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002

CR LI-YAU GRADIENT ESTIMATE FOR WITTEN LAPLACIAN VIA BAKRY-EMERY PSEUDOHERMITIAN RICCI CURVATURE*

DER-CHEN CHANG†, SHU-CHENG CHANG‡, TING-JUNG KUO§, AND SIN-HUA LAI¶

Dedicated to Professor Ngaiming Mok on the occasion of his 60th birthday

Abstract. In this paper, we derive the sub-gradient estimate of the CR heat equation associated with the Witten sub-Laplacian via the Bakry-Emery Pseudohermitian Ricci Curvature. With its applications, we first get a Harnack inequality for the positive solution of this CR heat equation in a closed pseudohermitian $(2n+1)$-manifold. Secondly, we obtain Perelman-type linear entropy formulae for this CR heat equation.

Key words. CR heat equation, Li-Yau gradient estimate, Harnack inequality, Perelman Entropy formulae, Bakry-Emery Pseudohermitian Ricci Curvature, Pseudohermitian manifold, Witten Laplacian.

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1. Introduction. In the seminal paper of [LY], P. Li and S.-T. Yau established the parabolic Li-Yau Harnack estimate

$$\frac{\partial(\ln u)}{\partial t} - |\nabla \ln u|^2 + \frac{l}{2t} \geq 0 \tag{1.1}$$

for the positive solution $u(x,t)$ of the time-dependent heat equation

$$\frac{\partial u\,(x,t)}{\partial t} = \Delta u\,(x,t)$$

in a complete Riemannian l-manifold with nonnegative Ricci curvature. Here Δ is the Laplace-Beltrami operator.

Recently, X.-D. Li extended the Li-Yau Harnack estimate to the heat equation associated with the Witten Laplacian via the so-called Bakry-Emery Ricci Curvature in a complete l-manifold. More precisely, let $\phi \in C^2(M)$ and μ be the weighted volume measure on M given by

$$d\mu = e^{-\phi(x)} dv\,(x)$$

and the weighted Laplacian Δ_ϕ be defined by

$$\Delta_\phi := \Delta - \nabla\phi \cdot \nabla$$

which is the infinitesimal generator of the Dirichlet form

$$E(f,g) = \int_M < \nabla f, \nabla g > d\mu$$

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†Department of Mathematics, Georgetown University, Washington DC 20057-0001, USA; Department of Mathematics, Fu Jen Catholic University, Taipei 242, Taiwan (chang@georgetown.edu).
‡Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan; Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan (scchang @math.ntu.edu.tw).
§Department of Mathematics, National Taiwan Normal University, Taipei, 11677, Taiwan (tjkuo 1215@ntnu.edu.tw).
¶Department of Mathematics, National Central University, Chungli 32054, Taiwan (shlai@ncu. edu.tw).

for $f,g\in C_0^\infty(M)$. One of very important motivations to study this symmetric diffusion operator can be illustrated by a deep connection between symmetric diffusion operators Δ_ϕ (called Nelson's diffusion operator in stochastic mechanics, see [Wu] and [Li1]) and Schrödinger operators

$$H=\Delta-V$$

naturally raised in quantum mechanics and quantum field theory. Here $V=\frac{1}{4}|\nabla\phi|^2-\frac{1}{2}\Delta\phi$.

In the other paper of Li ([Li2]), he derived the Li-Yau Harnack estimate

$$\frac{\partial(\ln u)}{\partial t}-|\nabla\ln u|^2+\frac{m}{2t}\geq 0$$

for the positive solution $u(x,t)$ of the heat equation

$$\frac{\partial u(x,t)}{\partial t}=\Delta_\phi u(x,t)$$

in a complete Riemannian l-manifold with nonnegative m-dimensional Bakry-Emery Ricci curvature ([BE])

$$Ric_{m,l}(\Delta_\phi):=Ric+\nabla^2\phi(x)-\frac{\nabla\phi\otimes\nabla\phi}{(m-l)}\geq 0,\ m>l.$$

Using the method of the Li-Yau gradient estimate, the very first paper of H.-D. Cao and S.-T.Yau ([CY]) considered the heat equation

$$\frac{\partial u(x,t)}{\partial t}=Lu(x,t) \tag{1.2}$$

in a closed l-manifold with a positive measure where L is a subelliptic operator that has the form as sum of squares of vector fields

$$L=\sum_{i=1}^{h}X_i^2-Y,\quad h\leq l,\quad Y=\sum_{i=1}^{h}c_iX_i.$$

Here $\mathscr{X}=\{X_1,\ X_2,\ ...,X_h\}$ are smooth vector fields which satisfy bracket generating property, *i.e.,* $\mathscr{X}$ together with their commutators up to finite order span the tangent space at every point of M. Suppose that $[X_i,[X_j,X_k]]$ can be expressed as linear combinations of $X_1,\ X_2,...,X_h$ and their brackets $[X_1,X_2],...,[X_{h-1},X_h]$. They showed that for the positive solution $u(x,t)$ of (1.2) on $M\times[0,\infty)$, there exist constants C_1,C_2,C_3 and $\frac{1}{2}<\lambda<\frac{2}{3}$, such that for any $\delta>1$, $f(x,t)=\ln u\,(x,t)$ satisfies the following gradient estimate

$$\sum_i|X_if|^2-\delta f_t+\sum_\alpha(1+|Y_\alpha f|^2)^\lambda-\delta Yf\leq\frac{C_1}{t}+C_2+C_3t^{\frac{\lambda}{\lambda-1}} \tag{1.3}$$

where $\{Y_\alpha\}=\{[X_i,X_j]\}$, $i,j=1,\ldots,h$.

In the paper of [CKL], we obtained the CR Cao-Yau type Harnack estimate

$$4\frac{\partial(\ln u)}{\partial t}-|\nabla_b\ln u|^2-\frac{1}{3}t[(\ln u)_0]^2+\frac{16}{t}\geq 0 \tag{1.4}$$

for the positive solution $u(x,t)$ of the CR heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta_b u(x,t)$$

in a closed pseudohermitian 3-manifold (M, J, θ) with nonnegative Tanaka-Webster curvature and vanishing torsion. Here Δ_b is the time-independent sub-Laplacian and ∇_b is the subgradient. We also denote $\varphi_0 = T\varphi$ for a smooth function φ and the Reeb vector field T.

In the present paper, via the Bakry-Emery pseudohermitian Ricci Curvature, we extended Chang-Kuo-Lai Harnack estimate (1.4) and Perelman-type entropy formulae to the heat equation

$$\frac{\partial u(x,t)}{\partial t} = Lu(x,t) \tag{1.5}$$

in a closed pseudohermitian $(2n+1)$-manifold $(M, J, \theta, d\mu)$ with

$$Lu(x,t) := \Delta_b u(x,t) - \nabla_b \phi(x) \cdot \nabla_b u(x,t).$$

Here $d\mu = e^{-\phi(x)}\theta \wedge (d\theta)^n$, $\phi \in C^2(M)$.

Note that L satisfies the following integration by parts formula

$$\int_M \langle \nabla_b u, \nabla_b v \rangle d\mu = -\int_M (Lu)v\, d\mu = -\int_M u(Lv)\, d\mu,\ \forall u, v \in C^2(M).$$

We first recall some notions as in section 2. Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_{\mathbb{R}} \xi = 2n$. A CR structure J compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the integrability condition (see next section). A CR structure J can extend to $\mathbb{C}\otimes\xi$ and decomposes $\mathbb{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to eigenvalues i and $-i$, respectively. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ and $\xi = \ker\theta$. Such a choice determines a unique real vector field T transverse to ξ which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$. Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$.

Before we go further, let us introduce an important notion in this paper, "*the m-dimensional Bakry-Emery pseudohermitian Ricci curvature associated with a diffusion operator*", which plays a crucial role in the study of our problems. We define the ∞-dimensional Bakry-Emery pseudohermitian Ricci curvature $Ric(L)$ by

$$Ric(L)(W,W) := R_{\alpha\bar{\beta}} W_{\bar{\alpha}} W_\beta + \mathrm{Re}[\phi_{\alpha\overline{\beta}} W_{\bar{\alpha}} W_\beta]$$

and the m-dimensional Bakry-Emery pseudohermitian Ricci curvature $Ric_{m,n}(L)$ by

$$Ric_{m,n}(L) := Ric(L) - \frac{\nabla_b\phi \otimes \nabla_b\phi}{2(m-4n)}$$

for $W = W^\alpha Z_\alpha + W^{\bar{\alpha}} Z_{\bar{\alpha}} \in T_{1,0} \oplus T_{0,1}$, $m > 4n$. We also define $Tor(L)$ by

$$Tor(L)(W,W) := 2\,\mathrm{Re}\left[\sum_{\alpha,\beta=1}^{n} (i(n-2)A_{\bar{\alpha}\bar{\beta}} - \phi_{\bar{\alpha}\bar{\beta}}) W_\alpha W_\beta\right].$$

Note that m is not necessarily an integer and we use the convention that $m = 4n$ if and only if $L = \Delta_b$ (i.e. ϕ is constant).

By using the arguments of ([LY]) and ([CKL]), we are able to derive the CR version of Li-Yau gradient estimate for the positive solution of CR heat equation (1.5) in a closed pseudohermitian $(2n+1)$-manifold.

THEOREM 1.1. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$2Ric_{m,n}(L)\,(X,\ X) - Tor(L)\,(X,\ X) \geq 0 \tag{1.6}$$

for all $X \in T_{1,0} \oplus T_{0,1}$. If $u\,(x,t)$ is the positive solution of

$$\left(L - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

on $M \times [0,\ \infty)$ with

$$[L,\ \mathbf{T}]\,u = 0. \tag{1.7}$$

Then $f\,(x,t) = \ln u\,(x,t)$ satisfies the following subgradient estimate

$$\left[|\nabla_b f|^2 - \left(1 + \frac{6}{n} + 4k^2\right) f_t\right] \leq \frac{d}{t} + J \tag{1.8}$$

on $M \times (0,\ \infty)$ where $d = m\left(1 + \frac{6}{n} + 4k^2\right)^2\left(k^2+1\right)$, $J = d \cdot \left(\frac{6}{n} + 4k^2\right)^{-1}$, $m > 4n$, and $k = 4n^2 \sup|\nabla_b \phi|$.

As a consequence of Theorem 1.1, for a constant function $\phi \in C^2(M)$, one obtains the CR version of Li-Yau gradient estimate for the positive solution of CR heat equation

$$\left(\Delta_b - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

in a closed pseudohermitian $(2n+1)$-manifold. Theorem 1.1 also recovers our previous results in ([CKL]) in a closed pseudohermitian 3-manifold.

COROLLARY 1.1. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$2Ric\,(X,\ X) - (n-2)Tor\,(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. If $u\,(x,t)$ is the positive solution of

$$\left(\Delta_b - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

with

$$[\Delta_b,\ \mathbf{T}]\,u = 0$$

on $M \times [0,\ \infty)$. Then $f\,(x,t) = \ln u\,(x,t)$ satisfies the following subgradient estimate

$$\left[|\nabla_b f|^2 - \left(1 + \frac{3}{n}\right) f_t + \frac{n}{3} t (f_0)^2\right] < \frac{\left(\frac{9}{n} + 6 + n\right)}{t}.$$

REMARK 1.1.

(1) Formula (1.6) is the CR analogue of the Bakry-Emery Ricci curvature tensors assumption in a closed Riemannian manifold [Li1].
(2) Our subgradient estimates are more delicate due to the fact that the symmetric diffusion operator L is only subelliptic. In fact, by comparing with Riemannian case, we obtain an extra gradient estimate in the so-called missing direction $\mathbf{T}$.
(3) We observe that the main difference between the usual Riemannian Laplacian and sub-Laplacian is the Reeb vector field $\mathbf{T}$. Then condition (1.7) is very natural due to the subellipticity of L in the method of Li-Yau gradient estimate. Furthermore, it follows from Lemma 3.7 (see section 3) that

$$[L,\ \mathbf{T}]\,u = 2\operatorname{Im} Qu - 4\operatorname{Re}\left(\phi_\alpha u_\beta A_{\bar{\alpha}\bar{\beta}}\right) + \langle \nabla_b \phi_0, \nabla u\rangle$$

and

$$[\Delta_b,\ \mathbf{T}]\,u = 2\operatorname{Im} Qu.$$

Here Q is the purely holomorphic second-order operator [GL] defined by

$$Qu = 2i\left(A_{\bar{\alpha}\bar{\beta}} u_\alpha\right)_\beta.$$

Then condition (1.7) holds if $A_{\alpha\beta} = 0$ and $\phi_0 = 0$. In particular, if $A_{\alpha\beta} = 0$, then $[\Delta_b,\ \mathbf{T}]\,u = 0$ as well.
(4) When $\phi(x)$ is a constant function and $n = 1$, we recover the main result of [CKL]. Furthermore, in view of Theorem 1.1, we still have the general subgradient estimate when we replace the lower bound of Bakry-Emery Ricci curvature condition (1.6) by a negative constant. We refer to [CKL] for some details when $\phi(x)$ is a constant function.

