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SUBMANIFOLDS WITH CONSTANT JORDAN ANGLES AND RIGIDITY OF THE LAWSON-OSSERMAN CONE*

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Abstract. The Lawson-Osserman cone [24] is a four dimensional coassociative submanifold in $\mathbb{R}^7$ in terms of Harvey-Lawson calibrated geometries [16] and the basic counterexample for Bernstein type results for minimal graphs of higher codimension in Euclidean space. We shall explore the geometry of this cone in terms of its basic property of constant Jordan angles and show a rigidity result within the class of coassociative submanifolds with constant Jordan angles. This will also shed new light on the higher codimension Bernstein problem.

Key words. Lawson-Osserman cone, constant Jordan angles, coassociative submanifolds.

Mathematics Subject Classification. 58E20, 53A10.

1. Introduction. The Lawson-Osserman cone [24] is a cone over the Hopf map $S^3 \to S^2$ (and analogous constructions are possible for the higher Hopf maps). Harvey-Lawson [16] showed that the Lawson-Osserman cone is a four dimensional coassociative submanifold in $\mathbb{R}^7$ which can be identified with the imaginary octonians. It thus is an important example in their theory of calibrated geometries. The Lawson-Osserman cone has also played an important role as a counterexample to the higher codimension version of the Bernstein problem for minimal submanifolds in Euclidean space. Its geometry is surprisingly rich, as we shall explore in this paper. This should ultimately also contribute to deeper insight into the higher codimension Bernstein problem.

The Lawson-Osserman cone is defined as follows [24]. Let $\mathbb{O}$ and $\mathbb{H}$ denote the octonions and quaternions, respectively. We have $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e$, with e a unit element orthogonal to $\mathbb{H}$, and for any $a, b, c, d \in \mathbb{H}$,

$$(a + be)(c + de) = (ac - \bar{d}b) + (da + b\bar{c})e. \tag{1.1}$$

Denote $\mathrm{Sp}_1 := \{q \in \mathbb{H} : |q| = 1\}$. Assume $a \in \mathrm{Im}\ \mathbb{H}$ is a fixed unit element, then

$$M(a) := \{r\left[(\sqrt{5}/2)qa\bar{q} + \bar{q}e\right] : q \in \mathrm{Sp}_1, r \in \mathbb{R}^+\} \tag{1.2}$$

is a 4-dimensional cone in $\mathrm{Im}\ \mathbb{O}$, which is the graph of the function $\eta : \mathbb{H}\backslash\{0\} \to \mathrm{Im}\ \mathbb{H}\backslash\{0\}$

$$\eta(x) = \frac{\sqrt{5}}{2|x|}\bar{x}\varepsilon x. \tag{1.3}$$

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Here $\varepsilon \in \mathrm{Im}\,\mathbb{H}$ and $|\varepsilon| = 1$. Note that η is a *cone-like function*, i.e. $\eta(tx) = t\eta(x)$ for any t and x. It was discovered by Lawson-Osserman [24] that η is a Lipschitz solution to the non-parametric minimal surface equations that is not C^1. While this thus is an entire solution of the minimal surface system $\mathbb{R}^4 \to \mathbb{R}^3$ that is not smooth at the origin, Ding-Yuan [11] constructed a family of 4-dimensional entire minimal graphs in Im $\mathbb{O}$; the tangent cone at infinity of each one is just the Lawson-Osserman cone. Thus, in contrast to the case of codimension 1, where by a theorem of Moser [26] every entire minimal graph with bounded gradient is planar, these constructions show that in higher codimension, nontrivial such entire minimal graphs exist. And as we shall explain below, understanding the Lawson-Osserman cone is a key for the analysis of the higher codimension Bernstein problem

The aim of the present paper then is to understand this cone in geometric terms, to explore its geometry and to show that it is rigid in some sense to be made precise in a moment. Our basic geometric tool will be the Jordan angles. Here, the Jordan angles between two linear subspaces P and Q are the critical values of the angle θ between the nonzero vectors u in P and their orthogonal projection u^* in Q. When these Jordan angles are constant for all the normal spaces of some submanifold M of Euclidean space and a fixed linear reference subspace, we say that M has constant Jordan angles. This is the fundamental concept of our paper, and we abbreviate it as CJA. For a precise statement, refer to Definition 1.1 below. Now it turns out the Lawson-Osserman counterexample has CJA relative to the imaginary quaternions when viewed as a subspace of the imaginary octonians. In view of the result of Harvey-Lawson [16] that the Lawson-Osserman cone is a four dimensional coassociative submanifold in $\mathbb{R}^7$, which can be identified with the imaginary octonians, we shall study such coassociative submanifolds with CJA and find that a coassociative graph with CJA relative to the imaginary quaternions and at most two different normal Jordan angles either is affine linear or a translate of a portion of the Lawson-Osserman cone.

