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ON THE CANONICAL MAPS OF NONSINGULAR THREEFOLDS OF GENERAL TYPE*

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Dedicate to Professor Ngaiming Mok on the occasion of his 60th Birthday

Abstract. Let S be a nonsingular minimal complex projective surface of general type and the canonical map of S is generically finite. Beauville showed that the geometric genus of the image of the canonical map is vanishing or equals the geometric genus of S and discussed the canonical degrees for these two cases. We generalize his results to nonsingular minimal complex projective threefolds.

Key words. Projective threefold, general type, canonical map, canonical degrees.

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1. Introduction. The study of the canonical maps of n dimensional projective varieties of general type is one of the central problems in algebraic geometry. For $n = 2$, Beauville ([Bea]) proved that the geometric genus of the image Σ of the canonical map of a nonsingular complex projective surface S of general type vanishes or equals to the geometric genus of S. Moreover, he showed that the canonical degree is less than or equal to 36 if $p_g(\Sigma) = 0$ and 9 if $p_g(\Sigma) = p_g(S)$.

For $n = 3$, M. Chen studied the canonical map of fiber type ([Ch1], [C-H]) and posted an open problem in [Ch1] as follows: Let X be a Gorenstein minimal projective 3-fold with at worst locally factorial terminal singularities. Suppose that the canonical map is generically finite onto its image. Is the generic degree of the canonical map universally upper bounded? Hacon gave a positive answer to Chen's problem. More precisely, he showed that the canonical degree is at most 576 and the statement is wrong without the Gorenstein condition. Recently, Y. Gao and the author improved Hacon's upper bound to 360 and the equality holds if and only if $p_g(X) = 4$, $q(X) = 2$, $\chi(\omega_X) = 5$, $K_X^3 = 360$ and $|K_X|$ is base point free ([D-G1]).

For $n \geq 4$, the situation is totally different. The reason is that, in case of $n < 4$, Miyaoka-Yau inequality plays a vital role in the proof while Miyaoka-Yau inequality is not effective enough to control K_X^n for $n \geq 4$. In [D-G3], the authors consider abelian canonical n-folds (see [D-G2] for $n = 2$ and [D-G1] for $n = 3$) such that the canonical degrees are universally upper bounded.

First, we list the following theorem due to Beauville.

Theorem 1.1 ([Bea]). *Let X be a nonsingular minimal complex projective threefold of general type. Suppose that the canonical map $\phi_X : X \dashrightarrow \mathbb{P}^{p_g(X)-1}$ is generically finite. Denote $\Sigma := \phi_X(X)$, then either:*

(1) *$p_g(\Sigma) = 0$, or*

(2) *$p_g(\Sigma) = p_g(X)$, where the canonical map of Σ is of degree 1.*

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In [Sup1], P. Supino produced a bound for the admissible degree of the canonical map of a threefold of general type X with the condition $h^2(\mathscr{O}_X) - h^1(\mathscr{O}_X) \geq 2$. In this paper, we generalize Beauville's other results to minimal nonsingular complex projective threefolds without the above condition.

THEOREM 1.2. *Let X be a nonsingular minimal complex projective threefold of general type. Suppose that the canonical map $\phi_X : X \dashrightarrow \mathbb{P}^{p_g(X)-1}$ is generically finite. Denote $\Sigma := \phi_X(X)$ and $d := deg\ \phi_X$.*

(1) *If $p_g(\Sigma) = 0$ and K_M is pseudo-effective, where M is a nonsingular model of Σ, then $d \leq 108$.*
(2) *If $p_g(\Sigma) = p_g(X)$, where the canonical map of Σ is of degree 1, then $d \leq 86$. Moreover, if $p_g(X) \geq 111$, then $d \leq 32$.*

REMARK 1.3. P. Supino also constructed some examples of threefolds of general type with canonical degree 3 ([Sup2]) and 4, 5, 6 with G. Casnati ([C-S]). Later, Cai ([Cai]) constructed some examples of threefolds with canonical degrees 32, 64 and 72. But his example of degree 72 depends on the existence of the surface of general type with canonical degree 36, which was first discovered by Yeung (cf. [Yeung], [L-Y]). Each example is a threefold which can be decomposed as Cartesian product of a surface and a curve. In [D-G1], Gao and the author use abelian covers to construct examples with canonical degree 2, 4, 8, 16, 32.

