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ON MINIMAL 3-FOLDS OF GENERAL TYPE WITH MAXIMAL PLURICANONICAL SECTION INDEX*

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Dedicated to Prof. Ngaiming Mok on his sixtieth birthday

Abstract. Let X be a minimal 3-fold of general type. The pluricanonical section index $\delta(X)$ is defined to be the minimal integer m so that $P_m(X) \geq 2$. According to Chen-Chen, one has either $1 \leq \delta(X) \leq 15$ or $\delta(X) = 18$. This note aims to intensively study those with maximal such index. A direct corollary is that the 57th canonical map of every minimal 3-fold of general type is stably birational.

Key words. Minimal threefolds of general type, pluricanonical section index, canonical stability index.

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1. Introduction.

One main task of birational geometry is to study the behavior of pluricanonical maps of projective varieties. In this paper, we restrict our interest to minimal projective 3-folds of general type. Without loss of generality, we may always study a minimal variety of general type over any algebraically closed field k of characteristic 0.

Let X be a minimal projective n-fold of general type. Traditionally X is always assumed to be $\mathbb{Q}$-factorial with at worst terminal singularities. We always denote by $\varphi_{m,X}$ the m-th canonical map corresponding to $|mK_X|$, where K_X is a canonical divisor of X. The *canonical stability index of* X is defined as

$$r_s(X) = \min\{m \in \mathbb{Z}_{>0} |\ \varphi_{l,X} \text{ is birational for all } l \geq m\}.$$

The *n-th canonical stability index* r_n is defined as

$$r_n = \sup\{r_s(X) |\ X \text{ is a minimal n-fold of general type}\}.$$

Such number r_n has the fundamental importance in explicit birational geometry (see, for instance, Hacon–McKernan [H-M06, Problem 1.5]). Here are the main results about r_n:

- ◇ $r_1 = 3$, as a well-known fact.
- ◇ $r_2 = 5$ according to Bombieri [Bom73].
- ◇ $r_n < +\infty$ for all $n \geq 3$ by Hacon–McKernan [H-M06] and Takayama [Tak06], independently.
- ◇ $r_3 \leq 61$ by Chen–Chen [EXP1, EXP2, EXP3].

With regard to the value of r_3, Hacon–McKernan [H-M06, Question 1.6] asked if $r_3 = 27$ is true. So far there is no known minimal 3-fold X satisfying $r_s(X) > 27$. In this short note, we are going to improve the known upper bound for r_3.

Let X be a minimal 3-fold of general type. As in Chen–Chen [EXP3], the *pluricanonical section index* (in short, *ps-index*) is defined as follows:

$$\delta(X) = \min\{m \in \mathbb{Z}_{>0} |\ P_m(X) \geq 2\}.$$

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By Chen–Chen [EXP3, Theorem 1.4], we know that either $1 \leq \delta(X) \leq 15$ or $\delta(X) = 18$. From our experience in studying the pluricanonical birationality, those 3-folds with largest ps-idex are the main obstacle for us to get better estimation of r_3. Thus we concentrate on studying those with maximal ps-index. Here is our main result:

Theorem 1.1. *Let X be a minimal 3-fold of general type with $\delta(X) = 18$. Then $r_s(X) \leq 57$.*

Together with Chen–Chen [EXP3, Theorem 6.2], Theorem 1.1 directly implies the following:

Corollary 1.2. *One has $r_3 \leq 57$.*

It is interesting to ask what the exact value of r_3 is. However, it seems to be very difficult to get an answer of this since the existence with $11 \leq \delta \leq 18$ has been in suspense. Theoretically it is clear from Chen-Chen [EXP3] that any minimal 3-fold X with $57 \leq r_s(X) \leq 61$ must have $\delta(X) = 18$. Thus to study those with maximal ps-index is a very natural choice. It is known that every minimal 3-fold with $\delta = 18$ has an induced fibration from $|18K|$ whose general fiber is a surface of general type with $p_g = 1$. A feature of this paper is that three maps φ_{18}, φ_{24} and φ_{36} and their mutual actions are considered simultaneously to prove the above mentioned main theorem.

