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INTERPOLATION FOR CURVES OF LARGE DEGREE*

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Dedicated to Professor Ngaiming Mok on the occasion of his sixtieth birthday

Abstract. In this paper, we establish an interpolation result involving higher-order conormal bundles of curves embedded by linear systems of large degree. As a consequence this gives evidence for the semistability conjecture due to Ein-Lazarsfeld.

Key words. Interpolation, semistablility, conomral bundle.

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1. Introduction. Throughout this paper we work over an algebraically closed field of characteristic zero. Let C be a nonsingular irreducible projective curve of genus $g \geq 0$ embedded by a complete linear system of a line bundle L into a projective space

$$\phi : C \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(L)).$$

For each nonnegative integer k, denote by $P^k(L)$ the bundle of k-th order principal part of L. When $k \leq \deg L - 2g$, lifting global sections of L to $P^k(L)$ induces a short exact sequence of vector bundles

$$0 \longrightarrow R^k(L) \longrightarrow H^0(L) \otimes \mathscr{O}_C \longrightarrow P^k(L) \longrightarrow 0.$$

The bundles $R^k(L)$ can be thought of as higher-order conormal bundles of C governing the geometry of the embedding ϕ. For instance, $R^0(L)^* \otimes L$ is the restricted tangent bundle $T_{\mathbb{P}^n}|_C$ and $R^1(L)^* \otimes L$ is the normal bundle $N_{C/\mathbb{P}^n}$ of C in $\mathbb{P}^n$.

The vector bundle $R^k(L)$ was studied by Lazarsfeld and the first author in [EL92], where they raised the following conjecture.

CONJECTURE 1.1 ([EL92, 4.2]). *There is an integer $d(g, k)$ such that the conormal bundle $R^k(L)$ is semistable for* $\deg L \geq d(g, k)$.

This has only been proved for $g = 0, 1$ and, as far as we know, has no known evidence for higher genus. Indeed, the case of $g = 0$ is trivial and when $g = 1$, $R^k(L)$ can be realized as the pullback of a Picard bundle under the étale morphism of C to the Jacobian. The desired semistability then follows from the result established in [EL92] that the Picard bundle is stable. However, this method fails for higher genus and the conjecture is widely open even for $k = 1$.

Recently, a couple of notions of interpolation for vector bundles on curves were introduced in [Ata]. Several results related to interpolation for normal bundles and restricted tangent bundles of general curves have been proved in [ALY], [Bal17] and [Lar]. It would be natural to understand interpolation for all $R^k(L)$ bundles on arbitrary curves when $\deg L$ is large. It turns out that the picture is quite clear in this case, as shown in the following main theorem of this paper.

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Theorem 1.2. *Let L be a very ample line bundle on a nonsingular projective curve C of genus $g \geq 0$. If there is a nonnegative integer k such that $\deg L \geq (k^2+2k+2)g+k$, then the vector bundle $R^k(L)^* \otimes L$ satisfies interpolation.*

As an immediate corollary, we have the interpolation for the restricted tangent bundle and the normal bundle for arbitrary curves of large degree.

Corollary 1.3. *Let L be a very ample line bundle on a nonsingular projective curve C of genus $g \geq 0$ and its complete linear system defines an embedding*

$$\phi : C \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(L)).$$

Then one has the following.

(1) *If $\deg L \geq 2g$, the restricted tangent bundle $T_{\mathbb{P}^n}|_C$ satisfies interpolation.*
(2) *If $\deg L \geq 5g+1$, the normal bundle $N_{C/\mathbb{P}^n}$ satisfies interpolation.*

There is a connection established in [Ata] between semistability and interpolation. Therefore, as another corollary, we obtain the following evidence for Conjecture 1.1.

Corollary 1.4. *Let L be a very ample line bundle on a nonsingular projective curve C of genus $g \geq 0$. If there is a nonnegative integer k such that $\deg L = (k^2+2k+2)g+k$ then the vector bundle $R^k(L)$ is semistable.*

The corollary also suggests that the potential value for the bound $d(g,k)$ in Conjecture 1.1 might be expected as $d(g,k) = (k^2+2k+2)g+k$.

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2. Interpolation and principal parts of vector bundles. In this section, we review the notion of interpolation introduced in [Ata] as well as the definition of the principal parts of vector bundles on curves.

