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HYPERSURFACES WITH CLOSED MÖBIUS FORM AND THREE DISTINCT CONSTANT MÖBIUS PRINCIPAL CURVATURES IN $\mathbb{S}^{m+1}$*

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Abstract. Let x be an m-dimensional umbilic-free hypersurface in an $(m+1)$-dimensional unit sphere $\mathbb{S}^{m+1}(m \geq 4)$. There are four basic Möbius invariants of x, i.e. Möbius metric $\mathbf{g}$, Möbius form $\mathbf{\Phi}$, Blaschke tensor $\mathbf{A}$ and Möbius second fundamental form $\mathbf{B}$. The eigenvalues of $\mathbf{B}$ are called Möbius principal curvatures. In this paper, we study hypersurfaces with closed Möbius form and three distinct constant Möbius principal curvatures, and give the Classification Theorem. Moreover, we give new Willmore hypersurfaces, which can be seen that they aren't Cartan minimal or Möbius isoparametric hypersurfaces.

Key words. Möbius geometry, Möbius form, Möbius principal curvature.

Mathematics Subject Classification. 53A30, 53C21, 53C40.

1. Introduction. Let $x : M^m \to \mathbb{S}^{m+1}$ be an m-dimensional umbilic-free hypersurface in an (m+1)-dimensional unit sphere $\mathbb{S}^{m+1}$ and $\{e_i\}$ be a local orthonormal tangent frame field of x for the induced metric $I = dx \cdot dx$ with dual frame field $\{\theta_i\}$. Let $II = \sum_{ij} h_{ij}\theta_i\theta_j$ be the second fundamental form and H be the mean curvature of x. Define $\rho^2 = \frac{m}{m-1}|II - \frac{1}{m}\mathbf{tr}(II)I|^2$, then the positive definite 2-form $\mathbf{g} = \rho^2 I$ is invariant under Möbius transformation group of $\mathbb{S}^{m+1}$ and is called Möbius metric of x. Three basic Möbius invariants of x, Möbius form $\mathbf{\Phi} = \rho\sum_i C_i\theta_i$, Blaschke tensor $\mathbf{A} = \rho^2\sum_{ij} A_{ij}\theta_i\theta_j$ and Möbius second fundamental form $\mathbf{B} = \rho^2\sum_{ij} B_{ij}\theta_i\theta_j$, are defined by([13])

$$C_i = -\rho^{-2}(e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\mathbf{log}\rho)), \tag{1.1}$$

$$\begin{aligned} A_{ij} = &- \rho^{-2}(\mathbf{Hess}_{ij}(\mathbf{log}\rho) - e_i(\mathbf{log}\rho)e_j(\mathbf{log}\rho) - Hh_{ij}) \\ &- \frac{1}{2}\rho^{-2}(\|\nabla\mathbf{log}\rho\|^2 - 1 + H^2)\delta_{ij}, \end{aligned} \tag{1.2}$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}), \tag{1.3}$$

where $\mathbf{Hess}_{ij}$ and ∇ are the Hessian matrix and the gradient with respect to the induced metric $I = dx \cdot dx$. We call the eigenvalues of $\mathbf{B}$ are Möbius principal curvatures of x.

A classical theorem of Möbius geometry states that x is in fact characterized by $\mathbf{g}$ and $\mathbf{B}$ up to Möbius equivalence. If all the Möbius principal curvatures of x are constant and $\mathbf{\Phi} = 0$, then the hypersurface is called Möbius isoparametric hypersurface. It can be proved that a Möbius isoparametric hypersurface is Dupin hypersurface. We refer readers to paper [4] for the important development of Lie sphere geometry

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of Dupin hypersurface. From (1.1) and (1.2) we can see that Euclidean isoparametric hypersurfaces are automatically Möbius isoparametric, but the converse is not true. Recently, these Möbius isoparametric hypersurfaces with two or three distinct constant Möbius principal curvatures are classified under Möbius group [7, 9, 10]. In addition, much has been known about hypersurfaces with vanishing Möbius form such as [8, 11]. However, many important hypersurfaces have constant Möbius principal curvatures but non-vanishing Möbius form. For instance, it was proved that these hypersurfaces with constant Möbius sectional curvature or isotropic Blaschke tensor have a closed Möbius form and two distinct constant Möbius principal curvatures [5, 6]. In the previous paper we present the classification theorem of hypersurfaces with closed Möbius form and two distinct principal curvatures [12]. It is natural to ask how about hypersurfaces with closed Möbius form and more distinct constant Möbius principal curvatures. In this paper, we are concerned with hypersurfaces with closed Möbius form and three distinct constant Möbius principal curvatures in a sphere space.

Let $M^2(c)$ be a two-dimensional space form of constant curvature c, and let $\gamma : I \to M^2(c)$ be an immersed curve. Define a functional $F(\gamma)$ as

$$F(\gamma) = \int_\gamma (\kappa^2 + \lambda) ds,$$

where s and κ are the arc parameter and geodesic curvature of γ, respectively. The critical point of $F(\gamma)$ is called an elastic curve. If constant $\lambda = 0$, the critical point of $F(\gamma)$ is called a free elastic curve. For a fixed number r, in [3], authors have defined a free hyperelastic curve (also call free r-elastic curve), which is a generalization of the classical Bernoulli's elastic curve, as a critical point of the following functional $F^r(\gamma)$

$$F^r(\gamma) = \int_\gamma \kappa^r ds.$$

Under suitable boundary conditions, γ is a free r-elastic curve if and only if the following Euler-Lagrange equation holds:

$$r(r-1)\kappa^{r-3}\left(\kappa \frac{d^2\kappa}{ds^2} + (r-2)\left(\frac{d\kappa}{ds}\right)^2 + \frac{\kappa^4}{r} + c\frac{\kappa^2}{r-1}\right) = 0.$$

In this paper, we will see that our results are closely related to free r-elastic curves.

For the purpose to make our main results intuitional, we use the following notations. $\mathbb{R}^{m+3}_1$ denotes Lorentz space with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle Y, Z\rangle = -y_0 z_0 + y_1 z_1 + \cdots + y_{m+2} z_{m+2},$$

where $Y = (y_0, y_1, \cdots, y_{m+2}), Z = (z_0, z_1, \cdots, z_{m+2}) \in \mathbb{R}^{m+3}$. C^{m+2}_+ denotes the positive light cone and $\mathbb{Q}^{m+1}$ denotes the quadric in the real projection space $\mathbb{R}P^{m+2}$, which are defined as follows:

$$C^{m+2}_+ = \{X \in \mathbb{R}^{m+3}_1 : \langle X, X\rangle = 0, x_0 > 0\},$$

$$\mathbb{Q}^{m+1} = \{[Y] \in \mathbb{R}P^{m+2} : \langle Y, Y\rangle = 0\}.$$

Let $\mathbb{H}^{m+1}$ be the hyperbolic space defined by

$$\mathbb{H}^{m+1} = \{(y_0, y_1) \in \mathbb{R}^+ \times \mathbb{R}^{m+1} | -y_0^2 + y_1 y_1 = -1\}.$$

Now let $\sigma : \mathbb{R}^{m+1} \to \mathbb{S}^{m+1}$ be the inverse of the stereographic projection given by

$$\sigma(u) = \left(\frac{1 - |u|^2}{1 + |u|^2}, \frac{2u}{1 + |u|^2} \right), \tag{1.3}$$

and $\tau : \mathbb{H}^{m+1} \to \mathbb{S}^{m+1}$ be the conformal map defined by

$$\tau(y_0, y_1) = \left(\frac{1}{y_0}, \frac{y_1}{y_0} \right), (y_0, y_1) \in \mathbb{H}^{m+1}. \tag{1.4}$$

The conformal maps σ and τ assign any hypersurface in $\mathbb{R}^{m+1}$ or $\mathbb{H}^{m+1}$ to a hypersurface in $\mathbb{S}^{m+1}$. We use map $\pi : C_+^{m+2} \to Q^{m+1}$ to denote the nature projection. A hypersurface $x : M^m \to \mathbb{S}^{m+1}$ determines map

$$X := \pi(1, x) : M^m \to \mathbb{Q}^{m+1}, \tag{1.5}$$

which is called the natural map of x. It is known that two hypersurfaces $x, \widetilde{x} : M^m \to \mathbb{S}^{m+1}$ are Möbius equivalent if and only if their respective natural maps $X, \widetilde{X} : M^m \to \mathbb{Q}^{m+1}$ are equivalent under the action of Lorentz group $O(m+2, 1)$.

In §2, we recall some fundamental theories about Möbius geometry of hypersurfaces in $\mathbb{S}^{m+1}$. In §3, we study the geometric structures of hypersurfaces with closed Möbius form and three distinct constant Möbius principal curvatures, and get the Classification Theorem 3.13. In §4, we give new examples of Willmore hypersurface with non-vanishing Möbius form.

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2. Möbius invariants for hypersurfaces in $\mathbb{S}^{m+1}$. In this section we define Möbius invariants and recall the structure equations for hypersurfaces in $\mathbb{S}^{m+1}$. For details we refer to [13].

Let $x : M^m \to \mathbb{S}^{m+1} \subset \mathbb{R}^{m+2}$ be an umbilic-free hypersurface immersed in $\mathbb{S}^{m+1}$. We define the Möbius position vector $Y : M^m \to \mathbb{R}_1^{m+3}$ of x by

$$Y = \rho(1, x) : M^m \to \mathbb{R}_1^{m+3}, \quad \rho^2 = \frac{m}{m-1}(\|II\|^2 - mH^2) > 0.$$

Then we have the following.

THEOREM 2.1 ([13]). *Two hypersurfaces $x, \widetilde{x} : M^m \to \mathbb{S}^{m+1}$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(m+2, 1)$ acting on $\mathbb{R}_1^{m+3}$ such that $Y = \widetilde{Y}T$.*

Since the Möbius transformation group in $\mathbb{S}^{m+1}$ is isomorphic to the subgroup $O^+(m+2, 1)$, which preserves the positive part of the light cone in $\mathbb{R}_1^{m+3}$, from Theorem 2.1 we know 2-form

$$\mathbf{g} = \langle dY, dY \rangle = \rho^2 dx \cdot dx \tag{2.1}$$

is a Möbius invariant and called the Möbius metric or Möbius first fundamental form induced by x (cf. [13]). Let $\triangle$ be the Laplacian with respect to $(M, \mathbf{g})$ and define

$$N = -\frac{1}{m}\triangle Y - \frac{1}{2m^2}\langle \triangle Y, \triangle Y \rangle Y. \tag{2.2}$$

Choose $\{E_1, \cdots, E_m\}$ as a local orthonormal basis for $(M, \mathbf{g})$ with dual basis $\{\omega_1, \cdots, \omega_m\}$ and write $E_i(Y) = Y_i$, then

$$\langle Y, Y\rangle = \langle N, N\rangle = 0, \quad \langle Y, N\rangle = 1, \quad \langle Y_i, Y_j\rangle = \delta_{ij}, \quad \langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0, \quad 1 \le i, j \le m.$$

Let V be the orthogonal complement of $\mathbf{span}\{Y, N, Y_1, \cdots, Y_m\}$ in $\mathbb{R}^{m+3}_1$. Then we have the orthogonal decomposition

$$\mathbb{R}^{m+3}_1 = \mathbf{span}\{Y, N\} \oplus \mathbf{span}\{Y_1, \cdots, Y_m\} \oplus V. \tag{2.3}$$

Let E be a unit vector field of V, then $\{Y, N, Y_1, \cdots, Y_m, E\}$ forms a moving frame in $\mathbb{R}^{m+3}_1$ along M. We will use the following range of indices in this section: $1 \le i, j, k, \cdots \le m$. Structure equations are given by

$$dY = \sum_i \omega_i Y_i, \tag{2.4}$$

$$dN = \sum_{ij} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i E, \tag{2.5}$$

$$dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j E, \tag{2.6}$$

$$dE = -\sum_i C_i \omega_i Y - \sum_{ij} B_{ij}\omega_j Y_i, \tag{2.7}$$

where $\{\omega_{ij}\}$ are connection forms of Möbius metric $\mathbf{g}$, A_{ij} and B_{ij} are symmetric with respect to (ij). It is clear that $\mathbf{A} = \sum_{ij} A_{ij}\omega_i \otimes \omega_j, \mathbf{B} = \sum_{ij} B_{ij}\omega_i \otimes \omega_j, \mathbf{\Phi} = \sum_i C_i\omega_i$ are Möbius invariants and called the Blaschke tensor, the Möbius second fundamental form and the Möbius form of x, respectively. The relations between $\mathbf{A}, \mathbf{B}, \mathbf{\Phi}$ and Euclidean invariants of x are given by (1.1), (1.2) and (1.3).

Let R_{ijkl}, R_{ij} and R be components of the curvature tensor, components of the Ricci curvature tensor and the scalar curvature of $\mathbf{g}$, respectively. By taking exterior differentiation of structure equations, we get these integrability conditions for structure equations as follows([13]):

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{2.8}$$

$$C_{i,j} - C_{j,i} = \sum_k B_{ik}A_{kj} - \sum_k B_{jk}A_{ki}, \tag{2.9}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.10}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2.11}$$

$$R_{ij} = -\sum_k B_{ik}B_{kj} + \mathbf{tr}(A)\delta_{ij} + (m-2)A_{ij}, \tag{2.12}$$

$$\mathbf{tr}(A) = \frac{1}{2m}(1 + \frac{m}{m-1}R), \sum_i B_{ii} = 0, \sum_{ij}(B_{ij})^2 = \frac{m-1}{m}, \tag{2.13}$$

where $\{A_{ij,k}\}$,$\{B_{ij,k}\}$ and $\{C_{i,j}\}$ are covariant derivatives of $\mathbf{A}, \mathbf{B}$ and $\mathbf{\Phi}$ with respect to the connection induced by $\mathbf{g}$. From (2.10) and (2.13) we have

$$\sum_i B_{ij,i} = -(m-1)C_j. \tag{2.14}$$

Let $\{B_{ij,kl}\}$ denote the second covariant derivatives of B_{ij}. They satisfy the Ricci identities

$$B_{ij,kl} - B_{ij,lk} = \sum_t B_{tj}R_{tikl} + \sum_t B_{it}R_{tjkl}. \tag{2.15}$$

If the hypersurface x is Möbius minimal, it is called a Willmore hypersurface and then it satisfies the following Euler-Lagrange equation.

$$\sum_i C_{i,i} + \sum_{ij}(\frac{1}{m-1}R_{ij} - A_{ij})B_{ij} = 0. \tag{2.16}$$

3. Hypersurfaces with closed Möbius form and three distinct constant Möbius principal curvatures in $\mathbb{S}^{m+1}$. Let $x : M^m \to \mathbb{S}^{m+1}(m \geq 4)$ be a hypersurface with three distinct constant Möbius principal curvatures λ, μ, ν. Then the multiplicities of λ, μ, ν are constant on M, without loss of generality, denoted by m_1,m_2,m_3, which satisfy $m_1 + m_2 + m_3 = m$. We prove the following proposition.