Throughout we will use the following symbols:

- ◇ "$\sim$" denotes linear equivalence or $\mathbb{Q}$-linear equivalence;
- ◇ "$\equiv$" denotes numerical equivalence;
- ◇ "$|M_1| \succcurlyeq |M_2|$" (or, equivalently, "$|M_2| \preccurlyeq |M_1|$") means, for linear systems $|M_1|$ and $|M_2|$ on a variety,

$$|M_1| \supseteq |M_2| + (\text{fixed effective divisor}).$$

- ◇ "$D \leq D'$" means that $D' - D$ is linearly (or $\mathbb{Q}$-linearly) equivalent to an effective divisor (or effective $\mathbb{Q}$-divisor) subject to the context for two divisors (or $\mathbb{Q}$-divisors) D and D'.

2. Preliminaries.

2.1. Convention. Let $|D|$ be any linear system of positive dimension on a normal projective variety Z.

(1) Denote by $\mathrm{Mov}|D|$ the moving part of $|D|$. We say that $|D|$ is *composed of a pencil* if $\dim \overline{Im(\Phi_{\mathrm{Mov}|D|})} = 1$.

(2) A *generic irreducible element of* $|D|$ means a general member of $\mathrm{Mov}|D|$ when $|D|$ is not composed of a pencil or, otherwise, an irreducible component in a general member of $\mathrm{Mov}|D|$.

(3) Let S_1, S_2 be two different irreducible and reduced divisors which are not contained in the fixed locus of $|D|$. If

$$\overline{\Phi_{|D|}(S_1)} \not\subseteq \overline{\Phi_{|D|}(S_2)} \text{ and } \overline{\Phi_{|D|}(S_2)} \not\subseteq \overline{\Phi_{|D|}(S_1)},$$

we say that $|D|$ *can distinguish S_1 and S_2.*

2.2. Induced fibration from φ_{m_0}. Let X be a minimal projective 3-fold of general type on which $P_{m_0}(X) \geq 2$ for an integer $m_0 > 0$. Fix an effective Weil divisor $K_{m_0} \sim m_0K_X$ on X. Take successive blow-ups, say $\pi\colon X' \to X$ along nonsingular centers, such that the following conditions are satisfied:

(i) X' is nonsingular and projective;
(ii) the moving part of $|m_0K_{X'}|$ is base point free and so that $g_{m_0} = \varphi_{m_0,X} \circ \pi$ is a non-constant morphism;
(iii) $\pi^*(K_{m_0}) \cup \{\pi - \text{exceptional divisors}\}$ has simple normal crossing supports.

Taking the Stein factorization of the morphism

$$g_{m_0}\colon X' \longrightarrow \overline{\varphi_{m_0,X}(X)} \subseteq \mathbb{P}^{P_{m_0}-1},$$

we have $X' \xrightarrow{f_{m_0}} \Gamma \xrightarrow{s} \overline{\varphi_{m_0,X}(X)}$ and the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\quad f_{m_0}\quad} & \Gamma \\ {\scriptstyle\pi}\downarrow & \overset{g_{m_0}}{\searrow} & \downarrow{\scriptstyle s} \\ X & \overset{\varphi_{m_0,X}}{\dashrightarrow} & \overline{\varphi_{m_0,X}(X)} \end{array}$$

We call $f_{m_0} : X' \longrightarrow \Gamma$ an *induced fibration from* φ_{m_0}. Denote by $r(X)$ the canonical index of X. We may write $m_0K_{X'} \sim_{\mathbb{Q}} \pi^*(m_0K_X)+E_{\pi,m_0}$ where E_{π,m_0} is an effective π-exceptional $\mathbb{Q}$-divisor. Denote by $|M_{m_0}|$ the moving part of $|m_0K_{X'}|$. Set $d_{m_0} = \dim(\Gamma)$. By the Bertini theorem, when $d_{m_0} \geq 2$, the general member $F \in |M_{m_0}|$ is irreducible and smooth and we set $a_{m_0} = 1$. When $d_{m_0} = 1$, $M_{m_0} \equiv a_{m_0}F$, where $a_{m_0} = \deg f_*\mathcal{O}_{X'}(M_{m_0})$ and F is a general fiber of f. By the above setting, we always have

$$m_0\pi^*(K_X) \equiv a_{m_0}F + E'_{m_0} \tag{2.1}$$

for some effective $\mathbb{Q}$-divisor E'_{m_0} on X'.

