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SMOOTHNESS OF THE RADON-NIKODYM DERIVATIVE OF A CONVOLUTION OF ORBITAL MEASURES ON COMPACT SYMMETRIC SPACES OF RANK ONE*

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Dedicated to Professor Ngaiming Mok on the occasion of his sixtieth birthday

Abstract. Let G/K be a compact symmetric space of rank one. The aim of this paper is to give sufficient conditions for the C^{ν}-smoothness of the Radon Nikodym derivative $f_{a_1,\dots,a_p} = d\left(\mu_{a_1} * \dots * \mu_{a_p}\right)/d\mu_{\mathsf{G}}$ of the convolution $\mu_{a_1} * \dots * \mu_{a_p}$ of some orbital measures μ_{a_i}, with respect to the Haar measure μ_{G} of G. This generalizes some of the main results in [12], in the case of compact rank one symmetric spaces, where the absolute continuity of the measure $\mu_{a_1} * \dots * \mu_{a_p}$ with respect to $d\mu_{\mathsf{G}}$ was considered. Our main result generalizes also the main results in [1] and [7], where the L^2-regularity was considered.

As a consequence of our main result, we give sufficient conditions for $f_{a_1,\dots,a_p}$ to be in $L^q\left(G, d\mu_{\mathsf{G}}\right)$ for all $q \geq 1$ and for the Fourier series of $f_{a_1,\dots,a_p}$ to converge absolutely and uniformly to $f_{a_1,\dots,a_p}$.

Key words. Orbital measures, Radon-Nikodym derivative, symmetric spaces.

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1. Introduction. In [11], Makarov proved that there exists a singular probability measure μ on $\mathbb{R}$ such that μ^n, the convolution of μ with itself n times, is singular for all positive integers n. In [4] Brown and Hewitt generalized Makarov's result to non-discrete locally compact abelian groups. Ragozin [12] proved that this phenomena does not occur for orbital measures on symmetric spaces G/K, more precisely he proved that a convolution of dim G/K number of continuous orbital measures is necessarily absolutely continuous with respect to the Haar measure of G. The question of the L^2-regularity of the Radon Nikodym derivative of a convolution of orbital measures was considered in [1]. The scheme developed in [1] works for any compact symmetric space, but some details were carried out only for $SU(2)/SO(2)$, and later, K. Hare and J. He [7], using the main results of [1], carried out the computations for the case of a compact, simply connected, symmetric space of rank one.

The aim of this paper is to investigate the C^{ν}-smoothness of the Radon-Nikodym derivative of a convolution of orbital measures on compact symmetric spaces of rank one. More precisely, let G/K be a compact symmetric space of rank one, $a_1, \dots, a_p$ be points in $G - N_G(K)$, where $N_G(K)$ is the normalizer of K in G, and let $\mu_{a_1}, \dots, \mu_{a_p}$ be the orbital measures supported on the double cosets Ka_1K,..., Ka_pK, (see the definition below). In this paper, we prove that the Radon-Nikodym derivative of $\mu_{a_1} * \dots * \mu_{a_p}$ with respect to the Haar measure μ_G is in $C^{\nu}(G)$ as soon as $p \geq c(\nu, G)$, where $c(\nu, G)$ is a constant which will be computed explicitly in terms of the integer ν and some data from the group G. As a corollary of this, we get some sufficient conditions for its L^q-regularity for $q \geq 1$. The paper is organized as follows: in section 2, we review a few basic facts about convolution of orbital measures, the Fourier transform of measures, and the Plancherel Theorem. In section 3 we introduce

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the appropriate Sobolev spaces. In section 4 we give some estimates of the spherical functions on rank one compact symmetric spaces. In section 5 we give a proof of the main result of the paper and state some of its corollaries.

2. Convolution of orbital measures, fourier transform, and Plancherel Theorem. The aim of this section is to introduce the notation, give some definitions, and state the results which will be needed in this paper. Let $(X, \mathcal{M}, \mu)$ be a measure space. A measure τ on $\mathcal{M}$ is said to be absolutely continuous with respect to μ, written $\tau \ll \mu$, if $\tau(E) = 0$ whenever $\mu(E) = 0$. We say that μ and τ are mutually singular, and we write $\mu \perp \tau$, if there exists A in $\mathcal{M}$ such that $\mu(A) = 0$ and $\tau(X - A) = 0$. If μ and τ are positive measures, then a Theorem of Radon-Nikodym, says that τ is absolutely continuous with respect to μ if and only if there exists a function f in $L^1(X, d\mu)$, $f \geq 0$, such that

$$\tau(A) = \int_A f(x)\, d\mu(x),$$

for all $A \in \mathcal{M}$. The function f, which is unique μ-a.e., is called the Radon-Nikodym derivative of τ with respect to μ, and is denoted by $d\tau/d\mu$.

