The definition does not depend on the choice of rigid CR frame.

Lemma 1.22. *There is a rigid Hermitian fiber metric on L. Moreover, for any Hermitian metric $\widetilde{h}^L$ on L, there is a rigid Hermitian metric h^L of L such that $\widetilde{\mathcal{R}}^L=R^L$ on X, where $\widetilde{\mathcal{R}}^L$ and R^L denote the curvatures induced by $\widetilde{h}^L$ and h^L respectively.*

We will prove Lemma 1.22 in the end of section 1.5. Until furthermore, we assume that h^L is a rigid Hermitian fiber metric on L. For $k>0$, $k\in\mathbb{Z}$, we shall consider (L^k,h^{L^k}). For $m\in\mathbb{Z}$, put

$$\Omega^{0,q}_m(X,L^k):=\{u\in\Omega^{0,q}(X,L^k): Tu=imu\}. \tag{1.27}$$

Let $(\,\cdot\,|\,\cdot\,)_{h^{L^k}}$ be the L^2 inner product on $\Omega^{0,q}(X,L^k)$ induced by h^{L^k}, $\langle\,\cdot\,|\,\cdot\,\rangle$ and let $\|\cdot\|_{h^{L^k}}$ denote the corresponding norm. Let s be a local rigid CR frame of L on an open set $D\subset X$. For $u=\widetilde{u}\otimes s^k, v=\widetilde{v}\otimes s^k\in\Omega^{0,q}_0(D,L^k)$, we have

$$(u|v)_{h^{L^k}}=\int_X\langle\tilde{u}|\tilde{v}\rangle e^{-k\Phi(x)}dv_X, \tag{1.28}$$

where dv_X is the volume form on X induced by the rigid Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$. Let $L^2_{(0,q),m}(X,L^k)$ be the completion of $\Omega^{0,q}_m(X,L^k)$ with respect to $(\cdot|\cdot)_{h^{L^k}}$. For $m\in\mathbb{Z}$, let

$$Q^q_{m,k}: L^2_{(0,q)}(X,L^k)\to L^2_{(0,q),m}(X,L^k) \tag{1.29}$$

be the orthogonal projection with respect to $(\cdot|\cdot)_{h^{L^k}}$. Fix $\delta>0$, let $F_{\delta,k}: L^2_{(0,q)}(X,L^k)\to L^2_{(0,q)}(X,L^k)$ be the continuous map given by

$$F_{\delta,k}(u):=\sum_{|m|\leq k\delta}Q^q_{m,k}u. \tag{1.30}$$

Let $\overline{\partial}^*_{b,k}:\Omega^{0,q+1}(X,L^k)\to\Omega^{0,q}(X,L^k)$ be the formal adjoint of $\overline{\partial}_b$ with respect to $(\cdot|\cdot)_{h^{L^k}}$. Since $\langle\cdot|\cdot\rangle$ and h^{L^k} are rigid, we can check that

$$T\overline{\partial}^*_{b,k}=\overline{\partial}^*_{b,k}T \text{ on } \Omega^{0,q}(X,L^k), q=0,1,\cdots,n-1, \tag{1.31}$$

and

$$\overline{\partial}^*_{b,k}:\Omega^{0,q+1}_m(X,L^k)\to\Omega^{0,q}_m(X,L^k), \forall m\in\mathbb{Z}. \tag{1.32}$$

Put

$$\Box^{(q)}_{b,k}:=\overline{\partial}_b\overline{\partial}^*_{b,k}+\overline{\partial}^*_{b,k}\overline{\partial}_b:\Omega^{0,q}(X,L^k)\to\Omega^{0,q}(X,L^k).$$

From (1.21), (1.31)and (1.32) we have

$$T\Box^{(q)}_{b,k} = \Box^{(q)}_{b,k}T \text{ on } \Omega^{0,q}(X, L^k), q = 0, 1, \cdots, n-1, \tag{1.33}$$

and

$$\Box^{(q)}_{b,k} : \Omega^{0,q}_m(X, L^k) \to \Omega^{0,q}_m(X, L^k), \forall m \in \mathbb{Z}. \tag{1.34}$$

We will write $\Box^{(q)}_{b,k,m}$ to denote the restriction of $\Box^{(q)}_{b,k}$ on the space $\Omega^{0,q}_m(X, L^k)$. For every $m \in \mathbb{Z}$, we extend $\Box^{(q)}_{b,k,m}$ to $L^2_{(0,q),m}(X, L^k)$ in the sense of distribution by

$$\Box^{(q)}_{b,k,m} : \text{Dom}(\Box^{(q)}_{b,k,m}) \subset L^2_{(0,q),m}(X, L^k) \to L^2_{(0,q),m}(X, L^k), \tag{1.35}$$

where $\text{Dom}(\Box^{(q)}_{b,k,m}) = \{u \in L^2_{(0,q),m}(X, L^k) : \Box^{(q)}_{b,k,m}u \in L^2_{(0,q),m}(X, L^k)\}$. The following follows from Kohn's L^2 estimate (see theorem 8.4.2 in [7]).

THEOREM 1.23. *For every $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there exists a constant $C_{s,k} > 0$ such that*

$$\|u\|_{s+1} \le C_{s,k}\left(\|\Box^{(q)}_{b,k}u\|_s + \|Tu\|_s + \|u\|_s\right), \forall u \in \Omega^{0,q}(X, L^k) \tag{1.36}$$

where $\|\cdot\|_s$ denotes the sobolev norm of order s on X.

From Theorem 1.23, we deduce that

THEOREM 1.24. *Fix $m \in \mathbb{Z}$, for every $s \in \mathbb{N}_0$, there is a constant $C_{s,k,m} > 0$ such that*

$$\|u\|_{s+1} \le C_{s,k,m}\left(\|\Box^{(q)}_{b,k,m}u\|_s + \|u\|_s\right), \forall u \in \Omega^{0,q}_m(X, L^k). \tag{1.37}$$

From Theorem 1.24 and some standard argument in functional analysis, we deduce the following Hodge theory for $\Box^{(q)}_{b,k,m}$.

THEOREM 1.25. *Fix $m \in \mathbb{Z}$. $\Box^{(q)}_{b,k,m} : \text{Dom}(\Box^{(q)}_{b,k,m}) \subset L^2_{(0,q),m}(X, L^k) \to L^2_{(0,q),m}(X, L^k)$ is a self-adjoint operator. The spectrum of $\Box^{(q)}_{b,k,m}$ denoted by $\text{Spec}(\Box^{(q)}_{b,k,m})$ is a discrete subset of $[0, \infty)$. For every $\lambda \in \text{Spec}(\Box^{(q)}_{b,k,m})$ the λ-eigenspace*

$$\mathcal{H}^q_{b,m,\lambda}(X, L^k) := \left\{u \in \text{Dom}\Box^{(q)}_{b,k,m} : \Box^{(q)}_{b,k,m}u = \lambda u\right\} \tag{1.38}$$

is finite dimensional with $\mathcal{H}^q_{b,m,\lambda}(X, L^k) \subset \Omega^{0,q}_m(X, L^k)$ and for $\lambda = 0$ we denote by $\mathcal{H}^q_{b,m}(X, L^k)$ the harmonic space $\mathcal{H}^q_{b,m,0}(X, L^k)$ for brevity and then we have the Dolbeault isomorphism

$$\mathcal{H}^q_{b,m}(X, L^k) \cong H^q_{b,m}(X, L^k). \tag{1.39}$$

From Theorem 1.25 and (1.39), we deduce that $\dim H^q_{b,m}(X, L^k) < \infty, \forall m \in \mathbb{Z}$.

1.4. Our strategy. Denote by $\det(\mathcal{R}_x^L+2s\mathcal{L}_x)$ the product of all the eigenvalues of $\mathcal{R}_x^L+2s\mathcal{L}_x$ with respect to the given rigid Hermitian metric. Since

$$\Omega_m^{0,q}(X,L^k)\perp\Omega_{m'}^{0,q}(X,L^k),$$

when $m,m'\in\mathbb{Z}$ and $m\neq m'$, we write

$$\Omega_{\leq k\delta}^{0,q}(X,L^k):=\bigoplus_{m\in\mathbb{Z},|m|\leq k\delta}\Omega_m^{0,q}(X,L^k) \tag{1.40}$$

and in particular,

$$\mathcal{H}_{b,\leq k\delta}^q(X,L^k):=\bigoplus_{m\in\mathbb{Z},|m|\leq k\delta}\mathcal{H}_{b,m}^q(X,L^k). \tag{1.41}$$

Here δ is a small constant. Then we have the following Hodge theory

$$\begin{aligned}&\dim\mathcal{H}_{b,\leq k\delta}^q(X,L^k)<\infty,\mathcal{H}_{b,\leq k\delta}^q(X,L^k)\\ &\subset\Omega_{\leq k\delta}^{0,q}(X,L^k),\mathcal{H}_{b,\leq k\delta}^q(X,L^k)\cong H_{b,\leq k\delta}^q(X,L^k).\end{aligned} \tag{1.42}$$

Let $f_j\in\Omega_{\leq k\delta}^{0,q}(X,L^k), j=1,\cdots,m_k$ be an orthonormal basis for the space $\mathcal{H}_{b,\leq k\delta}^q(X,L^k)$. The partial Szegő kernel function is defined by

$$\Pi_{\leq k\delta}^q(x):=\sum_{j=1}^{m_k}|f_j(x)|_{h^{L^k}}^2. \tag{1.43}$$

It is easy to see that $\Pi_{\leq k\delta}^q(x)$ is independent of the choice of the orthonormal basis and

$$\dim\mathcal{H}_{b,\leq k\delta}^q(X,L^k)=\int_X\Pi_{\leq k\delta}^q(x)dv_X. \tag{1.44}$$

The following is our first main technique result.

THEOREM 1.26.

$$\sup\{k^{-n}\Pi_{\leq k\delta}^q(x):k>0,x\in X\}<\infty. \tag{1.45}$$

Furthermore, we have

$$\limsup_{k\to\infty}k^{-n}\Pi_{\leq k\delta}^q(x)\leq(2\pi)^{-n}\int_{\mathbb{R}_{x,q}\bigcap[-\delta,\delta]}|\det(\mathcal{R}_x^L+2s\mathcal{L}_x)|ds \tag{1.46}$$

for all $x\in X$.

From Theorem 1.26 and by Fatou's lemma, we obtain Theorem 1.4. From Theorem 1.4 and some simple argument, we deduce

COROLLARY 1.27 (Demailly's weak morse inequalities). *Let M be a compact Hermitian manifold with* $\dim_{\mathbb{C}}\mathrm{M}=\mathrm{n}-1$ *and (L,h^L) be a Hermitian line bundle over M. Then $\forall q=0,1,2,\ldots,n-1$,*

$$\dim H_{\overline{\partial}}^q(M,L^k)\leq k^{n-1}(2\pi)^{-(n-1)}\int_{M(q)}|\det\mathcal{R}_x^L|dv_M(x)+o(k^{n-1}), \tag{1.47}$$

where $H^q_{\overline{\partial}}(M,L^k)$ denotes the q-th $\overline{\partial}$-cohomology group with values in L^k, dv_M is the induced volume form on M, $\mathcal{R}^L_x, x \in M$ is the Ricci curvature of the Hermitian line bundle (L,h^L) and $M(q)$ is a subset of M where $\mathcal{R}^L_x$ has exactly q negative eigenvalues and $n-1-q$ positive eigenvalues.

Proof. Let $X = M \times S^1$. Then X is a Levi-flat CR manifold of $\dim_{\mathbb{R}} \mathrm{X} = 2\mathrm{n}-1$ with S^1 action $e^{i\theta}$ and the global induced vector field is $T = \frac{\partial}{\partial\theta}$. Let $\pi_1 : X = M \times S^1 \to M$ be the natural projection. Then $L_1 := \pi_1^* L$ is naturally a rigid CR line bundle over X. It is easy to see that

$$\dim H^q_{\overline{\partial}}(M,L^k) = \frac{1}{2k+1}\dim H^q_{b,\leq k}(X,L_1^k). \tag{1.48}$$

From (1.9), we have

$$\begin{aligned}
&\dim H^q_{b,\leq k}(X,L_1^k)\\
&\leq (2\pi)^{-n}k^n\int_X\int_{\mathbb{R}_{x,q}\cap[-1,1]}|\det \mathcal{R}^{L_1}_x + 2s\mathcal{L}_x|ds dv_X + o(k^n)\\
&= (2\pi)^{-n}k^n\int_{M\times S^1}\int_{\mathbb{R}_{x,q}\cap[-1,1]}|\det \mathcal{R}^{L_1}_x|ds dv_X + o(k^n)\\
&= 2(2\pi)^{-n}k^n\int_{M(q)\times S^1}|\det \mathcal{R}^{L}_x|dv_M dv_{S^1} + o(k^n)\\
&= 2(2\pi)^{-(n-1)}k^n\int_{M(q)}|\det \mathcal{R}^{L}_x|dv_M + o(k^n).
\end{aligned}\tag{1.49}$$

From (1.49) and (1.48), we get the conclusion of the Corollary 1.27. □

For $\lambda \geq 0$ and $\lambda \in \mathbb{R}$, we define

$$\mathcal{H}^q_{b,\leq k\delta,\lambda}(X,L^k) := \left\{u \in \Omega^{0,q}_{\leq k\delta}(X,L^k) : \Box^{(q)}_{b,k}u = \lambda u\right\} \tag{1.50}$$

and

$$\mathcal{H}^q_{b,\leq k\delta,\leq k\sigma}(X,L^k) := \bigoplus_{\lambda\leq k\sigma}\mathcal{H}^q_{b,\leq k\delta,\lambda}(X,L^k). \tag{1.51}$$

Set $\Pi^q_{\leq k\delta,\leq k\sigma}(x) = \sum_{j=1}^{d_k}|g_j(x)|^2$, where $\{g_j(x)\}_{j=1}^{d_k} \subset \Omega^{0,q}_{\leq k\delta}(X,L^k)$ is any orthonormal basis of the space $\mathcal{H}^q_{b,\leq k\delta,\leq k\sigma}(X,L^k)$. Our second main technique result is the following

THEOREM 1.28. *For any sequence $v_k > 0$ with $v_k \to 0$ as $k \to \infty$, there exists a constant C_0' independent of k, such that*

$$k^{-n}\Pi^q_{\leq k\delta,\leq kv_k}(x) \leq C_0' \tag{1.52}$$

for all $x \in X$. Moreover, there is a sequence $\mu_k > 0, \mu_k \to 0$ as $k \to \infty$, such that for any sequence $v_k > 0$ with $\lim\limits_{k\to\infty}\frac{\mu_k}{v_k} = 0$, we have

$$\lim_{k\to\infty}k^{-n}\Pi^q_{\leq k\delta,\leq kv_k}(x) = (2\pi)^{-n}\int_{\mathbb{R}_{x,q}\bigcap[-\delta,\delta]}|\det(\mathcal{R}^L_x + 2s\mathcal{L}_x)|ds \tag{1.53}$$

for all $x \in X_{\mathrm{reg}}$.