LEMMA 2.1. *Keep the above notation. Denote by r_X the Cartier index of X. Then $r_X(\pi^*(K_X)|_F)^2$ is an integer.*

Proof. Set $\pi_*(F) = F_0$. Then $r_X(\pi^*(K_X)|_F)^2 = r_X(K_X^2 \cdot F_0)$ which is independent of the choice of the birational modification π. Take $\mu : Y \longrightarrow X$ to be a resolution of singularities of X which is assumed to have at worst terminal (hence isolated) singularities. Write $K_Y = \mu^*(K_X) + E_\mu$ where E_μ is the exceptional divisor. Denote by F_μ the strict transform of F_0. Then

$$r_X(K_X^2 \cdot F_0) = (r_X\mu^*(K_X) \cdot (K_Y - E_\mu) \cdot F_\mu) = (r_X\mu^*(K_X) \cdot K_Y \cdot F_\mu)$$

is an integer. ☐

2.3. Technical set up. Keep the same notation as in 2.2. Pick a generic irreducible element F of $|M_{m_0}|$. Assume that $|G|$ is a base point free linear system on F. Pick a generic irreducible element C of $|G|$. Since $\pi^*(K_X)|_F$ is nef and big, we may always assume that there exists a positive rational number $\beta > 0$ so that $\pi^*(K_X)|_F - \beta C$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor. Let m be a positive integer. Set

$$\xi = (\pi^*(K_X)|_F \cdot C),$$

$$\alpha_m = (m - 1 - \frac{m_0}{a_{m_0}} - \frac{1}{\beta})\xi.$$

2.4. Some frequently used results.

THEOREM 2.2 (see Chen-Chen [EXP3, 2.2, Theorem 2.7]). *Keep the same assumption as in 2.2 and 2.3. Let m be a positive integer. Let X be a minimal 3-fold of general type with $P_{m_0} \geq 2$ for some integer $m_0 > 0$. The following statements hold:*

(i) *One has the inequality*

$$\xi \geq \frac{\deg(K_C)}{1 + m_0/a_{m_0} + 1/\beta}. \tag{2.2}$$

(ii) *When $\alpha_m > 1$, one has the inequality*

$$m\xi \geq \deg(K_C) + \lceil \alpha_m \rceil. \tag{2.3}$$

When $\alpha_m > 0$ and C is an even divisor, one has the inequality

$$m\xi \geq \deg(K_C) + 2\lceil \frac{\alpha_m}{2} \rceil. \tag{2.4}$$

(iii) *Assume that $|mK_{X'}|$ distinguishes different generic irreducible elements of $|M_{m_0}|$ and that, on the generic irreducible element F of $|M_{m_0}|$, $|mK_{X'}||_F$ distinguishes different generic irreducible elements of $|G|$. When $\alpha_m > 2$, $\varphi_{m,X}$ is birational onto its image.*

LEMMA 2.3 (see Chen-Chen [EXP3, Lemma 2.4, Lemma 2.5]). *Let $\sigma : S \longrightarrow S_0$ be a birational contraction from a nonsingular projective surface of general type onto its minimal model S_0. Assume that $(K_{S_0}^2, p_g(S_0)) \neq (1, 2)$. Then one has*

(1) *$(\sigma^*(K_{S_0}) \cdot C) \geq 2$ for any moving irreducible curve C on S (i.e. C moves in a linear system of positive dimension);*

(2) *$(\sigma^*(K_{S_0}) \cdot \tilde{C}) \geq 2$ for any very general irreducible curve $\tilde{C}$ on S.*

LEMMA 2.4 ([Ch2014, Lemma 2.5]). *Let S be a nonsingular projective surface. Let $\mathcal{M}$ be a nef and big $\mathbb{Q}$-divisor on S satisfying the following conditions:*

(1) *$\mathcal{M}^2 > 8$;*

(2) *$(\mathcal{M} \cdot C_P) \geq 4$ for all irreducible curves C_P passing through each very general point $P \in S$.*

Then the linear system $|K_S + \lceil \mathcal{M} \rceil|$ separates two distinct points in very general positions. Consequently $\Phi_{|K_S + \lceil \mathcal{M} \rceil|}$ is a birational map onto its image.