For more precise statements, we now develop some notation and technical concepts.

1.1. Jordan angles and submanifolds with CJA. Let P and Q_0 be m-dimensional subspaces (i.e. m-planes) in $\mathbb{R}^{n+m}$. The *Jordan angles* between P and Q_0 are the critical values of the angle θ between a nonzero vector u in P and its orthogonal projection u^* in Q_0 as u runs through P. This concept was first introduced by Jordan [18] in 1875, and they are also called *principal angles* in some references, e.g. [12]. If θ is a nonzero Jordan angle between P and Q_0 determined by a unit vector u in P and its projection u^* in Q_0, then u is called an *angle direction* of P relative to Q_0, and the 2-plane spanned by u and u^* is called an *angle 2-plane* between P and Q_0 (see [30]).

Denote by $\mathcal{P}_0$ the orthogonal projection of $\mathbb{R}^{n+m}$ onto Q_0 and by $\mathcal{P}$ the orthogonal projection of $\mathbb{R}^{n+m}$ onto P, then $\mathcal{P} \circ \mathcal{P}_0$ is a nonnegative definite self-adjoint transformation on P. Moreover, θ is a Jordan angle between P and Q_0 if and only if $\mu := \cos^2\theta$ is an eigenvalue of $\mathcal{P} \circ \mathcal{P}_0$, and u is an angle direction with respect to θ if and only if u is an eigenvector associated to the eigenvalue μ. Therefore, all the angle directions with respect to θ constitute a linear subspace of P, which is called an *angle space* of P relative to Q_0 (see [22]) and we denote it by P_θ. In particular,

$$P_0 = P \cap Q_0, \qquad P_{\pi/2} = P \cap Q_0^\perp. \tag{1.4}$$

The dimension of P_θ is called the *multiplicity* of θ, which is denoted by m_θ. If we denote by $\mathrm{Arg}(P, Q_0)$ the set consisting of all distinct Jordan angles between P and

Q_0, then

$$P = \bigoplus_{\theta \in \mathrm{Arg}(P,Q_0)} P_\theta \tag{1.5}$$

and

$$m = \sum_{\theta \in \mathrm{Arg}(P,Q_0)} m_\theta. \tag{1.6}$$

The Jordan angles between two m-planes completely determine their relative positions. More precisely, if $\mathrm{Arg}(P_1, Q_1) = \mathrm{Arg}(P_2, Q_2)$ and the multiplicities of the corresponding Jordan angles are equivalent, then there exists a rigid motion of $\mathbb{R}^{n+m}$, carrying P_1, Q_1 onto P_2, Q_2, respectively. And vice versa (see [30]).

The following lemma reveals the close relationship between $\mathrm{Arg}(P, Q_0)$, $\mathrm{Arg}(Q_0, P)$ and $\mathrm{Arg}(P^\perp, Q_0^\perp)$.

LEMMA 1.1 ([22]). *Let P, Q_0 be m-planes in $\mathbb{R}^{n+m}$, then Arg(P, Q_0) = Arg(Q_0, P) and the multiplicities of each corresponding Jordan angles are equivalent. If we denote*

$$R_\theta := P_\theta + (Q_0)_\theta \tag{1.7}$$

for each $\theta \in$ Arg(P, Q_0), then $R_\theta \perp R_\sigma$ whenever $\theta \neq \sigma$, and

$$P + Q_0 = \bigoplus_{\theta \in Arg(P,Q_0)} R_\theta. \tag{1.8}$$

For any $\theta \in (0, \pi/2]$, $\theta \in$ Arg$(P^\perp, Q_0^\perp)$ if and only if $\theta \in$ Arg(P, Q_0), and $m_\theta^\perp = m_\theta$, $R_\theta = P_\theta \oplus P_\theta^\perp$. Moreover, for every $\theta \in$ Arg$(P, Q_0) \cap (0, \pi/2)$, there exists an isometric automorphism $\Phi_\theta : R_\theta \to R_\theta$, such that

(i) $\Phi_\theta(P_\theta) = P_\theta^\perp$, $\Phi_\theta(P_\theta^\perp) = P_\theta$;