Integrating (1.53), we have

THEOREM 1.29. *There is a sequence* $\mu_k > 0, \mu_k \to 0$ *as* $k \to \infty$, *such that for any sequence* $v_k > 0$ *with* $\lim_{k\to\infty} \frac{\mu_k}{v_k} = 0$, *we have*

$$\begin{aligned}&\dim \mathcal{H}^q_{b,\leq k\delta,\leq kv_k}(X, L^k)\\ &= (2\pi)^{-n}k^n \int_X \int_{\mathbb{R}_{x,q}\bigcap[-\delta,\delta]} |\det(\mathcal{R}^L_x + 2s\mathcal{L}_x)| ds dv_X + o(k^n).\end{aligned} \tag{1.54}$$

Proof of Theorem 1.9. Set $\mathcal{H}^q_{b,\leq k\delta,0<\lambda\leq k\sigma}(X, L^k) := \bigoplus_{0<\lambda\leq k\sigma} \mathcal{H}^q_{b,\leq k\delta,\lambda}(X, L^k)$. We define a map

$$\begin{aligned}P : \mathcal{H}^q_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k) &\to \mathcal{H}^{q-1}_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k) \oplus \mathcal{H}^{q+1}_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k)\\ u &\mapsto (\overline{\partial}^*_b u, \overline{\partial}_b u).\end{aligned} \tag{1.55}$$

Since map P is injective, it follows that

$$\begin{aligned}&\dim\mathcal{H}^q_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k)\\ &\leq \dim\mathcal{H}^{q-1}_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k) + \dim\mathcal{H}^{q+1}_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k).\end{aligned} \tag{1.56}$$

From Theorem 1.29, we have

$$\dim\mathcal{H}^{q-1}_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k) = o(k^n), \dim\mathcal{H}^{q+1}_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k) = o(k^n). \tag{1.57}$$

Since $\dim\mathcal{H}^q_{b,\leq k\delta,\leq kv_k}(X, L^k) = \dim\mathcal{H}^q_{b,\leq k\delta,0<\lambda\leq kv_k}(X, L^k) + \dim\mathcal{H}^q_{b,\leq k\delta}(X, L^k)$, combining Theorem 1.29 and (1.57), we get the conclusion of the Theorem 1.9. □

From Theorem 1.29 and the linear algebraic argument from Demailly in [8], [9] and [19], we obtain Theorem 1.5. From Theorem 1.5, we can repeat the proof of Corollary 1.27 and deduce

COROLLARY 1.30 (Demailly's strong Morse inequalities). *Let* M *be a compact Hermitian manifold with* $\dim_{\mathbb{C}}\mathrm{M} = \mathrm{n} - 1$ *and* (L, h^L) *be a Hermitian line bundle on* M. *Then for any* $0 \leq q \leq n-1$, *we have*

$$\begin{aligned}&\sum_{j=0}^{q}(-1)^{q-j} \dim H^j_{\overline{\partial}}(M, L^k)\\ &\leq k^{n-1}(2\pi)^{-(n-1)} \sum_{j=0}^{q}(-1)^{q-j} \int_{M(j)} |\det \mathcal{R}^L_x| dv_M + o(k^{n-1}).\end{aligned} \tag{1.58}$$

1.5. Canonical local coordinates. In this work, we need the following result due to Baouendi-Rothschild-Treves, (see [2]).

THEOREM 1.31. *For* $x_0 \in X$, *there exist local coordinates* $(x_1, \cdots, x_{2n-1}) = (z, \theta) = (z_1, \cdots, z_{n-1}, \theta), z_j = x_{2j-1} + ix_{2j}, j = 1, \cdots, n-1, x_{2n-1} = \theta$, *defined in some small neighborhood* $D = \{(z, \theta) : |z| < r, |\theta| < \varepsilon\}$ *centered at* x_0 *such that on* D

$$\begin{aligned}T &= \frac{\partial}{\partial\theta}\\ Z_j &= \frac{\partial}{\partial z_j} + i\frac{\partial\varphi}{\partial z_j}(z)\frac{\partial}{\partial\theta}, j = 1, \cdots, n-1,\end{aligned} \tag{1.59}$$

where $\{Z_j(x)\}$ form a basis of $T^{1,0}_x X$ for each $x \in D$, and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of θ.

The local coordinates defined in Theorem 1.31 are called canonical local coordinates. By using canonical local coordinates, we get another way to define $Tu, \forall u \in \Omega^{0,q}(X)$. Let (z,θ) be the canonical coordinates defined on $D \subset\subset X$. It is clearly that

$$\{d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}, 1 \leq j_1 < \cdots < j_q \leq n-1\}$$

is a basis for $T^{*0,q}_x X$, $\forall\, x \in D$. Let $u \in \Omega^{0,q}(X)$. On D, we write

$$u = \sum_{j_1 < \cdots < j_q} u_{j_1 \cdots j_q} d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

Then on D we can check that

$$Tu = \sum_{j_1 < \cdots < j_q} (Tu_{j_1 \cdots j_q}) d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}. \tag{1.60}$$

REMARK 1.32. *Since the Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ is rigid, we can find orthonormal frame $\{e^j\}_{j=1}^{n-1}$ of $T^{*0,1}X$ on D such that $e^j(x) = e^j(z), \forall x = (z,\theta) \in D, j = 1, \cdots, n-1$. Moreover, if we denote by dv_X the volume form with respect to the rigid Hermitian metric on $\mathbb{C}TX$, then on D,*

$$dv_X = m(z) dv(z) d\theta, \tag{1.61}$$

where $m(z) \in C^\infty(D, \mathbb{R})$ which does not depend on θ and $dv(z) = 2^{n-1} dx_1 \cdots dx_{2n-2}$.

With respect to the orthonormal frame defined in Remark 1.32, write

$$u = \sum\nolimits'_{|J|=q} u_J e^J, J = (j_1, \ldots, j_q), e^J = e^{j_1} \wedge \cdots \wedge e^{j_q}, \tag{1.62}$$

where the prime means the multi index in the summation is strictly increasing. Then from (1.13) and Remark 1.32, we can check that

$$Tu = \sum\nolimits'_{|J|=q} (Tu_J) e^J. \tag{1.63}$$

Proof of Lemma 1.22. Fix $p \in X$ and let (z,θ) be canonical coordinates defined in some neighbourhood of p such that $(z(p), \theta(p)) = (0,0)$ and (1.59) hold. Suppose that (z,θ) defined on $\{z \in \mathbb{C}^{n-1} : |z| < \delta\} \times \{\theta \in \mathbb{R} : |\theta| < \delta\}$, for some $\delta > 0$. For $z \in \mathbb{C}^{n-1}$, $|z| < \delta$, $\theta \in \mathbb{R}$, we identify (z,θ) with $e^{i\theta} \circ (z,0) \in X$. Thus, we may assume that θ is defined on $\mathbb{R}$. Put

$$\begin{aligned} A :=\{\lambda \in [0,2\pi] :\ & \text{There is a local trivializing section } s \text{ defined on} \\ & \big\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon\big\} \times [0, \lambda + \varepsilon), \text{ for some } 0 < \varepsilon < \delta\}. \end{aligned}$$

It is clearly that A is a non-empty open set in $[0,2\pi]$. We claim that A is closed. Let λ_0 be a limit point of A. Consider the point $(0,\lambda_0)$. For some $\varepsilon_1 > 0$, ε_1 small, there is a local trivializing section s_1 defined on $\big\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_1\big\} \times (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1]$. Since λ_0 is a limit point of A, we can find a local trivializing section $\widetilde{s}$ defined on $\big\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_2\big\} \times [0, \lambda_0 - \frac{\varepsilon_1}{2})$, for some $\varepsilon_2 > 0$. Now, $\widetilde{s} = g s_1$ on

$$\big\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_0\big\} \times (\lambda_0 - \varepsilon_1, \lambda_0 - \frac{\varepsilon_1}{2})$$

for some rigid CR function g, where $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Since g is independent of θ, g is well-defined on $\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_0\} \times \mathbb{R}$. Put $s = \widetilde{s}$ on $\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_0\} \times [0, \lambda_0 - \frac{\varepsilon_1}{2})$ and $s = gs_1$ on $\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_0\} \times [\lambda_0 - \frac{\varepsilon_1}{2}, \lambda_0 + \varepsilon_1)$. It is straightforward to check that s is well-defined as a local trivializing section on $\{z \in \mathbb{C}^{n-1} : |z| < \varepsilon_0\} \times [0, \lambda_0 + \varepsilon_1)$. Thus, $\lambda_0 \in A$ and hence $A = [0, 2\pi]$.

From the discussion above, we see that we can find local trivializations $W_1, \ldots, W_N$ such that $X = \bigcup_{j=1}^N W_j$ and $\bigcup_{0 \le \theta \le 2\pi} e^{i\theta} W_t \subset W_t$, $t = 1, \ldots, N$. Take any Hermitian fiber metric $\widetilde{h}^L$ on L and let $\widetilde{\Phi}$ denotes the corresponding local weight. Let h^L be the Hermitian fiber metric on L locally given by $|s|^2_{h^L} = e^{-\Phi}$, where $\Phi(x) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\Phi}(e^{i\theta}x) d\theta$. It is obviously that h^L is well-defined and $T\Phi = 0$. Moreover, it is easy to see that $\widetilde{\mathcal{R}}^L = \mathcal{R}^L$, where $\widetilde{\mathcal{R}}^L$ and $\mathcal{R}^L$ denote the curvatures of L induced by $\widetilde{h}^L$ and h^L respectively. The lemma follows. □

2. The estimates of the partial Szegő kernel function $\Pi^q_{\le k\delta}$. We first introduce some notations. For $x \in X$, we can choose an orthonormal frame $\{e^j\}_{j=1}^{n-1}$ of $T^{*0,1}X$ defined in section 1.5 over a neighborhood D of x. For $J = (j_1, \cdots, j_q)$ with $j_1 < \cdots < j_q$, we define $e^J = e^{j_1} \wedge \cdots \wedge e^{j_q}$. Then $\{e^J : |J| = q, J \text{ strictly increasing}\}$ is an orthonormal frame for $T^{*0,q}X$ over D. For any $f \in \Omega^{0,q}(X, L^k)$, on D, we may write

$$f = \sum\nolimits'_{|J|=q} f_J e^J, \text{ with } f_J = \langle f | e^J \rangle \in C^\infty(D, L^k). \tag{2.1}$$

The extramal function $S^q_{\le k\delta, J}(y)$ for $y \in D$ along the direction e^J is defined by

$$S^q_{\le k\delta, J}(y) := \sup_{\alpha \in \mathcal{H}^q_{b, \le k\delta}(X, L^k), \|\alpha\|_{h^{L^k}} = 1} |\alpha_J(y)|^2_{h^{L^k}}. \tag{2.2}$$

We can repeat the proof of Lemma 2.1 in [16] and conclude that

LEMMA 2.1. *For every local orthonormal frame $\{e^J : |J| = q,$ strict increasing$\}$ of $T^{*0,q}X$ over an open set D we have for $y \in D$,*

$$\Pi^q_{\le k\delta}(y) = \sum\nolimits'_{|J|=q} S^q_{\le k\delta, J}(y). \tag{2.3}$$

2.1. The scaling technique. Fix $p \in X$. Let $U_1, \cdots, U_{n-1}$ be the dual frame of $\overline{e^1}, \cdots, \overline{e^{n-1}}$ and for which the Levi-form is diagonal at p. Furthermore, let s be a rigid CR frame of L on an open neighborhood of p and $|s|^2_{h^L} = e^{-\Phi}$. We take canonical local coordinates $(z, \theta), z_j = x_{2j-1} + ix_{2j}, j = 1, \cdots, n-1$ defined in Theorem 1.31 such that $\omega_0(p) = -d\theta, (z(p), \theta(p)) = 0$,

$$\left\langle \frac{\partial}{\partial x_j}(p) \Big| \frac{\partial}{\partial x_t}(p) \right\rangle = 2\delta_{jt}, \left\langle \frac{\partial}{\partial x_j}(p) \Big| \frac{\partial}{\partial \theta}(p) \right\rangle = 0, \left\langle \frac{\partial}{\partial \theta}(p) \Big| \frac{\partial}{\partial \theta}(p) \right\rangle = 1 \tag{2.4}$$

for $j, t = 1, \ldots, 2n-2$, and with respect to the canonical coordinates (z, θ),

$$U_j = \frac{\partial}{\partial z_j} + i\lambda_j \overline{z}_j \frac{\partial}{\partial \theta} + O(|(z, \theta)|^2), j = 1, \ldots, n-1, \tag{2.5}$$

where $\{\lambda_j\}_{j=1}^{n-1}$ are the eigenvalues of Levi-form at p with respect to the given rigid Hermitian metric and $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}), j = 1, \ldots, n-1$. Moreover, by changing the local rigid CR frame of L we assume the local weight

$$\Phi = \sum_{j,t=1}^{n-1} \mu_{j,t}\overline{z}_j z_t + O(|z|^3). \tag{2.6}$$

In this section, we work with canonical local coordinates (z,θ) defined on an open neighbourhood D of p and we identify D with some open set in $\mathbb{R}^{2n-1}$. Let $(\cdot|\cdot)_{k\Phi}$ be the weighted inner product on the space $\Omega_0^{0,q}(D)$ defined as follows:

$$(f|g)_{k\Phi} = \int_D \langle f|g\rangle e^{-k\Phi(z)} dv_X(x) \tag{2.7}$$

where $f,g \in \Omega_0^{0,q}(D)$. We denote by $L^2_{(0,q)}(D,k\Phi)$ the completion of $\Omega_0^{0,q}(D)$ with respect to $(\cdot|\cdot)_{k\Phi}$. For $r>0$, let $D_r = \{(z,\theta)\in\mathbb{R}^{2n-1} : |z|<r, |\theta|<r\}$. Here $\{z=(z_1,\ldots,z_{n-1})\in\mathbb{C}^{n-1} : |z|<r\}$ means that $\{z\in\mathbb{C}^{n-1} : |z_j|<r, j=1,\cdots,n-1\}$. Let F_k be the scaling map $F_k(z,\theta)=(\frac{z}{\sqrt{k}},\frac{\theta}{k})$. From now on, we assume k is sufficiently large such that $F_k(D_{\log k})\Subset D$. We define the scaled bundle $F_k^*T^{*0,q}X$ on $D_{\log k}$ to be the bundle whose fiber at $(z,\theta)\in D_{\log k}$ is

$$F_k^*T^{*0,q}X|_{(z,\theta)} = \left\{\sum\nolimits'_{|J|=q} a_J e^J\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right) : a_J\in\mathbb{C}, |J|=q, J \text{ strictly increasing}\right\}. \tag{2.8}$$

We take the Hermitian metric $\langle\cdot|\cdot\rangle_{F_k^*}$ on $F_k^*T^{*0,q}X$ so that at each point $(z,\theta)\in D_{\log k}$,

$$\left\{e^J\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right) : |J|=q, J \text{ strictly increasing}\right\} \tag{2.9}$$

is an orthonormal frame for $F_k^*T^{*0,q}X$ on $D_{\log k}$. Let $F_k^*\Omega^{0,q}(D_r)$ denote the space of smooth sections of $F_k^*T^{*0,q}X$ over D_r and let $F_k^*\Omega_0^{0,q}(D_r)$ be the subspace of $F_k^*\Omega^{0,q}(D_r)$ whose elements have compact support in D_r. Given $f\in\Omega^{0,q}(D)$. We write $f=\sum'_{|J|=q} f_J e^J$. We define the scaled form $F_k^*f\in F_k^*\Omega^{0,q}(D_{\log k})$ by

$$F_k^*f = \sum\nolimits'_{|J|=q} f_J\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right) e^J\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right). \tag{2.10}$$

For brevity, we denote F_k^*f by $f(\frac{z}{\sqrt{k}},\frac{\theta}{k})$. Let P be a partial differential operator of order one on $F_k(D_{\log k})$ with C^∞ coefficients. We write $P = a(z,\theta)\frac{\partial}{\partial\theta} + \sum_{j=1}^{2n-2} a_j(z,\theta)\frac{\partial}{\partial x_j}$. The scaled partial differential operator $P_{(k)}$ on $D_{\log k}$ is given by

$$P_{(k)} = \sqrt{k}F_k^*a\frac{\partial}{\partial\theta} + \sum_{j=1}^{2n-2} F_k^*a_j\frac{\partial}{\partial x_j}. \tag{2.11}$$