Let $M = G/K$ be a compact symmetric space and let a be an element of G. The positive functional

$$I_a(h) = \int_K \int_K h(k_1 a k_2)\, d\mu_K(k_1)\, d\mu_K(k_2), \ h \in C(G),$$

where a is an element of G, $C(G)$ is the set of continuous functions on G, and μ_K is the Haar measure of K, defines, by the Radon-Riesz representation theorem, a positive orbital measure on G, which will be denoted by μ_a. If a is not in the normalizer of K in G, then μ_a is continuous. Since KaK has empty interior in G and since the support of the measure μ_a is KaK, the measure μ_a is singular with respect to the Haar measure μ_G of the group G.

Given two positive measures μ and τ on G, again, by the Radon-Riesz representation theorem, the positive functional

$$J_{\mu*\tau}(h) = \int_G \int_G h(xy)\, d\mu(x)\, d\tau(y),$$

defines a positive measure, called the convolution of μ and τ, which is denoted by $\mu * \tau$. We write

$$J_{\mu*\tau}(h) = \int_G h(x)\, d(\mu * \tau)(x).$$

Similarly, by induction, the convolution of any number of measures can be defined.

The Fourier transform of a measure μ evaluated at a unitary irreducible representation

$$\pi : G \to \mathsf{GL}(V_\pi)$$

of the group G, denoted by $\widehat{\mu}(\pi)$, is an element of $\mathsf{End}(V_\pi)$ defined by

$$\widehat{\mu}(\pi) X = \int_G \pi\left(g^{-1}\right) X d\mu(g), \ X \text{ in } V_\pi.$$

Properties of the Fourier transform can be found in [14].

Let $(.,.)_{V_\pi}$ be a $G-$invariant inner product in V_π. Then $(.,.)_{V_\pi}$ induces an inner product $(.,.)_{HS}$ on $\mathsf{End}\,(V_\pi)$, called the Hilbert-Schmidt product, which is defined as follows: Let $e_1,...,e_{d_\pi}$ be an orthonormal basis of V_π, where $d_\pi = \dim V_\pi$, and let T and S be two elements of $\mathsf{End}\,(V_\pi)$. Then

$$(T,S)_{HS} = \sum_{i=1}^{d_\pi} (Te_i, Se_i)_{V_\pi}.$$

It can be proved that the Hilbert-Schmidt inner product is independent of the choice of the orthonormal basis of E_π and

$$(S,T)_{\mathsf{HS}} = \mathsf{Tr}\,(T^* \circ S),$$

where T^* is the adjoint of T with respect to $(.,.)_{V_\pi}$. The corresponding norm will be denoted by $\|.\|_{HS}$. The Hilbert direct sum

$$\mathcal{O}p\left(\widehat{G}\right) = \widehat{\bigoplus_{[\pi]\in\widehat{G}}} \mathsf{End}\,(V_\pi)$$

of the Hilbert spaces $(\mathsf{End}\,(V_\pi))_{[\pi]\in\widehat{G}}$ with the inner product

$$\left((S_\pi)_{[\pi]\in\widehat{G}}, (T_\pi)_{[\pi]\in\widehat{G}}\right)_{\mathcal{O}p\left(\widehat{G}\right)} = \sum_{[\pi]\in\widehat{G}} (S_\pi, T_\pi)_{\mathsf{HS}} \tag{2.1}$$

is a Hilbert space which contains $\bigoplus_{[\pi]\in\widehat{G}} \mathsf{End}\,(V_\pi)$ as a dense subset.

Consider the map

$$\widehat{\cdot} : L^2(G) \longrightarrow \mathcal{O}p\left(\widehat{G}\right) \tag{2.2}$$

defined by

$$\widehat{f} = \left(\sqrt{d_\pi}\pi(f)\right)_{[\pi]\in\widehat{G}},$$

where

$$\pi(f)\,v = \int_G f(g)\,\pi(g)\,v\,d\mu(g),\ v \in V_\pi.$$

The Parseval-Plancherel Theorem, see [14], Chapter 3, states that the map (2.2) is an isomorphism, hence

$$(f_1, f_2)_{L^2(G)} = \left(\widehat{f_1}, \widehat{f_2}\right)_{\mathcal{O}p\left(\widehat{G}\right)}. \tag{2.3}$$

3. The Sobolev norm of the density of the convolution of orbital measures. Let G be a compact semisimple Lie group, and let $\langle .,. \rangle$ be the Killing form on the Lie algebra $\mathfrak{g}$ of G. Let $X_1, X_2, ..., X_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to the non-degenerate form $\langle .,. \rangle$. The Casimir operator is given by

$$-\Delta = \sum_{j=1}^{n} X_j^2.$$

The operator Δ is an element of the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$, and it can be proved that it is independent of the orthonormal basis chosen. It is also known that Δ is essentially self adjoint, see [16], hence Δ has a unique self adjoint extension, which will be denoted by Δ.