LEMMA 2.5 (cf. [Ch2014, Lemma 2.6]). *Let $X' \longrightarrow X$ be a birational morphism from a nonsingular projective model X' onto X, which is a minimal projective 3-fold of general type. Assume that $f : X' \longrightarrow \mathbb{P}^1$ is a fibration with the general fiber F. Denote by $\sigma : F \longrightarrow F_0$ the birational contraction onto the minimal model F_0. The following statements are true:*

(1) *Set $\mu_0 = \frac{K_X^3}{3(\pi^*(K_X)|_F)^2}$. Then*

$$\pi^*(K_X)|_F \geq (\frac{\mu_0}{\mu_0 + 1} - \varepsilon)\sigma^*(K_{F_0})$$

where $0 < \varepsilon \ll 1$ is any rational number.

(2) *If $\pi^*(K_X) \geq \nu_0 F$ for some positive rational number ν_0, then*

$$\pi^*(K_X)|_F \geq \frac{\nu_0}{\nu_0 + 1}\sigma^*(K_{F_0}).$$

Proof. (1) For any sufficiently large and divisible integer m (such that m is divisible by the canonical index $r(X)$), the Riemann-Roch formula on X' implies

$$P_m(X') = h^0(X', m\pi^*(K_X)) \approx \frac{1}{6}K_X^3 m^3.$$

On the other hand, the Riemann-Roch on F gives

$$h^0(F, m\pi^*(K_X)|_F) \approx \frac{1}{2}(\pi^*(K_X)|_F)^2 m^2$$

as m is sufficiently large. Therefore

$$P_m(X') > (\mu_0 - \delta) m h^0(F, m\pi^*(K_X)|_F)$$

for $m \gg 0$ and very small number $\delta > 0$. Consider the restriction maps:

$$H^0(X', M_m - tF) \xrightarrow{\theta_t} V_{m,t} \subset H^0(F, m\pi^*(K_X)|_F)$$

where $t \geq 0$, m is sufficiently divisible and $V_{m,t}$ is the image space. Since $\dim(V_{m,t}) \leq h^0(F, m\pi^*(K_X)|_F)$ for all t, we naturally have

$$m\pi^*(K_X) - (\mu_0 - \delta)mF \geq M_m - (\mu_0 - \delta)mF > 0$$

for all large and divisible integers m (such that $(\mu_0 - \delta)m$ is integral). Thus, up to $\mathbb{Q}$-linear equivalence, one has

$$\pi^*(K_X) \geq (\mu_0 - \delta)F.$$

The rest of (1) follows directly from the statement (2).

(2) We may find a very large and divisible integer N so that $N\pi^*(K_X)$ and $N\nu_0 F$ are Cartier divisors and that

$$N\pi^*(K_X) \geq N\nu_0 F$$

holds as Cartier divisors. Now we may apply Chen-Chen [EXP3, Lemma 2.1(ii)] by replacing $(m_0, \Lambda, \theta_\Lambda)$ with $(N, \pi(N\nu_0 F), N\nu_0)$. What we get is

$$\pi^*(K_X)|_F \geq \frac{N\nu_0}{N + N\nu_0}\sigma^*(K_{F_0}) = \frac{\nu_0}{\nu_0 + 1}\sigma^*(K_{F_0}).$$

□

3. Proof of the main theorem.

3.1. General setting. In this section we always assume X to be a minimal 3-fold of general type with $\delta(X) = 18$. Set $m_0 = 18$ and keep the same notation as in 2.2. We have an induced fibration $f = f_{18} : X' \longrightarrow \Gamma$. Pick up a general fiber F of f. For this kind of 3-folds, the following properties are known:

(R1) (see [EXP3, Theorem 5.1]) $K_X^3 = \frac{1}{1170}$, $P_2(X) = q(X) = 0$, $\chi(\mathcal{O}_X) = 2$ and Reid's singularity basket

$$B_X = \{4 \times (1,2), (4,9), (2,5), (5,13), 3 \times (1,3), 2 \times (1,4)\}.$$

(R2) Direct calculation shows that $P_8 = 1$, $P_{18} = 2$, $P_{19} = 0$, $P_{24} = 3$, $P_{36} = 8$ and $P_m > 0$ for all $m \geq 20$.

(R3) (see [EXP3, Theorem 6.1]) $p_g(F) = 1$ and $r_s(X) \leq 61$.

Set $m_1 = 24$ and $m_2 = 36$. For any positive integer l, we have defined $|M_l| = \mathrm{Mov}|lK_{X'}|$. Modulo further birational modification to π, we may assume that $|M_{18}|$, $|M_{24}|$ and $|M_{36}|$ are all base point free.