Given p, $q \in M$, by Chow's connectivity theorem [Cho], there always exists a horizontal curve (see definition 2.1) with finite length joining them. Now integrating (1.8) over $(\gamma(t), t)$ of a horizontal path $\gamma : [t_1, t_2] \to M$ joining points x_1, x_2 in M, we obtain the following CR version of Li-Yau Harnack inequality.

THEOREM 1.2. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$2Ric_{m,n}(L)\,(X,\ X) - Tor(L)\,(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. If $u(x,t)$ is the positive solution of

$$\left(L - \frac{\partial}{\partial t}\right) u(x,t) = 0$$

with

$$[L,\ \mathbf{T}]\,u = 0$$

on $M \times [0,\ \infty)$. Then for any x_1, x_2 in M and $0 < t_1 < t_2 < \infty$, we have the Harnack inequality

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-p} \exp\left\{-\frac{\left(1+\frac{6}{n}+4k^2\right)}{4}\left[\frac{d_{cc}(x_1,x_2)^2}{t_2-t_1}\right] - q\,(t_2-t_1)\right\}.$$

Here $m > 4n$, $k = 4n^2 \sup |\nabla_b \phi|$, $p = m\left(1 + \frac{6}{n} + 4k^2\right)\left(k^2+1\right)$, $q = p \cdot \left(\frac{6}{n} + 4k^2\right)^{-1}$, *and* d_{cc} *is the Carnot-Carathéodory distance (see Definition 2.1).*

As pointed out by R. Hamilton, the derivation of the entropy formula resembles Li-Yau gradient estimate for the heat equation. We will derive the monotonicity of CR Perelman-type entropy formula for the CR heat equation (1.5) from the subgradient estimate (1.8).

Let $u(x,t)$ be the positive solution of (1.5) on $M \times [0,\infty)$ and $g(x,t)$ be the function which satisfies

$$u(x,t) = \frac{e^{-g(x,t)-Jbt}}{(4\pi t)^{d\times a}}$$

with $m > 4n$, $k = 4n^2 \sup |\nabla_b \phi|$, $d = m\left(1 + \frac{6}{n} + 4k^2\right)^2\left(k^2+1\right)$, $J = d \cdot \left(\frac{6}{n} + 4k^2\right)^{-1}$, and $\int_M u d\mu = 1$. Here $a, b > 0$ to be determined later.

We first define the so-called Boltzmann-Shannon-Nash entropy

$$N(u,t) = -\int_M (\ln u) u d\mu \tag{1.9}$$

and

$$\widetilde{N}(u,t) = N(u,t) - d \times a(\ln 4\pi t + 1) - Jbt. \tag{1.10}$$

Next following a method developed by Perelman, we define

$$\begin{aligned}\mathcal{W}(u,t) &= \int_M \left[t|\nabla_b g|^2 + g - 2d\times a - Jbt\right] u d\mu \\ &= \frac{d}{dt}\left[t\widetilde{N}(u,t)\right]\end{aligned} \tag{1.11}$$

and

$$\widetilde{\mathcal{W}}(u,t) = \mathcal{W}(u,t) + \frac{n}{8}t^2\int_M g_0^2 u d\mu. \tag{1.12}$$

Then by applying Theorem 1.1, we obtain the following entropy formulae for $\widetilde{N}(u,t)$ and $\widetilde{\mathcal{W}}(u,t)$.

Theorem 1.3. *Let (M,J,θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$2Ric_{m,n}(L)(X,\ X) - Tor(L)(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. Let $u(x,t)$ be the positive solution of

$$\left(L - \frac{\partial}{\partial t}\right) u(x,t) = 0$$

and

$$[L,\ \mathbf{T}]\, u = 0$$

on $M \times (0,\ \infty)$ with $\int_M u d\mu = 1$. Then

$$\frac{d}{dt}\widetilde{N}(u,t) = \int_M \left[|\nabla_b g|^2 + \left(1 + \frac{6}{n} + 4k^2\right) g_t + \frac{d \times \left(\frac{6}{n} + 4k^2\right) a}{t} + \left(\frac{6}{n} + 4k^2\right) Jb\right] u d\mu \leq 0$$

for all $t \in (0, \infty)$, $m > 4n$, $k = 4n^2 \sup |\nabla_b \phi|$, $d = m\left(1 + \frac{6}{n} + 4k^2\right)^2 \left(k^2 + 1\right)$, $J = d \cdot \left(\frac{6}{n} + 4k^2\right)^{-1}$, *and* $a, b \geq 1$.

THEOREM 1.4. *Let* (M, J, θ) *be a closed pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$2Ric_{m,n}(L)\,(X,\ X) - Tor(L)\,(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. *Let* $u(x,t)$ *be the positive solution of*

$$\left(L - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

and

$$[L,\ \mathbf{T}]\,u = 0$$

on $M \times [0,\ \infty)$ *with* $\int_M u d\mu = 1$. *Then*

$$\begin{aligned} \frac{d}{dt}\widetilde{\mathcal{W}} \leq & -2t \int_M u \left(\sum_{\alpha,\beta=1}^{n} |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} \left|g_{\alpha\bar{\beta}}\right|^2 \right) d\mu - \frac{2t}{m} \int_M u|Lg|^2 d\mu \\ & -2t \int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b g, \nabla_b g) d\mu \ \leq\ 0 \end{aligned}$$

for $a \geq 2 + 2k^2 H^{-1} + \frac{32}{n}$, $b \geq 1 + k^2 H^{-1} + \frac{16}{n}$, *and* $t \leq \frac{1}{2H}$. *Where* $H = \sup |\nabla_b \phi|^2 \neq 0$, $m > 4n$.

For a constant function $\phi \in C^2(M)$, we define

$$\widetilde{\mathcal{W}}(u,t) = \int_M \left[t|\nabla_b g|^2 + g - \left(\frac{9}{n} + 6 + n\right) \times 2a \right] u d\mu + \frac{n}{2} t^2 \int_M g_0^2 u d\mu \tag{1.13}$$

and

$$u(x,t) = \frac{e^{-g(x,t)}}{(4\pi t)^{(\frac{9}{n}+6+n)\times a}}, \tag{1.14}$$

with $\int_M u d\mu = 1$ (say $\phi = 0$). As a consequence of Theorem 1.4, we have

COROLLARY 1.2. *Let* (M, J, θ) *be a closed pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$2Ric\,(X,\ X) - (n-2)Tor\,(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. *Let* $u(x,t)$ *be the positive solution of*

$$\left(\Delta_b - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

and

$$[\Delta_b,\ \mathbf{T}]\,u = 0$$

on $M \times [0, \infty)$ *with* $\int_M u d\mu = 1$. *Then*

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{W}} \leq & - 4t \int_M u \left(\sum_{\alpha,\beta=1}^{n} |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |g_{\alpha\overline{\beta}}|^2 \right) d\mu \\ & - t \int_M u[2Ric - (n-2)Tor](\nabla_b g, \nabla_b g) d\mu \\ & - \frac{t}{n} \int_M u(\Delta_b g)^2 d\mu \ \leq \ 0.\end{aligned}$$

for all $t \in (0, \infty)$, $a \geq 2 + \frac{4}{n}$.

We briefly describe the methods used in our proofs. In section 2, we first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold. In section 3, we will recall the CR version of the Bochner formula and derive some key Lemmas. In section 4, we derive the Li-Yau gradient estimate for the CR heat equation of the Witten Laplacian. In section 5, by using the subgradient estimate in the previous section, we derive entropy formulae for the CR heat equation (1.5).

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2. Preliminary. We first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold (see [L1], [L2] for more details). Let (M, ξ) be a $(2n+1)$ -dimensional, orientable, contact manifold with contact structure ξ. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$ and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} \tag{2.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. Actually we can always choose Z_α such that $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$; hence, throughout this note, we assume $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$.

The Levi form $\langle \ , \ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W \rangle_{L_\theta} = -i \left\langle d\theta, Z \wedge \overline{W} \right\rangle. \tag{2.2}$$

We can extend $\langle \ , \ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle \overline{Z}, \overline{W} \right\rangle_{L_\theta} = \overline{\langle Z, W \rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle\ ,\ \rangle$. For example

$$\langle u, v\rangle = \int_M u\overline{v}\ d\mu,$$

for functions u and v.

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \theta_\alpha{}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = \theta_{\bar\alpha}{}^{\bar\beta} \otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where $\theta_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \theta_\alpha{}^\beta + \theta\wedge\tau^\beta, \\ 0 &= \tau_\alpha \wedge \theta^\alpha, \\ 0 &= \theta_\alpha{}^\beta + \theta_{\bar\beta}{}^{\bar\alpha}. \end{aligned} \tag{2.3}$$

We can write $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Webster-Tanaka connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar\alpha}\}$, is

$$\begin{aligned} \Pi_\beta{}^\alpha &= \overline{\Pi_{\bar\beta}{}^{\bar\alpha}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha, \\ \Pi_0{}^\alpha &= \Pi_\alpha{}^0 = \Pi_0{}^{\bar\beta} = \Pi_{\bar\beta}{}^0 = \Pi_0{}^0 = 0. \end{aligned}$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho\wedge\theta^{\bar\sigma} + W_\beta{}^\alpha{}_\rho\theta^\rho\wedge\theta - W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho}\wedge\theta + i\theta_\beta\wedge\tau^\alpha - i\tau_\beta\wedge\theta^\alpha \tag{2.4}$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma}, \quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

Here $R_\gamma{}^\gamma{}_{\alpha\bar\beta}$ is the pseudohermitian curvature tensor, $R_{\alpha\bar\beta} = R_\gamma{}^\delta{}_{\alpha\bar\beta}$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the torsion tensor.

We define Ric and Tor by

$$Ric(X,Y) = R_{\alpha\bar\beta}X^\alpha Y^{\bar\beta} \tag{2.5}$$

and

$$Tor(X,Y) = i\sum_{\alpha,\beta}\left(A_{\bar\alpha\bar\beta}X^{\bar\alpha}Y^{\bar\beta} - A_{\alpha\beta}X^\alpha Y^\beta\right) \tag{2.6}$$

for $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$ on $T_{1,0}$.

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0,\alpha,\bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_\alpha = Z_\alpha u$, $u_{\alpha\bar\beta} = Z_{\bar\beta}Z_\alpha u - \omega_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma u$. For a real function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle_{L_\theta} = du(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b u = \sum_\alpha u_{\bar{\alpha}} Z_\alpha + u_\alpha Z_{\bar{\alpha}}$. We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}$$

by

$$(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$

In particular,

$$|\nabla_b u|^2 = 2u_\alpha u_{\overline{\alpha}}, \quad |\nabla_b^2 u|^2 = 2(u_{\alpha\beta} u_{\overline{\alpha}\overline{\beta}} + u_{\alpha\overline{\beta}} u_{\overline{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \sum_\alpha (u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha})$$

and

$$(\nabla^H)^2 u\,(W, U) := \langle (\nabla^H)^2 u\,(W)\,, U\rangle_{L_\theta}$$

where $W,\, U \in T_{1,0} \oplus T_{0,1}$.

For convenience, we will omit the L_θ for the Levi form $\langle\ ,\rangle_{L_\theta}$. That is in this paper all the computations about the form $\langle\ ,\rangle$ is Levi form $\langle\ ,\rangle_{L_\theta}$.