Let $f\in C^\infty(F_k(D_{\log k}))$. We can check that

$$P_{(k)}(F_k^*f) = \frac{1}{\sqrt{k}}F_k^*(Pf). \tag{2.12}$$

The scaled differential operator $\overline{\partial}_{b,(k)} : F_k^*\Omega^{0,q}(D_{\log k}) \to F_k^*\Omega^{0,q+1}(D_{\log k})$ is given by

$$\overline{\partial}_{b,(k)} = \sum_{j=1}^{n-1} e_j\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge\overline{U}_{j,(k)} + \sum_{j=1}^{n-1}\frac{1}{\sqrt{k}}(\overline{\partial}_b e_j)\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge\left(e_j\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge\right)^*, \tag{2.13}$$

where $\left(e_j\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge\right)^*: F_k^*T^{*0,q}X\to F_k^*T^{*0,q-1}X$ is the adjoint of $e_j\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge$ with respect to the $\langle\cdot|\cdot\rangle_{F_k^*}$, $j=1,\ldots,n-1$. That is,

$$\left\langle e_j\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge u\Big|v\right\rangle_{F_k^*}=\left\langle u\Big|\left(e_j\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge\right)^* v\right\rangle_{F_k^*}$$

for all $u\in F_k^*T^{*0,q-1}X$, $v\in F_k^*T^{*0,q}X$. From (2.13), $\overline{\partial}_{b,(k)}$ satisfies that

$$\overline{\partial}_{b,(k)}F_k^*f=\frac{1}{\sqrt{k}}F_k^*(\overline{\partial}_b f). \tag{2.14}$$

Let $(\cdot|\cdot)_{kF_k^*\Phi}$ be the inner product on the space $F_k^*\Omega_0^{0,q}(D_{\log k})$ defined as follows:

$$(f|g)_{kF_k^*\Phi}=\int_{D_{\log k}}\langle f|g\rangle_{F_k^*}e^{-kF_k^*\Phi}(F_k^*m)dv(z)d\theta. \tag{2.15}$$

Let

$$\overline{\partial}^*_{b,(k)}: F_k^*\Omega^{0,q+1}(D_{\log k})\to F_k^*\Omega^{0,q}(D_{\log k})$$

be the formal adjoint of $\overline{\partial}_{b,(k)}$ with respect to $(\cdot|\cdot)_{kF_k^*\Phi}$. Then we also have

$$\overline{\partial}^*_{b,(k)}F_k^*f=\frac{1}{\sqrt{k}}F_k^*(\overline{\partial}_b f). \tag{2.16}$$

We define now the scaled complex Laplacian $\Box^{(q)}_{b,(k)}: F_k^*\Omega^{0,q}(D_{\log k})\to F_k^*\Omega^{0,q}(D_{\log k})$ which is given by

$$\Box^{(q)}_{b,(k)}=\overline{\partial}^*_{b,(k)}\overline{\partial}_{b,(k)}+\overline{\partial}_{b,(k)}\overline{\partial}^*_{b,(k)}. \tag{2.17}$$

Then (2.14) and (2.16) imply that

$$\Box^{(q)}_{b,(k)}F_k^*f=\frac{1}{k}F_k^*(\Box^{(q)}_{b,k}f). \tag{2.18}$$

Similarly, as Proposition 2.3 in [12], Proposition 2.3 in [16], we have

Proposition 2.2.

$$\begin{aligned}\Box^{(q)}_{b,(k)}=&\sum_{j=1}^{n-1}\left[\left(-\frac{\partial}{\partial z_j}-i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta}+\sum_{t=1}^{n-1}\mu_{t,j}\overline{z}_t\right)\left(\frac{\partial}{\partial\overline{z}_j}-i\lambda_j z_j\frac{\partial}{\partial\theta}\right)\right]\\&+\sum_{j,t=1}^{n-1}e_j\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge\left(e_t\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\wedge\right)^*\left(\mu_{j,t}-2i\lambda_j\delta_{j,t}\frac{\partial}{\partial\theta}\right)+\varepsilon_k P_k\end{aligned} \tag{2.19}$$

on $D_{\log k}$, where ε_k is a sequence tending to zero with $k\to\infty$, P_k is a second order differential operator and all the derivatives of the coefficients of P_k are uniformly bounded in k on $D_{\log k}$.

Let $U \subset D_{\log k}$ be an open set and let $W^s_{kF_k^*\Phi}(U, F_k^*T^{*0,q}X), s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the sobolev space of order s of sections of $F_k^*T^{*0,q}X$ over U with respect to the weight $kF_k^*\Phi$. The sobolev norm on this space is given by

$$\|u\|^2_{kF_k^*\Phi,s,U} := \sum_{\alpha\in\mathbb{N}_0^{2n-1},|\alpha|\leq s} \sum\nolimits'_{|J|=q} \int_U |\partial^\alpha_{x,\theta} u_J|^2 e^{-kF_k^*\Phi}(F_k^*m)dv(z)d\theta \tag{2.20}$$

where $u = \sum'_{|J|=q} u_J e^J \left(\frac{z}{\sqrt{k}}, \frac{\theta}{k}\right) \in W^s_{kF_k^*\Phi}(U, F_k^*T^{*0,q}X)$.

PROPOSITION 2.3. *For every $r > 0$ with $D_{2r} \Subset D_{\log k}$, there exists a constant $C_{r,s} > 0$ independent of k and the point p such that for all $u \in F_k^*\Omega^{0,q}(D_{\log k})$, we have*

$$\begin{aligned} &\|u\|^2_{kF_k^*\Phi,s+1,D_r} \\ &\leq C_{r,s}\left(\|u\|^2_{kF_k^*\Phi,D_{2r}} + \|\Box^{(q)}_{b,(k)}u\|^2_{kF_k^*\Phi,s,D_{2r}} + \left\|\left(\frac{\partial}{\partial\theta}\right)^{s+1} u\right\|^2_{kF_k^*\Phi,D_{2r}}\right). \end{aligned} \tag{2.21}$$

Proof. We can repeat the procedure of Kohn's L^2 estimate with minor change (see Theorem 8.4.2 in [7]) and conclude that

$$\begin{aligned} &\|u\|^2_{kF_k^*\Phi,s+1,D_r} \\ &\leq C_{r,s,k}\left(\|u\|^2_{kF_k^*\Phi,D_{2r}} + \|\Box^{(q)}_{b,(k)}u\|^2_{kF_k^*\Phi,s,D_{2r}} + \left\|\left(\frac{\partial}{\partial\theta}\right)^{s+1} u\right\|^2_{kF_k^*\Phi,D_{2r}}\right), \end{aligned} \tag{2.22}$$

for every $u \in F_k^*\Omega^{0,q}(D_{\log k})$. Since all the derivatives of the coefficients of the operator $\Box^{(q)}_{b,(k)}$ are uniformly bounded in k, it is straightforward to see that $C_{r,\,s,k}$ can be taken to be independent of k and the point p. ∎

THEOREM 2.4. *There is a constant $C_0 > 0$ such that for all k and all $x \in X$, we have*

$$k^{-n}\Pi^q_{\leq k\delta}(x) \leq C_0. \tag{2.23}$$

Proof. For any $x \in X$, we choose canonical local coordinates (z, θ) defined in Theorem 1.31 in a neighborhood D centered at x. Let s be a local rigid CR frame of L over D. For $u_k \in \mathcal{H}^q_{b,\leq k\delta}(X, L^k), \|u_k\|_{h^{L^k}} = 1, u_k = \tilde{u}_k \otimes s^k$ on D. Set $\tilde{u}_{(k)} := k^{-\frac{n}{2}}F_k^*(\tilde{u}_k)$ on $D_{\log k}$. Write

$$u_k = \sum_{|m|\leq k\delta, m\in\mathbb{Z}} u_{k,m}, Tu_{k,m} = imu_{k,m} \tag{2.24}$$

which implies that

$$\tilde{u}_k = \sum_{|m|\leq k\delta, m\in\mathbb{Z}} \tilde{u}_{k,m}, T\tilde{u}_{k,m} = im\tilde{u}_{k,m}. \tag{2.25}$$

Then the scaling of $\tilde{u}_k$ given by

$$\tilde{u}_{(k)} = k^{-\frac{n}{2}} \sum_{|m|\leq k\delta, m\in\mathbb{Z}} F_k^*(\widetilde{u}_{k,m}) \tag{2.26}$$

satisfies

$$\|\tilde{u}_{(k)}\|^2_{kF_k^*\Phi,D_{\log k}} = \|\tilde{u}_k\|^2_{k\Phi,F_k(D_{\log k})} \le \|u_k\|^2_{h^{L^k}} = 1. \tag{2.27}$$

From (2.18) we have

$$\Box^{(q)}_{b,(k)}\tilde{u}_{(k)} = 0 \text{ on } D_{\log k}. \tag{2.28}$$

By Proposition 2.3 and combining (2.27), (2.28) we have

$$\|\tilde{u}_{(k)}\|^2_{kF_k^*\Phi,s+1,D_r} \le C_{r,s}(1 + \|(\frac{\partial}{\partial\theta})^{s+1}\tilde{u}_{(k)}\|^2_{kF_k^*\Phi,D_{2r}}) \tag{2.29}$$

for any $r > 0$ with $D_{2r} \Subset D_{\log k}$. Since

$$\frac{\partial}{\partial\theta}\tilde{u}_{(k)} = k^{-\frac{n}{2}}\frac{\partial}{\partial\theta}\sum_{|m|\le k\delta, m\in\mathbb{Z}} F_k^*(\tilde{u}_{k,m}) = k^{-\frac{n}{2}}\sum_{|m|\le k\delta, m\in\mathbb{Z}}\left(\frac{im}{k}\right)\tilde{u}_{k,m}\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right), \tag{2.30}$$

then

$$\left(\frac{\partial}{\partial\theta}\right)^{s+1}\tilde{u}_{(k)} = k^{-\frac{n}{2}}\sum_{|m|\le k\delta, m\in\mathbb{Z}}\left(\frac{im}{k}\right)^{s+1}\tilde{u}_{k,m}\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right). \tag{2.31}$$

Thus,

$$\left\|\left(\frac{\partial}{\partial\theta}\right)^{s+1}\tilde{u}_{(k)}\right\|^2_{kF_k^*\Phi,D_r} \le \delta^{s+1}(k\delta)\sum_{|m|\le k\delta, m\in\mathbb{Z}}\left\|k^{-\frac{n}{2}}\tilde{u}_{k,m}\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\right\|^2_{kF_k^*\Phi,D_{2r}}. \tag{2.32}$$

From (2.25), there is a function $\hat{\tilde{u}}_{k,m}(z) \in C^\infty$ such that

$$\tilde{u}_{k,m}(z,\theta) = \hat{\tilde{u}}_{k,m}(z)e^{im\theta} \text{ on } D. \tag{2.33}$$

Since

$$\begin{aligned}
& k\sum_{|m|\le k\delta, m\in\mathbb{Z}}\left\|k^{-\frac{n}{2}}\tilde{u}_{k,m}\left(\frac{z}{\sqrt{k}},\frac{\theta}{k}\right)\right\|^2_{kF_k^*\Phi,D_{2r}} \\
&\le \sum_{|m|\le k\delta}\int_{D_{2r}} k^{-(n-1)}\left|\hat{\tilde{u}}_{k,m}\left(\frac{z}{\sqrt{k}}\right)\right|^2 e^{-k\Phi(\frac{z}{\sqrt{k}})}m\left(\frac{z}{\sqrt{k}}\right)dv(z)d\theta \\
&\le \sum_{|m|\le k\delta}(4r)\int_{|z|\le\frac{2r}{\sqrt{k}}}|\hat{\tilde{u}}_{k,m}(z)|^2 e^{-k\Phi(z)}m(z)dv(z) \\
&\le \frac{4r}{\varepsilon}\sum_{|m|\le k\delta}\int_{|z|\le\frac{2r}{\sqrt{k}}}\int_{|\theta|<\varepsilon}|\tilde{u}_{k,m}(z,\theta)|^2 e^{-k\Phi(z)}m(z)dv(z)d\theta \\
&\le \frac{4r}{\varepsilon}\sum_{|m|\le k\delta}\|u_{k,m}\|^2_{h^{L^k}} \le \frac{4r}{\varepsilon}\|u_k\|^2_{h^{L^k}} \le \frac{4r}{\varepsilon},
\end{aligned} \tag{2.34}$$

where $\varepsilon > 0$ is a small constant. From (2.34) and (2.32), we deduce that

$$\left\|\left(\frac{\partial}{\partial\theta}\right)^{s+1}\tilde{u}_{(k)}\right\|^2_{kF_k^*\Phi,D_r}\leq\tilde{C}_{r,s},$$

where $\tilde{C}_{r,s}$ is a constant independent of k. Combining this with (2.29), there exists a constant $C'_{r,s} > 0$ independent of k such that

$$\|\tilde{u}_{(k)}\|^2_{kF_k^*\Phi,s+1,D_r}\leq C'_{r,s}. \tag{2.35}$$

From (2.35) and Sobolev embedding theorem, there exists a constant $C(x) > 0$ such that for all k, we have $k^{-n}|u_k(x)|^2_{h^{L^k}} = |\tilde{u}_{(k)}(0)|^2 \leq C(x)$. Since X is compact, we infer that $C' = \sup\{k^{-n}|u_k(x)|^2_{h^{L^k}} : k > 0, x \in X\} < \infty$. Thus, for a local orthonormal frame $\{e^J : |J| = q, J \text{ strictly increasing}\}$ we have that $\sup\{k^{-n}S^q_{\leq k\delta,J}(x) : x \in X, k > 0\} \leq C_0$. From Lemma 2.1, we get the conclusion of Theorem 2.4. □

2.2. The Heisenberg guoup H_n. We identify $\mathbb{R}^{2n-1}$ with the Heisenberg gruop $H_n := \mathbb{C}^{n-1}\times\mathbb{R}$. We also write (z,θ) to denote the coordinates of H_n, $z = (z_1,\cdots,z_{n-1}), \theta\in\mathbb{R}$, $z_j = x_{2j-1}+ix_{2j}, j = 1,\cdots,n-1$. Then

$$\left\{U_{j,H_n}=\frac{\partial}{\partial z_j}+i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta}; j=1,\cdots,n-1\right\} \tag{2.36}$$

and

$$\left\{U_{j,H_n},\overline{U_{j,H_n}},T=\frac{\partial}{\partial\theta}; j=1,\cdots,n-1\right\}$$

are local frames for the bundles of $T^{1,0}H_n$ and $\mathbb{C}TH_n$. Then

$$\left\{dz_j,d\overline{z}_j,\omega_0=-d\theta+\sum_{j=1}^{n-1}(i\lambda_j\overline{z}_jdz_j-i\lambda_jz_jd\overline{z}_j):j=1,\cdots,n-1\right\} \tag{2.37}$$

is the basis of $\mathbb{C}T^*H_n$ which are dual to $\{U_{j,H_n},\overline{U_{j,H_n}},-T\}$. Let $\langle\cdot|\cdot\rangle$ be the Hermitian metric defined on $T^{*0,q}H_n$ such that $\{d\overline{z}^J : |J| = q; J \text{ strictly increasing}\}$ is an orthonormal frame of $T^{*0,q}H_n$. Let

$$\overline{\partial}_{b,H_n}=\sum_{j=1}^{n-1}d\overline{z}_j\wedge\overline{U_{j,H_n}}:\Omega^{0,q}(H_n)\to\Omega^{0,q+1}(H_n) \tag{2.38}$$

be the Cauchy-Riemann operator defined on H_n. Put $\Phi_0(z) = \sum_{j,t=1}^{n-1}\mu_{j,t}\overline{z}_jz_t \in C^\infty(H_n,\mathbb{R})$. Let $(\cdot|\cdot)_{\Phi_0}$ be the inner product on $\Omega_0^{0,q}(H_n)$ with respect to the weight function $\Phi_0(z)$ defined as follows:

$$(f|g)_{\Phi_0}=\int_{H_n}\langle f|g\rangle e^{-\Phi_0(z)}dv(z)d\theta \tag{2.39}$$

where $dv(z) = 2^{n-1}dx_1\cdots dx_{2n-2}$. We denote by $\|\cdot\|_{\Phi_0}$ the norm on $\Omega_0^{0,q}(H_n,\Phi_0)$ induced by the inner product $(\cdot|\cdot)_{\Phi_0}$. Let us denote by $L^2_{(0,q)}(H_n,\Phi_0)$ the completion of $\Omega_0^{0,q}(H_n)$ with respect to the norm $\|\cdot\|_{\Phi_0}$. Let $\overline{\partial}^{*,\Phi_0}_{b,H_n}:\Omega^{0,q+1}(H_n)\to\Omega^{0,q}(H_n)$ be the formal adjoint of $\overline{\partial}_{b,H_n}$ respect to $(\cdot|\cdot)_{\Phi_0}$. The Kohn Laplacian on H_n is given by

$$\Box^{(q)}_{b,H_n}=\overline{\partial}_{b,H_n}\overline{\partial}^{*,\Phi_0}_{b,H_n}+\overline{\partial}^{*,\Phi_0}_{b,H_n}\overline{\partial}_{b,H_n}:\Omega^{0,q}(H_n)\to\Omega^{0,q}(H_n). \tag{2.40}$$