PROPOSITION 3.1. *Let $x : M^m \to \mathbb{S}^{m+1}$ be a hypersurface with three distinct constant Möbius principal curvatures, then there exist only two cases:*
(1) Möbius form $\mathbf{\Phi}$ of M doesn't vanish, and at least one of Möbius principal curvatures has the multiplicity 1;
(2) Möbius form $\mathbf{\Phi}$ of M vanishes.

Proof. For convenience, in this section we make:

$$1 \leq a, b, \cdots \leq m_1; \quad m_1 + 1 \leq p, q, \cdots \leq m_1 + m_2;$$

$$m_1 + m_2 + 1 \leq \alpha, \beta, \cdots \leq m; \quad 1 \leq i, j, \cdots \leq m.$$

Since M has three distinct constant Möbius principal curvatures λ, μ, ν, we define three distributions V_λ, V_μ and V_ν as follows:

$$V_\lambda = \bigcup_{P \in M} V_\lambda(P), \quad V_\mu = \bigcup_{P \in M} V_\mu(P), \quad V_\nu = \bigcup_{P \in M} V_\nu(P),$$

where $V_\lambda(P)$, $V_\mu(P)$ and $V_\nu(P)$ are eigenspaces corresponding to λ, μ and ν at a point $P \in M$, with $\mathbf{dim}(V_\lambda(P)) = m_1$, $\mathbf{dim}(V_\mu(P)) = m_2$ and $\mathbf{dim}(V_\nu(P)) = m_3$, respectively. Then

$$TM = V_\lambda \oplus V_\mu \oplus V_\nu.$$

Choose a local orthonormal tangent frame field $\{E_1, \cdots, E_m\}$ of $T(M)$ in a neighborhood of P, such that

$$V_\lambda = \mathbf{span}\{E_1, \cdots, E_{m_1}\}, \quad V_\mu = \mathbf{span}\{E_{m_1+1}, \cdots, E_{m_1+m_2}\},$$

$$V_\nu = \mathbf{span}\{E_{m_1+m_2+1}, \cdots, E_m\}.$$

Then, with respect to this basis,

$$B_{ai} = \lambda\delta_{ai}, \quad B_{pi} = \mu\delta_{pi}, \quad B_{\alpha i} = \nu\delta_{\alpha i}.$$

So

$$B_{ab,i} = 0, \quad B_{pq,i} = 0, \quad B_{\alpha\beta,i} = 0.$$

With (2.10), we get

$$B_{ap,q} = -\delta_{pq}C_a, \quad B_{ap,b} = -\delta_{ab}C_p, \quad B_{a\alpha,\beta} = -\delta_{\alpha\beta}C_a,$$

$$B_{a\alpha,b} = -\delta_{ab}C_\alpha, \quad B_{p\alpha,q} = -\delta_{pq}C_\alpha, \quad B_{p\alpha,\beta} = -\delta_{\alpha\beta}C_p.$$

So, from (2.14), we have

$$(m_1 - 1)C_a = 0, \quad (m_2 - 1)C_p = 0, \quad (m_3 - 1)C_\alpha = 0. \tag{3.1}$$

If $\mathbf{\Phi} \neq 0$, there exists a point $P \in M$ such that $\mathbf{\Phi}(P) \neq 0$. Without loss of generality, assume $C_1(P) \neq 0$, then (3.1) implies $m_1 = 1$. □

Hypersurfaces of Case (2) in Proposition 3.1 are Möbius isoparametric hypersurfaces and have been classified by Hu-Li and Hu-Zhai in [7] and [9]. Here assume x satisfies $\mathbf{\Phi} \neq 0, d\mathbf{\Phi} = 0$, then, from Proposition 3.1, there are two cases:

$$\text{(i)} \ \ m_1 = m_2 = 1, m_3 \geq 2; \quad \text{(ii)} \ \ m_1 = 1, m_2, m_3 \geq 2.$$

When $m = 4$, there exists only Case (i).

3.1. Case (i): $m_1 = m_2 = 1, \quad m_3 \geq 2$. Let x be a hypersurface with $\mathbf{\Phi} \neq 0, d\mathbf{\Phi} = 0$ and $m_1 = m_2 = 1, m_3 \geq 2$. Because $d\mathbf{\Phi} = 0$, there exists a local orthonormal tangent frame field such that $\mathbf{A}$ and $\mathbf{B}$ are diagonalized simultaneously. Firstly, we get $C_\alpha = 0$ from (3.1), and

$$B_{11,i} = B_{22,i} = B_{\alpha\beta,i} = B_{1\alpha,1} = B_{2\alpha,2} = 0, \quad B_{12,1} = -C_2, \quad B_{21,2} = -C_1,$$

$$B_{1\alpha,\beta} = -C_1\delta_{\alpha\beta}, \quad B_{2\alpha,\beta} = -C_2\delta_{\alpha\beta}, \quad B_{12,\alpha} = B_{1\alpha,2} = B_{2\alpha,1}.$$

So

$$\omega_{12} = \frac{1}{\lambda - \mu}\left(-C_2\omega_1 - C_1\omega_2 + \sum_\alpha B_{12,\alpha}\omega_\alpha\right), \tag{3.2}$$

$$\omega_{1\alpha} = \frac{1}{\lambda - \nu}(-C_1\omega_\alpha + B_{12,\alpha}\omega_2), \quad \omega_{2\alpha} = \frac{1}{\mu - \nu}(-C_2\omega_\alpha + B_{12,\alpha}\omega_1). \tag{3.3}$$

Lemma 3.2. *$B_{12,\alpha} = 0$, for all $\alpha = 3, \cdots, m$.*

Proof. Assume one of $B_{12,\alpha}$ such as $B_{12,3}$ doesn't vanish.

For $\alpha \geq 4$,

$$\sum_i B_{13,\alpha i}\omega_i = B_{12,\alpha}\omega_{23} + B_{12,3}\omega_{2\alpha}, \quad \sum_i B_{13,1i}\omega_i = 2B_{12,3}\omega_{21} - C_2\omega_{23} + C_1\omega_{13},$$

then

$$B_{13,\alpha 1} = \frac{2B_{12,3}B_{12,\alpha}}{\mu - \nu}, \quad B_{13,1\alpha} = -\frac{2B_{12,3}B_{12,\alpha}}{\lambda - \mu}. \tag{3.4}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are diagonalized simultaneously, $R_{ijkl} = 0$ when i, j, k, l are distinct with each other. From Ricci identities we have

$$B_{13,\alpha 1} = B_{13,1\alpha}. \tag{3.5}$$

Putting (3.4) into (3.5), we get

$$B_{12,3}B_{12,\alpha}\left(\frac{1}{\mu - \nu} - \frac{1}{\mu - \lambda}\right) = 0,$$

which implies $B_{12,\alpha} = 0$ for all $\alpha \geq 4$.

Moreover, after a direct calculation with (3.2)-(3.3), we obtain for $\alpha \geq 3$

$$R_{1\alpha 1\alpha} = -\frac{1}{\lambda - \nu}\left(-C_{1,1} - \frac{C_2^2}{\mu - \nu} + \frac{C_1^2}{\lambda - \nu}\right) - \frac{2B_{12,\alpha}^2}{(\lambda - \mu)(\mu - \nu)},$$

$$R_{2\alpha 2\alpha} = -\frac{1}{\mu - \nu}\left(-C_{2,2} - \frac{C_1^2}{\lambda - \nu} + \frac{C_2^2}{\mu - \nu}\right) - \frac{2B_{12,\alpha}^2}{(\mu - \lambda)(\lambda - \nu)}.$$

With the Gauss equation (2.11), we have

$$\lambda\nu + A_{11} + A_{\alpha\alpha} = -\frac{1}{\lambda - \nu}\left(-C_{1,1} - \frac{C_2^2}{\mu - \nu} + \frac{C_1^2}{\lambda - \nu}\right) - \frac{2B_{12,\alpha}^2}{(\lambda - \mu)(\mu - \nu)},$$

$$\mu\nu + A_{22} + A_{\alpha\alpha} = -\frac{1}{\mu - \nu}\left(-C_{2,2} - \frac{C_1^2}{\lambda - \nu} + \frac{C_2^2}{\mu - \nu}\right) - \frac{2B_{12,\alpha}^2}{(\mu - \lambda)(\lambda - \nu)},$$

which mean $B_{12,\alpha}^2(\lambda + \mu - 2\nu)$ is independent of α. If $\lambda + \mu - 2\nu \neq 0$, then $B_{12,3}^2 = B_{12,4}^2 = 0$ which contradicts the assumption. Hence $\lambda + \mu - 2\nu = 0$, which implies from (2.13) that $\lambda = -\mu, \nu = 0$. Then

$$\omega_{12} = \frac{1}{2\lambda}(-C_2\omega_1 - C_1\omega_2 + B_{12,3}\omega_3), \quad \omega_{1\alpha} = -\frac{C_1}{\lambda}\omega_\alpha, \quad \omega_{2\alpha} = \frac{C_2}{\lambda}\omega_\alpha, \quad \alpha \geq 4,$$

$$\omega_{13} = \frac{1}{\lambda}(B_{12,3}\omega_2 - C_1\omega_3), \quad \omega_{23} = -\frac{1}{\lambda}(B_{12,3}\omega_1 - C_2\omega_3).$$

Now for $\alpha, \beta \geq 4$, with a direct calculation, we obtain

$$C_{1,1} = E_1(C_1) + \frac{C_2^2}{2\lambda}, \quad C_{1,2} = E_2(C_1) + \frac{C_1C_2}{2\lambda}, \quad C_{1,3} = E_3(C_1) - \frac{B_{12,3}C_2}{2\lambda},$$

$$C_{2,1} = E_1(C_2) - \frac{C_1C_2}{2\lambda}, \quad C_{2,2} = E_2(C_2) - \frac{C_1^2}{2\lambda}, \quad C_{2,3} = E_3(C_2) + \frac{B_{12,3}C_1}{2\lambda},$$

$$C_{3,1} = -\frac{C_2B_{12,3}}{\lambda}, \quad C_{3,2} = \frac{C_1B_{12,3}}{\lambda}, \quad C_{3,3} = -\frac{-C_1^2 + C_2^2}{\lambda},$$

$$C_{1,\alpha} = E_\alpha(C_1), \quad C_{2,\alpha} = E_\alpha(C_2), \quad C_{3,\alpha} = 0,$$

$$C_{\alpha,\alpha} = -\frac{-C_1^2 + C_2^2}{\lambda}, \quad C_{\alpha,i} = 0, \quad i \neq \alpha, \quad \alpha \geq 4.$$

Hence

$$E_2(C_1) - E_1(C_2) = -\frac{C_1C_2}{\lambda}, \quad E_3(C_1^2 + C_2^2) = 0, \quad E_\alpha(C_1) = E_\alpha(C_2) = 0, \quad \alpha \geq 4. \tag{3.6}$$

According to

$$\begin{aligned}-R_{1212}\omega_1 \wedge \omega_2 = &\frac{1}{2\lambda}\Bigg(- dC_2 \wedge \omega_1 - dC_1 \wedge \omega_2 + dB_{12,3} \wedge \omega_3 + \left(\frac{C_1^2 + C_2^2}{2\lambda} + \frac{2B_{12,3}^2}{\lambda}\right)\omega_1 \wedge \omega_2 \\ &- \frac{C_1B_{12,3}}{2\lambda}\omega_1 \wedge \omega_3 + \frac{C_2B_{12,3}}{2\lambda}\omega_2 \wedge \omega_3 + B_{12,3}\sum_\alpha \omega_\alpha \wedge \omega_{\alpha 3}\Bigg) \\ &- \frac{1}{\lambda^2}\left(-B_{12,3}^2\omega_1 \wedge \omega_2 + C_1B_{12,3}\omega_1 \wedge \omega_3 - C_2B_{12,3}\omega_2 \wedge \omega_3\right),\end{aligned}$$

we get

$$\omega_{\alpha 3}(E_3) = -E_\alpha(B_{12,3}), \quad \sum_\alpha \omega_\alpha \wedge \omega_{\alpha 3}(E_\beta) = 0, \quad \omega_{\alpha 3}(E_i) = 0, \quad i = 1, 2, \tag{3.7}$$

$$R_{1212} = \frac{C_{1,1} - C_{2,2}}{2\lambda} - \frac{C_1^2 + C_2^2}{2\lambda} - \frac{2B_{12,3}^2}{\lambda^2}. \tag{3.8}$$

(3.7) implies that there exist locally smooth functions $\phi_{\alpha\beta}$ such that

$$\omega_{\alpha 3} = -E_\alpha(B_{12,3})\omega_3 + \sum_\beta \phi_{\alpha\beta}\omega_\beta, \quad \phi_{\alpha\beta} = \phi_{\beta\alpha}.$$