Given a vector bundle E on a projective nonsingular curve C, we write $H^i(E)$ for the cohomology group $H^i(C,E)$ and $h^i(E) = \dim_k H^i(E)$. If $S = x_1 + \cdots + x_q$ is an effective divisor, we write $E|_S = E \otimes \mathcal{O}_S$ as the restriction of E onto S. In particular, if $x \in C$ is a closed point $E|_x = E \otimes k(x)$ where $k(x)$ is the residual field of x. The slope $\mu(E)$ is defined by $\deg E/\operatorname{rank} E$. E is called semistable if $\mu(F) \leq \mu(E)$ for any subbundle $F \subseteq E$.

Definition 2.1. Let C be a nonsingular projective curve and let E be a vector bundle on C. Suppose that

$$h^0(E) = q \cdot \operatorname{rank} E + t, \qquad \text{with } 0 \leq t < \operatorname{rank} E.$$

E is said to satisfy interpolation if there exist $q+1$ distinct points $x_1 \cdots, x_q, x$ and a vector subspace $V \subseteq E|_x$ of codimension t such that the restriction morphism

$$H^0(E) \longrightarrow E|_S \oplus E|_x/V$$

is surjective, where $S = x_1 + \cdots + x_q$.

Remark 2.2. What we adopted here is called regular interpolation in [Ata, Definition 3.3]. In particular, Atanasov also proved that regular interpolation is equivalent to what he called strongly interpolation [Ata, Theorem 8.1]. By semicontinuity, one can actually choose the points x_i and x in the definition as general points.

Remark 2.3. It is also easy to see that E satisfies interpolation if and only if for every $m \geq 1$, there is a general effective divisor Z of degree m such that the restriction morphism

$$H^0(E) \longrightarrow E|_Z$$

has maximal rank.

If the vector bundle E is nonspecial, i.e., $h^1(E) = 0$, then one can verify interpolation by the following observation proved by several authors.

Proposition 2.4 ([Bal17, Lemma 1], [ALY, Proposition 4.5]). *Let E be a nonspecial vector bundle on a nonsingular curve C such that*

$$h^0(E) = q \cdot \operatorname{rank} E + t, \ \textit{with } 0 \leq t \leq \operatorname{rank} E.$$

Then E satisfies interpolation if and only if there exist general effective divisors S of degree q and S' of degree $q+1$ respectively such that

$$h^1(E(-S)) = 0 \ \textit{and} \ h^0(E(-S')) = 0.$$

Under certain conditions, the interpolation property will imply semistability of vector bundles. More precisely, let F be a subbundle of E and write $h^0(E) = q \cdot \operatorname{rank} E + t$ with $0 \leq t < \operatorname{rank} E$. It was showed in [Ata, Proposition 3.14] that

$$\frac{h^0(F)}{\operatorname{rank} F} \leq \frac{h^0(E)}{\operatorname{rank} E} + \min\left\{1, \frac{t}{\operatorname{rank} F}\right\} - \frac{t}{\operatorname{rank} E}.$$

Hence if E is nonspecial and $t = 0$ one immediately deduces that

$$\frac{\chi(F)}{\operatorname{rank} F} \leq \frac{\chi(E)}{\operatorname{rank} E},$$

which implies that E is semistable. We summarize this fact in the following proposition.

Proposition 2.5 ([Ata, Corollary 3.15]). *Let E be a nonspecial vector bundle on a nonsingular curve C such that* $\operatorname{rank} E$ *divides $h^0(E)$. If E satisfies interpolation then E is semistable.*

Example 2.6. In general, interpolation does not imply semistability. For example, consider a vector bundle $E = \mathscr{O}_C(2P) \oplus \mathscr{O}_C(3P)$ on an elliptic curve C, where P is a point. Then E satisfies interpolation but is not semistable.

Next we recall the notion of the principal parts of a line bundle. Let $\Delta \subset C \times C$ be the diagonal with the ideal sheaf I_Δ. Consider the diagram

$$\begin{array}{ccccc} \Delta & \hookrightarrow & C \times C & \xrightarrow{q} & C \\ & & \big\downarrow{\scriptstyle p} & & \\ & & C & & \end{array} \qquad (2.6.1)$$

where the morphism p and q are natural projections. For a line bundle $\mathscr{L}$ on C and an integer $k \geq 0$, the k-th order principal part of $\mathscr{L}$ is defined by

$$P^k(\mathscr{L}) = p_*(q^*\mathscr{L} \otimes \frac{\mathscr{O}_{C\times C}}{I_\Delta^{k+1}}).$$

It is easy to see that $P^k(\mathscr{L})$ is a vector bundle of rank $k+1$. Directly from the definition, we have the following simple observation.