We pause and introduce some notations. Choose $\chi(\theta)\in C_0^\infty(\mathbb{R})$ so that $\chi(\theta)=1$ when $|\theta|<1$ and $\chi(\theta)=0$ when $|\theta|>2$ and set $\chi_j(\theta)=\chi(\frac{\theta}{j}), j\in\mathbb{N}$. For any $u(z,\theta)\in\Omega^{0,q}(H_n)$ with $\|u\|_{\Phi_0}<\infty$. Let

$$\hat{u}_j(z,\eta)=\int_{\mathbb{R}}u(z,\theta)\chi_j(\theta)e^{-i\theta\eta}d\theta\in\Omega^{0,q}(H_n), j=1,2,\ldots. \tag{2.41}$$

From Parseval's formula, $\{\hat{u}_j(z,\eta)\}$ is a cauchy sequence in $L^2_{(0,q)}(H_n,\Phi_0)$. Thus there is $\hat{u}(z,\eta)\in L^2_{(0,q)}(H_n,\Phi_0)$ such that $\hat{u}_j(z,\eta)\to\hat{u}(z,\eta)$ in $L^2_{(0,q)}(H_n,\Phi_0)$. We call $\hat{u}(z,\eta)$ the partial Fourier transform of $u(z,\theta)$ with respect to θ. From Parseval's formula, we can check that

$$\int_{H_n}|\hat{u}(z,\eta)|^2e^{-\Phi_0(z)}dv(z)d\eta=2\pi\int_{H_n}|u(z,\theta)|^2e^{-\Phi_0(z)}dv(z)d\theta. \tag{2.42}$$

Let $s\in L^2_{(0,q)}(H_n,\Phi_0)$. Assume that $\int|s(z,\eta)|^2d\eta<\infty$ and $\int|s(z,\eta)|d\eta<\infty$ for all $z\in\mathbb{C}^{n-1}$. Then, from Parseval's formula, we can check that

$$\begin{aligned}&\iint\langle\,\hat{u}(z,\eta)\,|\,s(z,\eta)\,\rangle e^{-\Phi_0(z)}d\eta dv(z)\\=&\iint\langle\,u(z,\theta)\,|\,\int e^{i\theta\eta}s(z,\eta)d\eta\,\rangle e^{-\Phi_0(z)}d\theta dv(z).\end{aligned} \tag{2.43}$$

2.3. Proof of Theorem 1.26. Now we can prove the second part of Theorem 1.26.

THEOREM 2.5.

$$\limsup_{k\to\infty}k^{-n}\Pi^q_{\leq k\delta}(x)\leq(2\pi)^{-n}\int_{\mathbb{R}_{x,q}\bigcap[-\delta,\delta]}|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)|ds \tag{2.44}$$

for all $x\in X$.

Proof. Fix $x\in X$ and let s be a rigid CR frame of L on an open neighborhood D of x and $|s|^2_{h^L}=e^{-\Phi}$. We take canonical local coordinates $(z,\theta), z_j=x_{2j-1}+ix_{2j}, j=1,\cdots,n-1$ defined in Theorem 1.31 such that $\omega_0(x)=-d\theta,(z(x),\theta(x))=0$, and (2.4), (2.5), (2.6) hold . Until further notice, we work with canonical coordinates (z,θ) defined on an open neighbourhood D of x and we identify D with some open set in $\mathbb{R}^{2n-1}$. We will use the same notations as in section 2.1. Fix $|J|=q$, J is strictly increasing. First, from definition of extremal function, there exists a sequence $\alpha_{k_j}\in\mathcal{H}^q_{b,\leq k_j\delta}(X,L^{k_j})$, $0<k_1<k_2<\cdots$, such that $\|\alpha_{k_j}\|^2_{h^{L^{k_j}}}=1$ and

$$\lim_{j\to\infty}k_j^{-n}|\alpha_{k_j,J}(x)|^2_{h^{L^{k_j}}}=\limsup_{k\to\infty}k^{-n}S^q_{\leq k\delta,J}(x) \tag{2.45}$$

where $\alpha_{k_j,J}$ is the component of α_{k_j} along the direction e^J. Put $\alpha_{k_j}=\tilde{\alpha}_{k_j}\otimes s^{k_j}$, $\tilde{\alpha}_{k_j}\in\Omega^{0,q}(D)$. We will always use α_{k_j} to denote $\tilde{\alpha}_{k_j}$ if there is no misunderstanding, then

$$\alpha_{(k_j)}=k_j^{-\frac{n}{2}}F^*_{k_j}(\alpha_{k_j})\in F^*_{k_j}\Omega^{0,q}(D_{\log k_j}). \tag{2.46}$$

It is easy to see that for every j,

$$\|\alpha_{(k_j)}\|_{k_jF^*_{k_j}\Phi,D_{\log k_j}} \leq 1, \quad \Box^{(q)}_{b,(k_j)}\alpha_{(k_j)} = 0. \tag{2.47}$$

Moreover, we can repeat the procedure in the proof of Theorem 2.4 and obtain that for every $r > 0$ and $s \in \mathbb{N}_0$ there is a $C_{r,s}$ independent of k_j such that

$$\left\|(\frac{\partial}{\partial\theta})^{s+1}\alpha_{(k_j)}\right\|_{k_jF^*_{k_j}\Phi,D_r} \leq C_{r,s}, \quad \forall j. \tag{2.48}$$

From (2.47), (2.48) and Proposition 2.3, we can repeat the same argument in Theorem 2.9 of [16] and conclude that there is a subsequence $\{\alpha_{(k_{s_1})}, \alpha_{(k_{s_2})}, \ldots\}$ of $\{\alpha_{(k_j)}\}$, $0 < k_{s_1} < k_{s_2} < \cdots$, such that $\alpha_{(k_{s_t})}$ converges uniformly with all derivatives on any compact subset of H_n to a smooth form $u = \sum'_{|J|=q} u_J d\overline{z}^J \in \Omega^{0,q}(H_n)$ as $t \to \infty$. Thus,

$$\limsup_{k\to\infty} k^{-n} S^q_{\leq k\delta,J}(x) \leq |u_J(0)|^2. \tag{2.49}$$

Moreover, (2.47) implies that u satisfies

$$\|u\|_{\Phi_0} \leq 1, \quad \Box^{(q)}_{b,H_n} u = 0. \tag{2.50}$$

Then we will need

Lemma 2.6. *With the notations above, $\hat{u}(z,\eta) \equiv 0$ in $L^2_{(0,q)}(H_n,\Phi_0)$ when $|\eta| > \delta$.*

Proof. To prove $\hat{u}(z,\eta) \equiv 0$ when $|\eta| > \delta$, we only need to show that for any $\varphi(z,\eta) \in C^\infty_0(\mathbb{C}^{n-1} \times \{\eta \in \mathbb{R} : |\eta| > \delta\})$ and $|J| = q$, J is strictly increasing, we have

$$\int_{H_n} \hat{u}_J(z,\eta)\varphi(z,\eta)e^{-\Phi_0(z)}dv(z)d\eta = 0. \tag{2.51}$$

We assume that $\operatorname{supp}\varphi \Subset \{z \in \mathbb{C}^{n-1} : |z| \leq r_0\} \times \{\eta \in \mathbb{R} : |\eta| > \delta\}$. Here, $\{z \in \mathbb{C}^{n-1} : |z| < r_0\}$ means that $\{z \in \mathbb{C}^{n-1} : |z_j| < r_0, j = 1, \cdots, n-1\}$. Choose $\chi \in C^\infty_0(\mathbb{R})$ such that $\chi \equiv 1$ when $|\theta| \leq 1$ and $\operatorname{supp}\chi \Subset \{\theta \in \mathbb{R} : |\theta| < 2\}$. From (2.43), we have

$$\begin{aligned} &\frac{1}{2\pi}\int_{H_n} \hat{u}_J(z,\eta)\varphi(z,\eta)e^{-\Phi_0(z)}dv(z)d\eta \\ &= \int_{H_n} u_J(z,\theta)\check{\varphi}(z,\theta)e^{-\Phi_0(z)}dv(z)d\theta \\ &= \lim_{r\to\infty}\int_{H_n} u_J(z,\theta)\check{\varphi}(z,\theta)e^{-\Phi_0(z)}\chi(\frac{\theta}{r})dv(z)d\theta, \end{aligned} \tag{2.52}$$

where $\check{\varphi}(z,\theta) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\theta\eta}\varphi(z,\eta)d\eta$ is the inverse Fourier transform of $\varphi(z,\eta)$ respect to η. For simplicity, we may assume that $\alpha_{(k_j)}$ converges uniformly with all derivatives on any compact subset of H_n to u as $j \to \infty$. Note that $\alpha_{k_j} \in \Omega^{0,q}_{\leq k_j\delta}(X, L^k)$. For each j, on D, we can write

$$\alpha_{k_j} = s^{k_j} \otimes \sum_{m\in\mathbb{Z},|m|\leq k_j\delta} \widetilde{\alpha}_{k_j,m}, \quad \widetilde{\alpha}_{k_j,m} \in \Omega^{0,q}(D), \quad \forall m \in \mathbb{Z}, \ |m| \leq k_j\delta,$$

and for each $m \in \mathbb{Z}$, $|m| \leq k_j\delta$, we can write

$$\widetilde{\alpha}_{k_j,m} = \sum_{|J|=q}{}' \widetilde{\alpha}_{k_j,m,J}(z,\theta)e^J(z),$$
$$\widetilde{\alpha}_{k_j,m,J} = \hat{\alpha}_{k_j,m,J}(z)e^{im\theta},\ \hat{\alpha}_{k_j,m,J}(z) \in C^\infty(D),\ \forall\,|J| = q,\ J \text{ is strictly increasing}.$$

When r is fixed, by dominated convergence theorem

$$\begin{aligned}
&\int_{H_n} u_J(z,\theta)\check{\varphi}(z,\theta)e^{-\Phi_0(z)}\chi(\frac{\theta}{r})dv(z)d\theta\\
&= \lim_{j\to\infty}\sum_{|m|\leq k_j\delta}\int_{H_n} k_j^{-\frac{n}{2}}\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)e^{i\frac{m}{k_j}\theta}\check{\varphi}(z,\theta)\chi\left(\frac{\theta}{r}\right)e^{-\Phi_0(z)}dv(z)d\theta\\
&= \lim_{j\to\infty}\sum_{|m|\leq k_j\delta}\int_{|z|\leq r_0}\int_{\mathbb{R}} k_j^{-\frac{n}{2}}\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)e^{i\frac{m}{k_j}\theta}\check{\varphi}(z,\theta)\chi\left(\frac{\theta}{r}\right)e^{-\Phi_0(z)}dv(z)d\theta.
\end{aligned} \tag{2.53}$$

Since $\operatorname{supp}\varphi(z,\eta) \Subset \{z \in \mathbb{C}^{n-1} : |z| \leq r_0\} \times \{\theta \in \mathbb{R} : |\eta| > \delta\}$ and $|\frac{m}{k_j}| < \delta$, we have

$$\begin{aligned}
&\sum_{|m|\leq k_j\delta}\int_{H_n} k_j^{-\frac{n}{2}}\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)e^{i\frac{m}{k_j}\theta}\check{\varphi}(z,\theta)e^{-\Phi_0(z)}dv(z)d\theta\\
&= \sum_{|m|\leq k_j\delta}\int_{H_n} k_j^{-\frac{n}{2}}\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)\varphi(z,-\frac{m}{k_j})e^{-\Phi_0(z)}dv(z) = 0.
\end{aligned} \tag{2.54}$$

By (2.54)

$$\begin{aligned}
&\lim_{j\to\infty}\sum_{|m|\leq k_j\delta}\int_{|z|\leq r_0}\int_{\mathbb{R}} k_j^{-\frac{n}{2}}\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)e^{i\frac{m}{k_j}\theta}\check{\varphi}(z,\theta)\chi\left(\frac{\theta}{r}\right)e^{-\Phi_0(z)}dv(z)d\theta\\
&= \lim_{j\to\infty}\sum_{|m|\leq k_j\delta}\int_{|z|\leq r_0}\int_{\mathbb{R}} k_j^{-\frac{n}{2}}\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)e^{i\frac{m}{k_j}\theta}\check{\varphi}(z,\theta)\left(\chi\left(\frac{\theta}{r}\right)-1\right)e^{-\Phi_0(z)}dv(z)d\theta.
\end{aligned} \tag{2.55}$$

Now,

$$\begin{aligned}
&\left|\sum_{|m|\leq k_j\delta}\int_{|z|\leq r_0}\int_{\mathbb{R}} k_j^{-\frac{n}{2}}\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)e^{i\frac{m}{k_j}\theta}\check{\varphi}(z,\theta)\left(\chi\left(\frac{\theta}{r}\right)-1\right)e^{-\Phi_0(z)}dv(z)d\theta\right|\\
&\leq \sum_{|m|\leq k_j\delta}\int_{|z|\leq r_0}\int_{|\theta|\geq r} k_j^{-\frac{n}{2}}\left|\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)\right|\cdot|\check{\varphi}(z,\theta)|e^{-\Phi_0(z)}dv(z)d\theta.
\end{aligned} \tag{2.56}$$

By Hölder inequality, we have

$$\begin{aligned}
&\int_{|z|\leq r_0}\int_{|\theta|\geq r} k_j^{-\frac{n}{2}}|\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)|\cdot|\check{\varphi}(z,\theta)|e^{-\Phi_0(z)}dv(z)d\theta \\
\leq\ &\left(\int_{|z|\leq r_0}\int_{|\theta|\geq r} k_j^{-n}|\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)|^2\cdot|\check{\varphi}(z,\theta)|e^{-\Phi_0(z)}dv(z)d\theta\right)^{\frac{1}{2}} \\
&\times\left(\int_{|z|\leq r_0}\int_{|\theta|\geq r}|\check{\varphi}(z,\theta)|e^{-\Phi_0(z)}dv(z)d\theta\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.57}$$