Clearly we have $\Gamma \cong \mathbb{P}^1$ since $q(X) = 0$. We will analyze the interactions among $\varphi_{18,X}$, $\varphi_{24,X}$ and $\varphi_{36,X}$. On a general fiber F of $f : X' \longrightarrow \mathbb{P}^1$, we set $\mathcal{L} = \pi^*(K_X)|_F$, which is a nef and big $\mathbb{Q}$-divisor.

3.2. A sufficient condition for birationality of $\varphi_{m,X}$.

LEMMA 3.1. *Keep the above setting. Then $|mK_{X'}|$ can distinguish different generic irreducible elements of $|M_{18}|$ for all $m \geq 38$.*

Proof. By (R2), we have $mK_{X'} \geq M_{18} \sim F$ whenever $m \geq 38$. Since $q(X) = 0$, $|F|$ is a rational pencil. Thus $|mK_{X'}|$ naturally distinguishes different fibers of f. □

LEMMA 3.2. *Assume that $\mathcal{L} \geq \tilde{\beta}\sigma^*(K_{F_0})$ for certain positive rational number $\tilde{\beta}$. Then $|K_F + \lceil n\mathcal{L} \rceil|$ gives a birational map whenever*

$$n \geq max\{\lfloor \sqrt{\frac{8}{\mathcal{L}^2}} \rfloor + 1, \frac{2}{\tilde{\beta}}\}.$$

Proof. First, we have $(n\mathcal{L})^2 > 8$ for such numbers n. Secondly, for any very general curve $\tilde{C}$ on F, one has

$$(n\mathcal{L} \cdot \tilde{C}) \geq n\tilde{\beta}(\sigma^*(K_{F_0}) \cdot \tilde{C}) \geq 2n\tilde{\beta} \geq 4$$

by Lemma 2.3(2) and the fact that F_0 is not a $(1, 2)$ surface. Thus $|K_F + \lceil n\mathcal{L} \rceil|$ gives a birational map by Lemma 2.4. □

PROPOSITION 3.3. *Let X be a minimal 3-fold of general type with $\delta(X) = 18$. Keep the above notation. Assume that $\mathcal{L} \geq \tilde{\beta}\sigma^*(K_{F_0})$ for certain positive rational number $\tilde{\beta}$. Then $\varphi_{m,X}$ is birational for all*

$$m \geq max\{\lfloor \sqrt{\frac{8}{\mathcal{L}^2}} \rfloor + 20, \frac{2}{\tilde{\beta}} + 19, 38\}.$$

Proof. For any $n > 0$, we have

$$|K_{X'} + \lceil n\pi^*(K_X) \rceil + F| \preccurlyeq |(n + 19)K_{X'}|.$$

By Kawamata-Viehweg vanishing theorem ([Kaw82, V82]), we have

$$\begin{aligned} |K_{X'} + \lceil n\pi^*(K_X) \rceil + F||_F &= |K_F + \lceil n\pi^*(K_X) \rceil|_F| \\ &\succcurlyeq |K_F + \lceil n\mathcal{L} \rceil|. \end{aligned}$$

Now the statement clearly follows from Lemma 3.1 and Lemma 3.2. □

REMARK 3.4. In practice we may take $\tilde{\beta}$ to be $\frac{1}{19}$ by Lemma 2.5 while applying Proposition 3.3.

3.3. The case when $|18K|$ and $|24K|$ are composed of the same pencil.

Theorem 3.5. *Let X be a minimal 3-fold of general type with $\delta(X) = 18$. Assume that $|18K_X|$ and $|24K_X|$ are composed of the same pencil of surfaces. Then $\varphi_{m,X}$ is birational for all $m \geq 53$.*

Proof. By our assumption, both φ_{18} and φ_{24} induces the same fibration $f : X' \longrightarrow \mathbb{P}^1$. Since $P_{24}(X) = 3$, we may write

$$24\pi^*(K_X) \sim 2F + E'_{24}$$

where E'_{24} is an effective $\mathbb{Q}$-divisor. By Lemma 2.5(2), we have

$$\pi^*(K_X)|_F \geq \frac{1}{13}\sigma^*(K_{F_0})$$

which means that we have $\tilde{\beta} = \frac{1}{13}$. Clearly we have $\mathcal{L}^2 \geq \frac{1}{13^2}$.