PROPOSITION 2.7. *Consider two line bundles $\mathscr{L}$ and $\mathscr{L}'$ on a curve, then one has*

$$H^0(P^k(\mathscr{L}) \otimes \mathscr{L}') = H^0(P^k(\mathscr{L}') \otimes \mathscr{L})$$

and the natural morphisms

$$H^0(\mathscr{L}) \otimes H^0(\mathscr{L}') \longrightarrow H^0(P^k(\mathscr{L}) \otimes \mathscr{L}')$$

and

$$H^0(\mathscr{L}) \otimes H^0(\mathscr{L}') \longrightarrow H^0(P^k(\mathscr{L}') \otimes \mathscr{L})$$

are the same.

Proof. All cohomology groups in the proposition can be seen on the product $C \times C$. One can identify the two natural morphisms as

$$H^0(p^*\mathscr{L} \otimes q^*\mathscr{L}') \longrightarrow H^0(p^*\mathscr{L} \otimes q^*\mathscr{L}' \otimes \frac{\mathscr{O}_{C\times C}}{I_\Delta^{k+1}})$$

But the Kuneth formula gives $H^0(p^*\mathscr{L} \otimes q^*\mathscr{L}') = H^0(p^*\mathscr{L}) \otimes H^0(q^*\mathscr{L}')$. Then the result follows from the projection formula. □

Recall that a line bundle $\mathscr{L}$ on a projective variety X separates k-jets at a non-singular closed point $x \in X$ if the natural morphism

$$H^0(\mathscr{L}) \longrightarrow \mathscr{L} \otimes \mathscr{O}_X/\mathfrak{m}_x^{k+1}$$

is surjective, where $\mathfrak{m}_x \subset \mathscr{O}_X$ is the maximal ideal defining the point x. For further information on separation of jets, we refer to [Laz04, Chapter 5].

PROPOSITION 2.8. *Let $\mathscr{L}$ be a globally generated line bundle on a curve C. Let $\mathscr{Q}$ be the cokernel of the canonical morphism*

$$H^0(\mathscr{L}) \otimes \mathscr{O}_C \longrightarrow P^k(\mathscr{L}).$$

Then one has

$$\mathrm{Supp}(\mathscr{Q}) = \{x \in C \mid \mathscr{L} \textit{ does not separate } k\textit{-jets at } x\}$$

In particular, $\mathscr{Q} = 0$ if and only if $\mathscr{L}$ separates k-jets at any $x \in C$.

Proof. We use the diagram (2.6.1) in the proof. The question is local and hence the morphism is surjective at the point $x \in C$ if and only if the morphism

$$H^0(\mathscr{L}) \otimes k(x) \longrightarrow P^k(\mathscr{L}) \otimes k(x),$$

is surjective. But by base change, if we write $C_x = p^{-1}(x)$, we see that the above morphism is the same as

$$H^0(\mathscr{L}) \longrightarrow \mathscr{L} \otimes \mathscr{O}_{C_x}/\mathfrak{m}_x^{k+1}$$

where $\mathfrak{m}_x$ is the defining ideal of x in C_x (cutting by the diagonal). Then the result is clear. ☐

REMARK 2.9. It is clear that $\mathscr{L}$ separates 0-jets if and only if it is base-point-free. $\mathscr{L}$ separates 1-jets if and only if the complete linear system $|\mathscr{L}|$ defines a unramified morphism

$$\phi_{|\mathscr{L}|} : C \longrightarrow \mathbb{P}(H^0(\mathscr{L})).$$

PROPOSITION 2.10. *Let C be a nonsingular curve of genus $g \geq 0$. For $k \geq 0$, one has the following results:*

(1) *A line bundle of degree $\geq 2g + k$ separates k-jets at any point x of C.*
(2) *A general line bundle of degree $\geq g + k + 1$ separates k-jets at any point x of C.*
(3) *A general line bundle of degree $\geq g + k$ separates k-jets at some point x of C.*

Proof. If $g = 0$, then the results are trivial even for arbitrary line bundles instead of general ones. So in the sequel we assume $g \geq 1$.

(1) Let $\mathscr{L}$ be a line bundle on C of degree $\geq 2g + k$. Let $x \in C$ be a point. Since $\deg \mathscr{L}(-(k+1)x) \geq 2g - 1$, we see that $h^1(\mathscr{L}(-(k+1)x)) = 0$ which means that $\mathscr{L}$ separates k-jets at x.