Since $I - \Delta$ is a positive self adjoint operator, its square root $(I - \Delta)^{\frac{1}{2}}$ is well defined. For a positive real number s, we put

$$\mathcal{L}_s = (I - \Delta)^{\frac{s}{2}} := \left((I - \Delta)^{\frac{1}{2}}\right)^s.$$

Define an inner product on $C^\infty(G)$ as follows

$$\begin{aligned}(\varphi, \psi)_{H^s(G)} &= (\mathcal{L}_s\varphi, \mathcal{L}_s\psi)_{L^2(G)} \\ &= \int_G \mathcal{L}_s\varphi(g)\, \mathcal{L}_s\psi(g)\, d\mu_G(g),\end{aligned}$$

where $d\mu_G$ is the Haar measure of G. The completion of $C^\infty(G)$ with respect to the inner product $(\varphi, \psi)_{H^s(G)}$ is the Sobolev space, denoted by $H^s(G)$, and described by

$$H^s(G) = \left\{ f \in L^2(G, d\mu_G) \mid \|f\|^2_{H^s(G)} = \left\|(I - \Delta)^{\frac{s}{2}} f\right\|^2_{L^2(G, d\mu_G)} < \infty \right\},$$

where $\|\cdot\|_{H^s}$ is the norm corresponding to the inner product $(\cdot, \cdot)_{H^s(G)}$. For more detail, one can see chapter 10 of [13].

Let $\pi : G \longrightarrow GL(V_\pi)$ be an irreducible unitary representation of G, with highest weight λ_π. It is well-established, see [15], that

$$\Delta\left(\pi(g)_{ij}\right) = \kappa_\pi \pi(g)_{ij},$$

where

$$\kappa_\pi = \langle \lambda_\pi + 2\rho, \lambda_\pi \rangle,$$

is the Casimir constant associated with the representation π, 2ρ is the sum of all positive roots, and $\pi(g)_{ij}$ is the element in the i^{th} row and j^{th} column of the matrix $\pi(g)$.

Suppose that for some positive integer p, there exists a function $f_{a_1,...,a_p} \in L^2(G)$ such that

$$\widehat{\mu_{a_1} * ... * \mu_{a_p}} = \widehat{f_{a_1,...,a_p}}. \tag{3.1}$$

Then

$$\mu_{a_1} * ... * \mu_{a_p} = f_{a_1,...,a_p} d\mu_G.$$

From now on, the convolution $\mu_{a_1} * ... * \mu_{a_p}$ will be denoted by $\mu(a_1, ..., a_p)$.

PROPOSITION 1. *With the above notation defined, we have*

$$\left\| f_{a_1,\ldots,a_p} \right\|^2_{H^s(G)} = \sum_{[\pi]\in\widehat{G}_K} d_\pi \left(1+\kappa_\pi\right)^s \prod_{k=1}^{p} \left|\varphi_\pi\left(a_k\right)\right|^2,$$

where $\widehat{G}_K$ is the set of equivalence classes of spherical representations of the Gelfand pair (G, K) and φ_π is the spherical function corresponding to the spherical representation π.

Proof. By Plancheral-Parseval identity, we get

$$\begin{aligned} \left\| f_{a_1,\ldots,a_p} \right\|^2_{H^s(G)} &= \left\| (I-\Delta)^{\frac{s}{2}} f_{a_1,\ldots,a_p} \right\|^2_{L^2(G)} \\ &= \left((I-\Delta)^s f_{a_1,\ldots,a_p},\ f_{a_1,\ldots,a_p} \right)_{L^2(G)} \\ &= \left(\widehat{(I-\Delta)^s f_{a_1,\ldots,a_p}}, \widehat{f_{a_1,\ldots,a_p}} \right)_{\mathsf{HS}}. \end{aligned} \tag{3.2}$$

Moreover

$$\widehat{(I-\Delta)^s f_{a_1,\ldots,a_p}} = \left(1+\kappa_\pi\right)^s \widehat{f_{a_1,\ldots,a_p}}. \tag{3.3}$$