In fact, one may have a better estimation for $\mathcal{L}^2$. We have $m_1 = 24$, $a_{m_1} = 2$. On the general fiber F, set $|\tilde{G}| = |2\sigma^*(K_{F_0})|$, which is base point free by the surface theory. Pick a generic irreducible element C' of $|\tilde{G}|$. Clearly C' is an even divisor on F. Replacing $(m_0, a_{m_0}, f, F, |G|, C, \beta)$ with $(m_1, a_{m_1}, f, F, |\tilde{G}|, C', 2\tilde{\beta})$ while applying Theorem 2.2, one easily obtains that $(\mathcal{L} \cdot C') \geq \frac{1}{5}$. Thus

$$\mathcal{L}^2 = (\pi^*(K_X)|_F)^2 \geq \frac{1}{26}(\mathcal{L} \cdot C') \geq \frac{1}{130}.$$

By Proposition 3.3, $\varphi_{m,X}$ is birational for all $m \geq 53$. □

In this case, we can find the constraint for $p_g(F)$.

Corollary 3.6. *Under the same assumption as that of Theorem 3.5, one has $(K^2_{F_0}, p_g(F_0)) = (1, 1)$.*

Proof. Suppose $K^2_{F_0} \geq 2$. Then we have $\mathcal{L}^2 \geq \frac{2}{13^2}$. This implies

$$K_X^3 \geq \frac{1}{12}\mathcal{L}^2 \geq \frac{1}{1014} > \frac{1}{1170},$$

a contradiction. □

3.4. The case when $|18K|$ and $|24K|$ are not composed of the same pencil. We set $|G| = |M_{24}|_F|$. By our assumption, $|G|$ is base point free. Pick a generic irreducible element C in $|G|$. Recall that $\xi = (\mathcal{L} \cdot C)$. Clearly we may take $\beta = \frac{1}{24}$.

Lemma 3.7. *One has $\mathcal{L}^2 \geq \frac{1}{195}$.*

Proof. Assume that $(\pi^*(K_X)|_F)^2 \leq \frac{11}{2340}$. Then we have

$$\mu_0 = \frac{K_X^3}{3(\pi^*(K_X)|_F)^2} \geq \frac{2340}{33 \cdot 1170} = \frac{1}{16.5},$$

which implies, by Lemma 2.5 and Lemma 2.3, that

$$\xi = (\pi^*(K_X)|_F \cdot C) \geq \frac{1}{17.5}(\sigma^*(K_{F_0}) \cdot C) \geq \frac{2}{17.5}$$

and, on the other hand,

$$(\pi^*(K_X)|_F)^2 \geq \frac{1}{24}\xi \geq \frac{1}{210} > \frac{11}{2340},$$

a contradiction. Thus we have $\mathcal{L}^2 > \frac{11}{2340}$.

Noting that $r_X \cdot (\pi^*(K_X)|_F)^2$ is integral by Lemma 2.1 and $r_X = 2340$, we see

$$\mathcal{L}^2 = (\pi^*(K_X)|_F)^2 \geq \frac{12}{2340} = \frac{1}{195}.$$

□

We have actually proved the following weaker result:

Corollary 3.8. *Let X be a minimal 3-fold of general type with $\delta(X) = 18$. Then $\varphi_{m,X}$ is birational for all $m \geq 59$.*

Proof. By Theorem 3.5, we may assume that $|18K|$ and $|24K|$ are not composed of the same pencil. The statement directly follows from Proposition 3.3 and Lemma 3.7. □

In order to prove the birationality of $\varphi_{57,X}$, we start to study the behavior of $|36K_{X'}|$. Recall that we have $|M_{36}| = \mathrm{Mov}|36K_{X'}|$. Consider the natural map

$$H^0(X', M_{36}) \xrightarrow{\theta_{36}} V_{36} \subseteq H^0(F, M_{36}|_F)$$

where $V_{36} = \mathrm{Im}(\theta_{36})$.

Theorem 3.9. *Let X be a minimal 3-fold of general type with $\delta(X) = 18$. Assume that $|18K_X|$ and $|24K_X|$ are not composed of the same pencil. If $\dim_k(V_{36}) \geq 5$, then φ_m is birational for all $m \geq 57$.*

Proof. Clearly we have $h^0(F, M_{36}|_F) \geq 5$. Set $|G_{36}| = |M_{36}|_F|$. By our assumption, $|G_{36}|$ is base point free. Recall that we have already another curve family $|C|$ on F.