2. Canonical map of threefolds of general type. Let S be a nonsingular minimal surface of general type with geometric genus $p_g(S) \geq 3$. Denote by $\phi_S : S \dashrightarrow \mathbb{P}^{p_g(S)-1}$ the canonical map and let $d := \deg(\phi_S)$. The following Beauville's result is well-known.

THEOREM 2.1 ([Bea]). *If the canonical image $F := \phi_S(S)$ is a surface, then either:*

(1) *$p_g(F) = 0$, or*
(2) *$p_g(F) = p_g(S)$, where F is a canonical surface.*

Moreover, in case (1),

(i) *$d \leq 36$ and $d = 36$ if and only if $p_g(S) = 3$, $q(S) = 0$, $K_S^2 = 36$ and $|K_S|$ base point free,*
(ii) *$d \leq 9$ if $\chi(\mathscr{O}) \geq 31$,*
(iii) *$d \leq 4$ if $\chi(\mathscr{O}) \geq 31$ and F is not a ruled surface;*

and in case (2),

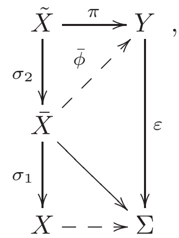
(i) *$d \leq 9$ and $d = 9$ if and only if $p_g(S) = 4$, $q(S) = 0$, $K_S^2 = 45$, $|K_S|$ base point free and F is a surface of degree 5 in $\mathbb{P}^3$,*
(ii) *$d \leq 3$ if $\chi(\mathscr{O}) \geq 14$.*

REMARK 2.2. The existence of a surface of general type with $p_g(S) = 3$, $q(S) = 0$, $K_S^2 = 36$ and $|K_S|$ base point free was first proved by Yeung (cf. [Yeung], [L-Y]).

The following theorem is due to Beauville. We mimic Beauville's proof to nonsingular minimal complex projective threefolds as follows in order to keep this note self-contained.

Proof of Theorem 1.1. Let $|K_X| = |S| + F$ such that S is the moving part and F

is the fixed part of $|K_X|$. Consider the following diagram



where σ_1 is the elimination of the indeterminacy of ϕ_X, σ_2 is the elimination of the indeterminacy of $\bar{\phi}$, ε is a resolution of Σ. Suppose that $\sigma = \sigma_1 \circ \sigma_2$. Let $|\sigma^* S| = |S'| + \sum_{i=1}^{r} a_i E_i$ such that $|S'|$ is base point free and

$$\tilde{\phi} = \varepsilon \circ \pi = \phi_{|S'|} : \tilde{X} \to \Sigma \subseteq \mathbb{P}^{p_g(X)-1}.$$

Denote $\mathscr{O}_Y(H) = \varepsilon^* \mathscr{O}_\Sigma(1)$, then $S' = \pi^* \circ \varepsilon^* \mathscr{O}_\Sigma(1) = \pi^*(\mathscr{O}_Y(H))$.

$$\begin{aligned} K_{\tilde{X}} &= \sigma^* K_X + \sum_{i=1}^{r} b_i E_i \\ &= \sigma^* S + \sigma^* F + \sum_{i=1}^{r} b_i E_i \\ &= S' + Z, \end{aligned} \tag{2.1}$$

where $b_i = 1$ or 2, $Z = \sum_{i=1}^{r}(a_i + b_i)E_i + \sigma^* F$. Then $|K_{\tilde{X}}| = |S'| + Z$. So

$$\begin{aligned} p_g(X) = h^0(\mathscr{O}_{\tilde{X}}(K_{\tilde{X}})) &= h^0(\mathscr{O}_{\tilde{X}}(S')) = h^0(\pi^* \mathscr{O}_Y(H)) \\ &\geq h^0(\mathscr{O}_Y(H)) \geq p_g(X). \end{aligned} \tag{2.2}$$

Therefore $h^0(\mathscr{O}_{\tilde{X}}(S')) = h^0(\mathscr{O}_Y(H)) = p_g(X)$ and $|S'| = \pi^*|H|$ from which we have $|K_{\tilde{X}}| = \pi^*|H| + Z$.