(ii) $\Phi_\theta^2 = -\mathbf{Id}$;

(iii) *For any nonzero vector $u \in P_\theta$ ($v \in P_\theta^\perp$), $\Phi_\theta(u)$ ($\Phi_\theta(v)$) lies in the angle 2-plane generated by u (v); more precisely,*

$$\begin{aligned} \sec\theta\ \mathcal{P}_0 u &= \cos\theta\ u - \sin\theta\ \Phi_\theta(u), \\ \sec\theta\ \mathcal{P}_0^\perp v &= \cos\theta\ v - \sin\theta\ \Phi_\theta(v). \end{aligned} \tag{1.9}$$

Denote

$$r(P) := \sum_{\theta \in \mathrm{Arg}(P,Q_0)\cap(0,\pi/2]} m_\theta = \sum_{\theta \in \mathrm{Arg}(P^\perp,Q_0^\perp)\cap(0,\pi/2]} m_\theta^\perp \tag{1.10}$$

then $0 \in \mathrm{Arg}(P, Q_0)$ if and only if $r(P) < m$, and $m_0 = m - r(P)$. Similarly $0 \in \mathrm{Arg}(P^\perp, Q_0^\perp)$ if and only if $r(P) < n$, and $m_0^\perp = n - r(P)$.

Let M be an n-dimensional submanifold in $\mathbf{R}^{n+m}$ and Q_0 be a fixed m-plane in $\mathbf{R}^{n+m}$. Denote by TM and NM the tangent bundle and the normal bundle along M, respectively. For any $p \in M$, denote by $\mathrm{Arg}(N_pM, Q_0)$ ($\mathrm{Arg}(T_pM, Q_0^\perp)$) the set consisting of all the *normal (tangent) Jordan angles* at p relative to Q_0 (Q_0^T).

Now we can formulate the definition of *submanifolds with constant Jordan angles (CJA)*, the main subject of this paper.

DEFINITION 1.1. *Let M be an n-dimensional submanifold of $\mathbf{R}^{n+m}$ and Q_0 be a fixed m-plane. If $Arg(N_pM, Q_0)$ is independent of $p \in M$, and the multiplicity of each normal Jordan angle θ (denoted by m_θ^N) is constant, then we say M has* ***constant Jordan angles (CJA)*** *relative to Q_0.*

By Lemma 1.1, M has CJA relative to Q_0 if and only if $\text{Arg}(T_pM, Q_0^T)$ is independent of $p \in M$, and multiplicity of each tangent Jordan angle θ (denoted by m_θ^T) is constant. In this case, we denote by Arg^N (Arg^T) the normal (tangent) Jordan angles relative to Q_0 (Q_0^T).

REMARKS.

- Let γ be an arc-length parameterized curve in $\mathbf{R}^3$. If γ is a *constant angle curve*, i.e. the unit tangent vector at every point makes a constant angle with a fixed straight line in $\mathbf{R}^3$, then γ is a helix, and vice versa. Let S be a smooth surface in $\mathbf{R}^3$, if the normal vector at every point makes a constant angle with a fixed straight line in $\mathbf{R}^3$, then S is said to be a *constant angle surface* in $\mathbf{R}^3$. A surface S in $\mathbf{R}^3$ is a constant angle surface if and only if it is locally isometric to either a cylinder, a right circular cone, or the tangential developable of a helix. Moreover, if we additionally assume S to be complete, then S has to be a cylinder. Recently, many geometers are interested in constant angle surfaces in other ambient spaces, e.g. $S^2 \times \mathbf{R}$ [8], $\mathbf{H}^2 \times \mathbf{R}$ [10], Heisenberg group [13], Minkowski space [25] and product spaces [9]. Our notion is a natural generalization of the classical constant angle curves and surfaces.
- If M^n is a hypersurface of $\mathbf{R}^{n+1}$, then M has CJA if and only if M is a helix hypersurface [7]. Hence the concept of submanifolds with CJA is a natural generalization of helix hypersurfaces to higher codimensional cases. Helix hypersurfaces are closed related to the shadow problem (see [15]) formulated by H. Wente, and another interesting motivation for the study of helix hypersurfaces comes from the physics of interfaces of liquid crystal (see [5]).
- Let S be a surface in $\mathbf{R}^4$, then S has CJA if and only if S is a surface in $\mathbf{R}^4$ with *constant principal angles with respect to a plane.* This concept was introduced by Bayard-Di Scala-Castro-Hernández in [2]. In this paper, the authors established a local existence theorem and classified all the complete surfaces in $\mathbf{R}^4$ with constant principal angles.