Combining (3.2) and (3.3), we deduce that

$$\begin{aligned} \left\| f_{a_1,\ldots,a_p} \right\|^2_{H^s(G)} &= \sum_{[\pi]\in\widehat{G}_K} d_\pi \left(1+\kappa_\pi\right)^s \left(\widehat{f_{a_1,\ldots,a_p}}(\pi), \widehat{f_{a_1,\ldots,a_p}}(\pi) \right)_{\mathsf{HS}} \\ &= \sum_{[\pi]\in\widehat{G}_K} d_\pi \left(1+\kappa_\pi\right)^s \mathsf{Tr}\left(\widehat{f_{a_1,\ldots,a_p}}(\pi)\, \widehat{f_{a_1,\ldots,a_p}}(\pi)^* \right). \end{aligned} \tag{3.4}$$

Combining (3.4), (3.1), and the definition of the Hilbert-Schmidt norm, we get

$$\begin{aligned} \left\| f_{a_1,\ldots,a_p} \right\|^2_{\mathrm{H}^s(G)} &= \sum_{[\pi]\in\widehat{G}_K} d_\pi \left(1+\kappa_\pi\right)^s \mathsf{Tr}\left(\widehat{\mu(a_1,...,a_p)}(\pi)\, \widehat{\mu(a_1,...,a_p)}(\pi)^* \right) \\ &= \sum_{[\pi]\in\widehat{G}_K} d_\pi \left(1+\kappa_\pi\right)^s \left\| \widehat{\mu(a_1,...,a_p)}(\pi) \right\|^2_{\mathsf{HS}}. \end{aligned}$$

Since (π, V_π) is an irreducible unitary spherical representation, we have $\dim V_\pi^K = 1$, where

$$V_\pi^K = \left\{ X \text{ in } E_\pi \mid \pi(k)\, X = X \quad \text{for all } k \text{ in } K \right\}.$$

For more details see [1]. Let $X_{\pi,1}$ be a unit vector in V_π^K. An argument similar to the one in Lemma 6 of [1], gives

$$\left\| \widehat{\mu(a_1,...,a_p)}(\pi) \right\|^2_{\mathsf{HS}} = \begin{cases} \left|\left(\pi(a_1)\, X_{\pi,1}, X_{\pi,1}\right)\right|^2 \ldots \left|\left(\pi(a_p)\, X_{\pi,1}, X_{\pi,1}\right)\right|^2 & \text{if } \pi \text{ is spherical} \\ 0 & \text{otherwise.} \end{cases}$$

The Proposition follows from

$$\varphi_\pi(a_i) = \left(\pi(a_i)\, X_{\pi,1}, X_{\pi,1}\right).$$

□

4. Spherical functions of compact symmetric spaces of rank one. Let M be a compact symmetric space of rank one, G the identity component of the isometry group of M, and K the isotropy subgroup of G at a fixed point o in M. Then G is semisimple and M can be identified with G/K, see [8]. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{k}$ (respectively $\mathfrak{p}$) is the Lie algebra of K (respectively the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $\langle .,. \rangle$ of G). The tangent space to M at $x_0 = eK$, where e is the identity element of G, can be identified with $\mathfrak{p}$ via the map which associates to an element X in $\mathfrak{p}$ the element $\widetilde{X}$, where

$$\widetilde{X} f(x_0) = \frac{d}{dt}\Big|_{t=0} f(\exp(tX)\, x_0).$$

The form $-\langle .,. \rangle$ is K-invariant and defines a Riemannian metric on M.

Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, where $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{a}_{\mathbb{C}}$) is the complexification of $\mathfrak{g}$ (resp. $\mathfrak{a}$). Choose a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$ and let $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$, or simply Σ^+, be the corresponding set of positive roots. It is known that Σ^+ is either of the form $\{\alpha\}$ or $\{\alpha, 2\alpha\}$, depending on the symmetric space G/K, see section 5. Let π be a spherical representation and let μ be the restriction of the highest weight of π to $\mathfrak{a}$. Then

$$\mu = \begin{cases} n\alpha & \text{if } \Sigma^+ = \{\alpha\}; \\ \\ 2n\alpha & \text{if } \Sigma^+ = \{\alpha, 2\alpha\}. \end{cases}$$