According to

$$\begin{aligned}-R_{1313}\omega_1 \wedge \omega_3 = &\frac{1}{\lambda}\Bigg(-dC_1 \wedge \omega_3 + dB_{12,3} \wedge \omega_2 - \frac{5C_1B_{12,3}}{2\lambda}\omega_1 \wedge \omega_2 \\ &+ \frac{2C_1^2 + B_{12,3}^2}{2\lambda}\omega_1 \wedge \omega_3 - \frac{C_1C_2}{\lambda}\omega_2 \wedge \omega_3\Bigg) \\ &+ \frac{1}{2\lambda^2}\left(C_1B_{12,3}\omega_1 \wedge \omega_2 + (C_2^2 - B_{12,3}^2)\omega_1 \wedge \omega_3 + C_1C_2\omega_2 \wedge \omega_3\right),\end{aligned}$$

we get

$$E_\alpha(B_{12,3}) = 0, \quad \omega_{\alpha 3} = \sum_\beta \phi_{\alpha\beta}\omega_\beta, \tag{3.9}$$

$$E_3(B_{12,3}) + E_2(C_1) + \frac{C_1C_2}{2\lambda} = 0, \quad E_1(B_{12,3}) = \frac{2C_1B_{12,3}}{\lambda}, \tag{3.10}$$

$$R_{1313} = \frac{B_{12,3}^2}{\lambda^2} + \frac{C_{1,1}}{\lambda} - \frac{C_1^2 + C_2^2}{\lambda^2}. \tag{3.11}$$

According to

$$\begin{aligned}
&-R_{1\alpha1\alpha}\omega_1 \wedge \omega_\alpha \\
=& \frac{1}{\lambda}\left(-dC_1 \wedge \omega_\alpha + \frac{C_1^2}{\lambda}\omega_1 \wedge \omega_\alpha - \frac{C_1C_2}{\lambda}\omega_2 \wedge \omega_\alpha + \sum_\beta C_1\phi_{\alpha\beta}\omega_3 \wedge \omega_\beta\right) \\
&+\frac{1}{2\lambda^2}\left(C_2\omega_1 \wedge \omega_\alpha + C_1\omega_2 \wedge \omega_\alpha - B_{12,3}\omega_3 \wedge \omega_\alpha\right) \\
&+\sum_\beta \frac{\phi_{\alpha\beta}}{\lambda}\left(B_{12,3}\omega_2 \wedge \omega_\beta - C_1\omega_3 \wedge \omega_\beta\right),
\end{aligned}$$

we get

$$\frac{B_{12,3}}{\lambda}\phi_{\alpha\beta} = 0, \quad i.e. \;\; \phi_{\alpha\beta} = 0, \quad \alpha \neq \beta. \tag{3.12}$$

$$E_2(C_1) + \frac{C_1C_2}{2\lambda} = -B_{12,3}\phi, \quad R_{1\alpha1\alpha} = \frac{C_{1,1}}{\lambda} - \frac{C_1^2 + C_2^2}{\lambda^2}. \tag{3.13}$$

So (3.12) implies there exists locally a smooth function ϕ such that $\omega_{\alpha3} = \phi\omega_\alpha$. (3.10) and (3.13) imply

$$\phi = \frac{E_3(B_{12,3})}{B_{12,3}}.$$

According to

$$\begin{aligned}
-R_{2323}\omega_2 \wedge \omega_3 = -\frac{1}{\lambda}\Bigg(& dB_{12,3} \wedge \omega_1 - dC_2 \wedge \omega_3 - \frac{B_{12,3}C_2}{\lambda}\omega_1 \wedge \omega_2 + \frac{C_1C_2}{2\lambda}\omega_1 \wedge \omega_3 \\
&+\left(\frac{B_{12,3}}{\lambda} - \frac{C_2^2}{\lambda} - \frac{C_1^2}{2\lambda}\right)\omega_2 \wedge \omega_3\Bigg),
\end{aligned}$$

we get

$$R_{2323} = \frac{B_{12,3}^2}{\lambda^2} - \frac{C_{2,2}}{\lambda} - \frac{C_1^2 + C_2^2}{\lambda^2}. \tag{3.14}$$

Because

$$A_{13,2} = (A_{11} - A_{33})\frac{B_{12,3}}{\lambda}, \quad A_{23,1} = (A_{33} - A_{22})\frac{B_{12,3}}{\lambda}, \quad A_{13,2} = A_{23,1},$$

we have $A_{11} + A_{22} = 2A_{33}$, i.e.

$$2R_{1212} + 2\lambda^2 = R_{1313} + R_{2323}. \tag{3.15}$$

Putting (3.8), (3.11) and (3.14) into (3.15), we have

$$2\lambda^2 = 6\frac{B_{12,3}^2}{\lambda^2} - \frac{C_1^2 + C_2^2}{\lambda^2},$$

which means

$$6E_3(B_{12,3}^2) = E_3(C_1^2 + C_2^2) = 0, \quad E_3(B_{12,3}) = 0, \quad \phi = 0.$$

Then

$$-R_{3\alpha 3\alpha}\omega_3 \wedge \omega_\alpha = \frac{C_1^2 + C_2^2}{\lambda^2}\omega_3 \wedge \omega_\alpha - \frac{C_2 B_{12,3}}{\lambda^2}\omega_1 \wedge \omega_\alpha - \frac{C_1 B_{12,3}}{\lambda^2}\omega_2 \wedge \omega_\alpha,$$

which means $C_1 = C_2 = 0$. It deduces a contradiction with $\mathbf{\Phi} \neq 0$ and the lemma holds. □

Lemma 3.3. *There exists locally a smooth single-variable function f such that $\mathbf{\Phi} = df$.*

Proof. Lemma 3.2 implies that (3.2)-(3.3) become as follows:

$$\omega_{12} = \frac{1}{\lambda - \mu}(-C_2\omega_1 - C_1\omega_2), \quad \omega_{1\alpha} = -\frac{C_1}{\lambda - \nu}\omega_\alpha, \quad \omega_{2\alpha} = -\frac{C_2}{\mu - \nu}\omega_\alpha. \tag{3.16}$$

Then

$$d\omega_1 \equiv 0\mathbf{mod}(\omega_1, \omega_2), \quad d\omega_2 \equiv 0\mathbf{mod}(\omega_1, \omega_2), \quad d\omega_\alpha \equiv 0\mathbf{mod}(\omega_\beta),$$

which means from Frobenius Theorem that the distributions $V_\lambda \oplus V_\mu$ and V_ν are integrable. Let M_1 and M_2 be their integrable submanifolds, of which local coordinates are (u, v) and $(w_3, \cdots, w_m)$ (denoted by (w_α) or (w)), respectively. Then it holds locally $M = M_1 \times M_2$.

Consider $\mathbf{\Psi} = -\frac{C_1}{\lambda-\nu}\omega_1 - \frac{C_2}{\mu-\nu}\omega_2$. With a direct calculation of $-R_{1\alpha 1\alpha}\omega_1 \wedge \omega_\alpha$ and $-R_{2\alpha 2\alpha}\omega_2 \wedge \omega_\alpha$, we get

$$C_{1,2} = E_2(C_1) + \frac{C_1 C_2}{\lambda - \mu} = 0, \quad C_{2,1} = E_1(C_2) - \frac{C_1 C_2}{\lambda - \mu} = 0, \tag{3.17}$$

$$C_{1,\alpha} = E_\alpha(C_1) = 0, \quad C_{2,\alpha} = E_\alpha(C_2) = 0. \tag{3.18}$$

Hence $d\mathbf{\Psi} = 0$, which means there exists locally a smooth function F such that $\mathbf{\Psi} = dF$. Then

$$F_1 := E_1(F) = -\frac{C_1}{\lambda - \nu}, \quad F_2 := E_2(F) = -\frac{C_2}{\mu - \nu}, \quad F_\alpha := E_\alpha(F) = 0. \tag{3.19}$$

Notice, from (3.17) and (3.18), F satisfies

$$F_{12} - pF_1F_2 = 0, \quad F_{21} - qF_1F_2 = 0, \tag{3.20}$$

where $F_{ij} = E_jE_i(F), p = \frac{\mu-\nu}{\lambda-\mu}, q = \frac{\nu-\lambda}{\lambda-\mu}$, then $d(e^{pF}\omega_1) = 0$ and $d(e^{qF}\omega_2) = 0$. So there exists locally a coordinate $(\tilde{u}, \tilde{v})$, for convenience, denoted also by (u, v), of M_1 such that

$$E_1 = e^{pF}\frac{\partial}{\partial u}, \quad E_2 = e^{qF}\frac{\partial}{\partial v}, \tag{3.21}$$

then (3.20) is equivalent to $\frac{\partial^2 F}{\partial u \partial v} = 0$. Hence there exists locally a coordinate (u, v) of M_1 and two single-variable smooth functions $\varphi = \varphi(u), \psi = \psi(v)$ such that

$$F(u, v) = \varphi(u) + \psi(v).$$

Put (3.16) into (2.6), we get

$$E_\alpha(Y_1)=\sum_i \omega_{1i}(E_\alpha)Y_i=-\frac{C_1}{\lambda-\nu}Y_\alpha, E_\alpha(Y_2)=\sum_i \omega_{2i}(E_\alpha)Y_i=-\frac{C_2}{\mu-\nu}Y_\alpha. \quad (3.22)$$

Let $\tilde{Y}=e^{-F}Y$. (3.22) is equivalent to

$$E_\alpha(E_1(\tilde{Y}))=0, \quad E_\alpha(E_2(\tilde{Y}))=0.$$

Hence there exist locally smooth vector functions $\varsigma=\varsigma(u,v)$ and $\tau=\tau(w)$ such that

$$\tilde{Y}=\varsigma+\tau, \quad Y=e^F(\varsigma+\tau).$$

Notice $\langle\tilde{Y},\tilde{Y}\rangle=0$, then $\langle dY,dY\rangle=e^{2F}(g_1+g_2)$, where $g_1=\langle d\varsigma,d\varsigma\rangle, g_2=\langle d\tau,d\tau\rangle$.

Consider relationships among $(M_1,g_1),(M_2,g_2)$ and $(M,\mathbf{g})$. Choose $\{\tilde{\omega}_1,\tilde{\omega}_2\}$ and $\{\tilde{\omega}_\alpha\}$ be local orthonormal tangent frame fields of (M_1,g_1) and (M_2,g_2), respectively. Let $\omega_i=e^F\tilde{\omega}_i$, then $\{\omega_i\}$ is a local orthonormal tangent frame field of $(M,\mathbf{g})$. Denote $\tilde{R}_{1212}$ and $\tilde{\omega}_{12}$ be the curvature and connection form of (M_1,g_1), $\tilde{R}_{\alpha\beta\gamma\sigma}$ and $\tilde{\omega}_{\alpha\beta}$ be the components of the curvature tensor and connection forms of (M_2,g_2).

On one hand,

$$d\omega_\alpha=e^F(dF\wedge\tilde{\omega}_\alpha+\sum_\beta\tilde{\omega}_{\alpha\beta}\wedge\tilde{\omega}_\beta).$$

On the other hand,

$$d\omega_\alpha=\sum_i\omega_i\wedge\omega_{i\alpha}=-e^F\left(\frac{C_1}{\lambda-\nu}\omega_1\wedge\tilde{\omega}_\alpha+\frac{C_2}{\mu-\nu}\omega_2\wedge\tilde{\omega}_\alpha-\sum_\beta\omega_{\alpha\beta}\wedge\tilde{\omega}_\beta\right).$$

Therefore $\omega_{\alpha\beta}=\tilde{\omega}_{\alpha\beta}$ and

$$R_{\alpha\beta\gamma\sigma}=e^{-2F}\tilde{R}_{\alpha\beta\gamma\sigma}-\left(\frac{C_1^2}{(\lambda-\nu)^2}+\frac{C_2^2}{(\mu-\nu)^2}\right)(\delta_{\alpha\gamma}\delta_{\beta\sigma}-\delta_{\alpha\sigma}\delta_{\beta\gamma}). \quad (3.23)$$

Noticing $B_{12,\alpha=0}$, from the Gauss equation, we get $A_{\alpha\alpha}=A_{\beta\beta}(\alpha\neq\beta)$. Denote $A_{11}=r_1,A_{22}=r_2,A_{\alpha\alpha}=r_3$. (3.23) and the Gauss equation imply

$$\tilde{R}_{\alpha\beta\alpha\beta}=e^{2F}\left(\frac{C_1^2}{(\lambda-\nu)^2}+\frac{C_2^2}{(\mu-\nu)^2}+\nu^2+2r_3\right). \quad (3.24)$$

Since $E_\gamma(r_3)=A_{\alpha\alpha,\gamma}=A_{\alpha\gamma,\alpha}=0$ and (3.18), we have $E_\gamma(\tilde{R}_{\alpha\beta\alpha\beta})=0$, which means $\tilde{R}_{\alpha\beta\alpha\beta}$ is a constant, denoted by a. Hence (M_2,g_2) is a space form with constant curvature a.