Denote

$$\begin{aligned} N_\theta M &:= \{\nu \in N_pM : p \in M, \nu \text{ is an angle direction associated to } \theta\}, \\ T_\theta M &:= \{v \in T_pM : p \in M, v \text{ is an angle direction associated to } \theta\}. \end{aligned} \tag{1.11}$$

Let $\mathcal{P}_0$ and $\mathcal{P}_0^\perp$ be orthogonal projections onto Q_0 and $Q_0^\perp$, $(\cdot)^T$ and $(\cdot)^N$ denote orthogonal projections onto T_pM and N_pM, respectively. Then $\nu \in N_\theta M$ if and only if

$$(\mathcal{P}_0\nu)^N = \cos^2\theta\nu \tag{1.12}$$

and similarly $u \in T_\theta M$ if and only if

$$(\mathcal{P}_0^\perp u)^T = \cos^2\theta u. \tag{1.13}$$

As shown in [22], $N_\theta M$ ($T_\theta M$) is a smooth subbundle of NM (TM), which is said to be a *normal (tangent) angle space distribution* associated to θ. NM and TM have the following vector bundle decompositions

$$NM = \bigoplus_{\theta\in \mathrm{Arg}^N} N_\theta M, \quad TM = \bigoplus_{\theta\in \mathrm{Arg}^T} T_\theta M. \tag{1.14}$$

A curve $\gamma : t \in (a,b) \mapsto \gamma(t) \in M$, all of whose tangent vectors belongs to a tangent angle space distribution, i.e. $\dot\gamma(t) \in T_\theta M$ for every $t \in (a,b)$, is called an *angle line* of M. More generally, an *angle surface* is a connected submanifold S of M, such that for any $p \in S$, $T_pS \subset T_\theta M$.

By Lemma 1.1, it is easy to derive the following conclusion:

Proposition 1.1. *Let $\theta \in Arg^N$ and $\theta \neq 0, \pi/2$, then there exists a smooth mapping $\Phi_\theta : R_\theta M \to R_\theta M$, where*

$$R_\theta M := N_\theta M \oplus T_\theta M, \tag{1.15}$$

such that:

(i) *Φ_θ keeps each fiber invariant;*
(ii) *the length of each vector in $R_\theta M$ is invariant under Φ_θ;*
(iii) *$\Phi_\theta^2 = -\mathbf{Id}$;*
(iv) *$\Phi_\theta(N_\theta M) = T_\theta M$, $\Phi_\theta(T_\theta M) = N_\theta M$;*
(v) *for any $\nu \in N_\theta M$ and $u \in T_\theta M$,*

$$\begin{aligned} \sec\theta\ \mathcal{P}_0\nu &= \cos\theta\ \nu - \sin\theta\ \Phi_\theta(\nu),\\ \sec\theta\ \mathcal{P}_0^\perp u &= \cos\theta\ u - \sin\theta\ \Phi_\theta(u). \end{aligned} \tag{1.16}$$

Φ_θ is called the anti-involution associated to θ.

Denote

$$r := \sum_{\theta\in \mathrm{Arg}^N, \theta\neq 0} m_\theta^N = \sum_{\theta\in \mathrm{Arg}^T, \theta\neq 0} m_\theta^T. \tag{1.17}$$

As shown above, $0 \in \mathrm{Arg}^N$ ($0 \in \mathrm{Arg}^T$) if and only if $r < m$ ($r < n$), and the multiplicity of 0 equals $m - r$ ($n - r$). Let

$$g^N := |\mathrm{Arg}^N| \qquad g^T := |\mathrm{Arg}^T| \tag{1.18}$$

be the numbers of distinct normal Jordan angles and tangent Jordan angles, respectively, then $g^N = g^T + 1$ whenever $r \equiv n < m$, $g^T = g^N + 1$ whenever $r \equiv m < n$, and otherwise $g^N = g^T$.