Consequently the highest weights of the spherical representations of G, when restricted to $\mathfrak{a}$ are of the form $\mu = n\beta$, where $n \in \mathbb{N}$ and β is the largest element of Σ^+. Put

$$m_\beta = \dim(\mathfrak{g}_{\mathbb{C}})_\beta, m_{\beta/2} = \dim(\mathfrak{g}_{\mathbb{C}})_{\beta/2} \text{ and } \rho = \frac{1}{2}\sum_{\gamma \in \Sigma^+} m_\gamma \gamma,$$

where

$$(\mathfrak{g}_{\mathbb{C}})_\gamma = \{X \text{ in } \mathfrak{g}_{\mathbb{C}} \mid ad(H)X = \gamma(H)X \text{ for all } H \text{ in } \mathfrak{a}_{\mathbb{C}}\}.$$

Since G/K is of rank one, $\mathfrak{a} = \mathbb{R}H_0$. In what follows, we chose H_0 such that $\alpha(H_0) = \sqrt{-1}$. The Killing form $\langle .,. \rangle$ induces an inner product on the set of roots, which will be denoted also by $\langle .,. \rangle$.

THEOREM 1 ([9], Chapter V, Theorem 4.5, or [18], Chapter 11, Theorem 11.4.21). *Let G/K be a compact connected, simply connected symmetric space of rank one. Then the spherical functions of the pair $(G,\ K)$ are parametrized by the classes $[j\beta]$, $j = 0, 1, 2, ...$ and β is the maximal element of Σ^+, and are given by*

$$\varphi_{j\beta}(\exp(tH_0)) = {}_2F_1\left(\frac{1}{2}m_{\beta/2} + m_\beta + j, -j; \frac{1}{2}(m_{\beta/2} + m_\beta + 1); \sin^2\left(\frac{\langle \beta, \alpha \rangle}{2\langle \alpha, \alpha \rangle}t\right)\right), \qquad (4.1)$$

where ${}_2F_1(.,.;.;.)$ is the Gaussian hypergeometric function.

REMARK 1. *Since $\varphi_{j\beta}$ is $K-$bi-invariant, and since $G = K\exp(\mathfrak{a})K$, it is enough to define $\varphi_{j\beta}$ on $\exp(\mathfrak{a})$.*

Let a and b be two real numbers, $a, b > -1$, and let

$$P_j^{(a,b)}(x) = \begin{cases} \dfrac{(-1)^j}{2^j j!}(1-x)^{-a}(1+x)^{-b}\left(\frac{d}{dx}\right)^j\left((1-x)^{j+a}(1+x)^{j+b}\right) & \text{if } x \in (-1,1); \\ (-1)^j \frac{(b+1)_j}{j!} & \text{if } x = -1; \\ \frac{(a+1)_j}{j!} & \text{if } x = 1. \end{cases}$$

be the Jacobi polynomial, where

$$(a+1)_j = \frac{\Gamma(a+1+j)}{\Gamma(a+1)}.$$

Corollary 1. *For j large enough, we have*

$$\varphi_{j\beta}(\exp(tH_0)) \sim \tfrac{\Gamma(a+1)}{j^a} P_j^{(a,b)}\left(\cos \tfrac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} t\right),$$

where

$$a = \tfrac{1}{2}\left(m_{\beta/2} + m_\beta - 1\right) \text{ and } b = \tfrac{1}{2}(m_\beta - 1).$$

Proof. From the following well known relation between Jacobi polynomial and Gauss hypergeometric series, see [3],

$$P_j^{(a,b)}(x) = \frac{(a+1)_j}{j!} {}_2F_1\left(a+b+j+1, -j; a+1; \frac{1-x}{2}\right). \tag{4.2}$$

The Corollary follows from (4.1), (4.2), and

$$\frac{j!}{(a+1)_j} \sim \frac{\Gamma(a+1)}{j^a}.$$

□

The following result gives the expansion of $P_j^{(a,b)}\left(\cos \frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} t\right)$ for j sufficiently large

Lemma 1 (Darboux [5]). *Let a and b be real numbers. Then*

$$P_j^{(a,b)}\left(\cos \tfrac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} t\right) = \frac{\cos\left(\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} jt + \frac{1}{2}(a+b+1)\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} t - \frac{1}{2}a\pi - \frac{1}{4}\pi\right) + O\left(j^{-\frac{1}{2}}\right)}{\sqrt{j\pi}\left(\sin \frac{\langle\beta,\alpha\rangle}{2\langle\alpha,\alpha\rangle} t\right)^{a+\frac{1}{2}}\left(\cos \frac{\langle\beta,\alpha\rangle}{2\langle\alpha,\alpha\rangle} t\right)^{b+\frac{1}{2}}},$$

as $j \to \infty$ uniformly on any subinterval $\delta \le \frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} t \le \pi - \delta$, $\delta > 0$.