Since $\operatorname{supp}\varphi(z,\eta)\Subset\{z\in\mathbb{C}^{n-1}:|z|\leq r_0\}\times\mathbb{R}$, we have

$$\sup_{|z|\leq r_0}|\check{\varphi}(z,\theta)|\leq C_{r_0}\frac{1}{|\theta|^p},\forall\ |\theta|>>1 \tag{2.58}$$

for some $p>3, p\in\mathbb{Z}$ and constant $C_{r_0}>0$. Combining (2.57) and (2.58),we have

$$\begin{aligned}
&\int_{|z|\leq r_0}\int_{|\theta|\geq r} k_j^{-\frac{n}{2}}\left|\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)\right|\cdot|\check{\varphi}(z,\theta)|e^{-\Phi_0(z)}dv(z)d\theta \\
&\leq C_{r_0}\frac{1}{r^{p-1}}\left(\int_{|z|\leq r_0} k_j^{-n}\left|\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)\right|^2 e^{-\Phi_0(z)}dv(z)\right)^{\frac{1}{2}} \\
&\leq C'_{r_0}\frac{1}{r^{p-1}}\left(\int_{|z|\leq r_0} k_j^{-n}\left|\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)\right|^2 e^{-k_jF^*_{k_j}\Phi(z)}(F^*_{k_j}m)dv(z)\right)^{\frac{1}{2}} \\
&\leq C'_{r_0}\frac{1}{r^{p-1}}\left(\int_{|z|\leq\frac{r_0}{\sqrt{k_j}}}\frac{1}{k_j}|\hat{\alpha}_{k_j,m,J}(z)|^2e^{-k_j\Phi(z)}m(z)dv(z)\right)^{\frac{1}{2}},
\end{aligned} \tag{2.59}$$

for $r>>1$, where $C_{r_0}>0$ and $C'_{r_0}>0$ are constants. Then from (2.59) and Cauchy-Schwartz inequality,

$$\begin{aligned}
&\sum_{|m|\leq k_j\delta}\int_{|z|\leq r_0}\int_{|\theta|\geq r} k_j^{-\frac{n}{2}}\left|\hat{\alpha}_{k_j,m,J}\left(\frac{z}{\sqrt{k_j}}\right)\right|\cdot|\check{\varphi}(z,\theta)|e^{-\Phi_0(z)}dv(z)d\theta \\
&\leq C'_{r_0}\frac{1}{r^{p-1}}\sqrt{\delta}\left(\sum_{|m|\leq k_j\delta}\int_{|z|\leq\frac{r_0}{\sqrt{k_j}}}|\hat{\alpha}_{k_j,m,J}(z)|^2e^{-k_j\Phi(z)}m(z)dv(z)\right)^{\frac{1}{2}} \\
&\leq C'_{r_0}\frac{1}{r^{p-1}}\frac{\sqrt{\delta}}{\sqrt{2\varepsilon}}\left(\sum_{|m|\leq k_j\delta}\int_{|z|\leq\frac{r_0}{\sqrt{k_j}},|\theta|\leq\varepsilon}|\widetilde{\alpha}_{k_j,m,J}(z,\theta)|^2e^{-k_j\Phi(z)}m(z)dv(z)d\theta\right)^{\frac{1}{2}} \\
&\leq C'_{r_0}\frac{1}{r^{p-1}}\frac{\sqrt{\delta}}{\sqrt{2\varepsilon}}\left(\sum_{|m|\leq k_j\delta}\|\alpha_{k_j,m}\|^2_{h^{L^{k_j}}}\right)^{\frac{1}{2}}\leq C'_{r_0}\frac{1}{r^{p-1}}\frac{\sqrt{\delta}}{\sqrt{2\varepsilon}}\|\alpha_{k_j}\|^2_{h^{L^{k_j}}}\leq C'_{r_0}\frac{1}{r^{p-1}}\frac{\sqrt{\delta}}{\sqrt{2\varepsilon}}.
\end{aligned} \tag{2.60}$$

From (2.53), (2.55), (2.56) and (2.60), we get

$$\left|\int_{H_n} u_J(z,\theta)\check{\varphi}(z,\theta)e^{-\Phi_0(z)}\chi(\frac{\theta}{r})dv(z)d\theta\right| \leq C'_{r_0}\frac{1}{r^{p-1}}\frac{\sqrt{\delta}}{\sqrt{2\varepsilon}}. \tag{2.61}$$

Letting $r \to \infty$, we get the conclusion of Lemma 2.6. ☐

We pause and introduce some notations. For fixed $\eta \in \mathbb{R}$, put $\Phi_\eta(z) = -2\eta \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + \sum_{j,t=1}^{n-1} \mu_{j,t}\overline{z}_j z_t$. We take the Hermitian metric $\langle\cdot|\cdot\rangle$ on the bundle $T^{*0,q}\mathbb{C}^{n-1}$ of $(0,q)$ forms on $\mathbb{C}^{n-1}$ so that $\{d\overline{z}^J; |J| = q,\ J \text{ strictly increasing}\}$ is an orthonormal frame. We also let $\Omega^{0,q}(\mathbb{C}^{n-1})$ denote the space of smooth sections of $T^{*0,q}\mathbb{C}^{n-1}$ over $\mathbb{C}^{n-1}$ and let $\Omega_0^{0,q}(\mathbb{C}^{n-1})$ be the subspace of $\Omega^{0,q}(\mathbb{C}^{n-1})$ whose elements have compact support in $\mathbb{C}^{n-1}$. Let $(\cdot|\cdot)_{\Phi_\eta}$ be the inner product on $\Omega_0^{0,q}(\mathbb{C}^{n-1})$ defined by

$$(f\,|\,g)_{\Phi_\eta} = \int_{\mathbb{C}^{n-1}} \langle f\,|\,g\rangle e^{-\Phi_\eta(z)}dv(z),\quad f,g \in \Omega_0^{0,q}(\mathbb{C}^{n-1}),$$

where $dv(z) = 2^{n-1}dx_1dx_2\cdots dx_{2n-2}$, and let $\|\cdot\|_{\Phi_\eta}$ denote the corresponding norm. Let us denote by $L^2_{(0,q)}(\mathbb{C}^{n-1},\Phi_\eta)$ the completion of $\Omega_0^{0,q}(\mathbb{C}^{n-1})$ with respect to the norm $\|\cdot\|_{\Phi_\eta}$. Let

$$\Box^{(q)}_{\Phi_\eta} = \overline{\partial}^{*,\Phi_\eta}\overline{\partial} + \overline{\partial}\,\overline{\partial}^{*,\Phi_\eta} : \Omega^{0,q}(\mathbb{C}^{n-1}) \to \Omega^{0,q}(\mathbb{C}^{n-1})$$

be the complex Laplacian with respect to $(\cdot|\cdot)_{\Phi_\eta}$, where $\overline{\partial}^{*,\Phi_\eta}$ is the formal adjoint of $\overline{\partial}$ with respect to $(\cdot|\cdot)_{\Phi_\eta}$. Let $B^{(q)}_{\Phi_\eta} : L^2_{(0,q)}(\mathbb{C}^{n-1},\Phi_\eta) \to \mathrm{Ker}\Box^{(q)}_{\Phi_\eta}$ be the Bergman projection and $B^{(q)}_{\Phi_\eta}(z,w)$ be the distribution kernel of $B^{(q)}_{\Phi_\eta}$ with respect to $(\cdot|\cdot)_{\Phi_\eta}$(see section 3.2 in [16]). Let $M_{\Phi_\eta} : T^{1,0}_z\mathbb{C}^{n-1} \to T^{1,0}_z\mathbb{C}^{n-1}, z \in \mathbb{C}^{n-1}$ be the linear map defined by

$$\langle M_{\Phi_\eta}U|V\rangle = \partial\overline{\partial}\Phi_\eta(U,\overline{V}), U,V \in T^{1,0}_z\mathbb{C}^{n-1}.$$

Put

$$\begin{aligned}\mathbb{R}_q =&\{\eta \in \mathbb{R} : M_{\Phi_\eta} \text{ has exactly } q \text{ negative eigenvalues}\\ &\text{and } n-1-q \text{ positive eigenvalues}\}.\end{aligned}$$

The following lemma is well known(see Berman [3], Hsiao and Marinescu [16], Ma and Marinescu [18]).

Lemma 2.7. *If $\eta \notin \mathbb{R}_q$, then $B^{(q)}_{\Phi_\eta}(z,z) = 0$ for all $z \in \mathbb{C}^{n-1}$. If $\eta \in \mathbb{R}_q$, then*

$$\sum\nolimits'_{|J|=q}\langle B^{(q)}_{\Phi_\eta}(z,z)d\overline{z}^J|d\overline{z}^J\rangle = e^{\Phi_\eta(z)}(2\pi)^{-n+1}|\det M_{\Phi_\eta}|\cdot 1_{\mathbb{R}_q}(\eta) \tag{2.62}$$

The following is also well-known.

Lemma 2.8 (See Theorem 3.1 and Lemma 3.5 in [16]). *For almost all $\eta \in \mathbb{R}$, $\hat{u}(z,\eta) \in \Omega^{0,q}(\mathbb{C}^{n-1})$, $\int_{\mathbb{C}^{n-1}} |\hat{u}(z,\eta)|^2 e^{-\Phi_0(z)}dv(z) < \infty$ and*

$$|\hat{u}_J(z,\eta)|^2 \leq \exp\,(\eta\sum_{j=1}^{n-1}\lambda_j|z_j|^2)\langle B^{(q)}_{\Phi_\eta}(z,z)d\overline{z}^J|d\overline{z}^J\rangle\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2e^{-\Phi_0(w)}dv(w),$$

for every strictly increasing index J, $|J| = q$.

Now, we can prove

PROPOSITION 2.9. *For every $|J| = q$, J is strictly increasing, we have*

$$u_J(0,0) = \frac{1}{2\pi}\int_{\eta\in\mathbb{R},|\eta|\leq\delta}\hat{u}_J(0,\eta)d\eta.$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}}\chi d\theta = 1$, $\chi \geqslant 0$ and $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$, $\chi_\varepsilon(\theta) = \frac{1}{\varepsilon}\chi(\frac{\theta}{\varepsilon})$. Then, $\chi_\varepsilon \to \delta_0$, $\varepsilon \to 0^+$ in the sense of distributions. Let $\hat{\chi}_\varepsilon := \int e^{-i\theta\eta}\chi_\varepsilon(\theta)d\theta$ be the Fourier transform of χ_ε. We can check that $|\hat{\chi}_\varepsilon(\eta)| \leqslant 1$ for all $\eta \in \mathbb{R}$, $\hat{\chi}_\varepsilon(\eta) = \hat{\chi}(\varepsilon\eta)$ and $\lim_{\varepsilon\to 0}\hat{\chi}_\varepsilon(\eta) = \lim_{\varepsilon\to 0}\hat{\chi}(\varepsilon\eta) = \hat{\chi}(0) = 1$. Let $\varphi \in C_0^\infty(\mathbb{C}^{n-1})$ such that $\int_{\mathbb{C}^{n-1}}\varphi(z)dv(z) = 1$, $\varphi \geqslant 0$, $\varphi(z) = 0$ if $|z| > 1$. Put $g_j(z) = j^{2n-2}\varphi(jz)e^{\Phi_0(z)}$, $j = 1, 2, \ldots$. Then, for J is strictly increasing, $|J| = q$, we have

$$u_J(0,0) = \lim_{j\to\infty}\lim_{\varepsilon\to 0^+}\int_{H_n}\langle\, u(z,\theta)\,|\,\chi_\varepsilon(\theta)g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}dv(z)d\theta. \tag{2.63}$$

From (2.43), we see that

$$\begin{aligned}&\iint\langle\, u(z,\theta)\,|\,\chi_\varepsilon(\theta)g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}dv(z)d\theta\\ &= \frac{1}{2\pi}\iint\langle\hat{u}(z,\eta)\,|\,\hat{\chi}_\varepsilon(\eta)g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}d\eta dv(z).\end{aligned} \tag{2.64}$$

From Lemma 2.6, it is not difficult to check that for every j and every $\varepsilon > 0$,

$$\begin{aligned}&\iint\langle\hat{u}(z,\eta)\,|\,\hat{\chi}_\varepsilon(\eta)g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}d\eta dv(z)\\ &= \iint_{|\eta|\leq\delta}\langle\hat{u}(z,\eta)\,|\,\hat{\chi}_\varepsilon(\eta)g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}d\eta dv(z).\end{aligned} \tag{2.65}$$

From Lemma 2.8, (2.65), we can apply Lebesque dominated convergence theorem and conclude that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\iint\langle\,\hat{u}(z,\eta)\,|\,\hat{\chi}_\varepsilon(\eta)g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}d\eta dv(z)\\ &= \iint_{|\eta|\leq\delta}\langle\,\hat{u}(z,\eta)\,|\,g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}d\eta dv(z).\end{aligned} \tag{2.66}$$

From (2.64) and (2.66), (2.63) becomes

$$u_J(0,0) = \lim_{j\to\infty}\frac{1}{2\pi}\iint_{|\eta|\leq\delta}\langle\,\hat{u}(z,\eta)\,|\,g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}d\eta dv(z). \tag{2.67}$$

Put $f_j(\eta) = \frac{1}{2\pi}\int\langle\hat{u}(z,\eta)\,|\,g_j(z)d\overline{z}^J\,\rangle e^{-\Phi_0(z)}dv(z)$. Since $\hat{u}(z,\eta) \in \Omega^{0,q}(\mathbb{C}^{n-1})$ for almost all η, we have $\lim_{j\to\infty}f_j(\eta) = \frac{1}{2\pi}\hat{u}_J(0,\eta)$ almost everywhere. Now, for almost every $\eta \in \mathbb{R}$,

$$
\begin{aligned}
|f_j(\eta)| &= \frac{1}{2\pi}\left|\int_{|z|\leqslant\frac{1}{j}}\langle\, \hat{u}(z,\eta)\,|\, j^{2n-2}\varphi(jz)d\overline{z}^J\,\rangle dv(z)\right| \\
&\leqslant \frac{1}{2\pi}\Big(\int_{|z|\leqslant\frac{1}{j}}|\hat{u}(z,\eta)|^2\, e^{-\Phi_0(z)} j^{2n-2}dv(z)\Big)^{\frac{1}{2}}\Big(\int_{|z|\leqslant\frac{1}{j}}|\varphi(jz)|^2\, e^{\Phi_0(z)} j^{2n-2}dv(z)\Big)^{\frac{1}{2}} \\
&\leqslant C_1\Big(\int_{|z|\leqslant 1}\left|\hat{u}(\frac{z}{j},\eta)\right|^2 e^{-\Phi_0(z/j)}dv(z)\Big)^{\frac{1}{2}} \\
&\leqslant C_2\Big(\int_{|z|\leqslant 1} e^{2\eta\sum_{t=1}^{n-1}\lambda_t\left|\frac{z_t}{j}\right|^2}\left|\mathrm{Tr}\, B^{(q)}_{\Phi_\eta}(\frac{z}{j},\frac{z}{j})\right| dv(z)\Big)^{\frac{1}{2}} \\
&\quad\times\Big(\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2\, e^{-\Phi_0(w)}dv(w)\Big)^{\frac{1}{2}} \quad \text{(here we used Lemma 2.8)} \\
&\leqslant C_3\Big(\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2\, e^{-\Phi_0(w)}dv(w)\Big)^{\frac{1}{2}},
\end{aligned}
\tag{2.68}
$$

where C_1, C_2, C_3 are positive constants. From this and the Lebesgue dominated convergence theorem, we conclude that

$$
u_J(0,0) = \lim_{j\to\infty}\int_{|\eta|\leq\delta} f_j(\eta)d\eta = \int_{|\eta|\leq\delta}\lim_{j\to\infty} f_j(\eta)d\eta = \frac{1}{2\pi}\int \hat{u}_J(0,\eta)d\eta\,.
$$