By a direct calculation, we get

$$r_1=-F_{11}-\frac{1}{2}F_1^2+\left(\frac{1}{2}+p\right)F_2^2-\frac{1}{2}ae^{-2F}-\lambda\nu+\frac{\nu^2}{2}, \quad (3.25)$$

$$r_2=-F_{22}-\frac{1}{2}F_2^2+\left(\frac{1}{2}+q\right)F_1^2-\frac{1}{2}ae^{-2F}-\mu\nu+\frac{\nu^2}{2}, \quad (3.26)$$

$$r_3 = \frac{1}{2}\left(ae^{-2F}F_1^2 - F_2^2 - \nu^2\right), \tag{3.27}$$

After calculating $d\omega_{12} - \sum_i \omega_{1i} \wedge \omega_{i2}$, we get with the Gauss equation that

$$r_1 + r_2 = qF_{11} + pF_{22} - q^2F_1^2 - p^2F_2^2 - \lambda\mu. \tag{3.28}$$

Put (3.25) and (3.26) into (3.28), then F needs to satisfy

$$pF_{11} + qF_{22} - pq(F_1^2 + F_2^2) + (\lambda-\nu)(\mu-\nu) = ae^{-2F}. \tag{3.29}$$

Under the chosen coordinate, we have

$$F_1 = e^{pF}\varphi', \quad F_2 = e^{qF}\psi',$$

$$F_{11} = e^{2pF}(\varphi'' + p\varphi'^2), \quad F_{22} = e^{2qF}(\psi'' + q\psi'^2),$$

$$F_{112} = 2p\psi' e^{(p-q)F}(\varphi'' + p\varphi'^2), \quad F_{221} = 2q\varphi' e^{(q-p)F}(\psi'' + q\psi'^2), \tag{3.30}$$

where $\varphi' = \frac{d\varphi}{du}, \varphi'' = \frac{d^2\varphi}{du^2}, \psi' = \frac{d\psi}{dv}$ and $\psi'' = \frac{d^2\psi}{dv^2}$. So (3.29) is equivalent to the following

$$p(\varphi'' + (p-q)\varphi'^2) + qe^{2(q-p)F}(\psi'' + (q-p)\psi'^2) + (\lambda-\nu)(\mu-\nu)e^{-2pF} = ae^{2qF}.$$

Differentiate the above equation, with respect to v firstly and u secondly, then

$$((\lambda-\nu)(\mu-\nu) - ae^{-2F})\varphi'\psi' = 0,$$

which implies that $\varphi' = 0$ or $\psi' = 0$. Without loss of generality, assume $\psi' = 0$, then $F_2 = 0$ and $C_2 = 0$, which mean F is a single-variable function and independent of v. Let $f = -(\lambda-\nu)F$, then $\mathbf{\Phi} = df$. $\square$

Now we have

$$\omega_{12} = -\frac{C_1}{\lambda-\mu}\omega_2, \quad \omega_{2\alpha} = 0, \tag{3.31}$$

and V_λ, V_μ are integrable. Denote L and N_1 as their integrable submanifolds, of which local coordinates are (u) and (v), respectively. Then it holds locally that $M_1 = L \times N_1$ and $M = L \times N_1 \times M_2$.

Next step we will study the differential geometry of M.

Choose u as the arc parameter of L. Then, from the proof of Lemma 3.3, $\mathbf{g}$ has a local coordinate expression

$$\mathbf{g} = du^2 + g_{22}(u,v)dv^2 + \sum_{\alpha,\beta} g_{\alpha\beta}(u,v,w)dw_\alpha dw_\beta.$$

Choose $E_1 = \frac{\partial}{\partial u}, \omega_1 = du$, then $f' = C_1$. Let $\bar{Y} = e^{\frac{f}{\lambda-\mu}}Y$. Because

$$E_2\left(\frac{\partial Y}{\partial u} + \frac{f'}{\lambda-\mu}Y\right) = E_\alpha\left(\frac{\partial Y}{\partial u} + \frac{f'}{\lambda-\nu}Y\right) = 0,$$

there exist locally smooth vector functions $\bar{\zeta} = \bar{\zeta}(u, w_\alpha)$ and $\bar{\eta} = \bar{\eta}(u,v)$, such that

$$\frac{\partial \bar{Y}}{\partial u} = \bar{\zeta}, \quad \frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f'\bar{Y} + \bar{\zeta} = \bar{\eta}.$$

Hence

$$f''\bar{\eta}-f'\frac{\partial\bar{\eta}}{\partial u}=f''\bar{\zeta}-\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f'^2\bar{\zeta}-f'\frac{\partial\bar{\zeta}}{\partial u},$$

which means there exists locally a smooth vector function $\bar{\xi}=\bar{\xi}(u)$ such that

$$f''\bar{\eta}-f'\frac{\partial\bar{\eta}}{\partial u}=-\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}\bar{\xi},$$

$$f''\bar{\zeta}-\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f'^2\bar{\zeta}-f'\frac{\partial\bar{\zeta}}{\partial u}=-\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}\bar{\xi}.$$

Then

$$\bar{\eta}=\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f'\left(\int\frac{\bar{\xi}}{(f')^2}du+\eta\right),$$

$$\bar{\zeta}=\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f'e^{-\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f}\left(\int e^{\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f}\frac{\bar{\xi}}{(f')^2}du-\zeta\right),$$

where $\eta=\eta(v),\zeta=\zeta(w_\alpha)$. Let

$$\xi=e^{\frac{f}{\mu-\lambda}}\int\frac{\bar{\xi}}{(f')^2}du-e^{\frac{f}{\nu-\lambda}}\int e^{\frac{\mu-\nu}{(\lambda-\mu)(\nu-\lambda)}f}\frac{\bar{\xi}}{(f')^2}du,$$

then

$$Y=\frac{(\lambda-\mu)(\nu-\lambda)}{(\mu-\nu)f'}e^{\frac{f}{\mu-\lambda}}(\bar{\eta}-\bar{\zeta})=\xi(u)+e^{\frac{f}{\mu-\lambda}}\eta(v)+e^{\frac{f}{\nu-\lambda}}\zeta(w). \tag{3.32}$$

Then

$$\mathbf{g}=du^2+e^{\frac{2f}{\mu-\lambda}}\langle d\eta,d\eta\rangle+e^{\frac{2f}{\nu-\lambda}}\langle d\zeta,d\zeta\rangle.$$

Remark 3.4. *Note that the function f needs to satisfy some conditions to guarantee integrability conditions* (2.8) – (2.14)*. Here we only need to check the information of* (2.8)*. Actually,* (3.31) *guarantees $A_{\alpha\beta,\gamma}=A_{\alpha\gamma,\beta}$ and $A_{\alpha\alpha,i}-A_{\alpha i,\alpha}=-\nu C_i(i\neq\alpha)$. Note that $A_{11,i}-A_{1i,1}=-\lambda C_i(i\neq 1)$ is equivalent to*

$$F_{112}-2pF_{11}F_2=0, \tag{3.33}$$

and $A_{22,i}-A_{2i,2}=-\mu C_i(i\neq 2)$ is equivalent to

$$F_{221}-2qF_{22}F_1=0. \tag{3.34}$$

From (3.30) *and $F_2=0$,* (3.33) *and* (3.34) *naturally hold on. So the condition of F is* (3.29)*, i.e. the condition of f is*

$$ae^{\frac{2f}{\lambda-\nu}}=\left(\frac{1}{\lambda-\nu}-\frac{1}{\lambda-\mu}\right)f''-\frac{1}{\lambda-\mu}\left(\frac{1}{\lambda-\nu}-\frac{1}{\lambda-\mu}\right)(f')^2+(\lambda-\nu)(\mu-\nu). \tag{3.35}$$

It can be checked that the first integral of (3.35) *is*

$$ae^{\frac{2f}{\lambda-\nu}}+be^{\frac{2f}{\lambda-\mu}}=(\mu-\nu)^2+\left(\frac{f'}{\lambda-\mu}-\frac{f'}{\lambda-\nu}\right)^2, \tag{3.36}$$

where b is a constant number.

Above all, we get the following theorem.

THEOREM 3.5. *There exists a local coordinate system* (u,v,w_α) *such that*

$$\mathbf{g}=du^2+e^{2h_1+\frac{2f}{\mu-\lambda}}dv^2+e^{2h_2+\frac{2f}{\nu-\lambda}}\|dw\|^2, \tag{3.37}$$

where $f=f(u)$ *satisfies* (3.35) (*i.e.* (3.36)),

$$h_1=-\mathbf{log}\left(1+\frac{b}{4}v^2\right),\quad h_2=-\mathbf{log}\left(1+\frac{a}{4}\|w\|^2\right),\quad \|dw\|^2=\sum_\alpha dw_\alpha^2,\quad \|w\|^2=\sum_\alpha w_\alpha^2,$$

and a,b *are constants.*

From Theorem 3.5, we choose $\{E_1=\frac{\partial}{\partial u},E_2=e^{\frac{f}{\lambda-\mu}-h_1}\frac{\partial}{\partial v},E_\alpha=e^{\frac{f}{\lambda-\nu}-h_2}\frac{\partial}{\partial w_\alpha}\}$ as an orthonormal tangent frame field of $(M,\mathbf{g})$, of which the dual frame field is $\{\omega_1=du,\omega_2=e^{\frac{f}{\mu-\lambda}+h_1}dv,\omega_\alpha=e^{\frac{f}{\nu-\lambda}+h_2}dw_\alpha\}$. Then

$$\omega_{\alpha\beta}=e^{\frac{f}{\lambda-\nu}-h_2}\left(\frac{\partial h_2}{\partial w_\alpha}\omega_\beta-\frac{\partial h_2}{\partial w_\beta}\omega_\alpha\right). \tag{3.38}$$

Structure equations (2.4)-(2.7) become

$$Y_{uu}=-r_1Y-N+\lambda E, \tag{3.39}$$

$$N_u=r_1Y_u+f'E,\quad \frac{\partial N}{\partial v}=r_2\frac{\partial Y}{\partial v},\quad \frac{\partial N}{\partial w_\alpha}=r_3\frac{\partial Y}{\partial w_\alpha}, \tag{3.40}$$

$$E_u=-\lambda Y_u-f'Y,\quad \frac{\partial E}{\partial v}=-\mu\frac{\partial Y}{\partial v},\quad \frac{\partial E}{\partial w_\alpha}=-\nu\frac{\partial Y}{\partial w_\alpha}, \tag{3.41}$$

$$r_2Y+N-\frac{f'}{\lambda-\mu}Y_u-\mu E=e^{\frac{2f}{\lambda-\mu}-2h_1}P_{22}, \tag{3.42}$$

$$(r_3Y+N-\frac{f'}{\lambda-\nu}Y_u-\nu E)\delta_{\alpha\beta}=e^{\frac{2f}{\lambda-\nu}-2h_2}P_{\alpha\beta}, \tag{3.43}$$

where

$$P_{22}=\frac{dh_1}{dv}\frac{\partial Y}{\partial v}-\frac{\partial^2 Y}{\partial v^2},$$

$$P_{\alpha\beta}=\frac{\partial h_2}{\partial w_\alpha}\frac{\partial Y}{\partial w_\beta}+\frac{\partial h_2}{\partial w_\beta}\frac{\partial Y}{\partial w_\alpha}-\frac{\partial^2 Y}{\partial w_\alpha\partial w_\beta}-\sum_\gamma\frac{\partial h_2}{\partial w_\gamma}\frac{\partial Y}{\partial w_\gamma}\delta_{\alpha\beta}.$$

After differentiating (3.42) by $\frac{\partial}{\partial v}$ with (3.40), (3.41) and (3.36), it can be checked directly that

$$\frac{d^3(e^{-h_1}\eta)}{dv^3} = 0,$$

which means that there exist constant vectors $\vec{a}, \vec{b}, \vec{c}$ such that

$$\eta = e^{h_1}(v^2\vec{a} + v\vec{b} + \vec{c}).$$

Notice $P_{\alpha\alpha} = P_{\beta\beta}$ and $P_{\alpha\beta} = 0$ when $\alpha \neq \beta$, hence

$$\frac{\partial^2(e^{-h_2}\zeta)}{\partial w_\alpha^2} = \frac{\partial^2(e^{-h_2}\zeta)}{\partial w_\beta^2}, \quad \frac{\partial^2(e^{-h_2}\zeta)}{\partial w_\alpha \partial w_\beta} = 0, \quad \alpha \neq \beta.$$

So there exist constant vectors $\vec{A}$, $\vec{B}_\alpha$, $\vec{C}$ such that

$$e^{-h_2}\zeta = \|w\|^2\vec{A} + \sum_\alpha w_\alpha \vec{B}_\alpha + \vec{C}.$$

Therefore,

$$\begin{aligned} Y =& \hat{\xi} + e^{\frac{f}{\mu-\lambda}+h_1}\left(\frac{1}{2}v^2\left(2\vec{a} - \frac{b}{2}\vec{c}\right) + v\vec{b}\right) \\ &+ e^{\frac{f}{\nu-\lambda}+h_2}\left(\frac{1}{2}\|w\|^2\left(2\vec{A} - \frac{a}{2}\vec{C}\right) + \sum_\alpha w_\alpha \vec{B}_\alpha\right), \end{aligned} \tag{3.44}$$

where $\hat{\xi} = \xi(u) + e^{\frac{f}{\mu-\lambda}}\vec{c} + e^{\frac{f}{\nu-\lambda}}\vec{C}$.

When $v = 0$ and $w = 0$, from (3.44), we have

$$\frac{\partial Y}{\partial u} = \hat{\xi}', \quad \frac{\partial Y}{\partial v} = e^{\frac{f}{\mu-\lambda}}\vec{b}, \quad \frac{\partial^2 Y}{\partial v^2} = e^{\frac{f}{\mu-\lambda}}\left(2\vec{a} - \frac{b}{2}\vec{c}\right),$$

$$\frac{\partial Y}{\partial w_\alpha} = e^{\frac{f}{\nu-\lambda}}\vec{B}_\alpha, \quad \frac{\partial^2 Y}{\partial w_\alpha^2} = e^{\frac{f}{\nu-\lambda}}\left(2\vec{A} - \frac{a}{2}\vec{C}\right).$$

According to these above formulas and (3.37), we have the following lemma with a similar proof as Lemma 5.1 in [12].

Lemma 3.6.

$$\langle\hat{\xi}, \hat{\xi}\rangle = 0, \quad \langle\hat{\xi}, \hat{\xi}'\rangle = 0, \quad \langle\hat{\xi}', \hat{\xi}'\rangle = 1,$$

$$\langle\hat{\xi}, \vec{b}\rangle = 0, \quad \langle\hat{\xi}', \vec{b}\rangle = 0, \quad \langle\hat{\xi}, \vec{B}_\alpha\rangle = 0, \quad \langle\hat{\xi}', \vec{B}_\alpha\rangle = 0,$$

$$\langle\vec{b}, \vec{b}\rangle = 1, \quad \langle\vec{B}_\alpha, \vec{B}_\beta\rangle = \delta_{\alpha\beta}, \ \langle\vec{b}, \vec{B}_\alpha\rangle = 0,$$

$$\left\langle 2\vec{a} - \frac{b}{2}\vec{c}, 2\vec{a} - \frac{b}{2}\vec{c}\right\rangle = b, \quad \left\langle 2\vec{a} - \frac{b}{2}\vec{c}, \vec{b}\right\rangle = \left\langle 2\vec{a} - \frac{b}{2}\vec{c}, \vec{B}_\alpha\right\rangle = 0,$$

$$\left\langle 2\vec{A}-\frac{a}{2}\vec{C},2\vec{A}-\frac{a}{2}\vec{C}\right\rangle = a,\ \ \left\langle 2\vec{A}-\frac{a}{2}\vec{C},\vec{B}_\alpha\right\rangle = \left\langle 2\vec{A}-\frac{a}{2}\vec{C},\vec{B}_\alpha\right\rangle = 0,$$

$$\left\langle 2\vec{a}-\frac{b}{2}\vec{c},\hat{\xi}\right\rangle = -e^{\frac{f}{\mu-\lambda}},\ \ \left\langle 2\vec{A}-\frac{a}{2}\vec{C},\hat{\xi}\right\rangle = -e^{\frac{f}{\nu-\lambda}}.$$

From (3.36), at least one of a, b must be positive. So there are five cases about signs of a, b.