Proof. See [5] or chapter 10 of [3]. □

Combining Corollary 1 and Lemma 1, we get

Proposition 2. *If $0 < t < \frac{\langle\alpha,\alpha\rangle}{\langle\beta,\alpha\rangle}\pi$, then there exists a positive constant C such that for j sufficiently large, we have*

$$|\varphi_{j\beta}(\exp(tH_0))| \le \frac{\sigma(\alpha,\beta,t)}{j^{\frac{1}{2}+a}},$$

where

$$\sigma(\alpha,\beta,t) = C\left|\left(\sin \tfrac{\langle\beta,\alpha\rangle}{2\langle\alpha,\alpha\rangle} t\right)^{-a-\frac{1}{2}}\left(\cos \tfrac{\langle\beta,\alpha\rangle}{2\langle\alpha,\alpha\rangle} t\right)^{-b-\frac{1}{2}}\right|.$$

5. Proof of the main Theorem. Riemannian symmetric spaces of rank one G/K are two point homogeneous in the sense that if p_1, p_2, q_1, q_2 are points of G/K such that $d(p_1, q_1) = d(p_2, q_2)$, where $d(.,.)$ is the distance induced from the Riemannian metric, then there exists an isometry $\varkappa$ such that $p_2 = \varkappa(p_1)$ and $q_2 = \varkappa(q_1)$. Conversely, every compact two-point homogeneous space is a symmetric space of rank one. It is well known that compact symmetric spaces of rank one are given by the following list (see [17] or [19], Theorem 8.12.2)

(1) the sphere $\mathbb{S}^n = \mathsf{SO}(n+1)/\mathsf{SO}(n)$.
(2) the real projective space $P^n(\mathbb{R}) = \mathsf{SO}(n+1)/\mathsf{O}(n)$.
(3) the complex projective space $P^n(\mathbb{C}) = \mathsf{SU}(n+1)/\mathsf{U}(n)$.
(4) the quaternionic projective space $P^n(\mathbb{H}) = \mathsf{Sp}(n+1)/\mathsf{Sp}(n) \times \mathsf{Sp}(1)$. and
(5) the octonian projective plane $P^2(\mathbb{O}) = \mathsf{F}_4/\mathsf{Spin}(9)$.

For low dimensions, we have the following isomorphisms:

$$P^1(\mathbb{R}) \cong \mathbb{S}^1,\ P^1(\mathbb{C}) \cong \mathbb{S}^2, \text{ and } P^1(\mathbb{H}) \cong \mathbb{S}^4.$$

So, without loss of generality, we can assume in the lists above that $n \geq 2$. The compact symmetric spaces of rank one mentioned above are simply connected except the real projective spaces.

Proposition 3. *Let $d_{m\beta} = \dim V_{m\beta}$, where $V_{m\beta}$ is the space in which the irreducible unitary representation with highest weight $m\beta$ is realized, and let $\kappa_{m\beta}$ be the Casimir constant corresponding to $m\beta$. Then there exist positive constants c_1 and c_2 such that for m large enough,*

(i)

$$c_1 m^{m_{\beta/2} + m_\beta} \leq d_{m\beta} \leq c_2 m^{m_{\beta/2} + m_\beta},$$

and

(ii)

$$c_1 m^2 \leq \kappa_{m\beta} \leq c_2 m^2.$$

Proof.

(i) This is a consequence of Weyl's dimension formula, see [6] or [7].
(ii) Follows from $\kappa_{m\beta} = \langle m\beta + 2\rho, m\beta \rangle$.

∎

Let H_0, in $\mathfrak{a}$, be as in section 4. As a a consequence of the fact that

$$\Sigma^+_{tH_0} = \Sigma^+ \Longleftrightarrow \exp(tH_0) \in N_G(K),$$

where

$$\Sigma_{tH_0} = \left\{ \gamma \text{ in } \Sigma^+ \mid \gamma(tH_0) \equiv 0 \pmod{\sqrt{-1}\pi} \right\},$$

we can easily see that if $\cos\left(\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} t\right) = 1$, then $\exp(tH_0) \in N_G(K)$, and if $\cos\left(\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} t\right) = -1$, then $\exp(tH_0) \notin N_G(K)$ if and only if $\Sigma^+ = \{\alpha, 2\alpha\}$.