We consider the natural restriction map

$$H^0(F, G_{36}) \longrightarrow W_{36} \subset H^0(C, D_{36})$$

where W_{36} is the image linear space and $D_{36} = G_{36}|_C$.

Case 1. If $\dim(W_{36}) \geq 4$, we have $h^0(C, D_{36}) \geq 4$. By Riemann-Roch formula and the Clifford theorem, one easily knows that $\deg(D_{36}) \geq 5$ since $g(C) \geq 2$. So

$$\mathcal{L}^2 = (\pi^*(K_X)|_F)^2 \geq \frac{1}{36 \cdot 24}(M_{36}|_F \cdot M_{24}|_F) \geq \frac{5}{36 \cdot 24} = \frac{1}{172.8}.$$

Case 2. If $\dim(W_{36}) \leq 3$, we have $h^0(F, G_{36} - C) \geq 2$. Pick up a generic irreducible element C'' in $\mathrm{Mov}|G_{36} - C|$. One has $G_{36} \geq C + C''$. Then, by Lemma 2.3 and Lemma 2.5, one has

$$\big(\pi^*(K_X)|_F \cdot G_{36}\big) \geq \frac{1}{19}\Big(\sigma^*(K_{F_0}) \cdot (C + C'')\Big) \geq \frac{4}{19}.$$

Thus one has

$$\mathcal{L}^2 = (\pi^*(K_X)|_F)^2 \geq \frac{1}{36}\big(\pi^*(K_X)|_F \cdot G_{36}\big) \geq \frac{1}{171}.$$

By Proposition 3.3, $\varphi_{m,X}$ is birational for all $m \geq 57$. □

THEOREM 3.10. *Let X be a minimal 3-fold of general type with $\delta(X) = 18$. Assume that $|18K_{X'}|$ and $|24K_{X'}|$ are not composed of the same pencil. If $\dim(V_{36}) \leq 4$, then φ_m is birational for all $m \geq 57$.*

Proof. Since $h^0(X', M_{36}) = P_{36}(X) = 8$, we have $h^0(X', M_{36} - F) \geq 4$. We may modify our previous π so that $\mathrm{Mov}|M_{36} - F|$ is also base point free. Set $|M_{36,-1}| = \mathrm{Mov}|M_{36} - F|$.

Case I. Assume that $|M_{36,-1}|$ is composed of a pencil of surfaces. Noting that $q(X) = 0$, such a pencil $|M_{36,-1}|$ must be over the rational curve (namely, a rational pencil). So

$$36\pi^*(K_X) - F \geq M_{36,-1} \geq 3F_1$$

where F_1 is a generic irreducible element of $|M_{36,-1}|$.

Subcase (I-1). Suppose that both $|F|$ and $|F_1|$ are the same pencil, we have $36\pi^*(K_X) \geq 4F$ and, by Lemma 2.5, the inequality

$$\pi^*(K_X)|_F \geq \frac{1}{10}\sigma^*(K_{F_0}).$$

Thus

$$\mathcal{L}^2 = (\pi^*(K_X)|_F)^2 \geq \frac{1}{100}.$$

Proposition 3.3 implies that φ_m is birational for all $m \geq 48$.

Subcase (I-2). If $|F|$ and $|F_1|$ are not the same pencil, then $|F_1|_F|$ is moving and we pick a generic irreducible element C_1 in $|F_1|_F|$. We are going to study the birationality of $\varphi_{m,X}$ directly. We consider the following linear system on X':

$$|K_{X'} + \lceil s\pi^*(K_X)\rceil + F + F_1| \preccurlyeq |(s+37)K_{X'}|$$

where $s \in \mathbb{Z}_{>0}$. By Kawamata-Viehweg vanishing theorem, we have

$$\begin{aligned} &|K_{X'} + \lceil s\pi^*(K_X)\rceil + F + F_1||_F \\ =\ &|K_F + \lceil s\pi^*(K_X)\rceil|_F + F_1|_F| \\ \succcurlyeq\ &|K_F + \lceil s\pi^*(K_X)|_F\rceil + C_1|. \end{aligned} \tag{3.1}$$