Suppose that $p_g(Y) = p_g(\Sigma) \neq 0$, then there is a nonzero holomorphic 3-form $\omega \in H^0(Y, \mathscr{O}_Y(K_Y))$.

Next we state that there exists $Q \in |H|$ such that $\mathrm{div}(\omega) \geq Q$. In fact by Hurwitz formula

$$\mathrm{div}(\pi^*\omega) = \pi^*\mathrm{div}(\omega) + \sum_i (e_i - 1)S_i + \sum_j r_j E_j,$$

where S_i's are branch loci, e_i's are the ramification indices and E_j's are contracted to points or curves by π. Since $\mathrm{div}(\pi^*(\omega)) \in |K_{\tilde{X}}| = \pi^*|H| + Z$, there exists $Q \in |H|$ such that

$$\pi^* Q + Z = \pi^*\mathrm{div}(\omega) + \sum_i (e_i - 1)S_i + \sum_j r_j E_j. \tag{2.3}$$

Let $Q = \sum h_\Gamma \Gamma$ and $\mathrm{div}(\omega) = \sum k_\Gamma \Gamma$. Suppose that $\pi^*\Gamma = \sum_t \Gamma_t + \sum r'_j E_j$, where $\Gamma = \pi(\Gamma_t)$. If $\pi^*\Gamma \not\supseteq S_i$, since

$$\pi^*(Q) = \pi^*(\sum h_\Gamma \Gamma) = \sum h_\Gamma \pi^* \Gamma$$

and

$$\mathrm{div}(\pi^*\omega) = \sum k_\Gamma \pi^*\Gamma + \sum_i (e_i - 1)S_i + \sum_j r_j E_j,$$

we have $h_\Gamma \le k_\Gamma$ by comparing the coefficients before Γ_t in the both sides of equality (2.3). If $\Gamma = \pi(S_i)$, suppose that $\pi^*\Gamma = e_i S_i + \sum_j r'_j E_j$. Similarly, we have $h_\Gamma e_i \le k_\Gamma e_i + (e_i - 1)$ by comparing the coefficients before Γ_t in the both sides of equality (2.3). So $h_\Gamma \le k_\Gamma$. Therefore we have $K_Y = \mathrm{div}(\omega) \ge Q$. So $h^0(\mathscr{O}_Y(K_Y)) \ge h^0(\mathscr{O}_Y(Q)) = p_g(X)$. On the other hand, $h^0(\mathscr{O}_Y(K_Y)) \le h^0(\mathscr{O}_{\tilde{X}}(K_{\tilde{X}})) = p_g(X)$. So

$$p_g(X) = p_g(Y) = p_g(\Sigma) = h^0(\mathscr{O}_Y(Q)).$$

So $|K_Y| = |Q| + Z'$ and $\phi_{|K_Y|} = \phi_{|Q|} = \varepsilon$ is birational. □

Lemma 2.3. *Let $\Sigma \subseteq \mathbb{P}^n$ be a non-degenerate projective variety of dimension 3. Take a resolution $\varepsilon : M \to \Sigma$. If K_M is pseudo-effective, then $deg\Sigma \ge 2n - 4$.*

Proof. Let $\mathscr{O}_M(H) = \varepsilon^*(\mathscr{O}_\Sigma(1))$. So $|H|$ is base point free. Take an element $S \in |H|$ such that S is irreducible and nonsingular surface. Then $S^3 = \deg\Sigma > 0$ and $\mathscr{O}_M(S)$ is nef and big. Notice that $|H|$ restrict to S, $|H|_S$, is also base point free, so we can take $C \in |H|_S$ a nonsingular irreducible curve such that $\mathscr{O}_S(C) := \mathscr{O}_M(H)|_S$. Then $C^2 = S^3 = \deg\Sigma$. Consider the exact sequence

$$0 \to \mathscr{O}_M \to \mathscr{O}_M(S) \to \mathscr{O}_S(S) \to 0.$$

Take cohomology of the exact sequence, we can get

$$h^0(\mathscr{O}_S(C)) = h^0(\mathscr{O}_S(S)) \ge h^0(\mathscr{O}_M(S)) - 1 \ge (n+1) - 1 = n.$$