Corollary 2. *Let $H_i = t_i H_0 \in \mathfrak{a}$, for $i = 1, ..., p$, and let $a_i = \exp(H_i)$. Under the conditions of Theorem 1, we have*

(i) *If* $0 < t_i < \frac{\langle\alpha,\alpha\rangle}{\langle\beta,\alpha\rangle}\pi$ *for* $i = 1, ..., p$, *then there exists a positive constant* $C(t_1, ..., t_p)$ *such that*

$$\left\|f_{a_1,...,a_p}\right\|^2_{H^s(G)} \leq C(t_1, ..., t_p) \sum_{m=1}^{\infty} \frac{1}{m^{(1+2a)p-\left(m_{\beta/2}+m_\beta+2s\right)}}.$$

More generally, we have:

(ii) *If* $\Sigma^+ = \{\alpha, 2\alpha\}$, $t_i = \frac{\langle\alpha,\alpha\rangle}{\langle\beta,\alpha\rangle}\pi$, *for all* $1 \leq t_i \leq j$, *and* $0 < t_i < \frac{\langle\alpha,\alpha\rangle}{\langle\beta,\alpha\rangle}\pi$ *for* $i = j+1, ..., p$, *then there exists a positive constant* $C(t_1, ..., t_p)$ *such that*

$$\left\|f_{a_1,...,a_p}\right\|^2_{H^s(G)} \leq C(t_1, ..., t_p) \sum_{m=1}^{\infty} \frac{1}{m^{(1+2a)(p-j)-\left(m_{\beta/2}+m_\beta+2s\right)-2jb}}.$$

Proof.

(i) From Proposition 3, we deduce that

$$\begin{aligned}\left\|f_{a_1,...,a_p}\right\|^2_{H^s(G)} &= \sum_{[\pi]\in\widehat{G}_K} d_\pi (1+\kappa_\pi)^s \prod_{k=l}^{p} \left|\varphi_\pi(a_k)\right|^2 \\ &\leq C \sum_{m=1}^{\infty} d_{m\beta}\kappa^s_{m\beta} \prod_{k=1}^{p} \left(\frac{C(H_k)}{m^{\frac{1}{2}+a}}\right)^2.\end{aligned}$$

Part (i) of Corollary 2 follows from Proposition 3.

(ii) From Proposition 2, and the fact that $\frac{j!}{(a+1)_j} \sim \frac{\Gamma(a+1)}{j^a}$, we deduce that

$$\begin{aligned}\left\|f_{a_1,...,a_p}\right\|^2_{H^s(G)} &= \sum_{[\pi]\in\widehat{G}_K} d_\pi (1+\kappa_\pi)^s \prod_{k=l}^{p} \left|\varphi_\pi(a_k)\right|^2 \\ &= \sum_{[\pi]\in\widehat{G}_K} d_\pi (1+\kappa_\pi)^s \prod_{k=1}^{j} \left|\varphi_\pi(a_k)\right|^2 \prod_{k=j+1}^{p} \left|\varphi_\pi(a_k)\right|^2 \\ &\leq C \sum_{m=1}^{\infty} d_{m\beta}\kappa^s_{m\beta} \prod_{k=1}^{j} m^{2b} \prod_{k=1+j}^{p} \left(\frac{C(H_k)}{m^{\frac{1}{2}+a}}\right)^2.\end{aligned}$$

Similarly, part (ii) of Corollary 2 follows from Proposition 3.

□

As a consequence of [18], table 11.4.16, we get the following

TABLE 1

	$m_{\beta/2}$	m_β	$a = \frac{1}{2}(m_{\beta/2} + m_\beta - 1)$	$b = \frac{1}{2}(m_\beta - 1)$	$(1+2a)p - (m_{\beta/2} + m_\beta + 2s)$
$\mathbb{S}^n$	$n-1$	0	$\frac{n-2}{2}$	$-\frac{1}{2}$	$(n-1)p - (n-1+2s)$
$P^n(\mathbb{C})$	$2n-2$	1	$n-1$	0	$(2n-1)p - (2n-1+2s)$
$P^n(\mathbb{H})$	$4n-4$	3	$2n-1$	1	$(4n-1)p - (4n-1+2s)$
$P^2(\mathbb{O})$	8	7	7	3	$15p - (15+2s)$

THEOREM 2 (Sobolev Embedding, [10], Theorem 1.2.1). *If* $s > k + \frac{\dim G}{2}$, *then*

$$H^s(G) \subset C^k(G),$$

where $C^k(G)$ is the set of k-differentiable functions on G.