Next we recall the following commutation relations ([L1]). Let φ be a scalar function and $\sigma = \sigma_\alpha \theta^\alpha$ be a $(1,0)$ form, then we have

$$\begin{aligned} \varphi_{\alpha\beta} &= \varphi_{\beta\alpha}, \\ \varphi_{\alpha\bar{\beta}} - \varphi_{\bar{\beta}\alpha} &= ih_{\alpha\overline{\beta}} \varphi_0, \\ \varphi_{0\alpha} - \varphi_{\alpha 0} &= A_{\alpha\beta} \varphi_{\bar{\beta}}, \\ \sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\bar{\gamma}} A_{\gamma\beta} - \sigma_\gamma A_{\alpha\beta,\bar{\gamma}}, \\ \sigma_{\alpha,0\bar{\beta}} - \sigma_{\alpha,\bar{\beta}0} &= \sigma_{\alpha,\gamma} A_{\bar{\gamma}\bar{\beta}} + \sigma_\gamma A_{\bar{\gamma}\bar{\beta},\alpha}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} &= iA_{\alpha\gamma}\sigma_\beta - iA_{\alpha\beta}\sigma_\gamma, \\ \sigma_{\alpha,\bar{\beta}\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\bar{\beta}} &= ih_{\alpha\overline{\beta}} A_{\bar{\gamma}\bar{\rho}}\sigma_\rho - ih_{\alpha\overline{\gamma}} A_{\bar{\beta}\bar{\rho}}\sigma_\rho, \\ \sigma_{\alpha,\beta\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\beta} &= ih_{\beta\overline{\gamma}}\sigma_{\alpha,0} + R_{\alpha\bar{\rho}\beta\bar{\gamma}}\sigma_\rho. \end{aligned} \tag{2.8}$$

Recall a lemma from A. Greenleaf ([Gr]) and also ([CC2]).

Definition 2.1. *Let (M, J, θ) be a closed pseudohermitian 3-manifold with $\xi = \ker\theta$. A piecewise smooth curve $\gamma : [0,1] \to M$ is said to be horizontal if $\gamma'(t) \in \xi$ whenever $\gamma'(t)$ exists. The length of γ is then defined by*

$$l(\gamma) = \int_0^1 (\langle \gamma'(t)\,, \gamma'(t)\rangle)^{\frac{1}{2}}\, dt.$$

The Carnot-Carathéodory distance d_{cc} between two points $p,\, q \in M$ is defined by

$$d_{cc}(p, q) = \inf\{l(\gamma)\,|\gamma \in C_{p,q}\},$$

where $C_{p,q}$ is the set of all horizontal curves which join p and q.

3. The CR Bochner Formulae. In this section, we will derive some CR Bochner formulae and obtain some key Lemmas in a closed pseudohermitian $(2n+1)$-manifold (M, J, θ). We first recall the CR Bochner formula for Δ_b.

LEMMA 3.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. For a (smooth) real function f on M, we have*

$$\begin{aligned}\frac{1}{2}\Delta_b|\nabla_b f|^2 =&|(\nabla^H)^2 f|^2 + \langle \nabla_b f, \nabla_b \Delta_b f\rangle \\ &+ [2Ric - (n-2)Tor]((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}) \\ &+ 2\langle J\nabla_b f, \nabla_b f_0\rangle\end{aligned} \tag{3.1}$$

where $(\nabla_b f)_{\mathbb{C}} = \sum_\alpha f_{\overline{\alpha}} Z_\alpha$ is the corresponding complex $(1,0)$-vector filed $\nabla_b f$.

Now we derive the following CR Bochner formula for L.

LEMMA 3.2. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. For a (smooth) real function f on M, we have*

$$\begin{aligned}\frac{1}{2}L|\nabla_b f|^2 = &|(\nabla^H)^2 f|^2 + \langle \nabla_b f, \nabla_b L f\rangle + (\nabla^H)^2\phi\,(\nabla_b f, \nabla_b f) \\ &+(\nabla^H)^2 f\,(\nabla_b f, \nabla_b \phi) - (\nabla^H)^2 f\,(\nabla_b \phi, \nabla_b f) \\ &+[2Ric - (n-2)\,Tor]((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}) \\ &+2\langle J\nabla_b f, \nabla_b f_0\rangle.\end{aligned}$$

Proof. By Lemma 3.1 and the definition of L, we have

$$\begin{aligned}\frac{1}{2}L|\nabla_b f|^2 &= \frac{1}{2}\Delta_b|\nabla_b f|^2 - \frac{1}{2}\nabla_b\phi\cdot\nabla_b|\nabla_b f|^2 \\ &= |(\nabla^H)^2 f|^2 + \langle \nabla_b f, \nabla_b \Delta_b f\rangle \\ &\quad +[2Ric - (n-2)Tor]((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}) \\ &\quad +2\langle J\nabla_b f, \nabla_b f_0\rangle - (\nabla^H)^2 f\,(\nabla_b \phi, \nabla_b f)\,.\end{aligned}$$

Since

$$\begin{aligned}\langle \nabla_b f, \nabla_b L f\rangle &= \langle \nabla_b f, \nabla_b \Delta_b f\rangle - \langle \nabla_b f, \nabla_b\,(\nabla_b\phi\cdot\nabla_b f)\rangle \\ &= \langle \nabla_b f, \nabla_b \Delta_b f\rangle - (\nabla^H)^2\phi\,(\nabla_b f, \nabla_b f) \\ &\quad -(\nabla^H)^2 f\,(\nabla_b f, \nabla_b \phi)\,,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{2}L|\nabla_b f|^2 = &|(\nabla^H)^2 f|^2 + \langle \nabla_b f, \nabla_b L f\rangle + (\nabla^H)^2\phi\,(\nabla_b f, \nabla_b f) \\ &+(\nabla^H)^2 f\,(\nabla_b f, \nabla_b \phi) - (\nabla^H)^2 f\,(\nabla_b \phi, \nabla_b f) \\ &+[2Ric - (n-2)\,Tor]((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}) \\ &+2\langle J\nabla_b f, \nabla_b f_0\rangle.\end{aligned}$$

□

Lemma 3.3. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. For a (smooth) real function f on M, $k = 4n^2 \sup |\nabla_b \phi|$, and any $\alpha > 0$, we have*

$$\begin{aligned}\frac{1}{2}L|\nabla_b f|^2 \geq & \sum_{\alpha,\beta=1}^{n} |f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |f_{\alpha\overline{\beta}}|^2 + \frac{1}{4n\,(1+\alpha)}|Lf|^2 \\ & -\frac{|\nabla_b \phi \cdot \nabla_b f|^2}{4n\alpha} + \frac{n}{4}f_0^2 + \langle \nabla_b f, \nabla_b Lf\rangle \\ & +(\nabla^H)^2\phi\,(\nabla_b f, \nabla_b f) - 2k^2|\nabla_b f|^2 \\ & +[2Ric - (n-2)\,Tor]((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}) \\ & +2\langle J\nabla_b f, \nabla_b f_0\rangle.\end{aligned}$$

Proof. Note that

$$|(\nabla^H)^2 f|^2 = 2\sum_{\alpha,\beta=1}^{n}\left(|f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2\right).$$

Hence

$$\begin{aligned}|(\nabla^H)^2 f|^2 &= 2\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\overline{\beta}}|^2 + \frac{1}{2}\sum_{\alpha=1}^{n}|f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}|^2 + \frac{n}{2}f_0^2 \\ &\geq 2\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\overline{\beta}}|^2 + \frac{1}{2n}\left(\Delta_b f\right)^2 + \frac{n}{2}f_0^2.\end{aligned}$$

Using the inequality:

$$(a+b)^2 \geq \frac{a^2}{1+\alpha} - \frac{b^2}{\alpha},\ \text{ for all } \alpha > 0,$$

and the definition of L, we obtain

$$\begin{aligned}\left(\Delta_b f\right)^2 &= \left(Lf + \nabla_b\phi\cdot\nabla_b f\right)^2 \\ &\geq \frac{(Lf)^2}{1+\alpha} - \frac{\left(\nabla_b\phi\cdot\nabla_b f\right)^2}{\alpha},\ \forall\alpha > 0.\end{aligned}$$

Hence

$$\begin{aligned}|(\nabla^H)^2 f|^2 \geq & 2\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\overline{\beta}}|^2 \\ & + \frac{(Lf)^2}{2n\,(1+\alpha)} - \frac{\left(\nabla_b\phi\cdot\nabla_b f\right)^2}{2n\alpha} + \frac{n}{2}f_0^2.\end{aligned} \tag{3.2}$$

Since

$$\begin{aligned}&|(\nabla^H)^2 f\,(\nabla_b f, \nabla_b\phi) - (\nabla^H)^2 f\,(\nabla_b\phi, \nabla_b f)\,| \\ &\leq |(\nabla^H)^2 f\,(\nabla_b f, \nabla_b\phi)\,| + |(\nabla^H)^2 f\,(\nabla_b\phi, \nabla_b f)\,|,\end{aligned}$$

and

$$\begin{aligned}|(\nabla^H)^2 f\,(\nabla_b f, \nabla_b\phi)\,| &\leq \frac{k}{2\lambda}|(\nabla^H)^2 f|^2 + \frac{k\lambda}{2}|\nabla_b f|^2, \\ |(\nabla^H)^2 f\,(\nabla_b\phi, \nabla_b f)\,| &\leq \frac{k}{2\lambda}|(\nabla^H)^2 f|^2 + \frac{k\lambda}{2}|\nabla_b f|^2, \forall\lambda > 0,\end{aligned}$$

one has

$$\begin{aligned}&|(\nabla^H)^2 f\,(\nabla_b f, \nabla_b \phi) - (\nabla^H)^2 f\,(\nabla_b \phi, \nabla_b f)\,|\\ \leq\ &\frac{k}{\lambda}|(\nabla^H)^2 f|^2 + k\lambda|\nabla_b f|^2, \quad \forall\lambda > 0.\end{aligned}$$

Let $\lambda = 2k$, we have

$$\begin{aligned}&|(\nabla^H)^2 f\,(\nabla_b f, \nabla_b \phi) - (\nabla^H)^2 f\,(\nabla_b \phi, \nabla_b f)\,|\\ \leq\ &\frac{1}{2}|(\nabla^H)^2 f|^2 + 2k^2|\nabla_b f|^2.\end{aligned}$$

Thus

$$\begin{aligned}&(\nabla^H)^2 f\,(\nabla_b f, \nabla_b \phi) - (\nabla^H)^2 f\,(\nabla_b \phi, \nabla_b f)\\ \geq\ &-\frac{1}{2}|(\nabla^H)^2 f|^2 - 2k^2|\nabla_b f|^2.\end{aligned} \tag{3.3}$$

By Lemma 3.2, (3.2) and (3.3), the proof of Lemma 3.3 is completed. □

Lemma 3.4. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$ -manifold. For a (smooth) real function f on M, $k = 4n^2 \sup|\nabla_b \phi|$, $m > 4n$ and for any $\upsilon > 0$, we have*

$$\begin{aligned}\frac{1}{2}L|\nabla_b f|^2 \geq &\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\overline{\beta}}|^2 + \frac{1}{m}|Lf|^2\\ &+\frac{n}{4}f_0^2 + \langle\nabla_b f, \nabla_b Lf\rangle\\ &+[2Ric_{m,n}\,(L) - Tor\,(L)](\nabla_b f, \nabla_b f)\\ &-\left(\frac{1}{\nu} + 2k^2\right)|\nabla_b f|^2 - \nu|\nabla_b f_0|^2.\end{aligned}$$

Proof. Let

$$m := 4n\,(1+\alpha)\,,$$

then by Lemma 3.3, one has

$$\begin{aligned}\frac{1}{2}L|\nabla_b f|^2 \geq &\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\overline{\beta}}|^2 + \frac{1}{m}|Lf|^2\\ &-\frac{|\nabla_b\phi\cdot\nabla_b f|^2}{m-4n} + \frac{n}{4}f_0^2 + \langle\nabla_b f, \nabla_b Lf\rangle\\ &+(\nabla^H)^2\phi\,(\nabla_b f, \nabla_b f) - 2k^2|\nabla_b f|^2\\ &+[2Ric - (n-2)\,Tor]((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}})\\ &+2\langle J\nabla_b f, \nabla_b f_0\rangle.\end{aligned}$$