Case 1: $a > 0, \quad b = 0$**.**

Take a frame of $\mathbb{R}^{m+3}_1$ up to a Lorentz transformation such that

$$\vec{a} = (1,-1,0,\cdots,0),\ \ 2\vec{A}-\frac{a}{2}\vec{C} = (0,0,0,0,-\sqrt{a},0,\cdots,0),$$

$$\vec{b} = (0,0,0,1,0,\cdots,0),\ \ \vec{B}_\alpha = (\underbrace{0,\cdots,0}_{\alpha+2},1,0,\cdots,0).$$

Assume $\hat{\xi} = (\xi_1,\xi_2,\xi_3,0,\xi_4,0,\cdots,0)$. It satisfies

$$-\xi_1-\xi_2 = -\frac{1}{2}e^{\frac{f}{\mu-\lambda}},\ \ -\sqrt{a}\xi_4 = -e^{\frac{f}{\nu-\lambda}},\ \ -\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2 = 0,$$

then

$$\xi_1 = \frac{1}{4}e^{\frac{f}{\mu-\lambda}} + e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{a}e^{\frac{2f}{\nu-\lambda}}+\xi_3^2\right),\ \ \xi_2 = \frac{1}{4}e^{\frac{f}{\mu-\lambda}} - e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{a}e^{\frac{2f}{\nu-\lambda}}+\xi_3^2\right),\ \ \xi_4 = \frac{1}{\sqrt{a}}e^{\frac{f}{\nu-\lambda}}.$$

Hence

$$\begin{aligned} Y = \Bigg(&\frac{1}{4}e^{\frac{f}{\mu-\lambda}} + e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{a}e^{\frac{2f}{\nu-\lambda}}+\xi_3^2\right) + e^{\frac{f}{\mu-\lambda}}v^2, \frac{1}{4}e^{\frac{f}{\mu-\lambda}} - e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{a}e^{\frac{2f}{\nu-\lambda}}+\xi_3^2\right) \\ &-e^{\frac{f}{\mu-\lambda}}v^2, \xi_3, e^{\frac{f}{\mu-\lambda}}v, e^{\frac{f}{\nu-\lambda}}\frac{1-\frac{a}{4}\|w\|^2}{\sqrt{a}(1+\frac{a}{4}\|w\|^2)}, e^{\frac{f}{\nu-\lambda}}\frac{w}{1+\frac{a}{4}\|w\|^2}\Bigg). \end{aligned} \tag{3.45}$$

Let $t = e^{\frac{f}{\lambda-\mu}}\xi_3$. $\langle\hat{\xi}',\hat{\xi}'\rangle = 1$ deduces

$$\frac{dt}{du} = e^{\frac{f}{\lambda-\mu}}\left(\frac{f'}{\lambda-\mu}\xi_3+\xi_3'\right) = \frac{\mu-\nu}{\sqrt{a}}e^{\frac{(\mu-\nu)f}{(\lambda-\mu)(\lambda-\nu)}} \neq 0, \tag{3.46}$$

which means $t = t(u)$ exists the inverse function $u = u(t)$. Let $\theta = \frac{1}{\mu-\nu}\frac{dt}{du}$. It can be checked from (3.36) that

$$1+\left(\frac{d\theta}{dt}\right)^2 = \frac{a^{\frac{\lambda-\nu}{\mu-\nu}}}{(\mu-\nu)^2}\theta^{\frac{2(\lambda-\mu)}{\mu-\nu}}. \tag{3.47}$$

Then

$$x = \sigma\circ\mathfrak{F},$$

where

$$\mathfrak{F}(t,y_1,y_2) = (t,y_1,\theta(t)y_2),\ \ t\in\mathbb{R}^1,\ \ y_1\in\mathbb{R}^1,\ \ y_2\in\mathbb{S}^{m-2}.$$

Consider the curve $C:\vec{r}=\{t,\theta(t)\}$ satisfying (3.47) in $\mathbb{R}^2$. It can be calculated directly from (3.46) and (3.47) that its arc differential and the geodesic curvature are the following

$$ds=\frac{a^{\frac{\lambda-\nu}{2(\mu-\nu)}}}{\mu-\nu}\theta^{\frac{\lambda-\mu}{\mu-\nu}}dt=e^{\frac{f}{\lambda-\mu}}du,\quad \kappa=(\lambda-\mu)e^{-\frac{f}{\lambda-\mu}}.$$

Because

$$\frac{d\kappa}{ds}=-e^{-\frac{2f}{\lambda-\mu}}f',\quad \frac{d^2\kappa}{ds^2}=-e^{-\frac{3f}{\lambda-\mu}}\left(\frac{2}{\lambda-\mu}f'^2-f''\right),$$

and (3.36), κ satisfies the following equation

$$\kappa\frac{d^2\kappa}{ds^2}+\left(\frac{\lambda-\mu}{\lambda-\nu}-2\right)\left(\frac{d\kappa}{ds}\right)^2+\frac{\lambda-\nu}{\lambda-\mu}\kappa^4=0,$$

which means the curve C is a free $\frac{\lambda-\mu}{\lambda-\nu}-$elastic curve in $\mathbb{R}^2$.

Case 2: $a>0,\quad b<0$**.**

Take a frame of $\mathbb{R}^{m+3}_1$ up to a Lorentz transformation such that

$$2\vec{a}-\frac{b}{2}\vec{c}=(\sqrt{-b},0,0,\cdots,0),\quad 2\vec{A}-\frac{a}{2}\vec{C}=(0,0,-\sqrt{a},0,\cdots,0),$$

$$\vec{b}=(0,1,0,\cdots,0),\quad \vec{B}_\alpha=(\underbrace{0,\cdots,0}_{\alpha},1,0,\cdots,0).$$

Assume $\hat{\xi}=(\xi_1,0,\xi_2,\underbrace{0,\cdots,0}_{m-2},\xi_3,\xi_4)$. It satisfies

$$-\sqrt{-b}\xi_1=-e^{\frac{f}{\mu-\lambda}},\quad -\sqrt{a}\xi_2=-e^{\frac{f}{\nu-\lambda}},\quad -\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2=0,$$

then

$$\xi_1=\frac{1}{\sqrt{-b}}e^{\frac{f}{\mu-\lambda}},\quad \xi_2=\frac{1}{\sqrt{a}}e^{\frac{f}{\nu-\lambda}},\quad \xi_3^2+\xi_4^2=-\frac{1}{b}e^{\frac{2f}{\mu-\lambda}}-\frac{1}{a}e^{\frac{2f}{\nu-\lambda}}.$$

Hence

$$Y=\left(e^{\frac{f}{\mu-\lambda}}\frac{1-\frac{b}{4}v^2}{\sqrt{-b}(1+\frac{b}{4}v^2)},e^{\frac{f}{\mu-\lambda}}\frac{v}{1+\frac{b}{4}v^2},e^{\frac{f}{\nu-\lambda}}\frac{1-\frac{a}{4}\|w\|^2}{\sqrt{a}(1+\frac{a}{4}\|w\|^2)},e^{\frac{f}{\nu-\lambda}}\frac{w}{1+\frac{a}{4}\|w\|^2},\xi_3,\xi_4\right).$$

Notice we can solve $\xi_3=\xi_3(u)$ and $\xi_4=\xi_4(u)$ from $\langle\hat{\xi}',\hat{\xi}'\rangle=1$. Let

$$\mathbf{cosh}P(u)=\frac{\xi_1}{\sqrt{\xi_1^2-\xi_2^2}},\quad \mathbf{cos}Q(u)=\frac{\xi_3}{\sqrt{\xi_1^2-\xi_2^2}}.$$

It can be checked that

$$P(u)=\mathbf{arctanh}\sqrt{-\frac{b}{a}}e^{\frac{f}{\lambda-\mu}-\frac{f}{\lambda-\nu}},\quad Q(u)=\int\frac{(\mu-\nu)\sqrt{-ab}e^{\frac{f}{\lambda-\mu}+\frac{f}{\lambda-\nu}}}{be^{\frac{2f}{\lambda-\mu}}+ae^{\frac{2f}{\lambda-\nu}}}du.$$

Choose a parameter expression of $\mathbb{S}^2$ as follows

$$\vec{r}(P,Q) = (\mathbf{tanh}P, \mathbf{sech}P\mathbf{cos}Q, \mathbf{sech}P\mathbf{sin}Q).$$

Consider the curve C in $\mathbb{S}^2$: $P = P(u), Q = Q(u)$, then with a direct calculation we get the geodesic curvature of C is

$$\kappa = \frac{\mu - \lambda}{\sqrt{-b}} e^{\frac{f}{\mu-\lambda}}. \tag{3.48}$$

Denote $fN(c)$ as the set $\{fy | y \in N(c)\}$ where f is a smooth function and $N(c)$ is a space form. Hence a hypersurface in this case is locally Möbius equivalent to

$$X(M) = \pi(\mathbb{H}^1 \times x_1(u)\mathbb{S}^{m-2} \times \{x_2(u), x_3(u)\}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a curve with the geodesic curvature (3.48) in $\mathbb{S}^2$.

Case 3: $a > 0, \quad b > 0$**.**

Take a frame of $\mathbb{R}^{m+3}_1$ up to a Lorentz transformation such that

$$2\vec{a} - \frac{b}{2}\vec{c} = (0, 0, -\sqrt{b}, 0, \cdots, 0), \quad 2\vec{A} - \frac{a}{2}\vec{C} = (0, 0, 0, 0, -\sqrt{a}, 0, \cdots, 0),$$

$$\vec{b} = (0, 0, 0, 1, 0, \cdots, 0), \quad \vec{B}_\alpha = (\underbrace{0, \cdots, 0}_{\alpha+2}, 1, 0, \cdots, 0).$$

Assume $\hat{\xi} = (\xi_1, \xi_2, \xi_3, 0, \xi_4, \underbrace{0, \cdots, 0}_{m-2})$. It satisfies

$$-\sqrt{b}\xi_3 = -e^{\frac{f}{\mu-\lambda}}, \quad -\sqrt{a}\xi_4 = -e^{\frac{f}{\nu-\lambda}}, \quad -\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 0,$$

then

$$\xi_3 = \frac{1}{\sqrt{b}} e^{\frac{f}{\mu-\lambda}}, \quad \xi_4 = \frac{1}{\sqrt{a}} e^{\frac{f}{\nu-\lambda}}, \quad -\xi_1^2 + \xi_2^2 = -\frac{1}{b} e^{\frac{2f}{\mu-\lambda}} - \frac{1}{a} e^{\frac{2f}{\nu-\lambda}}.$$

Hence

$$Y = \left(\xi_1, \xi_2, e^{\frac{f}{\mu-\lambda}} \frac{1 - \frac{b}{4}v^2}{\sqrt{b}(1 + \frac{b}{4}v^2)}, e^{\frac{f}{\mu-\lambda}} \frac{v}{1 + \frac{b}{4}v^2}, e^{\frac{f}{\nu-\lambda}} \frac{1 - \frac{a}{4}\|w\|^2}{\sqrt{a}(1 + \frac{a}{4}\|w\|^2)}, e^{\frac{f}{\nu-\lambda}} \frac{w}{1 + \frac{a}{4}\|w\|^2} \right).$$

We can also solve $\xi_1 = \xi_1(u)$ and $\xi_2 = \xi_2(u)$ from $\langle \hat{\xi}', \hat{\xi}' \rangle = 1$. Let

$$\mathbf{cosh}Q(u) = \frac{\xi_1}{\sqrt{\xi_3^2 + \xi_4^2}}, \quad \mathbf{cos}P(u) = \frac{\xi_3}{\sqrt{\xi_3^2 + \xi_4^2}}.$$

It can be also checked that

$$P(u) = \mathbf{arctan}\sqrt{\frac{b}{a}} e^{\frac{f}{\lambda-\mu} - \frac{f}{\lambda-\nu}}, \quad Q(u) = \int \frac{(\mu - \nu)\sqrt{ab} e^{\frac{f}{\lambda-\mu} + \frac{f}{\lambda-\nu}}}{be^{\frac{2f}{\lambda-\mu}} + ae^{\frac{2f}{\lambda-\nu}}} du.$$

Choose a parameter expression of $\mathbb{H}^2$ as follows

$$\vec{r}(P,Q) = (\mathbf{sec}P\mathbf{cosh}Q, \mathbf{sec}P\mathbf{sinh}Q, \mathbf{tan}P).$$

Consider the curve C in $\mathbb{H}^2$: $P = P(u), Q = Q(u)$, then with a direct calculation we get the geodesic curvature of C is

$$\kappa = \frac{\mu - \lambda}{\sqrt{b}} e^{\frac{f}{\mu-\lambda}}. \tag{3.49}$$

Hence a hypersurface in this case is locally Möbius equivalent to

$$X(M) = \pi(\{x_1(u), x_2(u)\} \times \mathbb{S}^1 \times x_3(u)\mathbb{S}^{m-2}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a curve with the geodesic curvature (3.49) in $\mathbb{H}^2$.