Let ν be a non-negative integer, and let

$$c(\nu, G) = \begin{cases} \frac{n+2s}{n-1} & \text{if } G = \mathsf{SO}(n+1),\ n \geq 2; \\ \frac{2n+2s}{2n-1} & \text{if } G = \mathsf{SU}(n+1),\ n \geq 2; \\ \frac{4n+2s}{4n-1} & \text{if } G = \mathsf{Sp}(n+1),\ n \geq 2; \\ \frac{16+2s}{15} & \text{if } G = \mathsf{F}_4, \end{cases}$$

and

$$d(\nu, j, G) = \begin{cases} \frac{2n+2s}{2n-1} + j & \text{if } G = \mathsf{SU}(n+1),\ n \geq 2; \\ \frac{4n+2s}{4n-1} + \left(\frac{4n+1}{4n-1}\right) j & \text{if } G = \mathsf{Sp}(n+1),\ n \geq 2; \\ \frac{16+2s}{15} + \frac{7}{5} j & \text{if } G = \mathsf{F}_4, \end{cases}$$

where

$$s = \lfloor \nu + \frac{\dim G}{2} \rfloor + 1 \left(\text{or any real number } s > \nu + \frac{\dim G}{2} \right).$$

From Table 1, Corollary 2, and Theorem 2, we deduce the following

Theorem 3. *Let ν be any non-negative integer.*
(i) *If $0 < t_i < \frac{\langle \alpha, \alpha \rangle}{\langle \beta, \alpha \rangle} \pi$, $a_i = \exp(t_i H_0)$ for all $1 \leq i \leq p$, and if $p > c(\nu, G)$, then*

$$\frac{d(\mu(a_1, ..., a_p))}{d\mu_G} = f_{a_1,...,a_p} \in C^\nu(G).$$

(ii) *If $\Sigma^+ = \{\alpha, 2\alpha\}$, $t_i = \frac{\langle \alpha, \alpha \rangle}{\langle \beta, \alpha \rangle} \pi$, for all $1 \leq t_i \leq j$, and $0 < t_i < \frac{\langle \alpha, \alpha \rangle}{\langle \beta, \alpha \rangle} \pi$ for $i = j+1, ..., p$, and if $p > d(\nu, j, G)$, then*

$$\frac{d\mu(a_1, ..., a_p)}{d\mu_G} = f_{a_1,...,a_p} \in C^\nu(G).$$

One consequence of our main theorem is the following:

Corollary 3. *Let $q \geq 1$ be any non-negative integer. If $0 < t_i < \frac{\langle \alpha, \alpha \rangle}{\langle \beta, \alpha \rangle} \pi$, $a_i = \exp(t_i H_0)$ for all $1 \leq i \leq p$, and if $p > c(0, G)$, then*

$$\frac{d\mu(a_1, ..., a_p)}{d\mu_G} = f_{a_1,...,a_p} \in L^q(G, d\mu_G),$$

where $c(0, G)$ is as above.

As a consequence of Corollary 2, and Table 1, we get the following

Corollary 4 ([1],[7]). *Under the notations of Corollary 2, part* (i) *we have*

$$\frac{d\mu(a_1,a_2)}{d\mu_G}\in L^2(G),$$

for all compact symmetric spaces of rank one, except the case

$$\mathbb{S}^2=P^1(\mathbb{C})=\mathsf{SU}(2)/\mathsf{SO}(2),$$

where we have

$$\frac{d\mu(a_1,...,a_p)}{d\mu_G}\in L^2(G)\ \ \text{if } p\geq 3.$$

A similar statement can be made if we assume part *ii*) of Theorem 3.

Corollary 5. *If*

$$k>\frac{\dim G}{4},\ \ \text{and } p>c(2k,G),$$

then the Fourier series

$$\sum_{[\pi]\in\widehat{G}_K} d_\pi \mathsf{Tr}\left(\widehat{\mu(a_1,...,a_p)}(\pi)\,\pi(g)\right)$$

of the density function $f_{a_1,...,a_p}$ *converges absolutely and uniformly to* $f_{a_1,...,a_p}(g)$.

Proof. This is a consequence of Theorem 3, (3.1), and part (1) of Theorem 1 in [15] which says that if a function $f\in C^{2k}(G)$ and if $2k>\frac{\dim G}{2}$, then the Fourier series of f converges absolutely and uniformly to f on G. □

Remark 2. *The real projective space* $P^n(\mathbb{R})=\mathsf{SO}(n+1)/\mathsf{O}(n)$ *is not covered by our main Theorem, since it is not simply connected. Using the explicit description of the spherical function of the Gelfand pair* $(\mathsf{SO}(n+1),\mathsf{O}(n))$, *which is given in [2], page* 65, *the same computations as above can be carried out.*

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