Since

$$[2Ric - (n-2)\,Tor]((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}) = \left[Ric - \frac{n-2}{2}Tor\right](\nabla_b f, \nabla_b f),$$

$$\begin{aligned} 2\langle J\nabla_b f,\nabla_b f_0\rangle &\geq -2|\nabla_b f||\nabla_b f_0| \\ &\geq -\frac{1}{\upsilon}|\nabla_b f|^2-\upsilon|\nabla_b f_0|^2, \quad \forall \upsilon>0, \end{aligned}$$

and

$$\begin{aligned} &\left[Ric+(\nabla^H)^2\phi-\frac{\nabla_b\phi\otimes\nabla_b\phi}{m-4n}-\frac{n-2}{2}Tor\right](\nabla_b f,\nabla_b f) \\ &=\left[2Ric_{m,n}(L)-Tor(L)\right](\nabla_b f,\nabla_b f), \end{aligned}$$

we get

$$\begin{aligned} \frac{1}{2}L|\nabla_b f|^2 \geq &\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2+\sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\overline{\beta}}|^2+\frac{1}{m}|Lf|^2 \\ &+\frac{n}{4}f_0^2+\langle\nabla_b f,\nabla_b Lf\rangle \\ &+\left[2Ric_{m,n}(L)-Tor(L)\right](\nabla_b f,\nabla_b f) \\ &-\left(\frac{1}{\upsilon}+2k^2\right)|\nabla_b f|^2-\upsilon|\nabla_b f_0|^2. \end{aligned}$$

□

Lemma 3.5. *Let (M,J,θ) be a closed pseudohermitian $(2n+1)$-manifold. For a smooth real-valued function $f(x)$ defined on M, then*

$$\begin{aligned} Lf_0 =& (Lf)_0+\langle\nabla_b\phi_0,\nabla_b f\rangle \\ &+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar\beta}\right)_{\bar\alpha}+\left(A_{\bar\alpha\bar\beta}f_{\beta}\right)_{\alpha}-\phi_\alpha f_\beta A_{\bar\alpha\bar\beta}-\phi_{\bar\alpha}f_{\bar\beta}A_{\alpha\beta}\right]. \end{aligned}$$

Proof. By direct computation and the commutation relation (2.7), we have

$$\begin{aligned} Lf_0 &= \Delta_b f_0-\langle\nabla_b\phi,\nabla_b f_0\rangle \\ &= (\Delta_b f)_0-\langle\nabla_b\phi,\nabla_b f_0\rangle+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar\beta}\right)_{\bar\alpha}+\left(A_{\bar\alpha\bar\beta}f_{\beta}\right)_{\alpha}\right] \\ &= (Lf)_0+\langle\nabla_b\phi,\nabla_b f\rangle_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar\beta}\right)_{\bar\alpha}+\left(A_{\bar\alpha\bar\beta}f_{\beta}\right)_{\alpha}\right] \\ &\quad-\sum_{\alpha=1}^{n}\left(\phi_\alpha f_{0\bar\alpha}+\phi_{\bar\alpha}f_{0\alpha}\right) \\ &= (Lf)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar\beta}\right)_{\bar\alpha}+\left(A_{\bar\alpha\bar\beta}f_{\beta}\right)_{\alpha}\right] \\ &\quad+\sum_{\alpha=1}^{n}\left(\phi_{\alpha0}f_{\bar\alpha}+\phi_{\bar\alpha0}f_\alpha+\phi_\alpha f_{\bar\alpha0}+\phi_{\bar\alpha}f_{\alpha0}-\phi_\alpha f_{0\bar\alpha}-\phi_{\bar\alpha}f_{0\alpha}\right). \end{aligned}$$

Since

$$\begin{aligned} \phi_{\alpha0}f_{\bar\alpha}+\phi_{\bar\alpha0}f_\alpha &= \phi_{0\alpha}f_{\bar\alpha}-A_{\alpha\beta}f_{\bar\alpha}\phi_{\bar\beta}-\phi_{0\bar\alpha}f_\alpha-A_{\bar\alpha\bar\beta}f_\alpha\phi_\beta \\ &= \langle\nabla_b\phi_0,\nabla_b f\rangle-A_{\alpha\beta}f_{\bar\alpha}\phi_{\bar\beta}-A_{\bar\alpha\bar\beta}f_\alpha\phi_\beta \end{aligned}$$

and

$$\begin{aligned}\phi_\alpha f_{\bar\alpha 0} + \phi_{\bar\alpha} f_{\alpha 0} - \phi_\alpha f_{0\bar\alpha} - \phi_{\bar\alpha} f_{0\alpha} &= \phi_\alpha \left(f_{\bar\alpha 0} - f_{0\bar\alpha}\right) + \phi_{\bar\alpha}\left(f_{\alpha 0} - f_{0\alpha}\right)\\ &= -\phi_\alpha f_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta},\end{aligned}$$

we get

$$\begin{aligned}Lf_0 = {} & (Lf)_0 + \langle \nabla_b \phi_0, \nabla_b f\rangle\\ & + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_\beta\right)_\alpha - \phi_\alpha f_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta}\right].\end{aligned}$$

∎

Now we define $V : C^\infty(M) \to C^\infty(M)$ by

$$\begin{aligned}V(f) = {} & \sum_{\alpha,\beta=1}^{n} 2\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_\beta\right)_\alpha - \phi_\alpha f_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta}\right]\\ & + \sum_{\alpha,\beta=1}^{n} 2\left(f_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta} + f_\alpha f_\beta A_{\bar\alpha\bar\beta}\right) + \langle \nabla_b \phi_0, \nabla_b f\rangle.\end{aligned}$$

Lemma 3.6. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. If $u(x,t)$ is the positive solution of (1.5) on $M \times [0,\infty)$, then $f(x,t) = \ln u(x,t)$ satisfies*

$$Lf_0 - f_{0t} = -2\langle \nabla_b f_0,\ \nabla_b f\rangle + V(f).$$

Proof. By Lemma 3.5, we have

$$\begin{aligned}Lf_0 = {} & (Lf)_0 + \langle \nabla_b \phi_0, \nabla_b f\rangle\\ & + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_\beta\right)_\alpha - \phi_\alpha f_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta}\right].\end{aligned}$$

Also we have

$$\left(L - \frac{\partial}{\partial t}\right) f(x,t) = -\left|\nabla_b f(x,t)\right|^2.$$

All these imply

$$\begin{aligned}Lf_0 - f_{0t} = {} & (Lf)_0 - f_{t0} + \langle \nabla_b \phi_0, \nabla_b f\rangle\\ & + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_\beta\right)_\alpha - \phi_\alpha f_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta}\right]\\ = {} & (Lf - f_t)_0 + \langle \nabla_b \phi_0, \nabla_b f\rangle\\ & + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_\beta\right)_\alpha - \phi_\alpha f_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta}\right]\\ = {} & \left(-\left|\nabla_b f\right|^2\right)_0 + \langle \nabla_b \phi_0, \nabla_b f\rangle\\ & + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_\beta\right)_\alpha - \phi_\alpha f_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} f_{\bar\beta} A_{\alpha\beta}\right]\\ = {} & -2\langle \nabla_b f_0,\ \nabla_b f\rangle + \langle \nabla_b \phi_0, \nabla_b f\rangle\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar\beta}\right)_{\bar\alpha}+\left(A_{\bar\alpha\bar\beta}f_{\beta}\right)_{\alpha}-\phi_\alpha f_\beta A_{\bar\alpha\bar\beta}-\phi_{\bar\alpha}f_{\bar\beta}A_{\alpha\beta}\right]\\
&+2\sum_{\alpha,\beta=1}^{n}\left(f_\alpha f_\beta A_{\bar\alpha\bar\beta}+f_{\bar\alpha}f_{\bar\beta}A_{\alpha\beta}\right)\\
=&-2\left\langle\nabla_b f_0,\ \nabla_b f\right\rangle+V(f).
\end{aligned}
$$

∎

Lemma 3.7. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$[L,\ \mathbf{T}]\,u=0.$$

Then $f(x,t)=\ln u(x,t)$ satisfies

$$V(f)=0.$$

Proof. By direct computation, we have

$$[L,\ \mathbf{T}]\,u=2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}u_{\bar\beta}\right)_{\bar\alpha}+\left(A_{\bar\alpha\bar\beta}u_{\beta}\right)_{\alpha}-\phi_\alpha u_\beta A_{\bar\alpha\bar\beta}-\phi_{\bar\alpha}u_{\bar\beta}A_{\alpha\beta}\right]+\langle\nabla_b\phi_0,\nabla_b u\rangle.$$

Then

$$
\begin{aligned}
V(f)=&\,2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar\beta}\right)_{\bar\alpha}+\left(A_{\bar\alpha\bar\beta}f_{\beta}\right)_{\alpha}-\phi_\alpha f_\beta A_{\bar\alpha\bar\beta}-\phi_{\bar\alpha}f_{\bar\beta}A_{\alpha\beta}\right]\\
&+2\sum_{\alpha,\beta=1}^{n}\left(f_\alpha f_\beta A_{\bar\alpha\bar\beta}+f_{\bar\alpha}f_{\bar\beta}A_{\alpha\beta}\right)+\langle\nabla_b\phi_0,\nabla_b f\rangle\\
=&\,2\sum_{\alpha,\beta=1}^{n}\left[A_{\alpha\beta,\bar\alpha}\frac{u_{\bar\beta}}{u}+A_{\alpha\beta}\left(\frac{u_{\bar\beta\bar\alpha}}{u}-\frac{u_{\bar\beta}u_{\bar\alpha}}{u^2}\right)\right]\\
&+2\sum_{\alpha,\beta=1}^{n}\left[A_{\bar\alpha\bar\beta,\alpha}\frac{u_{\beta}}{u}+A_{\bar\alpha\bar\beta}\left(\frac{u_{\beta\alpha}}{u}-\frac{u_{\beta}u_{\alpha}}{u^2}\right)\right]\\
&+2\sum_{\alpha,\beta=1}^{n}\left[A_{\bar\alpha\bar\beta}\frac{u_\alpha u_\beta}{u^2}+A_{\alpha\beta}\frac{u_{\bar\alpha}u_{\bar\beta}}{u^2}-A_{\bar\alpha\bar\beta}\phi_\alpha\frac{u_\beta}{u}-A_{\alpha\beta}\phi_{\bar\alpha}\frac{u_{\bar\beta}}{u}\right] \qquad (3.4)\\
=&\,\frac{1}{u}\langle\nabla_b\phi_0,\nabla_b u\rangle\\
&+\frac{1}{u}\left\{2\sum_{\alpha,\beta=1}^{n}\left[A_{\alpha\beta,\bar\alpha}u_{\bar\beta}+A_{\alpha\beta}u_{\bar\beta\bar\alpha}+A_{\bar\alpha\bar\beta,\alpha}u_\beta+A_{\bar\alpha\bar\beta}u_{\beta\alpha}\right]\right\}\\
&-\frac{1}{u}\left\{2\sum_{\alpha,\beta=1}^{n}\left[A_{\bar\alpha\bar\beta}u_\beta\phi_\alpha+A_{\alpha\beta}u_{\bar\beta}\phi_{\bar\alpha}\right]\right\}\\
=&\,\frac{1}{u}\,[L,T]\,u\\
=&\,0.
\end{aligned}
$$

∎

4. Li-Yau Subgradient Estimate. In this section, we derive CR version of Li-Yau gradient estimate for the CR heat equation of the Witten Laplacian in a closed pseudohermitian $(2n+1)$-manifold.

Let u be the positive solution of (1.5) and denote

$$f(x,t) = \ln u(x,t).$$

Then $f(x,t)$ satisfies the equation

$$\left(L - \frac{\partial}{\partial t}\right) f(x,t) = -\left|\nabla_b f(x,t)\right|^2. \tag{4.1}$$

Now we define a real-valued function $F(x,\ t,\ a,\ c) : M \times [0,\ T] \times \mathbf{R}^* \times \mathbf{R}^+ \to \mathbf{R}$ by

$$F(x,\ t,\ a,\ c) = t\left(\left|\nabla_b f\right|^2(x) + af_t + ctf_0^2(x)\right), \tag{4.2}$$

where $\mathbf{R}^*{=}\mathbf{R}\backslash\{0\}$ and $\mathbf{R}^+ = (0,\ \infty)$.