□

Now we turn to our situation. Fix $|J| = q$, J is strictly increasing. By Proposition 2.9, Lemma 2.8 and notice that (see (2.42))

$$
\int_{|\eta|\leq\delta}|\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w)d\eta \leq \int |\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w)d\eta \leq 2\pi,
$$

we have

$$
\begin{aligned}
|u_J(0,0)| &= \frac{1}{2\pi}\left|\int_{|\eta|\leq\delta}\hat{u}_J(0,\eta)d\eta\right| \\
&\leq \frac{1}{2\pi}\int_{|\eta|\leq\delta}|\hat{u}_J(0,\eta)|\frac{(\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w))^{\frac{1}{2}}}{(\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w))^{\frac{1}{2}}}d\eta \\
&\leq \frac{1}{2\pi}\left(\int_{|\eta|\leq\delta}\frac{|\hat{u}_J(0,\eta)|^2}{\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w)}d\eta\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{|\eta|\leq\delta}|\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w)d\eta\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2\pi}}\left(\int_{|\eta|\leq\delta}\frac{\langle B^{(q)}_{\Phi_\eta}(0,0)d\overline{z}^J|d\overline{z}^J\rangle\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w)}{\int_{\mathbb{C}^{n-1}}|\hat{u}(w,\eta)|^2 e^{-\Phi_0(w)}dv(w)}d\eta\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2\pi}}\left(\int_{|\eta|\leq\delta}\langle B^{(q)}_{\Phi_\eta}(0,0)d\overline{z}^J|d\overline{z}^J\rangle d\eta\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.69}
$$

Combining (2.49) and (2.69), we have

$$\limsup_{k\to\infty} k^{-n} S^{q}_{\leq k\delta, J}(p) \leq \frac{1}{2\pi}\int_{|\eta|\leq\delta} \langle B^{(q)}_{\Phi_\eta}(0,0)d\overline{z}^J | d\overline{z}^J\rangle d\eta. \tag{2.70}$$

Then Lemma 2.1, Lemma 2.7 and (2.70) imply that

$$\begin{aligned}\limsup_{k\to\infty} k^{-n}\Pi^{q}_{\leq k\delta}(x) &\leq \frac{1}{2\pi}\int_{|\eta|\leq\delta} {\sum}'_{|J|=q}\langle B^{(q)}_{\Phi_\eta}(0,0)d\overline{z}^J | d\overline{z}^J\rangle d\eta \\ &\leq \frac{1}{(2\pi)^n}\int_{|\eta|\leq\delta} |\det M_{\Phi_\eta}|\cdot 1_{\mathbb{R}_q}(\eta)d\eta \\ &\leq \frac{1}{(2\pi)^n}\int_{\mathbb{R}_q\cap[-\delta,\delta]} |\det M_{\Phi_\eta}| d\eta \\ &\leq \frac{1}{(2\pi)^n}\int_{\mathbb{R}_{x,q}\cap[-\delta,\delta]} |\det(\mathcal{R}^L_p + 2s\mathcal{L}_p)| ds.\end{aligned} \tag{2.71}$$

Thus we get the conclusion of Theorem 2.5. ∎

3. Strong Morse inequalities on CR manifolds with S^1 action. In this section, we will establish the strong Morse inequalities on compact CR manifolds with S^1 action. Following the same argument as in Proposition 3.8, Proposition 3.9 in [16] and by some minor change we have

PROPOSITION 3.1. *There exists $u \in \Omega^{0,q}(H_n)$ such that*

$$\Box^{(q)}_{b,H_n} u = 0, \|u\|_{\Phi_0} = 1, \tag{3.1}$$

$$|u(0,0)|^2 = (2\pi)^{-n}\int_{\mathbb{R}_q\cap[-\delta,\delta]} |\det M_{\Phi_\eta}| d\eta, \tag{3.2}$$

and

$$\hat{u}(z,\eta) \equiv 0 \text{ when } |\eta| > \delta. \tag{3.3}$$

Proof. Since some notations have been changed from Proposition 3.8 and 3.9 in [16], we will outline the proof here for the convenient of readers. For any $\eta \in \mathbb{R}$, we can find a unitary matrix $(a_{ij}(\eta))_{1\leq i,j\leq n-1}$ such that $z_i(\eta) = \sum_{i,j=1}^{n-1} a_{ij}(\eta) z_j$ and $\Phi_\eta(z) = \sum_{j=1}^{n-1} v_j(\eta)|z_j(\eta)|^2$, where $v_j(\eta), j = 1, \cdots, n-1$ are the eigenvalues of M_{Φ_η}. If $\eta \in \mathbb{R}_q$, we assume $v_1(\eta) < 0, \cdots, v_q(\eta) < 0, v_{q+1}(\eta) > 0, \cdots, v_{n-1}(\eta) > 0$. Put

$$\alpha(z,\eta) = C_0 |\det M_{\Phi_\eta}| 1_{\mathbb{R}_q\cap[-\delta,\delta]}(\eta)\exp\left(\sum_{j=1}^{q} v_j(\eta)|z_j(\eta)|^2\right) d\overline{z_1(\eta)}\wedge\cdots d\overline{z_q(\eta)}, \tag{3.4}$$

where $C_0 = (2\pi)^{1-\frac{n}{2}}\left(\int_{\mathbb{R}_q\cap[-\delta,\delta]} |\det M_{\Phi_\eta}| d\eta\right)^{-\frac{1}{2}}$. Then $\Box^{(q)}_{\Phi_\eta}\alpha(z,\eta) = 0$. Moreover, we have

$$\begin{aligned}&\int_{\mathbb{C}^{n-1}} |\alpha(z,\eta)|^2 e^{-\Phi_\eta(z)} dv(z) \\ &= 2\pi\left(\int_{\mathbb{R}_q\cap[-\delta,\delta]} |\det M_{\Phi_\eta}| d\eta\right)^{-1} |\det M_{\Phi_\eta}|\cdot 1_{\mathbb{R}_q\cap[-\delta,\delta]}(\eta).\end{aligned} \tag{3.5}$$

Set

$$u(z,\theta)=\frac{1}{2\pi}\int_{\mathbb{R}}\exp\Big(i\theta\eta+\eta\sum_{j=1}^{n-1}\lambda_j|z_j|^2\Big)\alpha(z,\eta)d\eta\in\Omega^{0,q}(H_n). \tag{3.6}$$

Using Lemma 3.2 in [16], we can check that $u(z,\theta)$ satifies the properties in Proposition 3.1. ◻

We will use the same notation as in section 2.1. Fix $x\in X_{\rm reg}$, choose canonical local coordinates (z,θ) near x such that $x\leftrightarrow 0$, $D=\{(z,\theta)\in\mathbb{C}^{n-1}\times\mathbb{R}:|z|<1,|\theta|<\pi\}$. It should be noticed that since $x\in X_{\rm reg}$, θ can be defined on $|\theta|<\pi$. Choose two cut-off functions $\chi\in C_0^\infty(\mathbb{C}^{n-1}),\tau\in C_0^\infty(\mathbb{R})$ in such that $\chi(z)\equiv 1$ when $|z|\le\frac{1}{2}$, $\chi(z)=0$ when $|z|>1$ and $\tau(\theta)\equiv 1,|\theta|\le\frac{1}{2};\tau(\theta)\equiv 0,|\theta|>1$. Let u be given as in Proposition 3.1. Put $\chi_k(z)=\chi(\frac{z}{\log k}),\tau_k(\theta)=\tau(\frac{\theta}{\log k})$. Put $u_k=\chi_k(\sqrt{k}z)\tau_k(k\theta)\sum'_{|J|=q}u_J(\sqrt{k}z,k\theta)e^J(z)\in\Omega^{0,q}(X)$. Then $\operatorname{supp}u_k\Subset D_{\frac{\log k}{\sqrt{k}}}$. Write $\alpha_k=k^{\frac{n}{2}}u_k(z,\theta)\otimes s^k\in\Omega^{0,q}(X,L^k)$. Then

$$\begin{aligned}\|\alpha_k\|^2_{h^{L^k}}&=k^n\int_X e^{-k\Phi(z)}|u_k(z,\theta)|^2dv_X\\&=k^n\int_X e^{-k\Phi(z)}\chi_k^2(\sqrt{k}z)\tau_k^2(k\theta)|u(\sqrt{k}z,k\theta)|^2dv_X\\&=k^n\int_{|z|\le\frac{\log k}{\sqrt{k}},|\theta|\le\frac{\log k}{k}}e^{-k\Phi(z)}\chi_k^2(\sqrt{k}z)\tau_k^2(k\theta)|u(\sqrt{k}z,k\theta)|^2m(z)dv(z)d\theta\\&=\int_{D_{\log k}}e^{-k\Phi(\frac{z}{\sqrt{k}})}\chi_k^2(z)\tau_k^2(\theta)|u(z,\theta)|^2m(\frac{z}{\sqrt{k}})dv(z)d\theta,\end{aligned} \tag{3.7}$$

where $m(z)dv(z)d\theta=dv_X$ on D. Then

$$\lim_{k\to\infty}\|\alpha_k\|^2_{h^{L^k}}=\int_{H_n}e^{-\Phi_0(z)}|u(z,\theta)|^2dv(z)d\theta=1. \tag{3.8}$$

Second,

$$k^{-n}|\alpha_k(0,0)|^2_{h^{L^k}}=|u(0,0)|^2=(2\pi)^{-n}\int_{\mathbb{R}_{x,q}\cap[-\delta,\delta]}|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)|ds. \tag{3.9}$$

Third, from $\Box^{(q)}_{b,H_n}u=0$, it is easy to see that there exists a sequence $\mu_k>0$, independent of p and tending to zero such that

$$\Big(\frac{1}{k}\Box^{(q)}_{b,k}\alpha_k\,\big|\,\alpha_k\Big)_{h^{L^k}}\le\mu_k. \tag{3.10}$$

Moreover, for every $j\in\mathbb{N}$,

$$\Big((\tfrac{1}{k}\Box^{(q)}_{b,k})^j\alpha_k\,\big|\,\alpha_k\Big)_{h^{L^k}}\to 0\text{ as }k\to\infty. \tag{3.11}$$

Theorem 3.2. *Set* $\beta_k=F_{\delta,k}\alpha_k:=\sum_{|m|\le k\delta}\alpha_{k,m}$, $T\alpha_{k,m}=(im)\alpha_{k,m}$. *Then we* *will have*

$$
\begin{aligned}
&(1)\ \beta_k\in\Omega^{0,q}_{\leq k\delta}(X,L^k),\ \lim_{k\to\infty}\|\beta_k\|^2_{h^{L^k}}=1,\\
&(2)\ \lim_{k\to\infty}k^{-n}|\beta_k(x)|^2_{h^{L^k}}=(2\pi)^{-n}\int_{\mathbb{R}_{x,q}\cap[-\delta,\delta]}|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)|ds,\\
&(3)\ \left(\frac{1}{k}\Box^{(q)}_{b,k}\beta_k|\beta_k\right)_{h^{L^k}}\leq\mu_k,\\
&(4)\ \left((\tfrac{1}{k}\Box^{(q)}_{b,k})^j\beta_k\big|\beta_k\right)_{h^{L^k}}\to 0\ \textit{as}\ k\to\infty,\ \forall j\in\mathbb{N}.
\end{aligned}
\tag{3.12}
$$

We postpone the proof of Theorem 3.2 until the end of this section.

PROPOSITION 3.3. *Let $v_k>0$ be any sequence with $\lim_{k\to\infty}\frac{\mu_k}{v_k}=0$. Then*

$$
\liminf_{k\to\infty}k^{-n}\Pi^q_{\leq k\delta,\leq kv_k}(x)\geq(2\pi)^{-n}\int_{\mathbb{R}_{x,q}\cap[-\delta,\delta]}|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)|ds,\quad\forall x\in X_{\mathrm{reg}}\,.
\tag{3.13}
$$

Proof. We will follow the argument of proposition 5.1 in [16] to prove this proposition. Let $\mathcal{H}^q_{b,\leq k\delta,>kv_k}(X,L^k)$ denote the space spanned by the eigenforms of $\Box^{(q)}_{b,k}$ restricting to $\Omega^{0,q}_{\leq k\delta}(X,L^k)$ whose eigenvalues are $>kv_k$. Fix $x\in X_{\mathrm{reg}}$ and let β_k be defined as in Theorem 3.2. $\beta_k=\beta^1_k+\beta^2_k$, where $\beta^1_k\in\mathcal{H}^q_{\leq k\delta,\leq kv_k}(X,L^k)$, $\beta^2_k\in\overline{\mathcal{H}^q_{\leq k\delta,>kv_k}(X,L^k)}$. Here the closure of $\mathcal{H}^q_{\leq k\delta,>kv_k}(X,L^k)$ is under the Q_b-norm defined in Proposition 5.1 [16]. Then

$$
\|\beta^2_k\|^2_{h^{L^k}}=(\beta^2_k|\beta^2_k)_{h^{L^k}}\leq\frac{1}{kv_k}(\Box^{(q)}_{b,k}\beta^2_k|\beta^2_k)_{h^{L^k}}\leq\frac{1}{kv_k}(\Box^{(q)}_{b,k}\beta_k|\beta_k)_{h^{L^k}}\leq\frac{\mu_k}{v_k}\to 0.
\tag{3.14}
$$

Since $\lim_{k\to\infty}\|\beta_k\|_{h^{L^k}}=1$, we get $\lim_{k\to\infty}\|\beta^1_k\|_{h^{L^k}}=1$. Now, we claim that

$$
\lim_{k\to\infty}k^{-n}|\beta^2_k(x)|^2_{h^{L^k}}=0.
\tag{3.15}
$$

On D with canonical local coordinates (z,θ), $x\leftrightarrow 0$ and $\Phi(0)=0$, we write $\beta^2_k=k^{\frac{n}{2}}\alpha^2_k\otimes s^k$, $\alpha^2_k\in\Omega^{0,q}(D)$. Then

$$
\lim_{k\to\infty}k^{-n}|\beta^2_k(0)|^2_{h^{L^k}}=\lim_{k\to\infty}|\alpha^2_k(0)|^2.
\tag{3.16}
$$

By Proposition 2.3, we have

$$
\begin{aligned}
|F^*_k\alpha^2_k(0)|^2\leq C_{n,r}\Bigg(&\|F^*_k\alpha^2_k\|^2_{kF^*_k\Phi,D_{2r}}+\|\Box^{(q)}_{b,(k)}F^*_k\alpha^2_k\|^2_{kF^*_k\Phi,n,D_{2r}}\\
&+\left\|\left(\frac{\partial}{\partial\theta}\right)^nF^*_k\alpha^2_k\right\|_{kF^*_k\Phi,D_{2r}}\Bigg).
\end{aligned}
\tag{3.17}
$$

From the proof of Theorem 2.4, we see that

$$
\left\|\left(\frac{\partial}{\partial\theta}\right)^nF^*_k\alpha^2_k\right\|_{kF^*_k\Phi,D_{2r}}\leq C\|\beta^2_k\|^2_{h^{L^k}}
\tag{3.18}
$$

where $C > 0$ is a constant which does not depend on k. Moreover, from (4) in Theorem 3.2,

$$\begin{aligned}\|\Box^{(q)}_{b,k}F^*_k\alpha^2_k\|^2_{kF^*_k,n,D_{2r}} &\leq C_1\sum_{m=1}^{n+1}\left\|\left(\frac{1}{k}\Box^{(q)}_{b,k}\right)^m\beta^2_k\right\|^2_{h^{L^k}}\\ &\leq C_1\sum_{m=1}^{n+1}\left\|\left(\frac{1}{k}\Box^{(q)}_{b,k}\right)^m\beta_k\right\|^2_{h^{L^k}}\to 0,\end{aligned} \tag{3.19}$$

where $C_1 > 0$ is a constant independent of k. Combining (3.14), (3.16), (3.17), (3.18) and (3.19), we get that

$$\lim_{k\to\infty}k^{-n}|\beta^2_k(0)|^2_{h^{L^k}} = \lim_{k\to\infty}|\alpha^2_k(0)|^2 = 0. \tag{3.20}$$

Then

$$\begin{aligned}\lim_{k\to\infty}k^{-n}|\beta^1_k(0)|^2_{h^{L^k}} &= \lim_{k\to\infty}k^{-n}|\beta_k(0)|^2_{h^{L^k}}\\ &= (2\pi)^{-n}\int_{\mathbb{R}_{x,q}\cap[-\delta,\delta]}|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)|ds.\end{aligned} \tag{3.21}$$

Now,

$$k^{-n}\Pi^q_{\leq k\delta,\leq kv_k}(x)\geq k^{-n}\frac{|\beta^1_k(x)|^2_{h^{L^k}}}{\|\beta^1_k\|^2_{h^{L^k}}}\to(2\pi)^{-n}\int_{\mathbb{R}_{x,q}\cap[-\delta,\delta]}|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)|ds. \tag{3.22}$$

The Proposition follows. ∎

From a simple modification of the proofs of Theorem 2.4 and Theorem 1.26, we get the following proposition

Proposition 3.4. *Let $v_k > 0$ be any sequence with $v_k \to 0$, as $k \to \infty$. Then there is a constant $C'_0 > 0$ independent of k such that $k^{-n}\Pi^q_{\leq k\delta,\leq kv_k}(x) \leq C'_0$, $\forall x \in X$, and*

$$\limsup_{k\to\infty}k^{-n}\Pi^q_{\leq k\delta,\leq kv_k}(x)\leq(2\pi)^{-n}\int_{\mathbb{R}_{x,q}\cap[-\delta,\delta]}|\det(\mathcal{R}^L_x+2s\mathcal{L}_x)|ds. \tag{3.23}$$

Combining Proposition 3.3 and Proposition 3.4, we get the conclusion of Theorem 1.28.