Whenever $s \geq 19$, $(K_{X'} + \lceil s\pi^*(K_X)\rceil)|_F$ is effective. So

$$|K_{X'} + \lceil s\pi^*(K_X)\rceil + F + F_1||_F$$

can distinguish different generic irreducible elements of $|F_1|_F|$. Applying the vanishing theorem once more, we have

$$\begin{aligned} &|K_F + \lceil s\pi^*(K_X)|_F\rceil + C_1||_{C_1} \\ =\ &|K_{C_1} + \lceil s\pi^*(K_X)|_F\rceil|_{C_1}| \\ =\ &|K_{C_1} + D_1|. \end{aligned}$$

Noting that

$$(\pi^*(K_X)|_F \cdot C_1) \geq \frac{2}{19},$$

we have

$$\deg(D_1) \geq s\big(\pi^*(K_X)|_F \cdot C_1\big) > 2$$

whenever $s \geq 20$. By Lemma 3.1, we have seen that φ_m is birational for all $m \geq 57$.

Case II. Assume that $|M_{36,-1}|$ is not composed of a pencil of surfaces. Pick a generic irreducible element $S_{-1} \in |M_{36,-1}|$. We have

$$36\pi^*(K_X) \geq F + S_{-1}.$$

Set $|G_{-1}| = |S_{-1}|_F|$. We use the parallel argument to that for Subcase (I-2). In fact, we have

$$|K_{X'} + \lceil s\pi^*(K_X)\rceil + F + S_{-1}| \preccurlyeq |(s+37)K_{X'}|$$

for any $s \in \mathbb{Z}_{>0}$. By Kawamata-Viehweg vanishing theorem, we have

$$\begin{aligned} & |K_{X'} + \lceil s\pi^*(K_X)\rceil + F + S_{-1}||_F \\ = & |K_F + \lceil s\pi^*(K_X)\rceil|_F + S_{-1}|_F| \\ \succcurlyeq & |K_F + \lceil s\pi^*(K_X)|_F\rceil + G_{-1}|. \end{aligned} \tag{3.2}$$

No matter whether $|G_{-1}|$ is composed of a pencil or not, $|K_{X'}+\lceil s\pi^*(K_X)\rceil+F+S_{-1}||_F$ can distinguish different generic irreducible elements of $|G_{-1}|$ whenever $s \geq 20$ (which is an easy exercise as an application of Kawamata-Viehweg vanishing theorem!).

Finally, we pick a generic irreducible element C_{-1} of $|G_{-1}|$. Applying the vanishing theorem once more, we have

$$\begin{aligned} & |K_F + \lceil s\pi^*(K_X)|_F\rceil + G_{-1}||_{C_{-1}} \\ = & |K_{C_{-1}} + \lceil s\pi^*(K_X)|_F\rceil|_{C_{-1}} + (G_{-1} - C_{-1})|_{C_{-1}}| \\ \succcurlyeq & |K_{C_{-1}} + D_{-1}| \end{aligned}$$

where

$$D_{-1} = \lceil s\pi^*(K_X)|_F\rceil|_{C_{-1}}$$

since $(G_{-1} - C_{-1})|_{C_{-1}}) \geq 0$. Noting that, by Lemma 2.3,

$$\big(\pi^*(K_X)|_F \cdot C_{-1}\big) \geq \frac{2}{19},$$

we have

$$\deg(D_{-1}) \geq s\Big(\pi^*(K_X)|_F \cdot C_{-1}\Big) > 2$$

whenever $s \geq 20$. By Lemma 3.1, we have seen that φ_m is birational for all $m \geq 57$. □

3.5. Proof of Theorem 1.1.

Proof. Theorem 1.1 follows directly from Theorem 3.5, Theorem 3.9 and Theorem 3.10. □

3.6. Proof of Corollary 1.2.

Proof. Recall the following:

THEOREM 3.11. *(see Chen–Chen [EXP3, Theorem 6.2]) Let X be a minimal 3-fold of general type with $\delta(X) \leq 15$. Then φ_m is birational for all $m \geq 56$.*

Now Corollary 1.2 follows directly from Theorem 3.11, [EXP3, Theorem 1.4(2)] ($\delta(X) \neq 16{,}17$) and Theorem 1.1. □

3.7. Open question. It is natural and interesting to ask: is $r_3 \leq 57$ optimal?

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