From the exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{O}_S(C) \to \mathscr{O}_C(C) \to 0,$$

we can also have $h^0(\mathscr{O}_S(C)) \le h^0(\mathscr{O}_C(C)) + 1$. If $h^1(\mathscr{O}_C(C)) \ne 0$, then by Clifford theorem, we have

$$n - 1 \le h^0(\mathscr{O}_C(C)) \le \frac{1}{2}C^2 + 1.$$

So $\deg\Sigma \ge 2n - 4$. If $h^1(\mathscr{O}_C(C)) = 0$, by Riemann-Roch theorem, we have $h^0(\mathscr{O}_C(C)) = C^2 - g(C) + 1$. So $\deg\Sigma \ge n - 2 + g(C)$. On the other hand,

$$\begin{aligned} 2g(C) - 2 &= C^2 + CK_S \\ &= C^2 + C(K_M + S)|_S \\ &= 2C^2 + K_M S^2 \\ &\ge 2C^2. \end{aligned} \tag{2.4}$$

So $g(C) \ge \deg\Sigma + 1$, a contradiction. Therefore $\deg\Sigma \ge 2n - 4$. □

With the notations as above, in [D-G1], Gao and the author showed that the canonical degree $d \le 360$ and the equality holds if and only if $p_g(X) = 4$, $q(X) = 2$, $\chi(\omega_X) = 5$, $K_X^3 = 360$ and $|K_X|$ is base point free, which is the generalization of Beauville's result of case (1) (i). In [Cai], Cai showed that if $p_g(X) \ge 105412$, then

$d \le 72$ which is the generalization of Beauville's result of case (1) (ii). Next, we are going to generalize Beaville's other results in Theorem 2.1.

Proof of Theorem 1.2. By the condition, one has that $p_g(X) \ge 5$.

For the case (1), by Lemma 2.3 and Miyaoka-Yau inequality ([Mi]), we have

$$d(2p_g(X) - 6) \le d \cdot \deg\Sigma \le K_X^3 \le 72\chi(\omega_X).$$

If we can show $\chi(\omega_X) \le p_g(X) + 1$, then

$$d \le 36\frac{\chi(\omega_X)}{p_g(X) - 3} \le 36\frac{p_g(X) + 1}{p_g(X) - 3} = 36(1 + \frac{4}{p_g(X) - 3}) \le 108. \tag{2.5}$$

If $q(X) \le 2$, then $\chi(\omega_X) \le p_g(X) + q(X) - 1 \le p_g(X) + 1$.

Now we can assume hereafter that $q(X) \ge 3$. Consider the Albanese map alb_X of X and the Stein factorization f of alb_X as follows:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle alb_X}\searrow & \downarrow \\ & & Alb(X) \end{array} .$$

By Hacon's argument (see the proof of [Ha], Theorem 1.1), one has

(1) $\chi(\omega_X) \le p_g(X)$, if $\dim Y \ge 2$;

(2) $\chi(\omega_X) \le p_g(X) + \chi(\omega_Y)$ and $\chi(\omega_Y)p_g(F) \le p_g(X)$, where F is the general fiber of f, if $\dim Y = 1$.

Hence if $\dim Y \ge 2$, by (2.5), the statement holds. More precisely,

$$d \le 36\frac{p_g(X)}{p_g(X) - 3} \le 90.$$

We only need to consider $\dim Y = 1$. By the argument in the main theorem of [D-G1], we can get that $p_g(F) \ge \dim X = 3$ and $p_g(X) \ge 6$.