(2) Let $\mathscr{L}$ be a general line bundle on C of degree $\geq g + k + 1$. For a point $x \in C$, it is sufficient to show the vanishing $H^1(\mathscr{L}(-(k+1)x)) = 0$, which by duality is equivalent to the vanishing $H^0(\omega_C \otimes \mathscr{L}^*((k+1)x)) = 0$. Notice that $d = \deg \omega_C \otimes \mathscr{L}^*((k+1)x) \leq g - 2$. We consider the following morphism

$$\alpha_d : C_d \times C \longrightarrow \mathrm{Pic}^{d-(k+1)}(C)$$

which maps the pair (D, x) to $\mathscr{O}_C(D - (k+1)x)$. Notice that the image of α_d contains all such line bundles A of degree $d - (k+1)$ that $A((k+1)x)$ is effective for some point x. But $\dim C_d \times C \leq g - 1$. Hence a general choice of $\mathscr{L}$ will make the line bundles $\omega_C \otimes \mathscr{L}^*$ out of the image of α_d. Indeed, consider the isomorphism

$$\beta : \mathrm{Pic}^{d-(k+1)-(2g-2)}(C) \longrightarrow \mathrm{Pic}^{d-(k+1)}(C)$$

defined as $\beta(\mathscr{R}) = \mathscr{R}^* \otimes \omega_C$ for $\mathscr{R} \in \mathrm{Pic}^{d-(k+1)-(2g-2)}(C)$. Then we can simply choose $\mathscr{L}$ not in $\beta^{-1}(\mathrm{Im}(\alpha_d))$.

(3) Let $\mathscr{L}$ be a general line bundle on C of degree $\geq g + k$. If $\deg \mathscr{L} > g + k$, then we can use (2). Thus in the sequel, we assume that $\deg \mathscr{L} = g + k$. Consider the morphism

$$v : \mathrm{Pic}^{g+k}(C) \times C \longrightarrow \mathrm{Pic}^{g-1}(C)$$

which maps $(\mathscr{R}, x) \in \mathrm{Pic}^{g+k} \times C$ to $\mathscr{R}(-(k+1)x) \in \mathrm{Pic}^{g-1}(C)$. Also consider the image $\mathrm{Im}\, u$ of the canonical morphism $u : C_{g-1} \longrightarrow \mathrm{Pic}^{g-1}(C)$. Clearly, $\dim \mathrm{Im}\, u < g$. Let $U = \mathrm{Pic}^{g-1}(C) - \mathrm{Im}\, u$. Then $v^{-1}(U)$ is an open set of $\mathrm{Pic}^{g+k}(C) \times C$. We project this open set to $\mathrm{Pic}^{g+k}(C)$ to obtain an open set W. Then if we take $\mathscr{L} \in W$, the construction of v shows that there exists $x \in C$ such that $\mathscr{L}(-(k+1)x)$ is not in the image of u. This means that $H^0(\mathscr{L}(-(k+1)x)) = 0$ and therefore $\mathscr{L}$ separates k-jets at x. ☐

3. Main theorem. In this section, we prove our main theorem. Recall that L is a very ample nonspecial line bundle (i.e., $h^1(L) = 0$) of degree d on a nonsingular projective curve C of genus $g \geq 0$. The complete linear system $|L|$ defines an embedding

$$\phi : C \longrightarrow \mathbb{P}^n = \mathbb{P}(H^0(L)), \text{ where } n = d - g.$$

For a nonnegative integer k, the global sections of L lift canonically to the global sections of the vector bundle $P^k(L)$. Assume that $0 \leq k \leq \deg L - 2g$. By Proposition 2.10 such lifting of sections is surjective so one obtains a short exact sequence

$$0 \longrightarrow R^k(L) \longrightarrow H^0(L) \otimes \mathscr{O}_C \longrightarrow P^k(L) \longrightarrow 0, \tag{3.0.1}$$

where $R^k(L)$ is defined as the kernel bundle. Taking the dual of the sequence (3.0.1) and tensoring it with L, we obtain a surjective morphism

$$H^0(L)^* \otimes L \twoheadrightarrow R^k(L)^* \otimes L.$$

Since L is nonspecial then so is the bundle $R^k(L)^* \otimes L$.