PROPOSITION 4.1. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$2Ric_{m,n}(L)(X,\ X) - Tor(L)(X,\ X) \geq 0 \tag{4.3}$$

for all $X \in T_{1,0} \oplus T_{0,1}$ and $k = 4n^2 \sup|\nabla_b \phi|$. If $u(x,t)$ is the positive solution of (1.5) on $M \times [0,\infty)$. Then

$$\begin{aligned}\left(L - \frac{\partial}{\partial t}\right) F \geq & -\frac{1}{t}F - 2\left\langle \nabla_b f,\ \nabla_b F\right\rangle \\ & + t\Bigg[2\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2 + 2\sum_{\alpha\neq\beta=1}^{n}\left|f_{\alpha\overline{\beta}}\right|^2 + \frac{2}{m}(Lf)^2 \\ & + \left(\frac{n}{2} - c\right)f_0^2 - \left(\frac{2}{ct} + 4k^2\right)\left|\nabla_b f\right|^2 + 2ctf_0V(f)\Bigg].\end{aligned} \tag{4.4}$$

Proof. First we differentiate F with respect to the t-variable.

$$F_t = \frac{1}{t}F + t\left[2(1+a)\left\langle \nabla_b f,\ \nabla_b f_t\right\rangle + cf_0^2 + 2ctf_0f_{0t} + aLf_t\right]. \tag{4.5}$$

By the assumption (4.3) and Lemma 3.4, one can compute

$$\begin{aligned} LF &= t\left[L\left|\nabla_b f\right|^2 + aLf_t + ctL\left(f_0^2\right)\right] \\ &= t\left[L\left|\nabla_b f\right|^2 + aLf_t + 2ctf_0Lf_0 + 2ct\left|\nabla_b f_0\right|^2\right] \\ &\geq t\Bigg[2\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2 + 2\sum_{\alpha\neq\beta=1}^{n}\left|f_{\alpha\overline{\beta}}\right|^2 + \frac{2}{m}(Lf)^2 + \frac{n}{2}f_0^2 \\ &\quad +2\left\langle \nabla_b f,\ \nabla_b Lf\right\rangle + 2(2Ric_{m,n}(L) - Tor(L))(\nabla_b f, \nabla_b f) \\ &\quad -\left(\frac{2}{\nu} + 4k^2\right)\left|\nabla_b f\right|^2 - 2\upsilon\left|\nabla_b f_0\right|^2 + aLf_t + 2ctf_0Lf_0 + 2ct\left|\nabla_b f_0\right|^2\Bigg]. \end{aligned}$$

Then taking $\nu = ct$

$$
\begin{aligned}
LF \geq t \Bigg[& 2\sum_{\alpha,\beta=1}^{n} |f_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n} |f_{\alpha\overline{\beta}}|^2 + \frac{2}{m}(Lf)^2 \\
& + \frac{n}{2} f_0^2 + 2\langle \nabla_b f,\ \nabla_b Lf\rangle - \left(\frac{2}{ct} + 4k^2\right)|\nabla_b f|^2 + aLf_t + 2ctf_0 Lf_0 \Bigg].
\end{aligned} \tag{4.6}
$$

It follows from (4.5) and (4.6) that

$$
\begin{aligned}
\left(L - \frac{\partial}{\partial t}\right) F \geq & -\frac{1}{t}F + t\Bigg[2\sum_{\alpha,\beta=1}^{n} |f_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n} |f_{\alpha\overline{\beta}}|^2 \\
& + \frac{2}{m}(Lf)^2 + \left(\frac{n}{2} - c\right) f_0^2 - \left(\frac{2}{ct} + 4k^2\right)|\nabla_b f|^2 \\
& - 2(1+a)\langle \nabla_b f,\ \nabla_b f_t\rangle + 2\langle \nabla_b f,\ \nabla_b Lf\rangle \\
& + 2ctf_0 (Lf_0 - f_{0t}) \Bigg].
\end{aligned} \tag{4.7}
$$

By Lemma 3.6 and definition of F, we have

$$
\begin{aligned}
& 2\langle \nabla_b f,\ \nabla_b Lf\rangle + 2ctf_0(Lf_0 - f_{0t}) - 2(1+a)\langle \nabla_b f,\ \nabla_b f_t\rangle \\
= & 2\left\langle \nabla_b f,\ \nabla_b \left(f_t - |\nabla_b f|^2\right)\right\rangle - 2(1+a)\langle \nabla_b f,\ \nabla_b f_t\rangle \\
& + 2ctf_0\left(-2\langle \nabla_b f_0,\ \nabla_b f\rangle + V(f)\right) \\
= & -2a\langle \nabla_b f,\ \nabla_b f_t\rangle - 2\left\langle \nabla_b f,\ \nabla_b |\nabla_b f|^2\right\rangle \\
& - 4ctf_0\langle \nabla_b f_0,\ \nabla_b f\rangle + 2ctf_0 V(f) \\
= & -2a\left\langle \nabla_b f,\ \nabla_b\left(\frac{1}{at}F - \frac{1}{a}|\nabla_b f|^2 - \frac{ct}{a}f_0^2\right)\right\rangle \\
& - 2\left\langle \nabla_b f,\ \nabla_b |\nabla_b f|^2\right\rangle - 4ctf_0\langle \nabla_b f_0,\ \nabla_b f\rangle + 2ctf_0 V(f) \\
= & -\frac{2}{t}\langle \nabla_b f,\ \nabla_b F\rangle + 2ctf_0 V(f).
\end{aligned} \tag{4.8}
$$

Substitute (4.8) into (4.7), we have

$$
\begin{aligned}
\left(L - \frac{\partial}{\partial t}\right) F \geq & -\frac{1}{t}F - 2\langle \nabla_b f,\ \nabla_b F\rangle \\
& + t\Bigg[2\sum_{\alpha,\beta=1}^{n} |f_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n} |f_{\alpha\overline{\beta}}|^2 + \frac{2}{m}(Lf)^2 \\
& + \left(\frac{n}{2} - c\right) f_0^2 - \left(\frac{2}{ct} + 4k^2\right)|\nabla_b f|^2 + 2ctf_0 V(f)\Bigg].
\end{aligned}
$$

∎

PROPOSITION 4.2. *The same assumption of Proposition 4.1, then*

$$\begin{aligned}\left(L-\frac{\partial}{\partial t}\right)F &\geq \frac{2}{ma^2t}F\left(F-\frac{ma^2}{2}\right)-2\left\langle\nabla_b f,\ \nabla_b F\right\rangle\\ &\quad+t\left[2\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2+2\sum_{\alpha\neq\beta=1}^{n}\left|f_{\alpha\overline{\beta}}\right|^2+\left(\frac{n}{2}-c-\frac{4c}{ma^2}F\right)f_0^2\right.\\ &\quad\left.+\left(-\frac{4\left(a+1\right)}{ma^2t}F-\frac{2}{ct}-4k^2\right)\left|\nabla_b f\right|^2+2ctf_0V\left(f\right)\right].\end{aligned}\tag{4.9}$$

Proof. By definition of F and (4.1),

$$\begin{aligned}Lf &= f_t-\left|\nabla_b f\right|^2\\ &= \frac{1}{at}F-\frac{a+1}{a}\left|\nabla_b f\right|^2-\frac{ct}{a}f_0^2.\end{aligned}$$

Then

$$\begin{aligned}\left(Lf\right)^2 &= \left[\frac{1}{at}F-\left(\frac{a+1}{a}\left|\nabla_b f\right|^2+\frac{ct}{a}f_0^2\right)\right]^2\\ &= \frac{1}{a^2t^2}F^2+\left(\frac{a+1}{a}\left|\nabla_b f\right|^2+\frac{ct}{a}f_0^2\right)^2\\ &\quad-\frac{2\left(a+1\right)}{a^2t}F\left|\nabla_b f\right|^2-\frac{2c}{a^2}Ff_0^2\\ &\geq \frac{1}{a^2t^2}F^2-\frac{2\left(a+1\right)}{a^2t}F\left|\nabla_b f\right|^2-\frac{2c}{a^2}Ff_0^2.\end{aligned}$$

It follows from (4.4),

$$\begin{aligned}\left(L-\frac{\partial}{\partial t}\right)F &\geq \frac{2}{ma^2t}F\left(F-\frac{ma^2}{2}\right)-2\left\langle\nabla_b f,\ \nabla_b F\right\rangle\\ &\quad+t\left[2\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2+2\sum_{\alpha\neq\beta=1}^{n}\left|f_{\alpha\overline{\beta}}\right|^2+\left(\frac{n}{2}-c-\frac{4c}{ma^2}F\right)f_0^2\right.\\ &\quad\left.+\left(-\frac{4\left(a+1\right)}{ma^2t}F-\frac{2}{ct}-4k^2\right)\left|\nabla_b f\right|^2+2ctf_0V\left(f\right)\right].\end{aligned}$$

∎

PROPOSITION 4.3. *The same assumption of Proposition 4.1. Let a, c, $T<\infty$ be fixed. For each $t\in[0,T]$, let $(p(t),s\left(t\right))\in M\times[0,t]$ be the maximal point of F on $M\times[0,t]$. Then at $(p(t),s\left(t\right))$, we have*

$$\begin{aligned}0 &\geq \frac{2}{ma^2t}F\left(F-\frac{ma^2}{2}\right)\\ &\quad+t\left[2\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2+2\sum_{\alpha\neq\beta=1}^{n}\left|f_{\alpha\overline{\beta}}\right|^2+\left(\frac{n}{2}-c-\frac{4c}{ma^2}F\right)f_0^2\right.\\ &\quad\left.+\left(-\frac{4\left(a+1\right)}{ma^2t}F-\frac{2}{ct}-4k^2\right)\left|\nabla_b f\right|^2+2ctf_0V\left(f\right)\right].\end{aligned}\tag{4.10}$$

Proof. Since $F(p(t), s(t), a, c) = \max_{(x,\mu)\in M\times[0,\ t]} F(x,\mu, a, c)$, the point $(p(t), s(t))$ is a critical point of $F(x, s(t), a, c)$. Then

$$\nabla_b F(p(t), s(t), a, c) = 0.$$

On the other hand, since $(p(t), s(t))$ is a maximal point, we can apply maximum principle at $(p(t), s(t))$ on $M \times [0, t]$

$$LF(p(t), s(t), a, c) = [\Delta_b F - \nabla_b \phi \cdot \nabla_b F](p(t), s(t), a, c) \leq 0 \tag{4.11}$$

and

$$\frac{\partial}{\partial t} F(p(t), s(t), a, c) \geq 0. \tag{4.12}$$

Now it follows from (4.11), (4.12) and (4.9) that

$$\begin{aligned} 0 \geq & \frac{2}{ma^2 t} F\left(F - \frac{ma^2}{2}\right) \\ & + t\left[2\sum_{\alpha,\beta=1}^{n} |f_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n} |f_{\alpha\overline{\beta}}|^2 + \left(\frac{n}{2} - c - \frac{4c}{ma^2}F\right) f_0^2\right. \\ & \left. + \left(-\frac{4(a+1)}{ma^2 t}F - \frac{2}{ct} - 4k^2\right)|\nabla_b f|^2 + 2ctf_0 V(f)\right]. \end{aligned}$$

∎

Now we are ready to prove our main Theorem.

Proof of Theorem 1.1. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that

$$2Ric_{m,n}(L)(X, X) - Tor(L)(X, X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$, and

$$[L, \mathbf{T}]\, u = 0.$$

Recall that

$$F(x, t, a, c) = t\left(|\nabla_b f|^2(x) + af_t + ctf_0^2(x)\right).$$

We separate the proof into two parts:

(i) We first claim that for each fixed $T > \frac{6}{n} + 4k^2$,

$$F\left(p(T), s(T), -1 - \frac{6}{n} - 4k^2, \frac{1}{2T}\right) < \frac{m\left(1 + \frac{6}{n} + 4k^2\right)^2 (k^2+1)\, T}{\left(\frac{6}{n} + 4k^2\right)},$$

where we choose $a = -1 - \frac{6}{n} - 4k^2$ and $c = \frac{1}{2T}$. (Here c depends on T and $(P(T), s(T)) \in M \times [0, T]$ is the maximal point of F on $M \times [0, T]$).