Therefore

$$d \le 36\frac{\chi(\omega_X)}{p_g(X) - 3} \le 36\frac{p_g(X) + \chi(\omega_Y)}{p_g(X) - 3} \le 36(1 + \frac{1}{p_g(F)})\frac{p_g(X)}{p_g(X) - 3} \le 96.$$

For the case (2), by M. Chen's result ([Ch2]) we have $\deg\Sigma \ge 3p_g(X) - 10$. So

$$d(3p_g(X) - 10) \le K_X^3 \le 72\chi(\omega_X).$$

Using the same arguments as above, we have, if $q(X) \le 2$,

$$d \le 72\frac{p_g(X) + 1}{3p_g(X) - 10}, \tag{2.6}$$

then

$$d \le 86.$$

If $q(X) \ge 3$ and

(1) if $\dim Y \geq 2$, then

$$d \leq 72\frac{p_g(X)}{3p_g(X)-10} \leq 72;$$

(2) if $\dim Y = 1$, then

$$d \leq 72(1+\frac{1}{p_g(F)})\frac{p_g(X)}{3p_g(X)-10} \leq 72.$$

Moreover, suppose that $p_g(X) \geq 111$.
If $q(X) \leq 2$, then

$$d \leq 72\frac{p_g(X)+1}{3p_g(X)-10}. \tag{2.7}$$

So

$$d \leq 24.$$

If $q(X) \geq 3$ and
(1) if $\dim Y \geq 2$, then

$$d \leq 72\frac{p_g(X)}{3p_g(X)-10}.$$

So

$$d \leq 24.$$

(2) if $\dim Y = 1$, then

$$d \leq 72(1+\frac{1}{p_g(F)})\frac{p_g(X)}{3p_g(X)-10}.$$

So

$$d \leq 32.$$

Therefore Theorem 1.2 has been proved. □

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REFERENCES

[Bea] A. Beauville, *L'application canonique pour les surfaces de type général*, Invention Math., 55 (1979), pp. 121–140.

[Cai] J. Cai, *Degree of the canonical map of a Georenstein 3-fold of general type*, Proc. AMS, 136:5 (2008), pp. 1565–1574.

[Ch1] M. Chen, *Weak Boundedness Theorems for Canonically Fibered Gorenstein Minimal 3-Folds*, Proc. Amer. Math. Soc., 133:5 (2005), pp. 1291–1298.

[Ch2] M. Chen, *Inequalities of Noether type for 3-folds of general type*, J. Math. Soc. Japan, 56:4 (2004), pp. 1131–1155.

[C-H] M. Chen and C. Hacon, *On the geography of Gorenstein minimal 3-folds of general type*, Asian J. Math., 10:4 (2006), pp. 757–764.

[C-S] G. Casnati and P. Supino, *Construction of threefolds with finite canonical map*, Glasg. Math. J., 44:1 (2002), pp. 65–79.

[D-G1] R. Du and Y. Gao, *On the canonical degrees of Gorenstein threefolds of general type*, Geom. Dedicata, 18:1 (2016), pp. 123–130.

[D-G2] R. Du and Y. Gao, *Canonical maps of surfaces defined by abelian covers*, Asian J. Math., 18:1 (2014), pp. 219–228.

[D-G3] R. Du and Y. Gao, *On abelian canonical n-folds of general type*, Taiwan. J. Math., 21:3 (2017), pp. 653–664.

[Ha] C. Hacon, *On the degree of the canonical maps of 3-folds*, Proc. Japan Acad., 80 Ser. A (2004), pp. 166–167.

[L-Y] C. Lai and S.-K. Yeung, *Examples of surfaces with canonical maps of maximal degree*, arXiv:1510.07097.

[Mi] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic geometry, Sendai, 1985, pp. 449–476, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[Sup1] P. Supino, *On Threefolds with* $K^3 = 2p_g - 6$, Kodai Math. J., 27:1 (2004), pp. 7–29.

[Sup2] P. Supino, *Triple covers of 3-folds as canonical maps*, Comm. Algebra, 26:5 (1998), pp. 1475–1487.

[Yeung] S. Yeung, *A surface of maximal canonical degree*, Math. Ann., 368 (2017), no. 3-4, pp. 1171-1189.