Proof of Theorem 3.2. We will use the same notations as in Theorem 3.2. Note that $\|\beta_k\|^2_{h^{L^k}} = \sum_{|m|\leq k\delta}\|\alpha_{k,m}\|^2_{h^{L^k}} \leq \|\alpha_k\|^2_{h^{L^k}}$. On canonical local coordinates $D =$

$\{(z,\theta):|z_j|<r,|\theta|<\pi,j=1,\cdots,n-1\}$, $\alpha_{k,m}$ can be expressed as following:

$$\begin{aligned}\alpha_{k,m}(z,\theta)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\alpha_k(z,t)e^{-imt}dte^{im\theta}\\&=\frac{1}{2\pi}k^{\frac{n}{2}}\int_{-\pi}^{\pi}u_k(z,t)e^{-imt}dte^{im\theta}\otimes s^k\\&=\frac{1}{2\pi}k^{\frac{n}{2}}\sum_{|J|=q}{}'\int_{-\pi}^{\pi}\chi_k(\sqrt{k}z)\tau_k(kt)u_J(\sqrt{k}z,kt)e^{-imt}dte^{im\theta}e^J(z)\otimes s^k\\&=\frac{1}{4\pi^2}k^{\frac{n}{2}}\int_{-\pi}^{\pi}\int_{|\eta|\le\delta}\chi_k(\sqrt{k}z)\hat{u}(\sqrt{k}z,\eta)\tau_k(kt)e^{-i(m-k\eta)t}dtd\eta e^{im\theta}\otimes s^k\\&=\frac{1}{4\pi^2}k^{\frac{n}{2}}\int_{|\eta|\le\delta}\chi_k(\sqrt{k}z)\hat{u}(\sqrt{k}z,\eta)\hat{\tau}\left((m-k\eta)\frac{\log k}{k}\right)\frac{\log k}{k}d\eta e^{im\theta}\otimes s^k.\end{aligned}\tag{3.24}$$

Assume that $|m|>k\delta$. Then $|m-k\eta|\neq 0$, for all $|\eta|\le\delta$. There exists a constant $C>0$ independent of k such that

$$\left|\hat{\tau}\left((m-k\eta)\frac{\log k}{k}\right)\right|\le\frac{C}{|m-k\eta|^3}\frac{k^3}{(\log k)^3}.\tag{3.25}$$

By Hölder inequality

$$\begin{aligned}&\int_{|\eta|\le\delta}|\hat{u}(\sqrt{k}z,\eta)|\frac{1}{|m-k\eta|^3}d\eta\\&\le\left(\int_{|\eta|\le\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta\right)^{\frac{1}{2}}\cdot\left(\int_{|\eta|\le\delta}\frac{1}{(m-k\eta)^6}d\eta\right)^{\frac{1}{2}}.\end{aligned}\tag{3.26}$$

Note that

$$\left(\int_{|\eta|\le\delta}\frac{1}{(m-k\eta)^6}d\eta\right)^{\frac{1}{2}}=\frac{1}{\sqrt{5k}}\left(\left[\frac{1}{(m-k\delta)^5}-\frac{1}{(m+k\delta)^5}\right]\right)^{\frac{1}{2}}.\tag{3.27}$$

From (3.24), (3.25), (3.26) and (3.27), we have

$$\begin{aligned}|\alpha_{k,m}(z,\theta)|_{h^{L^k}}\le\frac{C}{4\pi^2}k^{\frac{n}{2}}\chi_k(\sqrt{k}z)\frac{k^2}{(\log k)^2\sqrt{5k}}e^{-\frac{k\Phi(z)}{2}}\left(\int_{|\eta|\le\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta\right)^{\frac{1}{2}}\\\times\left[\frac{1}{(m-k\delta)^5}-\frac{1}{(m+k\delta)^5}\right]^{\frac{1}{2}}.\end{aligned}\tag{3.28}$$

Let $0 < \varepsilon < \frac{1}{3}$ be a small constant. Since

$$\begin{aligned}
&\sum_{|m|>k\delta+\frac{k}{(\log k)^{1+\varepsilon}}+1}\left[\frac{1}{(m-k\delta)^5}-\frac{1}{(m+k\delta)^5}\right]^{\frac{1}{2}}\\
&=2\sum_{m>k\delta+\frac{k}{(\log k)^{1+\varepsilon}}+1}\left[\frac{1}{(m-k\delta)^5}-\frac{1}{(m+k\delta)^5}\right]^{\frac{1}{2}}\\
&\leq 2\sum_{m>k\delta+\frac{k}{(\log k)^{1+\varepsilon}}+1}\left[\frac{1}{(m-k\delta)}\right]^{\frac{5}{2}}\\
&\leq 2\int_{\frac{k}{(\log k)^{1+\varepsilon}}}^{\infty}\frac{1}{\sigma^{\frac{5}{2}}}d\sigma=\frac{4}{3}\frac{(\log k)^{\frac{3}{2}(1+\varepsilon)}}{k^{\frac{3}{2}}},
\end{aligned} \tag{3.29}$$

we have

$$\begin{aligned}
&\Big|\sum_{|m|>k\delta+\frac{k}{(\log k)^{1+\varepsilon}}+1}\alpha_{k,m}(z,\theta)\Big|_{h^{L^k}}\\
&\leq \frac{C}{3\pi^2}k^{\frac{n}{2}}\chi_k(\sqrt{k}z)\frac{k^2}{(\log k)^2}\frac{1}{\sqrt{5k}}\frac{(\log k)^{\frac{3}{2}(1+\varepsilon)}}{k^{\frac{3}{2}}}e^{-\frac{k\Phi(z)}{2}}\left(\int_{|\eta|\leq\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta\right)^{\frac{1}{2}}\\
&\leq C_1k^{\frac{n}{2}}\chi_k(\sqrt{k}z)\frac{1}{(\log k)^{\frac{1-3\varepsilon}{2}}}e^{-\frac{k\Phi(z)}{2}}\left(\int_{|\eta|\leq\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta\right)^{\frac{1}{2}},
\end{aligned} \tag{3.30}$$

where $C_1 > 0$ is a constant independent of k. Let $\gamma_k(z,\theta) = \sum_{|m|>k\delta+\frac{k}{(\log k)^{1+\varepsilon}}+1}\alpha_{k,m}(z,\theta)$. Then

$$|\gamma_k(z,\theta)|^2_{h^{L^k}}\leq C_1k^n\chi_k^2(\sqrt{k}z)\frac{1}{(\log k)^{1-3\varepsilon}}e^{-k\Phi(z)}\int_{|\eta|\leq\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta. \tag{3.31}$$

For any $M > 0$,

$$\begin{aligned}
&\int_{|z|\leq\frac{\log k}{k}}\int_{|\theta|\leq\frac{M}{k}}|\gamma_k(z,\theta)|^2_{h^{L^k}}m(z)dv(z)d\theta\\
&\leq C_2\frac{M}{(\log k)^{1-3\varepsilon}}\int_{|z|\leq\log k}\int_{|\eta|\leq\delta}\chi_k^2(z)e^{-k\Phi(\frac{z}{\sqrt{k}})}|\hat{u}(z,\eta)|^2m(\frac{z}{\sqrt{k}})dv(z)d\theta.
\end{aligned} \tag{3.32}$$

Letting $k\to\infty$, we have

$$\lim_{k\to\infty}\int_{|z|\leq\frac{\log k}{\sqrt{k}}}\int_{|\theta|\leq\frac{M}{k}}|\gamma_k(z,\theta)|^2_{h^{L^k}}m(z)dv(z)d\theta=0. \tag{3.33}$$

On the other hand,

$$\begin{aligned}
|\alpha_{k,m}(z,\theta)|_{h^{L^k}} &\leq \frac{1}{2\pi}k^{\frac{n}{2}}\int_{-\pi}^{\pi}\int_{|\eta|\leq\delta}\chi_k(\sqrt{k}z)\tau_k(kt)|\hat{u}(\sqrt{k}z,\eta)|dtd\eta e^{-\frac{k\Phi(z)}{2}} \\
&\leq \frac{1}{2\pi}k^{\frac{n}{2}}\int_{|\eta|\leq\delta}|\hat{u}(\sqrt{k}z,\eta)|d\eta\chi_k(\sqrt{k}z)\left(\frac{\log k}{k}\right)d\eta e^{-\frac{k\Phi(z)}{2}} \\
&\leq Ck^{\frac{n}{2}}\left(\frac{\log k}{k}\right)\left(\int_{|\eta|\leq\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta\right)^{\frac{1}{2}}d\eta\chi_k(\sqrt{k}z)e^{\frac{-k\Phi(z)}{2}},
\end{aligned} \tag{3.34}$$

where $C>0$ is a constant independent of k. Let $\sigma_k = \sum_{k\delta<|m|\leq k\delta+\frac{k}{(\log k)^{1+\varepsilon}}}\alpha_{k,m}(z,\theta)$.

Then

$$\begin{aligned}
|\sigma_k|_{h^{L^k}} &\leq C\frac{\log k}{k}\frac{k}{(\log k)^{1+\varepsilon}}k^{\frac{n}{2}}\left(\int_{|\eta|\leq\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta\right)^{\frac{1}{2}}\chi_k(\sqrt{k}z)e^{\frac{-k\Phi(z)}{2}} \\
&\leq C\frac{1}{(\log k)^{\varepsilon}}k^{\frac{n}{2}}\left(\int_{|\eta|\leq\delta}|\hat{u}(\sqrt{k}z,\eta)|^2d\eta\right)^{\frac{1}{2}}\chi_k(\sqrt{k}z)e^{-\frac{k\Phi(z)}{2}}.
\end{aligned} \tag{3.35}$$

From (3.35), we can check that

$$\lim_{k\to\infty}\int_{|z|\leq\frac{\log k}{\sqrt{k}},|\theta|\leq\frac{M}{k}}|\sigma_k|^2m(z)dv(z)=0,\quad\forall M>0. \tag{3.36}$$

Write $\alpha_k=\beta_k+\gamma_k+\sigma_k$. Here, $\beta_k=\sum_{|m|\leq k\delta}\alpha_{k,m}$. Then

$$\begin{aligned}
&\int_{|z|\leq\frac{\log k}{\sqrt{k}},|\theta|\leq\frac{M}{k}}|\beta_k|^2_{h^{L^k}}m(z)dv(z)d\theta \\
&=\int_{|z|\leq\frac{\log k}{\sqrt{k}},|\theta|\leq\frac{M}{k}}|\alpha_k-\gamma_k-\sigma_k|^2_{h^{L^k}}m(z)dv(z)d\theta.
\end{aligned} \tag{3.37}$$

Since

$$\begin{aligned}
&\lim_{k\to\infty}\int_{|z|\leq\frac{\log k}{\sqrt{k}},|\theta|\leq\frac{M}{k}}|\alpha_k|^2_{h^{L^k}}m(z)dv(z)d\theta \\
&=\int_{\mathbb{C}^{n-1}\times\{\theta\in\mathbb{R}:|\theta|\leq M\}}|u(z,\theta)|^2e^{-\Phi_0(z)}dv(z)d\theta
\end{aligned} \tag{3.38}$$

and $\|u\|^2=1$, for any $\epsilon>0$, we can choose a constant $M>0$ such that

$$\int_{\mathbb{C}^{n-1}\times\{\theta\in\mathbb{R}:|\theta|\leq M\}}|u(z,\theta)|^2e^{-\Phi_0(z)}dv(z)d\theta\geq 1-\epsilon. \tag{3.39}$$

From (3.33), (3.36), (3.37), (3.38) and (3.39), we deduce that

$$\liminf_{k\to\infty}\|\beta_k\|^2_{h^{L^k}}\geq\lim_{k\to\infty}\int_{|z|\leq\frac{\log k}{\sqrt{k}},|\theta|\leq\frac{M}{k}}|\beta_k|^2_{h^{L^k}}m(z)dv(z)d\theta\geq 1-\epsilon,\quad\forall\epsilon>0. \tag{3.40}$$

Thus,

$$\liminf_{k\to\infty} \|\beta_k\|^2_{h^{L^k}} \geq 1.$$

On the other hand, $\|\beta_k\|^2_{h^{L^k}} \leq \|\alpha_k\|^2_{h^{L^k}} \leq 1$, then we have

$$\lim_{k\to\infty} \|\beta_k\|^2_{h^{L^k}} = 1. \tag{3.41}$$

Proof of (2) in Theorem 3.2. Recall that $\alpha_k = \beta_k + \gamma_k + \sigma_k$. From (3.31), we have

$$k^{-n}|\gamma_k(0,0)|^2_{h^{L^k}} \leq C_1 \frac{1}{(\log k)^{1-3\varepsilon}} \int_{|\eta|\leq\delta} |\hat{u}(0,\eta)|^2 d\eta. \tag{3.42}$$

Then

$$\lim_{k\to\infty} k^{-n}|\gamma_k(0,0)|^2_{h^{L^k}} = 0. \tag{3.43}$$

From (3.35), we have

$$k^{-n}|\sigma_k(0,0)|^2_{h^{L^k}} \leq \int_{|\eta|\leq\delta} |\hat{u}(0,\eta)|^2 d\eta \frac{1}{(\log k)^{2\varepsilon}}. \tag{3.44}$$

Then

$$\lim_{k\to\infty} k^{-n}|\sigma_k(0,0)|^2_{h^{L^k}} = 0. \tag{3.45}$$

Combining (3.9), (3.43) and (3.45) we get the conclusion of the second part of Theorem 3.2.