We prove by contradiction. Suppose not; that is,

$$F(p(T), s(T), \ -1-\frac{6}{n}-4k^2, \ \frac{1}{2T}) \geq \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)}.$$

Since $F(p(t), s(t))$ is continuous in the variable t when a, c are fixed and $F(p(0), s(0)) = 0$, by Intermediate-value theorem there exists a $t_0 \in (0,\ T]$ such that

$$F\left(p(t_0), s(t_0), \ -1-\frac{6}{n}-4k^2, \ \frac{1}{2T}\right) = \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)}. \tag{4.13}$$

By assumption (1.7) and Lemma 3.7, we have $V(f) = 0$. Now we substitute (4.13) into (4.10) at the point $(p(t_0), s(t_0))$ where $a = -1-\frac{6}{n}-4k^2$, $m > 4n$. Hence

$$\begin{aligned}
0 \geq & \frac{2}{m\left(-1-\frac{6}{n}-4k^2\right)^2 s(t_0)} \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)} \\
& \times\left(\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)} - \frac{m\left(-1-\frac{6}{n}-4k^2\right)^2}{2}\right) \\
& + \left(\frac{n}{2}-\frac{1}{2T}-\frac{2\frac{1}{T}}{m\left(-1-\frac{6}{n}-4k^2\right)^2}\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)}\right) s(t_0) f_0^2 \\
& + \left(\frac{4\left(\frac{6}{n}+4k^2\right)}{m\left(-1-\frac{6}{n}-4k^2\right)^2 s(t_0)} \times \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)} - \frac{4T}{s(t_0)} - 4k^2\right) s(t_0) |\nabla_b f|^2 \\
= & \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)s(t_0)}\left[\frac{\left(2k^2+2\right)T}{\left(\frac{6}{n}+4k^2\right)}-1\right] + \left[\frac{1+2nk^2-2k^2}{\frac{6}{n}+4k^2}-\frac{1}{T}\right] s(t_0) f_0^2 \\
& + 4k^2\left(T-s(t_0)\right)|\nabla_b f|^2.
\end{aligned} \tag{4.14}$$

Since $T > \frac{6}{n}+4k^2$, one has

$$\frac{\left(2k^2+2\right)T}{\left(\frac{6}{n}+4k^2\right)} - 1 > 0$$

and

$$\frac{1+2nk^2-2k^2}{\frac{6}{n}+4k^2} - \frac{1}{2T} \geq 0.$$

Also $T > s(t_0)$, we obtain

$$4k^2\left(T-s(t_0)\right) > 0.$$

This leads to a contradiction to (4.14). Hence

$$F\left(p(T), s(T), \ -1-\frac{6}{n}-4k^2, \ \frac{1}{2T}\right) < \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)}.$$

This implies that

$$\max_{(x,t)\in M\times[0,\ T]} t\left[|\nabla_b f|^2(x) - \left(1+\frac{6}{n}+4k^2\right)f_t + \frac{t}{2T}f_0^2(x)\right] < \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)}.$$

When we fix on the set $M\times\{T\}$, we have

$$T\left[\left|\nabla_b f\right|^2(x)-\left(1+\frac{6}{n}+4k^2\right)f_t+\frac{1}{2}f_0^2(x)\right]<\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)T}{\left(\frac{6}{n}+4k^2\right)}.$$

Hence for any $t>\frac{6}{n}+4k^2$, we have

$$\frac{\left|\nabla_b u\right|^2}{u^2}-\left(1+\frac{6}{n}+4k^2\right)\frac{u_t}{u}<\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{\left(\frac{6}{n}+4k^2\right)}. \tag{4.15}$$

(ii) Secondly, we consider the case when

$$T\leq\frac{6}{n}+4k^2.$$

We claim that

$$F\left(p(T),s\left(T\right),\ -1-\frac{6}{n}-4k^2,\ c\right)<\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c},$$

where we also choose $a=-1-\frac{6}{n}-4k^2$ and $c<\frac{1}{2\left(\frac{6}{n}+4k^2\right)}$ (here c dose *not* depend on T).

We prove by contradiction. Suppose not; that is,

$$F\left(p(T),s\left(T\right),\ -1-\frac{6}{n}-4k^2,\ c\right)\geq\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}.$$

Since $F\left(p(t),s\left(t\right)\right)$ is continuous in the variable t when a, c are fixed and $F\left(p(0),s\left(0\right)\right)=0$, by Intermediate-value theorem there exists a $t_0\in(0,\ T]$ such that

$$F\left(p(t_0),s\left(t_0\right),\ -1-\frac{6}{n}-4k^2,\ c\right)=\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}. \tag{4.16}$$

By assumption (1.7) and Lemma 3.7, we have $V\left(f\right)=0$. Now we substitute (4.16) into (4.10) at the point $\left(p(t_0),s\left(t_0\right)\right)$ where $a=-1-\frac{6}{n}-4k^2$, $m>4n$. Hence

$$\begin{aligned}
0\geq&\frac{2}{m\left(-1-\frac{6}{n}-4k^2\right)^2s\left(t_0\right)}\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}\\
&\times\left(\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}-\frac{m\left(-1-\frac{6}{n}-4k^2\right)^2}{2}\right)\\
&+\left(\frac{n}{2}-c-\frac{4c}{m\left(-1-\frac{6}{n}-4k^2\right)^2}\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}\right)s\left(t_0\right)f_0^2\\
&+\left(\frac{4\left(\frac{6}{n}+4k^2\right)}{m\left(-1-\frac{6}{n}-4k^2\right)^2s\left(t_0\right)}\times\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}-\frac{2}{cs\left(t_0\right)}-4k^2\right)s\left(t_0\right)\left|\nabla_b f\right|^2\\
=&\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{c\left(\frac{6}{n}+4k^2\right)s\left(t_0\right)}\left[\frac{\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}-\frac{1}{2}\right]+\left[\frac{2+2nk^2-k^2}{\frac{6}{n}+4k^2}-c\right]s\left(t_0\right)f_0^2\\
&+4k^2\left(\frac{1}{2c}-s\left(t_0\right)\right)\left|\nabla_b f\right|^2.
\end{aligned} \tag{4.17}$$

Since $c < \frac{1}{2\left(\frac{6}{n}+4k^2\right)}$, one has

$$\frac{\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c} - \frac{1}{2} > 0,$$

$$\frac{1+2nk^2-2k^2}{\frac{6}{n}+4k^2} - c \geq 0.$$

Also $\frac{1}{2c} > \frac{6}{n} + 4k^2 \geq T$ and $T > s\left(t_0\right)$, we obtain

$$4k^2\left(\frac{1}{2c} - s\left(t_0\right)\right) > 0.$$

This leads to a contadiction to (4.17). Hence

$$F\left(p(T), s\left(T\right),\ -1-\frac{6}{n}-4k^2,\ c\right) < \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)c}$$

for $c < \frac{1}{2\left(\frac{6}{n}+4k^2\right)}$ and $T \leq \frac{6}{n}+4k^2$.

By the same argument as above, we have

$$\frac{\left|\nabla_b u\right|^2}{u^2} - \left(1+\frac{6}{n}+4k^2\right)\frac{u_t}{u} < \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)ct} \tag{4.18}$$

for $c < \frac{1}{2\left(\frac{6}{n}+4k^2\right)}$ and $t \leq \frac{6}{n}+4k^2$.

(iii) Combining (4.15) and (4.18), we obtain that for any fixed $c < \frac{1}{2\left(\frac{6}{n}+4k^2\right)}$,

$$\begin{aligned}\frac{\left|\nabla_b u\right|^2}{u^2} - \left(1+\frac{6}{n}+4k^2\right)\frac{u_t}{u} &< \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{2\left(\frac{6}{n}+4k^2\right)ct} \\ &\quad + \frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{\left(\frac{6}{n}+4k^2\right)}\end{aligned}$$

for any $t > 0$.

Finally, let $c \to \frac{1}{2\left(\frac{6}{n}+4k^2\right)}$; we are done. □

Proof of Theorem 1.2. Let γ be a horizontal curve with $\gamma\left(t_1\right) = x_1$ and $\gamma\left(t_2\right) = x_2$. We define $\eta : [t_1, t_2] \to M \times [t_1, t_2]$ by

$$\eta\left(t\right) = \left(\gamma\left(t\right), t\right).$$

Clearly $\eta\left(t_1\right) = \left(x_1, t_1\right)$ and $\eta\left(t_2\right) = \left(x_2, t_2\right)$. Let $f = \ln u\left(x, t\right)$, integrate $\frac{d}{dt}f$ along η, we get

$$f\left(x_2, t_2\right) - f\left(x_1, t_1\right) = \int_{t_1}^{t_2} \frac{d}{dt} f\, dt = \int_{t_1}^{t_2} \{\langle\dot{\gamma}, \nabla_b f\rangle + f_t\} dt.$$

Applying Theorem 1.1 to f_t, this yields

$$\begin{aligned} f\left(x_2,t_2\right)-f\left(x_1,t_1\right) &\geq \int_{t_1}^{t_2}\left\{\frac{|\nabla_b f|^2}{\left(1+\frac{6}{n}+4k^2\right)}-\frac{p}{t}-q+\langle\dot{\gamma},\nabla_b f\rangle\right\}dt \\ &\geq -\frac{1+\frac{6}{n}+4k^2}{4}\int_{t_1}^{t_2}|\dot{\gamma}|^2dt-p\ln\left(\frac{t_2}{t_1}\right)-q\left(t_2-t_1\right). \end{aligned}$$

Here $p=m\left(1+\frac{6}{n}+4k^2\right)\left(k^2+1\right)$ and $q=\frac{m\left(1+\frac{6}{n}+4k^2\right)\left(k^2+1\right)}{\left(\frac{6}{n}+4k^2\right)}$.

Now we choose

$$|\dot{\gamma}|=\frac{d_{cc}\left(x_1,x_2\right)}{t_2-t_1}.$$

Then the inequality in Theorem 1.2 follows by taking exponential of the above inequality. □

5. Perelman-Type Entropy Formulae. In this section, we prove the monotonicity formulae for $\widetilde{N}(u,t)$ and $\widetilde{\mathcal{W}}(u,t)$ under the CR heat equation

$$\left(L-\frac{\partial}{\partial t}\right)u\left(x,t\right)=0$$

on $M\times[0,\infty)$.