So, in the case $b \neq 0$, the geodesic curvature κ of C satisfies

$$\frac{d\kappa}{ds} = e^{\frac{2f}{\mu-\lambda}} f', \quad \frac{d^2\kappa}{ds^2} = \frac{e^{\frac{3f}{\mu-\lambda}}}{|b|^{\frac{3}{2}}} \left(\frac{2f'^2}{\mu - \lambda} + f'' \right).$$

With (3.36), it can be checked that κ satisfies

$$\kappa \frac{d^2\kappa}{ds^2} + \left(\frac{\lambda - \mu}{\lambda - \nu} - 2 \right) \left(\frac{d\kappa}{ds} \right)^2 + \frac{\lambda - \nu}{\lambda - \mu} \kappa^4 + \delta \frac{\lambda - \nu}{\mu - \nu} \kappa^2 = 0, \tag{3.50}$$

where $\delta = 1$ for the curve C in Case 2 and $\delta = -1$ for the curve C in Case 3. Therefore, the curves C are free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curves in two-dimensional space forms.

Case 4: $a = 0, \quad b > 0$**.**

With a similar analysis in Case 1, it can be proved that in this case the hypersurface is locally Möbius equivalent to

$$x = \sigma \circ \tilde{\mathfrak{F}},$$

where

$$\tilde{\mathfrak{F}}(t, y_1, y_2) = (t, \theta y_1, y_2), \quad t \in \mathbb{R}^1, \quad y_1 \in \mathbb{S}^1, \quad y_2 \in \mathbb{R}^{m-2},$$

and $\theta = \theta(t)$ satisfies the following.

$$1 + \left(\frac{d\theta}{dt} \right)^2 = \frac{a^{\frac{\lambda-\mu}{\nu-\mu}}}{(\mu - \nu)^2} \theta^{\frac{2(\lambda-\nu)}{\nu-\mu}}.$$

Case 5: $a < 0, \quad b > 0$**.**

With a similar analysis in Case 2, it can be proved that in this case the hypersurface is locally Möbius equivalent to

$$X(M) = \pi(\mathbb{H}^{m-2} \times x_1(u)\mathbb{S}^1 \times \{x_2(u), x_3(u)\}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a curve with geodesic curvature $\kappa = \frac{\nu-\lambda}{\sqrt{-a}} e^{\frac{f}{\nu-\lambda}}$ in $\mathbb{S}^2$. Here the condition (3.36) of f is equivalent to the following condition of κ.

$$\kappa \frac{d^2\kappa}{ds^2} + \left(\frac{\lambda - \nu}{\lambda - \mu} - 2 \right) \left(\frac{d\kappa}{ds} \right)^2 + \frac{\lambda - \mu}{\lambda - \nu} \kappa^4 + \frac{\lambda - \mu}{\nu - \mu} \kappa^2 = 0.$$

By summing up, we get the following classification theorem.

THEOREM 3.7. *Let $x : M^m \to \mathbb{S}^{m+1} (m \geq 4)$ be a hypersurface with non-vanishing closed Möbius form $\mathbf{\Phi}$ and three constant Möbius principal curvatures λ, μ, ν, where λ, μ have single-multiplicity. Then there exists locally a smooth single variable function $f = f(u)$ satisfied (3.36) such that $\mathbf{\Phi} = df$. Moreover, x is locally Möbius equivalent to one of the following five hypersurfaces.*
(1) *For constants $a > 0, \quad b = 0$,*

$$x(M) = \sigma(\{x_1(u)\} \times \mathbb{R}^1 \times x_2(u)\mathbb{S}^{m-2}), \quad t \in \mathbb{R}^1,$$

where $\vec{r}(u) = \{x_1(u), x_2(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{R}^2$;
(2) *For constants $a > 0, \quad b < 0$,*

$$X(M) = \pi(\mathbb{H}^1 \times x_1(u)\mathbb{S}^{m-2} \times \{x_2(u), x_3(u)\}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{S}^2$;
(3) *For constants $a > 0, \quad b > 0$,*

$$X(M) = \pi(\{x_1(u), x_2(u)\} \times \mathbb{S}^1 \times x_3(u)\mathbb{S}^{m-2}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{H}^2$;
(4) *For constants $a = 0, \quad b > 0$,*

$$x(M) = \sigma(\{x_1(u)\} \times x_2(u)\mathbb{S}^1 \times \mathbb{R}^{m-2}), \quad t \in \mathbb{R}^1,$$

where $\vec{r}(u) = \{x_1(u), x_2(u)\}$ is a free $\frac{\lambda-\nu}{\lambda-\mu}$−elastic curve in $\mathbb{R}^2$;
(5) *For constants $a < 0, \quad b > 0$,*

$$X(M) = \pi(\mathbb{H}^{m-2} \times x_1(u)\mathbb{S}^1 \times \{x_2(u), x_3(u)\}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a free $\frac{\lambda-\nu}{\lambda-\mu}$−elastic curve in $\mathbb{S}^2$.

3.2. Case (ii): $m_1 = 1, \quad m_2, m_3 \geq 2$. Let $x : M^m \to \mathbb{S}^{m+1} (m \geq 4)$ be a hypersurface with $\mathbf{\Phi} \neq 0, d\mathbf{\Phi} = 0, m_1 = 1, m_2, m_3 \geq 2$. There exists locally a smooth function f such that $\mathbf{\Phi} = df$. Firstly, from (3.1), we have $C_p = 0, C_\alpha = 0$ and

$$B_{11,i} = B_{pq,i} = B_{\alpha\beta,i} = B_{1p,1} = B_{1\alpha,1} = B_{p\alpha,q} = B_{p\alpha,\beta} = 0,$$

$$B_{1p,q} = -\delta_{pq}C_1, \quad B_{1\alpha,\beta} = -\delta_{\alpha\beta}C_1, \quad B_{1p,\alpha} = B_{1\alpha,p} = B_{p\alpha,1},$$

$$f_1 := E_1(f) = C_1, \ f_p := E_p(f) = 0, \ f_\alpha := E_\alpha(f) = 0, \ \omega_{p\alpha} = \frac{1}{\mu - \nu} B_{1p,\alpha}\omega_1,$$

$$\omega_{1p} = \frac{1}{\lambda - \mu}\left(-C_1\omega_p + \sum_\alpha B_{1p,\alpha}\omega_\alpha\right), \quad \omega_{1\alpha} = \frac{1}{\lambda - \nu}\left(-C_1\omega_\alpha + \sum_p B_{1p,\alpha}\omega_p\right).$$

Because

$$C_{p,\alpha} = \frac{C_1 B_{1p,\alpha}}{\lambda - \mu}, \quad C_{\alpha,p} = \frac{C_1 B_{1p,\alpha}}{\lambda - \nu}, \quad C_{p,\alpha} = C_{\alpha,p},$$

it holds $B_{1p,\alpha} = 0$ and

$$\omega_{1p} = -\frac{C_1}{\lambda-\mu}\omega_p, \quad \omega_{1\alpha} = -\frac{C_1}{\lambda-\nu}\omega_\alpha, \quad \omega_{p\alpha} = 0. \tag{3.51}$$

Then

$$d\omega_p \equiv 0\mathbf{mod}(\omega_q), \quad d\omega_\alpha \equiv 0\mathbf{mod}(\omega_\beta),$$

which mean from Frobenius Theorem that the distributions V_λ, V_μ, V_ν are integrable. Denote L, N_1 and N_2 as their integrable submanifolds, of which local coordinates are $(u), (v_2, \cdots, v_{m_2+1})(i.e.(v)), (w_{m_2+2}, \cdots, w_m)(i.e.(w))$, respectively. Then it holds locally that $M = L \times N_1 \times N_2$.

Next step we will study the differential geometry of M.

Choose u as the arc parameter of L. $\mathbf{g}$ has a local coordinate expression

$$\mathbf{g} = du^2 + \sum_{p,q} g_{pq}(u,v,w)dv_p dv_q + \sum_{\alpha,\beta} g_{\alpha\beta}(u,v,w)dw_\alpha dw_\beta.$$

Choose $E_1 = \frac{\partial}{\partial u}, \omega_1 = du$, then $f' = C_1$.

With a similar analysis in (3.32), we get there exist locally smooth vector functions $\xi = \xi(u), \eta = \eta(v)$ and $\zeta = \zeta(w)$ such that

$$Y = \xi(u) + e^{\frac{f}{\mu-\lambda}}\eta(v) + e^{\frac{f}{\nu-\lambda}}\zeta(w). \tag{3.52}$$

Hence

$$\mathbf{g} = du^2 + e^{\frac{2f}{\mu-\lambda}}g_1 + e^{\frac{2f}{\nu-\lambda}}g_2,$$

where $g_1 = \langle d\eta, d\eta\rangle, g_2 = \langle d\zeta, d\zeta\rangle$.

THEOREM 3.8. *There exists a local coordinate system* (u, v_p, w_α) *such that*

$$\mathbf{g} = du^2 + e^{2h_1 + \frac{2f}{\mu-\lambda}}\|dv\|^2 + e^{2h_2+\frac{2f}{\nu-\lambda}}\|dw\|^2, \tag{3.53}$$

where

$$h_1 = -\mathbf{log}(1 + \frac{a}{4}\|v\|^2), \quad h_2 = -\mathbf{log}(1 + \frac{b}{4}\|w\|^2),$$

a, b *are constants and* f *satisfies*

$$\begin{aligned} ae^{\frac{2f}{\lambda-\mu}} = & \left(\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\nu}\right)f'' \\ & - \frac{1}{\lambda-\nu}\left(\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\nu}\right)(f')^2 - (\lambda-\mu)(\mu-\nu), \end{aligned} \tag{3.54}$$

$$\begin{aligned} be^{\frac{2f}{\lambda-\nu}} = & -\left(\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\nu}\right)f'' \\ & + \frac{1}{\lambda-\mu}\left(\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\nu}\right)(f')^2 + (\lambda-\nu)(\mu-\nu). \end{aligned} \tag{3.55}$$

Proof. With a direct calculation from (3.51), we get

$$R_{1p1p} = \frac{E_1(C_1)}{\lambda-\mu} - \frac{C_1^2}{(\lambda-\mu)^2}, \quad R_{1\alpha1\alpha} = \frac{E_1(C_1)}{\lambda-\nu} - \frac{C_1^2}{(\lambda-\nu)^2},$$

$$R_{p\alpha p\alpha} = -\frac{C_1^2}{(\lambda-\mu)(\lambda-\nu)}, \quad E_p(C_1) = E_\alpha(C_1) = 0.$$

With the Gauss equation, we get

$$\begin{aligned} r_1 := A_{11} =& \frac{1}{2}\Bigg(\left(\frac{1}{\lambda-\mu}+\frac{1}{\lambda-\nu}\right)f'' \\ &- \left(\frac{1}{(\lambda-\mu)^2}+\frac{1}{(\lambda-\nu)^2}-\frac{1}{(\lambda-\mu)(\lambda-\nu)}\right)(f')^2 - \lambda\mu+\mu\nu-\lambda\nu\Bigg), \end{aligned}$$

$$\begin{aligned} r_2 := A_{pp} =& \frac{1}{2}\Bigg(\left(\frac{1}{\lambda-\mu}-\frac{1}{\lambda-\nu}\right)f'' \\ &- \left(\frac{1}{(\lambda-\mu)^2}-\frac{1}{(\lambda-\nu)^2}+\frac{1}{(\lambda-\mu)(\lambda-\nu)}\right)(f')^2 - \lambda\mu-\mu\nu+\lambda\nu\Bigg), \end{aligned}$$

$$\begin{aligned} r_3 := A_{\alpha\alpha} =& \frac{1}{2}\Bigg(-\left(\frac{1}{\lambda-\mu}-\frac{1}{\lambda-\nu}\right)f'' \\ &+ \left(\frac{1}{(\lambda-\mu)^2}-\frac{1}{(\lambda-\nu)^2}-\frac{1}{(\lambda-\mu)(\lambda-\nu)}\right)(f')^2 + \lambda\mu-\mu\nu-\lambda\nu\Bigg). \end{aligned}$$

It can be checked that (M_1, g_1) and (M_2, g_2) are space forms. In fact, choose cotangent frame fields $\{\tilde{\omega}_p\}$ and $\{\tilde{\omega}_\alpha\}$ such that $g_1 = \sum_p \tilde{\omega}_p^2, g_2 = \sum_\alpha \tilde{\omega}_\alpha^2$. Let $\omega_p = e^{\frac{f}{\mu-\lambda}}\tilde{\omega}_p, \omega_\alpha = e^{\frac{f}{\nu-\lambda}}\tilde{\omega}_\alpha$, then $\{du, \omega_p, \omega_\alpha\}$ forms a cotangent frame field on $(M, \mathbf{g})$. Denote $\tilde{R}_{pqst}$ and $\tilde{\omega}_{pq}$ be components of the curvature tensor and connection forms of (M_1, g_1), $\tilde{R}_{\alpha\beta\gamma\sigma}$ and $\tilde{\omega}_{\alpha\beta}$ be components of the curvature tensor and connection forms of (M_2, g_2). Then

$$\omega_{pq} = \tilde{\omega}_{pq}, \quad \omega_{\alpha\beta} = \tilde{\omega}_{\alpha\beta},$$

$$R_{pqst} = e^{\frac{2f}{\lambda-\mu}}\tilde{R}_{pqst} - \left(\frac{f'}{\lambda-\mu}\right)^2(\delta_{ps}\delta_{qt} - \delta_{pt}\delta_{qs}),$$

$$R_{\alpha\beta\gamma\sigma} = e^{\frac{2f}{\lambda-\nu}}\tilde{R}_{\alpha\beta\gamma\sigma} - \left(\frac{f'}{\lambda-\nu}\right)^2(\delta_{\alpha\gamma}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}).$$

Combine with the Gauss equation, then

$$e^{\frac{2f}{\lambda-\mu}}\tilde{R}_{pqpq} = \left(\frac{f'}{\lambda-\mu}\right)^2 + 2r_2 + \mu^2,$$

$$e^{\frac{2f}{\lambda-\nu}}\tilde{R}_{\alpha\beta\alpha\beta} = \left(\frac{f'}{\lambda-\nu}\right)^2 + 2r_3 + \nu^2.$$

Therefore,

$$E_s(\tilde{R}_{pqpq}) = E_\gamma(\tilde{R}_{\alpha\beta\alpha\beta}) = 0,$$

which means $\tilde{R}_{pqpq}$ and $\tilde{R}_{\alpha\beta\alpha\beta}$ are constants, denoted by a and b. So the theorem holds. □