PROPOSITION 3.1. *As setting above, one has the following.*
(1) $\deg P^k(L) = (k+1)d + k(k+1)(g-1)$ *and* $\operatorname{rank} P^k(L) = k+1$.
(2) $\deg R^k(L)^* \otimes L = k(k+1)(g-1) + (n+1)d$ *and* $\operatorname{rank} R^k(L)^* \otimes L = n - k$.
(3) $\chi(R^k(L)^* \otimes L) = (n+k+2) \cdot \operatorname{rank}(R^k(L)^* \otimes L) + (k+1)^2 g$.

Proof. For (1), we need to use the canonical exact sequence

$$0 \longrightarrow S^k(\Omega^1_C) \otimes L \longrightarrow P^k(L) \longrightarrow P^{k-1}(L) \longrightarrow 0.$$

But for C a curve, $S^k(\Omega^1_C) = \omega^k_C$. Hence we deduce that

$$\deg P^k(L) = \deg P^{k-1}(L) + k(2g-2) + d.$$

Then the formula comes from the induction on k and $P^0(L) = L$.

(2) is straightforward from (1).

For (3), denote by $r = \operatorname{rank} R^k(L)^* \otimes L$ and apply Riemann-Roch theorem so that

$$\begin{aligned} \chi(R^k(L)^* \otimes L) &= \deg R^k(L)^* \otimes L + r\chi(\mathscr{O}_C) \\ &= k(k+1)(g-1) + (n+1)d + r\chi(\mathscr{O}_C) \\ &= (n+k+2)r + (k+1)^2 g \end{aligned}$$

as claimed. ∎

REMARK 3.2. We note that $R^k(L)^* \otimes L$ is nonspecial. Hence we can write

$$h^0(R^k(L)^* \otimes L) = q \cdot r + t,$$

where $q = (n+k+2)$, $r = \operatorname{rank} R^k(L)^* \otimes L = n - k$ and $t = (k+1)^2 g$. In oder to get general results on interpolation of $R^k(L)^* \otimes L$, we assume in the sequel that $r \geq t$, or equivalently

$$\deg L \geq (k^2 + 2k + 2)g + k.$$

In particular, if $\deg L = (k^2 + 2k + 2)g + k$, then the rank of $R^k(L)^* \otimes L$ divides $h^0(R^k(L)^* \otimes L)$.

Proof of Theorem 1.2. Recall that $d = \deg L$ and note that L is nonspecial with $n = d - g$. The proof contains two steps.

Step 1. Choose S as a general effective divisor of

$$\deg S = n + k + 2.$$

Clearly, $\deg S \geq g$ by the assumption on d. In this step, we show that the natural morphism

$$H^0(R^k(L)^* \otimes L) \longrightarrow H^0(R^k(L)^* \otimes L|_S)$$

is surjective. It is equivalent to show that $H^1(R^k(L)^* \otimes L(-S)) = 0$. By duality, we just need to show $H^0(\omega_C \otimes R^k(L) \otimes L^*(S)) = 0$. Write

$$L_1 = \omega_C \otimes L^*(S).$$

Tensoring L_1 with the short exact sequence (3.0.1), it suffices to show the canonical morphism

$$H^0(L) \otimes H^0(L_1) \longrightarrow H^0(P^k(L) \otimes L_1)$$

is injective. But by Proposition 2.7 this morphism is the same as the canonical morphism

$$H^0(L) \otimes H^0(L_1) \longrightarrow H^0(P^k(L_1) \otimes L). \tag{3.2.1}$$

In the rest of this part, we will show the morphism in (3.2.1) is injective.

It is easy to calculate that $\deg L_1 = g + k$. The general choice of S will make L_1 a general line bundle of degree $g + k$. Hence it is base point free and nonspecial and the canonical morphism

$$e_1 : H^0(L_1) \otimes \mathscr{O}_C \longrightarrow P^k(L_1)$$

is generically surjective by Proposition 2.10(3). But $\operatorname{rank} P^k(L_1) = h^0(L_1) = k + 1$. Thus the morphism e_1 is also generically injective and therefore injective. Now tensoring with L, we obtain an injection

$$0 \longrightarrow H^0(L_1) \otimes L \longrightarrow P^k(L_1) \otimes L.$$

Taking global sections immediately shows that the morphism in (3.2.1) is injective, which completes Step 1.