Proof of (3) in Theorem 3.2. $\beta_k = F_{\delta,k}\alpha_k$. Since $\alpha_k = F_{\delta,k}\alpha_k + (I - F_{\delta,k})\alpha_k$. Since $\Box^{(q)}_{b,k} F^{(q)}_{\delta,k} = F^{(q)}_{\delta,k}\Box^{(q)}_{b,k}$, then

$$\Box^{(q)}_{b,k}\alpha_k = \Box^{(q)}_{b,k}F_{\delta,k}\alpha_k + (I - F_{\delta,k})\Box^{(q)}_{b,k}\alpha_k \tag{3.46}$$

and

$$\begin{aligned} \left(\frac{1}{k}\Box^{(q)}_{b,k}\beta_k | \beta_k\right)_{h^{L^k}} &= \left(\frac{1}{k}\Box^{(q)}_{b,k}\alpha_k | F_{\delta,k}\alpha_k\right)_{h^{L^k}} = \left(F_{\delta,k}\frac{1}{k}\Box^{(q)}_{b,k}\alpha_k | F_{\delta,k}\alpha_k\right)_{h^{L^k}} \\ &\leq \left(\frac{1}{k}\Box^{(q)}_{b,k}\alpha_k | \alpha_k\right)_{h^{L^k}} \leq \mu_k. \end{aligned} \tag{3.47}$$

for some μ_k tending to zero. Similarly, we can repeat the procedure above and get (4) in Theorem 3.2. Thus, we get the conclusion of Theorem of 3.2. □

4. Examples. In this section, some examples are collected. The aim is to illustrate the main results in some simple situations.

4.1. CR manifolds in projective spaces. We consider $\mathbb{CP}^{N-1}$, $N \geq 3$. Let $[z] = [z_1, \ldots, z_N]$ be the homogeneous coordinates of $\mathbb{CP}^{N-1}$. Put

$$X := \left\{[z_1, \ldots, z_N] \in \mathbb{CP}^{N-1};\ \lambda_1 |z_1|^2 + \cdots + \lambda_m |z_m|^2 + \cdots + \lambda_N |z_N|^2 = 0\right\},$$

where $m \in \mathbb{N}$ and $\lambda_j \in \mathbb{R}$ $j = 1, \ldots, N$. We assume that $\lambda_1 < 0, \ldots, \lambda_m < 0$, $\lambda_{m+1} > 0, \lambda_{m+2} > 0, \ldots, \lambda_N > 0$. Then X is a compact CR manifold of dimension $2(N-1)-1$ with CR structure $T^{1,0}X := T^{1,0}\mathbb{CP}^{N-1} \bigcap \mathbb{C}TX$. X admits a S^1 action:

$$\begin{gathered} S^1 \times X \to X, \\ e^{i\theta} \circ [z_1, \ldots, z_m, z_{m+1}, \ldots, z_N] \to [e^{i\theta} z_1, \ldots, e^{i\theta} z_m, z_{m+1}, \ldots, z_N], \quad \theta \in [-\pi, \pi). \end{gathered} \tag{4.1}$$

Since $(z_1, \ldots, z_m) \neq 0$ on X, this S^1 action is well-defined. Moreover, it is straightforward to check that this S^1 action is CR and transversal. Let T be the global vector field induced by the S^1 action.

Let $E \to \mathbb{CP}^{N-1}$ be the hyperplane line bundle with respect to the Fubini-Study metric. For $j = 1, 2, \ldots, N$, put $W_j = \big\{[z_1, \ldots, z_N] \in \mathbb{CP}^{N-1};\ z_j \neq 0\big\}$. Then, E is trivial on W_j, $j = 1, \ldots, N$, and we can find local trivializing section e_j of E on W_j, $j = 1, \ldots, N$, such that for every $j, t = 1, \ldots, N$,

$$e_j(z) = \frac{z_j}{z_t} e_t(z) \ \text{ on } W_j \bigcap W_t, \quad z = [z_1, \ldots, z_N] \in W_j \bigcap W_t. \tag{4.2}$$

Consider $L := E|_X$. Then, L is a CR line bundle over $(X, T^{1,0}X)$. It is easy to see that X can be covered with open sets $U_j := W_j|_X$, $j = 1, 2, \ldots, m$, with trivializing sections $s_j := e_j|_X$, $j = 1, 2, \ldots, m$, such that the corresponding transition functions are rigid CR functions. Thus, L is a rigid CR line bundle over $(X, T^{1,0}X)$. Let h^L be the Hermitian fiber metric on L given by

$$|s_j(z_1, \ldots, z_N)|^2_{h^L} := e^{-\log\left(\frac{|z_1|^2 + \cdots + |z_N|^2}{|z_j|^2}\right)}, \quad j = 1, \ldots, m.$$

It is not difficult to check that h^L is well-defined and h^L is a rigid positive CR line bundle. From this and Theorem 1.7, we conclude that L is a big line bundle over X.

4.2. Compact Heisenberg groups. Let $\lambda_1, \ldots, \lambda_{n-1}$ be given non-zero integers. Let $\mathscr{C}H_n = (\mathbb{C}^{n-1} \times \mathbb{R})/_\sim$, where $(z, t) \sim (\widetilde{z}, \widetilde{t})$ if

$$\begin{aligned} &\widetilde{z} - z = (\alpha_1, \ldots, \alpha_{n-1}) \in \sqrt{2\pi}\mathbb{Z}^{n-1} + i\sqrt{2\pi}\mathbb{Z}^{n-1}, \\ &\widetilde{t} - t - i\sum_{j=1}^{n-1} \lambda_j (z_j \overline{\alpha}_j - \overline{z}_j \alpha_j) \in 2\pi\mathbb{Z}. \end{aligned}$$

We can check that $\sim$ is an equivalence relation and $\mathscr{C}H_n$ is a compact manifold of dimension $2n-1$. The equivalence class of $(z, t) \in \mathbb{C}^{n-1} \times \mathbb{R}$ is denoted by $[(z, t)]$. For a given point $p = [(z, t)]$, we define $T^{1,0}_p\mathscr{C}H_n$ to be the space spanned by

$$\big\{\tfrac{\partial}{\partial z_j} + i\lambda_j \overline{z}_j \tfrac{\partial}{\partial t}, \ \ j = 1, \ldots, n-1\big\}.$$

It is easy to see that the definition above is independent of the choice of a representative (z, t) for $[(z, t)]$. Moreover, we can check that $T^{1,0}\mathscr{C}H_n$ is a CR structure. $\mathscr{C}H_n$ admits the natural S^1 action: $e^{i\theta} \circ [z, t] \to [z, t + \theta]$, $0 \leq \theta < 2\pi$. Let T be the global vector field induced by this S^1 action. We can check that this S^1 action is CR and transversal and $T = \frac{\partial}{\partial t}$. We take a Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on the complexified tangent bundle $\mathbb{C}T\mathscr{C}H_n$ such that

$$\Big\{ \tfrac{\partial}{\partial z_j} + i\lambda_j \overline{z}_j \tfrac{\partial}{\partial t}\,, \tfrac{\partial}{\partial \overline{z}_j} - i\lambda_j z_j \tfrac{\partial}{\partial t}\,, -\tfrac{\partial}{\partial t}\,; j = 1, \ldots, n-1 \Big\}$$

is an orthonormal basis. The dual basis of the complexified cotangent bundle is

$$\Big\{dz_j\,,\,d\overline{z}_j\,,\,\omega_0:=-dt+\textstyle\sum_{j=1}^{n-1}(i\lambda_j\overline{z}_jdz_j-i\lambda_jz_jd\overline{z}_j);j=1,\ldots,n-1\Big\}.$$

The Levi form $\mathcal{L}_p$ of $\mathscr{C}H_n$ at $p\in\mathscr{C}H_n$ is given by $\mathcal{L}_p=\sum_{j=1}^{n-1}\lambda_jdz_j\wedge d\overline{z}_j$.

Now, we construct a rigid CR line bundle L over $\mathscr{C}H_n$. Let $L=(\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C})/_{\equiv}$ where $(z,\theta,\eta)\equiv(\widetilde{z},\widetilde{\theta},\widetilde{\eta})$ if

$$\begin{aligned}&(z,\theta)\sim(\widetilde{z},\widetilde{\theta}),\\&\widetilde{\eta}=\eta\exp(\sum_{j,t=1}^{n-1}\mu_{j,t}(z_j\overline{\alpha}_t+\frac{1}{2}\alpha_j\overline{\alpha}_t)),\end{aligned}$$

where $\alpha=(\alpha_1,\ldots,\alpha_{n-1})=\widetilde{z}-z$, $\mu_{j,t}=\mu_{t,j}$, $j,t=1,\ldots,n-1$, are given integers. We can check that $\equiv$ is an equivalence relation and L is a rigid CR line bundle over $\mathscr{C}H_n$. For $(z,\theta,\eta)\in\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C}$, we denote $[(z,\theta,\eta)]$ its equivalence class. It is straightforward to see that the pointwise norm

$$\big|[(z,\theta,\eta)]\big|^2_{h^L}:=|\eta|^2\exp\big(-\textstyle\sum_{j,t=1}^{n-1}\mu_{j,t}z_j\overline{z}_t\big)$$

is well defined. In local coordinates (z,θ,η), the weight function of this metric is

$$\phi=\sum_{j,t=1}^{n-1}\mu_{j,t}z_j\overline{z}_t.$$

Thus, L is a rigid CR line bundle over $\mathscr{C}H_n$ with rigid Hermitian metric h^L. Note that

$$\overline{\partial}_b=\textstyle\sum_{j=1}^{n-1}d\overline{z}_j\wedge(\frac{\partial}{\partial\overline{z}_j}-i\lambda_jz_j\frac{\partial}{\partial\theta}),\quad\partial_b=\sum_{j=1}^{n-1}dz_j\wedge(\frac{\partial}{\partial z_j}+i\lambda_j\overline{z}_j\frac{\partial}{\partial\theta}).$$

Thus $d(\overline{\partial}_b\phi-\partial_b\phi)=2\sum_{j,t=1}^{n-1}\mu_{j,t}dz_j\wedge d\overline{z}_t$ and for any $p\in\mathscr{C}H_n$,

$$\mathcal{R}^L_p=\sum_{j,t=1}^{n-1}\mu_{j,t}dz_j\wedge d\overline{z}_t.$$

From this and Theorem 1.7, we conclude that

THEOREM 4.1. *If $(\mu_{j,t})_{j,t=1}^{n-1}$ is positive definite, then L is a big line bundle on $\mathscr{C}H_n$.*

4.3. Holomorphic line bundles over a complex torus. Let

$$T_n:=\mathbb{C}^n/(\sqrt{2\pi}\mathbb{Z}^n+i\sqrt{2\pi}\mathbb{Z}^n)$$

be the flat torus. Let $\lambda=(\lambda_{j,t})_{j,t=1}^n$, where $\lambda_{j,t}=\lambda_{t,j}$, $j,t=1,\ldots,n$, are given integers. Let L_λ be the holomorphic line bundle over T_n with curvature the $(1,1)$-form $\Theta_\lambda=\sum_{j,t=1}^n\lambda_{j,t}dz_j\wedge d\overline{z}_t$. More precisely, $L_\lambda:=(\mathbb{C}^n\times\mathbb{C})/_\sim$, where $(z,\theta)\sim(\widetilde{z},\widetilde{\theta})$ if

$$\widetilde{z}-z=(\alpha_1,\ldots,\alpha_n)\in\sqrt{2\pi}\mathbb{Z}^n+i\sqrt{2\pi}\mathbb{Z}^n\,,\quad\widetilde{\theta}=\exp\big(\textstyle\sum_{j,t=1}^n\lambda_{j,t}(z_j\overline{\alpha}_t+\frac{1}{2}\alpha_j\overline{\alpha}_t)\big)\theta\,.$$

We can check that $\sim$ is an equivalence relation and L_λ is a holomorphic line bundle over T_n. For $[(z,\theta)] \in L_\lambda$ we define the Hermitian metric by

$$\left|[(z,\theta)]\right|^2 := |\theta|^2 \exp(-\textstyle\sum_{j,t=1}^n \lambda_{j,t} z_j \overline{z}_t)$$

and it is easy to see that this definition is independent of the choice of a representative (z,θ) of $[(z,\theta)]$. We denote by $\phi_\lambda(z)$ the weight of this Hermitian fiber metric. Note that $\partial\overline{\partial}\phi_\lambda = \Theta_\lambda$.

Let L_λ^* be the dual bundle of L_λ and let $\|\cdot\|_{L_\lambda^*}$ be the norm of L_λ^* induced by the Hermitian fiber metric on L_λ. Consider the compact CR manifold of dimension $2n+1$: $X = \{v \in L_\lambda^* : \|v\|_{L_\lambda^*} = 1\}$; this is the boundary of the Grauert tube associated to L_λ^*. The manifold X is equipped with a natural S^1-action. Locally X can be represented in local holomorphic coordinates (z,η), where η is the fiber coordinate, as the set of all (z,η) such that $|\eta|^2 e^{\phi_\lambda(z)} = 1$. The S^1-action on X is given by $e^{i\theta} \circ (z,\eta) = (z, e^{i\theta}\eta)$, $e^{i\theta} \in S^1$, $(z,\eta) \in X$. Let T be the global vector field on X induced by this S^1 action. We can check that this S^1 action is CR and transversal.

Let $\pi : L_\lambda^* \to T_n$ be the natural projection from L_λ^* onto T_n. Let $\mu = (\mu_{j,t})_{j,t=1}^n$, where $\mu_{j,t} = \mu_{t,j}$, $j,t = 1,\ldots,n$, are given integers. Let L_μ be another holomorphic line bundle over T_n determined by the constant curvature form $\Theta_\mu = \sum_{j,t=1}^n \mu_{j,t} dz_j \wedge d\overline{z}_t$ as above. The pullback line bundle $\pi^* L_\mu$ is a holomorphic line bundle over L_λ^*. If we restrict $\pi^* L_\mu$ on X, then we can check that $\pi^* L_\mu$ is a rigid CR line bundle over X.

The Hermitian fiber metric on L_μ induced by ϕ_μ induces a Hermitian fiber metric on $\pi^* L_\mu$ that we shall denote by $h^{\pi^* L_\mu}$. We let ψ to denote the weight of $h^{\pi^* L_\mu}$. The part of X that lies over a fundamental domain of T_n can be represented in local holomorphic coordinates (z,ξ), where ξ is the fiber coordinate, as the set of all (z,ξ) such that $r(z,\xi) := |\xi|^2 \exp(\sum_{j,t=1}^n \lambda_{j,t} z_j \overline{z}_t) - 1 = 0$ and the weight ψ may be written as $\psi(z,\xi) = \sum_{j,t=1}^n \mu_{j,t} z_j \overline{z}_t$. For convenient we denote $\pi^* L_\mu$ by L. From this we see that L is a $T-$rigid CR line bundle over X with rigid Hermitian fiber metric h^L. It is straightforward to check that for any $p \in X$, we have $\mathcal{R}_p^L = \frac{1}{2} d(\overline{\partial}_b \psi - \partial_b \psi)(p)|_{T^{1,0}X} = \sum_{j,t=1}^n \mu_{j,t} dz_j \wedge d\overline{z}_t$. Thus, if $(\mu_{j,t})_{j,t=1}^{n-1}$ is positive definite, then L is a rigid positive CR line bundle. From this and Theorem 1.7, we conclude that

THEOREM 4.2. *If $(\mu_{j,t})_{j,t=1}^{n-1}$ is positive definite, then L is a big line bundle over X.*

Acknowledgement. The authors would like to thank the referee for many detailed remarks that have helped to improve the presentation.

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