Proof of Theorem 1.3. Let g be the function which satisfies

$$u(x,t)=\frac{e^{-g(x,t)-Jbt}}{(4\pi t)^{d\times a}}$$

with $d=m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)$, $J=\frac{m\left(1+\frac{6}{n}+4k^2\right)^2\left(k^2+1\right)}{\left(\frac{6}{n}+4k^2\right)}$, $m>4n$, and $\int_M ud\mu=1$. Here $a,b>0$ to be determined. Denote $f=\ln u$. Then

$$f=-g-d\times a\ln(4\pi t)-Jbt.$$

Since $|\nabla_b f|^2=|\nabla_b g|^2$ and $f_t=-g_t-\frac{d\times a}{t}-Jb$, one has from Theorem 1.1,

$$|\nabla_b f|^2-\left(1+\frac{6}{n}+4k^2\right)f_t\leq\frac{d}{t}+J.$$

It follows that

$$\begin{aligned} &|\nabla_b g|^2+\left(1+\frac{6}{n}+4k^2\right)g_t+\frac{d\times\left[\left(1+\frac{6}{n}+4k^2\right)a-1\right]}{t} \\ &+J\left[\left(1+\frac{6}{n}+4k^2\right)b-1\right]\leq 0. \end{aligned} \tag{5.1}$$

Note that

$$\int_M\langle\nabla_b u,\nabla_b v\rangle d\mu=-\int_M Luvd\mu=-\int_M uLvd\mu,\quad\forall\ u,v\in C^2(M).$$

Hence we have

$$\begin{aligned}\frac{d}{dt}N(u,t) &= -\int_M u_t \ln u d\mu - \int_M u\frac{u_t}{u}d\mu \\ &= -\int_M Lu \ln u d\mu \\ &= \int_M \langle \nabla_b u, \nabla_b \ln u\rangle d\mu \\ &= \int_M (\frac{|\nabla_b u|^2}{u})d\mu \\ &= \int_M u|\nabla_b g|^2 d\mu\end{aligned}$$

and

$$\frac{d}{dt}\widetilde{N}(u,t) = -\int_M Lu \ln u d\mu - \frac{d\times a}{t} - Jb = \int_M u|\nabla_b g|^2 d\mu - \frac{d\times a}{t} - Jb. \quad (5.2)$$

But

$$\begin{aligned}\int_M g_t u d\mu &= -\int_M f_t u d\mu - \left(\frac{d\times a}{t} + Jb\right)\int_M u d\mu \\ &= -\int_M u_t d\mu - \frac{d\times a}{t} - Jb \\ &= -\frac{d\times a}{t} - Jb.\end{aligned}$$

Thus we have

$$\frac{d}{dt}\widetilde{N}(u,t) = \int_M u\left[|\nabla_b g|^2 + \left(1+\frac{6}{n}+4k^2\right)g_t + \frac{d\times\left(\frac{6}{n}+4k^2\right)a}{t} + \left(\frac{6}{n}+4k^2\right)Jb\right]d\mu.$$

Now we choose a, b such that

$$\begin{aligned}0 &< \left(\frac{6}{n}+4k^2\right)a < \left(1+\frac{6}{n}+4k^2\right)a - 1, \\ 0 &< \left(\frac{6}{n}+4k^2\right)b < \left(1+\frac{6}{n}+4k^2\right)b - 1.\end{aligned}$$

In other words, a, $b > 1$. It follows from (5.1) that

$$\frac{d}{dt}\widetilde{N}(u,t) < 0.$$

□

Proposition 5.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$2Ric_{m,n}(L)(X,\ X) - Tor(L)(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. Let $u(x,t)$ be the positive solution of

$$\left(L - \frac{\partial}{\partial t}\right)u(x,t) = 0$$

and

$$[L,\ \mathbf{T}]\, u = 0$$

on $M \times [0,\ \infty)$ *with* $\int_M u d\mu = 1$. *Then*

$$\begin{aligned} \frac{d}{dt}\mathcal{W} \leq & -2t \int_M u \left(\sum_{\alpha,\beta=1}^{n} |\left(\ln u\right)_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |\left(\ln u\right)_{\alpha\overline{\beta}}|^2 \right) d\mu \\ & - \frac{n}{2} t \int_M (\ln u)_0^2\, u d\mu - \frac{2}{m} t \int_M u |L \ln u|^2 d\mu \\ & - 2t \int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b \ln u, \nabla_b \ln u) d\mu \\ & + 2t\upsilon \int_M u |\nabla_b \left(\ln u\right)_0|^2 d\mu + \left(\frac{2}{\upsilon} + 4k^2\right) t \int_M u |\nabla_b \ln u|^2 d\mu \\ & + 2 \int_M u |\nabla_b g|^2 d\mu - \frac{d \times a}{t} - 2Jb. \end{aligned} \tag{5.3}$$

Where $d = m\left(1 + \frac{6}{n} + 4k^2\right)^2 \left(k^2 + 1\right)$, $J = d \cdot \left(\frac{6}{n} + 4k^2\right)^{-1}$, $\upsilon > 0$, $a, b \in \mathbf{R}$, *and* $m > 4n$.

Proof. Direct computation gives us

$$\begin{aligned} \mathcal{W} &= \int_M \left[t|\nabla_b g|^2 + g - 2d \times a - Jbt\right] u d\mu \\ &= \int_M \left[t|\nabla_b g|^2 - \ln u - d \times a \ln(4\pi t) - d \times 2a - 2Jbt\right] u d\mu \\ &= t\left[\int_M (|\nabla_b g|^2 - \frac{d \times a}{t} - Jb) u d\mu\right] \\ &\quad - \left[\int_M u \ln u d\mu + \int_M d \times a(\ln(4\pi t) + 1) u d\mu + Jbt \int_M u d\mu\right] \\ &= t\frac{d}{dt}\widetilde{N}(u,t) + \widetilde{N}(u,t) \ = \ \frac{d}{dt}\left(t\widetilde{N}(u,t)\right). \end{aligned} \tag{5.4}$$

Hence $\frac{d}{dt}\mathcal{W} = t\frac{d^2}{dt^2}\widetilde{N}(u,t) + 2\frac{d}{dt}\widetilde{N}(u,t)$. It follows from (5.2) that

$$\begin{aligned} \frac{d^2}{dt^2}\widetilde{N}(u,t) =& \frac{d}{dt}\left[-\int_M uL \ln u d\mu - \frac{d \times a}{t} - Jb\right] \\ =& -\int_M u_t L \ln u d\mu - \int_M u \frac{\partial}{\partial t}(L \ln u) d\mu + \frac{d \times a}{t^2} \\ =& -\int_M LuL \ln u d\mu - \int_M u \frac{\partial}{\partial t}(L \ln u) d\mu + \frac{d \times a}{t^2}. \end{aligned} \tag{5.5}$$

Note that

$$\begin{aligned} (\frac{\partial}{\partial t} - L)(L \ln u) = L(\frac{\partial}{\partial t} - L) \ln u \ &= \ L[\frac{u_t}{u} - (\frac{Lu}{u} - \frac{|\nabla_b u|^2}{u^2})] \\ &= L(|\nabla_b \ln u|^2). \end{aligned} \tag{5.6}$$

Again, from the CR Bochner formula for L, we have

$$\begin{aligned}
L(|\nabla_b \ln u|^2) \geq 2\left(\sum_{\alpha,\beta=1}^{n} |\left(\ln u\right)_{\alpha\beta}|^2 + 2\sum_{\alpha\neq\beta=1}^{n} |\left(\ln u\right)_{\alpha\overline{\beta}}|^2\right) + \frac{2}{m}|L\ln u|^2 + \frac{n}{2}\left(\ln u\right)_0^2 \\
+ 2\langle\nabla_b \ln u, \nabla_b L\ln u\rangle - 2\nu\left|\nabla_b\left(\ln u\right)_0\right|^2 - \left(\frac{2}{\upsilon} + 4k^2\right)|\nabla_b \ln u|^2 \\
+ 2[2Ric_{m,n}(L) - Tor(L)](\nabla_b \ln u, \nabla_b \ln u)
\end{aligned} \tag{5.7}$$

for all $\nu > 0$ and $m > 4n$. It follows from (5.5), (5.6), and (5.7) that

$$\begin{aligned}
\frac{d^2}{dt^2}\widetilde{N}(u,t) &= -\int_M LuL\ln u d\mu + \frac{d\times a}{t^2} \\
&\quad -\int_M uL|\nabla_b \ln u|^2 d\mu - \int_M uL(L\ln u)d\mu \\
&\leq -\int_M LuL\ln u d\mu - \int_M uL(L\ln u)d\mu + \frac{d\times a}{t^2} \\
&\quad -2\int_M u\left(\sum_{\alpha,\beta=1}^{n} |\left(\ln u\right)_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |\left(\ln u\right)_{\alpha\overline{\beta}}|^2\right)d\mu \\
&\quad -\int_M \frac{n}{2}\left(\ln u\right)_0^2 u d\mu - \int_M \frac{2}{m}|L\ln u|^2 u d\mu \\
&\quad +2\upsilon\int_M u|\nabla_b\left(\ln u\right)_0|^2 d\mu + \left(\frac{2}{\upsilon} + 4k^2\right)\int_M u|\nabla_b \ln u|^2 d\mu \\
&\quad -2\int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b \ln u, \nabla_b \ln u)d\mu \\
&\quad -2\int_M u\langle\nabla_b \ln u,\ \nabla_b L\ln u\rangle d\mu.
\end{aligned}$$

Since

$$\int_M LuL\ln u d\mu = \int_M uL\left(L\ln u\right)d\mu = -\int_M \langle\nabla_b u, \nabla_b L\ln u\rangle d\mu$$

and

$$\int_M u\left\langle\nabla_b \ln u,\ \nabla_b L\ln u\right\rangle d\mu = \int_M \left\langle\nabla_b u,\ \nabla_b L\ln u\right\rangle d\mu,$$

we have

$$-\int_M Lu(L\ln u)d\mu - \int_M uL(L\ln u)d\mu - 2\int_M u\left\langle\nabla_b \ln u,\ \nabla_b L\ln u\right\rangle d\mu = 0.$$

Hence, we conclude that

$$\begin{aligned}\frac{d^2}{dt^2}\widetilde{N}(u,t) \leq & -2\int_M u\left(\sum_{\alpha,\beta=1}^{n} |\left(\ln u\right)_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |\left(\ln u\right)_{\alpha\overline{\beta}}|^2\right) d\mu \\ & -\int_M \frac{n}{2}\left(\ln u\right)_0^2 u d\mu - \int_M \frac{2}{m}|L\ln u|^2 u d\mu + \frac{d\times a}{t^2} \\ & -2\int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b \ln u, \nabla_b \ln u) d\mu \\ & +2\upsilon\int_M u|\nabla_b\left(\ln u\right)_0|^2 d\mu + \left(\frac{2}{\upsilon}+4k^2\right)\int_M u|\nabla_b \ln u|^2 d\mu.\end{aligned}$$

Hence, we can derive (5.3) and the proof of the Proposition is therefore complete. ∎

LEMMA 5.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that $u(x,t)$ be the positive solution of*

$$\left(L - \frac{\partial}{\partial t}\right) u\left(x,t\right) = 0$$

on $M\times[0,\infty)$ and let g be the function which satisfies

$$u\left(x,t\right) = \frac{e^{-g(x,t)-Jbt}}{(4\pi t)^{d\times a}}$$

where $a, b, d, J \in \mathbf{R}$. Then

$$\begin{aligned}& 2\int_M u g_0 g_{0t} d\mu + \int_M g_0^2 L u d\mu \\ = & -2\int_M u|\nabla_b g_0|^2 d\mu - 4\int_M u g_0\langle\nabla_b\phi, \nabla_b g_0\rangle d\mu \\ & -2\int_M u g_0\langle\nabla_b\phi_0, \nabla_b g\rangle d\mu \\ & -4\int_M u g_0\left(\sum_{\alpha,\beta=1}^{n}\left(A_{\alpha\beta}g_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta}g_{\beta}\right)_{\alpha} - \phi_\alpha g_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha}g_{\bar\beta}A_{\alpha\beta}\right) d\mu \\ & +4\int_M u g_0\left(\sum_{\alpha,\beta=1}^{n}\left(g_{\bar\alpha}g_{\bar\beta}A_{\alpha\beta} + g_\alpha g_\beta A_{\bar\alpha\bar\beta}\right)\right) d\mu.\end{aligned} \tag{5.8}$$

Proof. Since $u\left(x,t\right) = \frac{e^{-g(x,t)-Jbt}}{(4\pi t)^{d\times a}}$, we have

$$Lg = g_t + |\nabla_b g|^2 + \frac{d\times a}{t} + Jb. \tag{5.9}$$

Note that

$$\begin{aligned}\left(|\nabla_b g|^2\right)_0 &= \left(2g_\alpha g_{\bar\alpha}\right)_0 = 2\left(g_{\alpha 0}g_{\bar\alpha} + g_\alpha g_{\bar\alpha 0}\right) \\ &= 2\left(g_{0\alpha} - A_{\alpha\beta}g_{\bar\beta}\right)g_{\bar\alpha} + 2g_\alpha\left(g_{0\bar\alpha} - A_{\bar\alpha\bar\beta}g_\beta\right) \\ &= 2\langle\nabla_b g_0, \nabla_b g\rangle - 2\left(g_{\bar\alpha}g_{\bar\beta}A_{\alpha\beta} + g_\alpha g_\beta A_{\bar\alpha\bar\beta}\right).\end{aligned} \tag{5.10}$$

By Lemma 3.5, we have

$$
\begin{aligned}
Lg_0 = & (Lg)_0 + \langle \nabla_b \phi_0, \nabla_b g \rangle \\
& + 2\sum_{\alpha,\beta=1}^{n} \left[\left(A_{\alpha\beta} g_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} g_{\beta}\right)_{\alpha} - \phi_\alpha g_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} g_{\bar\beta} A_{\alpha\beta} \right].
\end{aligned}
\tag{5.11}
$$