Step 2. Choose S' as a general effective divisor such that

$$\deg S' = n + k + 3.$$

The goal in this step is to show that the natural morphism

$$H^0(R^k(L)^* \otimes L) \longrightarrow H^0(R^k(L)^* \otimes L|_{S'})$$

is injective. It is equivalent to show that $H^0(R^k(L)^* \otimes L(-S')) = 0$. By duality, we just need to show $H^1(\omega_C \otimes R^k(L) \otimes L^*(S')) = 0$. Write

$$L_2 = \omega_C \otimes L^*(S').$$

Tensoring L_2 with the short exact sequence (3.0.1), it suffices to show the canonical morphism

$$H^0(L) \otimes H^0(L_2) \longrightarrow H^0(P^k(L) \otimes L_2)$$

is surjective and that $H^0(L) \otimes H^1(L_2) = 0$, which holds since L_2 is a general line bundle of degree $g+k+1$ and so has vanishing H^1. By Proposition 2.7 this morphism is the same as the canonical morphism

$$H^0(L) \otimes H^0(L_2) \longrightarrow H^0(P^k(L_2) \otimes L). \tag{3.2.2}$$

We will show in the sequel that the above morphism is surjective.

Note that $\deg L_2 = g + k + 1$. The general choice of S' makes L_2 a general line bundle of degree $g+k+1$. Hence it is base point free and nonspecial and the canonical morphism

$$e_2 : H^0(L_2) \otimes \mathscr{O}_C \longrightarrow P^k(L_2)$$

is surjective by Proposition 2.10(2). Write $\mathscr{K}$ as the kernel of the morphism e_2 and note that $\mathscr{K}$ is a line bundle. So we obtain a short exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow H^0(L_2) \otimes \mathscr{O}_C \longrightarrow P^k(L_2) \longrightarrow 0.$$

Tensor it with L. Then, since $H^1(L) = 0$, the surjectivity of (3.2.2) is equivalent to the vanishing of

$$H^1(\mathscr{K} \otimes L) = 0.$$

But by the assumption that $\deg L \geq (k^2+2k+2)g+k$, we see that $\deg \mathscr{K} \otimes L \geq g-1$. The general choice of S' also makes $\mathscr{K} \otimes L$ as a general line bundle of degree $\geq g-1$. Hence it is nonspecial. Therefore $H^1(\mathscr{K} \otimes L) = 0$, which finishes the proof. □

Remark 3.3. The bound of $\deg L \geq (k^2 + 2k + 2)g + k$ is crucial in the proof. There are two reasons for this. First, if $\deg L$ is small, for instance when $k = 0$ and $\deg L < 2g$, then L may not be able to separate 0-jets and therefore $R^0(L)$ may not be defined. Secondly, even if $R^k(L)$ is defined, if $\deg L$ is small, then the number $(k + 1)^2 g$ in Proposition 3.1 (3) may not be smaller than the rank of $R^k(L)^* \otimes L$ so that the degree of S in the proof should be increased.

To be more precise on the second point, we consider $g = 2$, $k = 1$ and $\deg L = 5 \cdot g = 10$. In this case, $h^0(L) = 9$ so we have an embedding

$$\phi : C \longrightarrow \mathbb{P}^n, \quad \text{with } n = 8$$

and $R^1(L)^* \otimes L = N$ is of rank 7. We can compute that

$$\chi(N) = h^0(N) = 12 \cdot \operatorname{rank} N + 1.$$

Thus the degrees of the effective divisors involved to check interpolation are 12 and 13, instead of 11 and 12 as indicated by Proposition 3.1 (3).

Remark 3.4. If $g = 0$, i.e., C is a rational normal curve, one can actually compute $R^k(L)^* \otimes L$. Precisely, suppose $C = \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ is embedded by the complete linear system of $L = \mathscr{O}_{\mathbb{P}^1}(d)$. For $d \geq \max(k, 1)$, one has

$$R^k(L)^* \otimes L = \bigoplus^{d-k} \mathscr{O}_{\mathbb{P}^1}(d + k + 1),$$

which is semistable and satisfies interpolation.

Proof of Corollary 1.4. Under the assumption that $\deg L = (k^2 + 2k + 2)g + k$, we see that $\operatorname{rank} R^k(L)^* \otimes L$ divides $h^0(R^k(L)^* \otimes L)$ (see Remark 3.2). Then by Proposition 2.5, $R^k(L)^* \otimes L$ is semistable and therefore so is $R^k(L)$. □

Remark 3.5. It was pointed out by the referee that one of possible geometric consequences of interpolation for $R^2(L)^* \otimes L$ is to determine how many general lines a curve of degree d and genus g in $\mathbb{P}^r$ is tangent to. We hope that the method we developed in the paper would be useful for this direction of the study.

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