From Theorem 3.8, we choose $\{E_1 = \frac{\partial}{\partial u}, E_p = e^{\frac{f}{\lambda-\mu}-h_1}\frac{\partial}{\partial v_p}, E_\alpha = e^{\frac{f}{\lambda-\nu}-h_2}\frac{\partial}{\partial w_\alpha}\}$ as an orthonormal tangent frame field of $(M, \mathbf{g})$, of which the dual frame field is $\{\omega_1 = du, \omega_p = e^{\frac{f}{\mu-\lambda}+h_1}dv_p, \omega_\alpha = e^{\frac{f}{\nu-\lambda}+h_2}dw_\alpha\}$. Then

$$\omega_{pq} = e^{\frac{f}{\lambda-\mu}-h_1}\left(\frac{\partial h_1}{\partial v_p}\omega_q - \frac{\partial h_1}{\partial v_q}\omega_p\right), \quad \omega_{\alpha\beta} = e^{\frac{f}{\lambda-\nu}-h_2}\left(\frac{\partial h_2}{\partial w_\alpha}\omega_\beta - \frac{\partial h_2}{\partial w_\beta}\omega_\alpha\right).$$

Structure equations (2.4)-(2.7) become

$$Y_{uu} = -r_1 Y - N + \lambda E,$$

$$N_u = r_1 Y_u + f'E, \quad \frac{\partial N}{\partial v_p} = r_2\frac{\partial Y}{\partial v_p}, \quad \frac{\partial N}{\partial w_\alpha} = r_3\frac{\partial Y}{\partial w_\alpha},$$

$$E_u = -\lambda Y_u - f'Y, \quad \frac{\partial E}{\partial v_p} = -\mu\frac{\partial Y}{\partial v_p}, \quad \frac{\partial E}{\partial w_\alpha} = -\nu\frac{\partial Y}{\partial w_\alpha},$$

$$\left(r_2 Y + N - \frac{f'}{\lambda-\mu}Y_u - \mu E\right)\delta_{pq} = e^{-\frac{2f}{\mu-\lambda}-2h_1}Q_{pq},$$

$$\left(r_3 Y + N - \frac{f'}{\lambda-\nu}Y_u - \nu E\right)\delta_{\alpha\beta} = e^{-\frac{2f}{\nu-\lambda}-2h_2}Q_{\alpha\beta},$$

where

$$Q_{pq} = \frac{\partial h_1}{\partial v_p}\frac{\partial Y}{\partial v_q} + \frac{\partial h_1}{\partial v_q}\frac{\partial Y}{\partial v_p} - \frac{\partial^2 Y}{\partial v_p\partial v_q} - \sum_s \frac{\partial h_1}{\partial v_s}\frac{\partial Y}{\partial v_s}\delta_{pq},$$

$$Q_{\alpha\beta} = \frac{\partial h_2}{\partial w_\alpha}\frac{\partial Y}{\partial w_\beta} + \frac{\partial h_2}{\partial w_\beta}\frac{\partial Y}{\partial w_\alpha} - \frac{\partial^2 Y}{\partial w_\alpha\partial w_\beta} - \sum_\gamma \frac{\partial h_2}{\partial w_\gamma}\frac{\partial Y}{\partial w_\gamma}\delta_{\alpha\beta}.$$

Notice

$$Q_{pp} = Q_{qq},\ Q_{pq} = 0,\ Q_{\alpha\alpha} = Q_{\beta\beta},\ Q_{\alpha\beta} = 0,\ if \quad p \neq q, \quad \alpha \neq \beta,$$

which mean

$$\frac{\partial^2(e^{-h_1}\eta)}{\partial v_p^2} = \frac{\partial^2(e^{-h_1}\eta)}{\partial v_q^2}, \quad \frac{\partial^2(e^{-h_1}\eta)}{\partial v_p\partial v_q} = 0,\ if \quad p \neq q,$$

$$\frac{\partial^2(e^{-h_2}\zeta)}{\partial w_\alpha^2} = \frac{\partial^2(e^{-h_2}\zeta)}{\partial w_\beta^2}, \quad \frac{\partial^2(e^{-h_2}\zeta)}{\partial w_\alpha\partial w_\beta} = 0,\ if \quad \alpha \neq \beta.$$

Then there exist locally constant vectors $\vec{A}_1, \vec{A}_2, \vec{B}_p, \vec{B}_\alpha, \vec{C}_1, \vec{C}_2$ such that

$$\eta(v) = e^{h_1(v)}\left(\|v\|^2\vec{A}_1 + \sum_p v_p\vec{B}_p + \vec{C}_1\right),\ \zeta(w) = e^{h_2(w)}\left(\|w\|^2\vec{A}_2 + \sum_\alpha w_\alpha\vec{B}_\alpha + \vec{C}_2\right).$$

Put them into (3.52) and then

$$\begin{aligned} Y =&\hat{\xi} + e^{\frac{f}{\mu-\lambda}+h_1}\left(\frac{1}{2}\|v\|^2\left(2\vec{A}_1 - \frac{a}{2}\vec{C}_1\right) + \sum_p v_p\vec{B}_p\right) \\ &+ e^{\frac{f}{\nu-\lambda}+h_2}\left(\frac{1}{2}\|w\|^2\left(2\vec{A}_2 - \frac{b}{2}\vec{C}_2\right) + \sum_\alpha v_\alpha\vec{B}_\alpha\right), \end{aligned}$$

where $\hat{\xi} = \xi(u) + e^{\frac{f}{\mu-\lambda}}\vec{C}_1 + e^{\frac{f}{\nu-\lambda}}\vec{C}_2$.

Remark 3.9. *It can be claimed that* (3.54) *and* (3.55) *are consistent conditions of structure equations. Actually,*

$$A_{pp,1} = \frac{\partial}{\partial u}(r_2), \quad A_{\alpha\alpha,1} = \frac{\partial}{\partial u}(r_3),$$

$$A_{p1,p} = -\frac{f'}{\lambda-\mu}(r_1 - r_2), \quad A_{\alpha1,\alpha} = -\frac{f'}{\lambda-\nu}(r_1 - r_3),$$

then (2.8) *becomes*

$$r_2' + \frac{f'}{\lambda-\mu}(r_1 - r_2) = -\mu f', \quad r_3' + \frac{f'}{\lambda-\nu}(r_1 - r_3) = -\nu f'.$$

Both the above equations are equivalent with

$$f''' - 2\left(\frac{1}{\lambda-\mu} + \frac{1}{\lambda-\nu}\right)f'f'' + \frac{2(f')^3}{(\lambda-\mu)(\lambda-\nu)} + 2(\lambda-\mu)(\lambda-\nu)f' = 0. \tag{3.56}$$

It's easily seen that both (3.54) *and* (3.55) *are the first integrals of* (3.56)*, and they are equivalent with the same equation as* (3.36).

Remark 3.10. (3.36) *implies that at least one of a and b is positive. Without loss of generality, we assume $b > 0$ and discuss the hypersurfaces in these cases of a.*

When $v = 0$ and $w = 0$, we have

$$\frac{\partial Y}{\partial u} = \hat{\xi}', \quad \frac{\partial Y}{\partial v_p} = e^{\frac{f}{\mu-\lambda}}\vec{B}_p, \quad \frac{\partial Y}{\partial w_\alpha} = e^{\frac{f}{\nu-\lambda}}\vec{B}_\alpha,$$

$$\frac{\partial^2 Y}{\partial v_p^2} = e^{\frac{f}{\mu-\lambda}}\left(2\vec{A}_1 - \frac{a}{2}\vec{C}_1\right), \quad \frac{\partial^2 Y}{\partial w_\alpha^2} = e^{\frac{f}{\nu-\lambda}}\left(2\vec{A}_2 - \frac{b}{2}\vec{C}_2\right).$$

Then we also have the following lemma according to Theorem 3.8 with a similar proof in Lemma 5.1 in [12].

Lemma 3.11.

$$\langle\hat{\xi}, \hat{\xi}\rangle = 0, \quad \langle\hat{\xi}, \hat{\xi}'\rangle = 0, \quad \langle\hat{\xi}', \hat{\xi}'\rangle = 1,$$

$$\langle \hat{\xi}, \vec{B}_p \rangle = 0, \quad \langle \hat{\xi}', \vec{B}_p \rangle = 0, \quad \langle \hat{\xi}, \vec{B}_\alpha \rangle = 0, \quad \langle \hat{\xi}', \vec{B}_\alpha \rangle = 0,$$

$$\langle \vec{B}_p, \vec{B}_q \rangle = \delta_{pq}, \quad \langle \vec{B}_\alpha, \vec{B}_\beta \rangle = \delta_{\alpha\beta}, \ \langle \vec{B}_p, \vec{B}_\alpha \rangle = 0,$$

$$\left\langle 2\vec{A}_1 - \frac{a}{2}\vec{C}_1, 2\vec{A}_1 - \frac{a}{2}\vec{C}_1 \right\rangle = a, \quad \left\langle 2\vec{A}_1 - \frac{a}{2}\vec{C}_1, \vec{B}_p \right\rangle = \left\langle 2\vec{A}_1 - \frac{a}{2}\vec{C}_1, \vec{B}_\alpha \right\rangle = 0,$$

$$\left\langle 2\vec{A}_2 - \frac{b}{2}\vec{C}_2, 2\vec{A}_2 - \frac{b}{2}\vec{C}_2 \right\rangle = b, \quad \left\langle 2\vec{A}_2 - \frac{b}{2}\vec{C}_2, \vec{B}_\alpha \right\rangle = \left\langle 2\vec{A}_2 - \frac{b}{2}\vec{C}_2, \vec{B}_\alpha \right\rangle = 0,$$

$$\left\langle 2\vec{A}_1 - \frac{a}{2}\vec{C}_1, \hat{\xi} \right\rangle = -e^{\frac{f}{\mu-\lambda}}, \quad \left\langle 2\vec{A}_2 - \frac{b}{2}\vec{C}_2, \hat{\xi} \right\rangle = -e^{\frac{f}{\nu-\lambda}}.$$

Case 1: $a = 0$**.**
Take a frame of $\mathbb{R}^{m+3}_1$ up to a Lorentz transformation such that

$$\vec{A}_1 = (1, -1, 0, \cdots, 0), \quad 2\vec{A}_2 - \frac{b}{2}\vec{C}_2 = (\underbrace{0, \cdots, 0}_{m_2+3}, -\sqrt{b}, 0, \cdots, 0),$$

$$\vec{B}_p = (\underbrace{0, \cdots, 0}_{p+1}, 1, 0, \cdots, 0), \quad \vec{B}_\alpha = (\underbrace{0, \cdots, 0}_{\alpha+2}, 1, 0, \cdots, 0).$$

Assume $\hat{\xi} = (\xi_1, \xi_2, \xi_3, \underbrace{0, \cdots, 0}_{m_2}, \xi_4, 0, \cdots, 0)$. It satisfies

$$-\xi_1 - \xi_2 = -\frac{1}{2}e^{\frac{f}{\mu-\lambda}}, \quad -\sqrt{b}\xi_4 = -e^{\frac{f}{\nu-\lambda}}, \quad -\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 0,$$

then

$$\xi_1 = \frac{1}{4}e^{\frac{f}{\mu-\lambda}} + e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{b}e^{\frac{2f}{\nu-\lambda}} + \xi_3^2\right), \quad \xi_2 = \frac{1}{4}e^{\frac{f}{\mu-\lambda}} - e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{b}e^{\frac{2f}{\nu-\lambda}} + \xi_3^2\right), \quad \xi_4 = \frac{1}{\sqrt{b}}e^{\frac{f}{\nu-\lambda}}.$$

Hence

$$\begin{aligned} Y = \Bigg(&\frac{1}{4}e^{\frac{f}{\mu-\lambda}} + e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{b}e^{\frac{2f}{\nu-\lambda}} + \xi_3^2\right) + e^{\frac{f}{\mu-\lambda}}\|v\|^2, \frac{1}{4}e^{\frac{f}{\mu-\lambda}} - e^{\frac{f}{\lambda-\mu}}\left(\frac{1}{b}e^{\frac{2f}{\nu-\lambda}} + \xi_3^2\right) \\ &-e^{\frac{f}{\mu-\lambda}}\|v\|^2, \xi_3, e^{\frac{f}{\mu-\lambda}}v, e^{\frac{f}{\nu-\lambda}}\frac{1-\frac{b}{4}\|w\|^2}{\sqrt{b}(1+\frac{b}{4}\|w\|^2)}, e^{\frac{f}{\nu-\lambda}}\frac{w}{1+\frac{b}{4}\|w\|^2}\Bigg). \end{aligned}$$

Let $t = e^{\frac{f}{\lambda-\mu}}\xi_3$, then $\langle \hat{\xi}', \hat{\xi}' \rangle = 1$ implies

$$\frac{dt}{du} = e^{\frac{f}{\lambda-\mu}}\left(\frac{f'}{\lambda-\mu}\xi_3 + \xi_3'\right) = \frac{\mu-\nu}{\sqrt{b}}e^{\frac{(\mu-\nu)f}{(\lambda-\mu)(\lambda-\nu)}} \neq 0.$$

Let $t = e^{\frac{f}{\lambda-\mu}}\xi_3$, $\theta = \frac{1}{\mu-\nu}\frac{dt}{du}$, then

$$x = \sigma \circ \mathfrak{F},$$

where

$$\mathfrak{F}(t, y_1, y_2) = (t, y_1, \theta(t)y_2), \quad t \in \mathbb{R}^1, \quad y_1 \in \mathbb{R}^{m_2}, \quad y_2 \in \mathbb{S}^{m_3}.$$

Consider the curve $C : \vec{r} = \{t, \theta(t)\}$. It can be checked similarly in Case 1 of the subsection 3.1 that the curve C is a free $\frac{\lambda-\mu}{\lambda-\nu}-$elastic curve in $\mathbb{R}^2$.

Case 2: $a < 0$.