Then by (5.9), (5.10), (5.11), one has

$$
\begin{aligned}
& 2\int_M u g_0 g_{0t} d\mu + \int_M g_0^2 Lu d\mu \\
= & 2\int_M u g_0 g_{0t} d\mu + \int_M uL\left(g_0^2\right) d\mu \\
= & 2\int_M u g_0 g_{0t} d\mu + 2\int_M u g_0 L g_0 d\mu + 2\int_M u|\nabla_b g_0|^2 d\mu \\
= & 2\int_M u g_0 \Bigg\{ 2Lg_0 - \langle \nabla_b \phi_0, \nabla_b g \rangle \\
& - 2\sum_{\alpha,\beta=1}^{n} \left[\left(A_{\alpha\beta} g_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} g_{\beta}\right)_{\alpha} - \phi_\alpha g_\beta A_{\bar\alpha\bar\beta} - \phi_{\bar\alpha} g_{\bar\beta} A_{\alpha\beta} \right] \Bigg\} d\mu \\
& + 2\int_M u g_0 \left(-2\langle \nabla_b g, \nabla_b g_0 \rangle + 2 g_{\bar\alpha} g_{\bar\beta} A_{\alpha\beta} + 2 g_\alpha g_\beta A_{\bar\alpha\bar\beta} \right) d\mu \\
& + 2\int_M u|\nabla_b g_0|^2 d\mu.
\end{aligned}
\tag{5.12}
$$

Note that

$$
\begin{aligned}
& 2\int_M u g_0 \left(2Lg_0 - 2\langle \nabla_b g, \nabla_b g_0 \rangle\right) d\mu + 2\int_M u|\nabla_b g_0|^2 d\mu \\
= & 4\int_M u g_0 \Delta_b g_0 d\mu - 4\int_M u g_0 \langle \nabla_b \phi, \nabla_b g_0 \rangle d\mu \\
& - 4\int_M u g_0 \langle \nabla_b g, \nabla_b g_0 \rangle d\mu + 2\int_M u|\nabla_b g_0|^2 d\mu \\
= & -4\int_M u|\nabla_b g_0|^2 d\mu - 4\int_M g_0 \langle \nabla_b u, \nabla_b g_0 \rangle d\mu - 4\int_M u g_0 \langle \nabla_b \phi, \nabla_b g_0 \rangle d\mu \\
& - 4\int_M u g_0 \langle \nabla_b g, \nabla_b g_0 \rangle d\mu + 2\int_M u|\nabla_b g_0|^2 d\mu \\
= & -2\int_M u|\nabla_b g_0|^2 d\mu - 4\int_M u g_0 \langle \nabla_b \phi, \nabla_b g_0 \rangle d\mu.
\end{aligned}
\tag{5.13}
$$

Substituting (5.13) into (5.12), we can derive (5.8). ☐

Proof of Theorem 1.4. Now for some $\alpha > 0$ which to be determined later such that $\frac{d}{dt}\widetilde{\mathcal{W}_\alpha} \leq 0$, we consider

$$
\widetilde{\mathcal{W}_\alpha} = \mathcal{W} + \alpha t^2 \int_M g_0^2 \frac{e^{-g(x,t)-Jbt}}{(4\pi t)^{d\times a}} d\mu.
$$

Since $\ln u = -g - d \times a \ln(4\pi t) - Jbt$, by Proposition 5.1 and Lemma (5.1), we get

$$\begin{aligned}
\frac{d}{dt}\widetilde{\mathcal{W}_\alpha} \leq & -2t\int_M u\left(\sum_{\alpha,\beta=1}^n |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |g_{\alpha\overline{\beta}}|^2\right) d\mu \\
& -\frac{n}{2}t\int_M ug_0^2 d\mu - \frac{2}{m}t\int_M u|Lg|^2 d\mu - \frac{d\times a}{t} - 2Jb \\
& -2t\int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b g, \nabla_b g) d\mu \\
& + \left(2t\upsilon - 2\alpha t^2\right)\int_M u|\nabla_b g_0|^2 d\mu + \left(\frac{2}{\upsilon}t + 4k^2 t + 2\right) \\
& \times \int_M u|\nabla_b g|^2 d\mu + 2\alpha t\int_M ug_0^2 d\mu \\
& -4\alpha t^2\int_M ug_0\langle\nabla_b\phi, \nabla_b g_0\rangle d\mu - 2\alpha t^2\int_M ug_0\langle\nabla_b\phi_0, \nabla_b g\rangle d\mu \\
& -4\alpha t^2\int_M ug_0\sum_{\alpha,\beta=1}^n\left[\left(A_{\alpha\beta}g_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta}g_\beta\right)_\alpha\right] d\mu \\
& +4\alpha t^2\int_M ug_0\left[\sum_{\alpha,\beta=1}^n\left(g_{\bar\alpha}g_{\bar\beta}A_{\alpha\beta} + g_\alpha g_\beta A_{\bar\alpha\bar\beta} + \phi_\alpha g_\beta A_{\bar\alpha\bar\beta} + \phi_{\bar\alpha}g_{\bar\beta}A_{\alpha\beta}\right)\right] d\mu.
\end{aligned}$$

Using the inequality:

$$xy \leq ax^2 + \frac{1}{4a}y^2, a > 0.$$

Then

$$\begin{aligned}
& -4\alpha t^2\int_M ug_0\langle\nabla_b\phi, \nabla_b g_0\rangle d\mu \\
\leq & \ 4\alpha t^2\int_M u|g_0|^2|\nabla_b\phi|^2 d\mu + \alpha t^2\int_M u|\nabla_b g_0|^2 d\mu \\
\leq & \ 4\alpha t^2 \sup|\nabla_b\phi|^2\int_M u|g_0|^2 d\mu + \alpha t^2\int_M u|\nabla_b g_0|^2 d\mu
\end{aligned}$$

and

$$\begin{aligned}
& 2\sum_{\alpha,\beta=1}^n\left[g_{\bar\alpha}g_{\bar\beta}A_{\alpha\beta} + g_\alpha g_\beta A_{\bar\alpha\bar\beta} + \phi_\alpha g_\beta A_{\bar\alpha\bar\beta} + \phi_{\bar\alpha}g_{\bar\beta}A_{\alpha\beta} - \left(A_{\alpha\beta}g_{\bar\beta}\right)_{\bar\alpha} - \left(A_{\bar\alpha\bar\beta}g_\beta\right)_\alpha\right] \\
& -\langle\nabla_b\phi_0, \nabla_b g\rangle = \frac{1}{u}[L, T]u,
\end{aligned}$$

we have

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{W}_\alpha} \leq& -2t\int_M u\left(\sum_{\alpha,\beta=1}^n |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |g_{\alpha\overline{\beta}}|^2\right)d\mu \\ &+\left(2\alpha t - \frac{n}{2}t + 4\alpha t^2 \sup|\nabla_b\phi|^2\right)\int_M ug_0^2 d\mu - \frac{2}{m}t\int_M u|Lg|^2 d\mu \\ &-\frac{d\times a}{t} - 2Jb - 2t\int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b g, \nabla_b g)d\mu \\ &+(2\upsilon t - \alpha t^2)\int_M u\left|\nabla_b g_0\right|^2 d\mu + \left(\frac{2}{\upsilon}t + 4k^2 t + 2\right)\int_M u|\nabla_b g|^2 d\mu \\ &+2\alpha t^2\int_M g_0\,[L,T]\,ud\mu.\end{aligned}$$

Let $H = \sup|\nabla_b\phi|^2 \neq 0$. We may choose $\alpha = \frac{n}{8}$ and $\nu = \frac{n}{16}t$, and $t \leq \frac{1}{2H}$, then

$$\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}_{\frac{n}{8}}}$$

and

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{W}} \leq& -2t\int_M u\left(\sum_{\alpha,\beta=1}^n |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |g_{\alpha\overline{\beta}}|^2\right)d\mu - \frac{2}{m}t\int_M u|Lg|^2 d\mu \\ &-2t\int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b g, \nabla_b g)d\mu \\ &+\left(\frac{32}{n} + 4k^2 t + 2\right)\int_M u|\nabla_b g|^2 d\mu - \frac{d\times a}{t} - 2Jb.\end{aligned} \tag{5.14}$$

However, we know that from Theorem 1.1

$$\begin{aligned}\int_M u|\nabla_b g|^2 d\mu &= \int_M u|\nabla_b \ln u|^2 d\mu \\ &\leq \int_M \left(1 + \frac{6}{n} + 4k^2\right)u\frac{u_t}{u}d\mu + \left(\frac{d}{t} + J\right)\int_M ud\mu \\ &\leq \left(1 + \frac{6}{n} + 4k^2\right)\int_M Lud\mu + \left(\frac{d}{t} + J\right) \\ &\leq \frac{d}{t} + J.\end{aligned}$$

Then

$$\left(2 + \frac{32}{n} + 4k^2 t\right)\int_M u|\nabla_b g|^2 d\mu \leq \frac{d\times\left(2 + \frac{32}{n} + 4k^2 t\right)}{t} + \left(2 + \frac{32}{n} + 4k^2 t\right)J.$$

Now if we choose

$$a \geq 2 + \frac{32}{n} + 2k^2 H^{-1} \qquad \text{and} \qquad b \geq 1 + \frac{16}{n} + k^2 H^{-1},$$

then (5.14) implies that

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{W}} \leq& -2t\int_M u\left(\sum_{\alpha,\beta=1}^n |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |g_{\alpha\overline{\beta}}|^2\right)d\mu - \frac{2}{m}t\int_M u|Lg|^2 d\mu \\ &-2t\int_M u[2Ric_{m,n}(L) - Tor(L)](\nabla_b g, \nabla_b g)d\mu \,\leq\, 0.\end{aligned}$$

Proof of Corollary 1.2. If ϕ is a constant function ($m = 4n$), we have

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{W}_\alpha} \leq -4t \int_M u \left(\sum_{\alpha,\beta=1}^{n} |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |g_{\alpha\overline{\beta}}|^2 \right) d\mu \\ +(2\alpha t - nt) \int_M u g_0^2 d\mu - \frac{t}{n} \int_M u(\Delta_b g)^2 d\mu \\ -t \int_M u[2Ric - (n-2)Tor](\nabla_b g, \nabla_b g) d\mu \\ +(2\upsilon t - 2\alpha t^2) \int_M u \left|\nabla_b g_0\right|^2 d\mu + \left(\frac{2}{\upsilon} t + 2\right) \int_M u|\nabla_b g|^2 d\mu - \frac{d \times a}{t},\end{aligned}$$

here $d = \left(\frac{9}{n} + 6 + n\right)$. We choose $\alpha = \frac{n}{2}$ and $\nu = \frac{n}{2}t$, then

$$\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}_{\frac{n}{2}}}$$

and

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{W}} \leq & -4t \int_M u \left(\sum_{\alpha,\beta=1}^{n} |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |g_{\alpha\overline{\beta}}|^2 \right) d\mu - \frac{t}{n} \int_M u(\Delta_b g)^2 d\mu \\ & - t \int_M u[2Ric - (n-2)Tor](\nabla_b g, \nabla_b g) d\mu \\ & + \left(\frac{4}{n} + 2\right) \int_M u|\nabla_b g|^2 d\mu - \frac{d \times a}{t}.\end{aligned} \tag{5.15}$$

But

$$\left(2 + \frac{4}{n}\right) \int_M u|\nabla_b g|^2 d\mu \leq \frac{d \times \left(2 + \frac{4}{n}\right)}{t}.$$

If we choose $a \geq 2 + \frac{4}{n}$, then

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{W}} \leq -4t \int_M u \left(\sum_{\alpha,\beta=1}^{n} |g_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^{n} |g_{\alpha\overline{\beta}}|^2 \right) d\mu \\ -t \int_M u[2Ric - (n-2)Tor](\nabla_b g, \nabla_b g) d\mu - \frac{t}{n} \int_M u(\Delta_b g)^2 d\mu \\ \leq 0.\end{aligned}$$

This completes the proof of the corollary.

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