Take a frame of $\mathbb{R}^{m+3}_1$ up to a Lorentz transformation such that

$$2\vec{A}_1 - \frac{a}{2}\vec{C}_1 = (\sqrt{-a}, 0, 0, \cdots, 0), \quad 2\vec{A}_2 - \frac{b}{2}\vec{C}_2 = (\underbrace{0, \cdots, 0}_{m_2+1}, -\sqrt{b}, 0, \cdots, 0),$$

$$\vec{B}_p = (\underbrace{0, \cdots, 0}_{p-1}, 1, 0, \cdots, 0), \quad \vec{B}_\alpha = (\underbrace{0, \cdots, 0}_{\alpha}, 1, 0, \cdots, 0).$$

Assume $\hat{\xi} = (\xi_1, \underbrace{0, \cdots, 0}_{m_2}, \xi_2, \underbrace{0, \cdots, 0}_{m_3}, \xi_3, \xi_4)$. It satisfies

$$\xi_1 = \frac{1}{\sqrt{-a}} e^{\frac{f}{\mu-\lambda}}, \quad \xi_2 = \frac{1}{\sqrt{b}} e^{\frac{f}{\nu-\lambda}}, \quad \xi_3^2 + \xi_4^2 = -\frac{1}{a} e^{\frac{2f}{\mu-\lambda}} - \frac{1}{b} e^{\frac{2f}{\nu-\lambda}}.$$

Hence

$$Y = \left(e^{\frac{f}{\mu-\lambda}} \frac{1 - \frac{a}{4}\|v\|^2}{\sqrt{-a}(1 + \frac{a}{4}\|v\|^2)}, e^{\frac{f}{\mu-\lambda}} \frac{v}{1 + \frac{a}{4}\|v\|^2}, e^{\frac{f}{\nu-\lambda}} \frac{1 - \frac{b}{4}\|w\|^2}{\sqrt{b}(1 + \frac{b}{4}\|w\|^2)}, e^{\frac{f}{\nu-\lambda}} \frac{w}{1 + \frac{b}{4}\|w\|^2}, \xi_3, \xi_4 \right).$$

It can be checked similarly in Case 2 of the subsection 3.1 that a hypersurface in this case is locally Möbius equivalent to

$$X(M) = \pi(\mathbb{H}^{m_2} \times x_1(u)\mathbb{S}^{m_3} \times \{x_2(u), x_3(u)\}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a curve with geodesic curvature $\kappa = \frac{\mu-\lambda}{\sqrt{-a}} e^{\frac{f}{\mu-\lambda}}$ in $\mathbb{S}^2$.

Case 3: $a > 0$.

Take a frame of $\mathbb{R}^{m+3}_1$ up to a Lorentz transformation such that

$$2\vec{A}_1 - \frac{a}{2}\vec{C}_1 = (0, 0, -\sqrt{a}, 0, \cdots, 0), \quad 2\vec{A}_2 - \frac{b}{2}\vec{C}_2 = (\underbrace{0, \cdots, 0}_{m_2+3}, -\sqrt{b}, 0, \cdots, 0),$$

$$\vec{B}_p = (\underbrace{0, \cdots, 0}_{p+1}, 1, 0, \cdots, 0), \quad \vec{B}_\alpha = (\underbrace{0, \cdots, 0}_{\alpha+2}, 1, 0, \cdots, 0).$$

Assume $\hat{\xi} = (\xi_1, \xi_2, \xi_3, \underbrace{0, \cdots, 0}_{m_2}, \xi_4, \underbrace{0, \cdots, 0}_{m_3})$. It satisfies

$$\xi_3 = \frac{1}{\sqrt{a}} e^{\frac{f}{\mu-\lambda}}, \quad \xi_4 = \frac{1}{\sqrt{b}} e^{\frac{f}{\nu-\lambda}}, \quad -\xi_1^2 + \xi_2^2 = -\frac{1}{a} e^{\frac{2f}{\mu-\lambda}} - \frac{1}{b} e^{\frac{2f}{\nu-\lambda}}.$$

Hence

$$Y = \left(\xi_1, \xi_2, e^{\frac{f}{\mu-\lambda}} \frac{1 - \frac{a}{4}\|v\|^2}{\sqrt{a}(1 + \frac{a}{4}\|v\|^2)}, e^{\frac{f}{\mu-\lambda}} \frac{v}{1 + \frac{a}{4}\|v\|^2}, e^{\frac{f}{\nu-\lambda}} \frac{1 - \frac{b}{4}\|w\|^2}{\sqrt{b}(1 + \frac{b}{4}\|w\|^2)}, e^{\frac{f}{\nu-\lambda}} \frac{w}{1 + \frac{b}{4}\|w\|^2} \right).$$

It can be checked similarly in Case 3 of the subsection 3.1 that a hypersurface in this case is locally Möbius equivalent to

$$X(M) = \pi(\{x_1(u), x_2(u)\} \times \mathbb{S}^{m_2} \times x_3(u)\mathbb{S}^{m_3}),$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a curve with geodesic curvature $\kappa = \frac{\mu-\lambda}{\sqrt{a}} e^{\frac{f}{\mu-\lambda}}$ in $\mathbb{H}^2$.

So, in the case $a \neq 0$, it can be checked similarly in (3.50) that κ satisfies (3.50) where $\delta = 1$ for the curve C in Case 2 and $\delta = -1$ for the curve C in Case 3. Therefore, the curves C are free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curves in two-dimensional space forms.

By summing up, we get the following classification theorem.

THEOREM 3.12. *Let $x : M^m \to \mathbb{S}^{m+1}(m \geq 5)$ be a hypersurface with three distinct constant Möbius principal curvatures λ, μ, ν, only one of which has one multiplicity. Its Möbius form satisfies $d\mathbf{\Phi} = 0, \mathbf{\Phi} \neq 0$. Then x is locally Möbius equivalent to one of the following three hypersurfaces.*

(1)

$$x(M) = \sigma(\{x_1(u)\} \times x_2(u)\mathbb{S}^{m_3} \times \mathbb{R}^{m_2}), \quad u \in \mathbb{R},$$

where $\vec{r}(u) = \{x_1(u), x_2(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{R}^2$;

(2)

$$X(M) = \pi(\mathbb{H}^{m_2} \times x_1(u)\mathbb{S}^{m_3} \times \{x_2(u), x_3(u)\}), \quad u \in \mathbb{R},$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{S}^2$;

(3)

$$X(M) = \pi(\{x_1(u), x_2(u)\} \times x_3(u)\mathbb{S}^{m_3} \times \mathbb{S}^{m_2}), \quad u \in \mathbb{R},$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{H}^2$. Here the integers $m_2, m_3 \geq 2, m_2 + m_3 = m - 1$.

From Theorem 3.7 and Theorem 3.12, we have the complete classification as follows.

THEOREM 3.13. *Let $x : M^m \to \mathbb{S}^{m+1}(m \geq 4)$ be a hypersurface with constant Möbius principal curvatures λ, μ, ν, of which multiplicities are $1, k, m-k-1$. Its Möbius form satisfies $d\mathbf{\Phi} = 0, \mathbf{\Phi} \neq 0$. Then x is locally Möbius equivalent to one of the following three hypersurfaces.*

(1)

$$x(M) = \sigma(\{x_1(u)\} \times x_2(u)\mathbb{S}^{m-k-1} \times \mathbb{R}^k), \quad u \in \mathbb{R},$$

where $\vec{r}(u) = \{x_1(u), x_2(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{R}^2$;

(2)

$$X(M) = \pi(\mathbb{H}^k \times x_1(u)\mathbb{S}^{m-k-1} \times \{x_2(u), x_3(u)\}), \quad u \in \mathbb{R},$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{S}^2$;

(3)

$$X(M) = \pi(\{x_1(u), x_2(u)\} \times x_3(u)\mathbb{S}^{m-k-1} \times \mathbb{S}^k), \quad u \in \mathbb{R},$$

where $\vec{r}(u) = \{x_1(u), x_2(u), x_3(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{H}^2$.

4. New examples of Willmore hypersurface. Let $x : M^m \to \mathbb{S}^{m+1}(m \geq 4)$ be a hypersurface with constant Möbius principal curvatures λ, μ, ν, of which multiplicities are $1, k, m-k-1$. Its Möbius form satisfies $d\mathbf{\Phi} = 0, \mathbf{\Phi} \neq 0$. Under the chosen basis in the discussion of Section 3, we have

$$C_{1,1} = f'', \quad C_{p,p} = -\frac{f'^2}{\lambda - \mu}, \quad C_{\alpha,\alpha} = -\frac{f'^2}{\lambda - \nu},$$

$$r_1 - r_3 = \frac{f''}{\lambda - \mu} - \frac{(\mu - \nu) f'^2}{(\lambda - \mu)^2 (\lambda - \nu)} - \lambda\mu + \mu\nu,$$

$$r_1 - r_2 = \frac{f''}{\lambda - \nu} + \frac{(\mu - \nu) f'^2}{(\lambda - \nu)^2 (\lambda - \mu)} - \lambda\nu + \mu\nu.$$

Hence, according to (2.16), x is Willmore hypersurface if and only if x satisfies

$$Af'' + Bf'^2 + C = 0, \tag{4.1}$$

where

$$A = (m-1)\left(\frac{1}{m(\lambda - \mu)(\lambda - \nu)} - 1\right), \quad C = m\lambda\mu\nu + \lambda^3 + k\mu^3 + (m-k-1)\nu^3,$$

$$B = (m-1)\left(\frac{k}{\lambda - \mu} + \frac{m-k-1}{\lambda - \nu}\right) + \frac{(\mu - \nu)}{(\lambda - \mu)(\lambda - \nu)}\left(\frac{-k\mu}{\lambda - \nu} + \frac{(m-k-1)\nu}{\lambda - \mu}\right).$$

From (3.36), it can be checked that

$$(f')^2 = \frac{(\lambda - \mu)^2 (\lambda - \nu)^2}{(\mu - \nu)^2}\left(ae^{\frac{2f}{\lambda - \mu}} + be^{\frac{2f}{\lambda - \nu}}\right) - (\lambda - \mu)^2 (\lambda - \nu)^2, \tag{4.2}$$

$$f'' = \frac{(\lambda - \mu)(\lambda - \nu)^2}{(\mu - \nu)^2} ae^{\frac{2f}{\lambda - \mu}} + \frac{(\lambda - \mu)^2 (\lambda - \nu)}{(\mu - \nu)^2} be^{\frac{2f}{\lambda - \nu}}. \tag{4.3}$$

Put (4.2) and (4.3) into (4.1), then

$$\begin{aligned} &(A(\lambda - \nu) + B(\lambda - \mu)(\lambda - \nu)) ae^{\frac{2f}{\lambda - \mu}} \\ &\quad + (A(\lambda - \mu) + B(\lambda - \mu)(\lambda - \nu)) be^{\frac{2f}{\lambda - \nu}} \\ &\quad - B(\lambda - \mu)(\lambda - \nu)(\mu - \nu)^2 + \frac{C(\mu - \nu)^2}{(\lambda - \mu)(\lambda - \nu)} = 0. \end{aligned} \tag{4.4}$$

Notice $f' \neq 0$ and λ, μ, ν are distinct with each other, then (4.4) implies

$$(A + B(\lambda - \mu))a = 0, \quad (A + B(\lambda - \nu))b = 0, \quad C = B(\lambda - \mu)^2 (\lambda - \nu)^2.$$

Because at least one of a, b is positive, without loss of generality, we assume $b > 0$. Hence

$$B = -\frac{A}{\lambda - \nu}, \quad C = -(\lambda - \mu)^2 (\lambda - \nu) A, \quad \frac{\mu - \nu}{\lambda - \nu} Aa = 0. \tag{4.5}$$

If $a \neq 0$, then $A = B = C = 0$, i.e.

$$\begin{cases} (\lambda-\mu)(\lambda-\nu) = \frac{1}{m} \\ (m-1)\left(\frac{k}{\lambda-\mu} + \frac{m-k-1}{\lambda-\nu}\right) + \frac{(\mu-\nu)}{(\lambda-\mu)(\lambda-\nu)}\left(\frac{-k\mu}{\lambda-\nu} + \frac{(m-k-1)\nu}{\lambda-\mu}\right) = 0 \\ m\lambda\mu\nu + \lambda^3 + k\mu^3 + (m-k-1)\nu^3 = 0 \\ \lambda + k\mu + (m-k-1)\nu = 0 \\ \lambda^2 + k\mu^2 + (m-k-1)\nu^2 = \frac{m-1}{m} \end{cases} \tag{4.6}$$

But (4.6) has no solution.

Above all, the following theorem holds.

THEOREM 4.1. *Let $x : M^m \to \mathbb{S}^{m+1} (m \geq 4)$ be a hypersurface with constant Möbius principal curvatures λ, μ, ν, of which multiplicities are $1, k, m-k-1$. Then x is a Willmore hypersurface if and only if x satisfies*

$$\begin{cases} (m-1)\left(\frac{k}{\lambda-\mu} + \frac{m-k-1}{\lambda-\nu}\right) + \frac{(\mu-\nu)}{(\lambda-\mu)(\lambda-\nu)}\left(\frac{-k\mu}{\lambda-\nu} + \frac{(m-k-1)\nu}{\lambda-\mu}\right) = \frac{m-1}{\lambda-\nu}\left(1 - \frac{1}{m(\lambda-\mu)(\lambda-\nu)}\right) \\ m\lambda\mu\nu + \lambda^3 + k\mu^3 + (m-k-1)\nu^3 = (m-1)(\lambda-\nu)(\lambda-\mu)^2\left(1 - \frac{1}{m(\lambda-\mu)(\lambda-\nu)}\right) \\ \lambda + k\mu + (m-k-1)\nu = 0 \\ \lambda^2 + k\mu^2 + (m-k-1)\nu^2 = \frac{m-1}{m} \end{cases}$$

and x is locally Möbius equivalent to

$$x(M) = \sigma(\{x_1(u)\} \times x_2(u)\mathbb{S}^k \times \mathbb{R}^{m-k-1}), \quad u \in \mathbb{R},$$

where $\vec{r}(u) = \{x_1(u), x_2(u)\}$ is a free $\frac{\lambda-\mu}{\lambda-\nu}$−elastic curve in $\mathbb{R}^2$.

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