1.2. Minimal submanifolds with CJA and the Bernstein problem. In this section, we shall explain the connection with the higher codimension Bernstein theorem. While this is not strictly necessary for the technical aspects of the present paper, it may put them into some perspective. In fact, the concept of CJA submanifolds that we have just introduced arises from our systematic investigation of the Bernstein problem in higher codimension. The classical Bernstein theorem [3] states that any entire minimal graph in $\mathbf{R}^3$ has to be affine linear. This result has been extended by J. Simons [28] to such entire minimal graphs in $\mathbf{R}^{n+1}$ for $n \leq 7$, whereas Bombieri-de Giorgi-Giusti [4] constructed counterexamples in higher dimensions. But for any dimensions, there is a weak version of the Bernstein type theorem, obtained by J. Moser [26] who proved that any entire solution $f:\mathbf{R}^n\to\mathbf{R}$ to the minimal surface equation

$$\mathrm{div}\Big(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\Big)=0 \tag{1.19}$$

has to be affine linear, provided that

$$v:=\sqrt{1+|\nabla f|^2} \tag{1.20}$$

is a bounded function. v is a significant quantity here for various reasons. First, the boundedness of v ensures that (1.19) is a uniformly elliptic equation, so that a Bernstein type result can be obtained by Moser's iteration. Secondly, for any $f:\mathbf{R}^n\to\mathbf{R}$, $x=(x^1,\cdots,x^n)\in\mathbf{R}^n\mapsto(x,f(x))\in\text{graph } f$ is a global coordinate chart of the graph of f, and a straightforward calculation shows that the volume form of graph f is $v dx^1\wedge\cdots\wedge dx^n$, i.e. v equals the ratio of the volume form of graph f and the coordinate plane. Thirdly, v has a close relationship with Jordan angles. A direct computation shows

$$\nu:=w(-\frac{\partial f}{\partial x^1},\cdots,-\frac{\partial f}{\partial x^n},1)\qquad\text{where } w:=v^{-1} \tag{1.21}$$

is a unit normal vector field on graph f. Thus the angle between ν and the x^{n+1}-axis is $\arccos w$, which is smaller than a fixed acute angle whenever the v-function is bounded. Therefore, Moser's theorem can be restated as: Let M be a complete minimal hypersurface in $\mathbf{R}^{n+1}$ and $\theta_0\in(0,\pi/2)$. If the angle between the normal vector and x^{n+1}-axis is smaller than θ_0 everywhere, then M has to be an affine n-plane.

Now we consider an n-dimensional entire minimal graph M in $\mathbf{R}^{n+m}$, generated by a smooth vector-valued function $f:\mathbf{R}^n\to\mathbf{R}^m$

$$x=(x^1,\cdots,x^n)\mapsto f(x)=(f^1(x),\cdots,f^m(x)).$$

Then f satisfies the minimal surface equations

$$\begin{aligned}\sum_i\frac{\partial}{\partial x^i}(vg^{ij})&=0\qquad\forall j=1,\cdots,n,\\ \sum_{i,j}\frac{\partial}{\partial x^i}(vg^{ij}\frac{\partial f^\alpha}{\partial x^j})&=0\qquad\forall\alpha=1,\cdots,m.\end{aligned} \tag{1.22}$$

Here $g_{ij}dx^idx^j$ is the induced metric on M, (g^{ij}) denotes the inverse matrix of (g_{ij}), and $v dx^1\wedge\cdots\wedge dx^n:=\det(g_{ij})^{1/2}dx^1\wedge\cdots\wedge dx^n$ is the volume form of M. More precisely,

$$v=\Big[\det\big(\delta_{ij}+\sum_\alpha\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}\big)\Big]^{1/2}. \tag{1.23}$$

Similarly to the case of codimension 1, the v-function has a close relationship with Jordan angles. At any point $p\in M$, denote by

$$0\le\theta_1\le\theta_2\le\cdots\le\theta_m<\pi/2 \tag{1.24}$$

the Jordan angles between N_pM and the coordinate m-plane, then a calculation shows (see [32][20])

$$v = \prod_{i=1}^{m} \sec \theta_m. \tag{1.25}$$

We note that

$$w := v^{-1} = \prod_{i=1}^{m} \cos \theta_m \tag{1.26}$$

is the inner product of the normal m-plane and the coordinate m-plane. Here all the m-planes are viewed as vectors in a Euclidean space of larger dimension, via Plücker embedding (see [21]).

It is natural to ask whether Moser's theorem can be generalized to the higher codimensional case. In other words, given an entire minimal graph $M = \text{graph } f \subset \mathbf{R}^{n+m}$ with $f : \mathbf{R}^n \to \mathbf{R}^m$, does the boundedness of the v-function ensure that M has to be an affine n-plane? The answer is 'Yes' for the cases of dimension 2 [6][23] and dimension 3 [1][14], but it is 'No' for dimension 4, according to the works of Lawson-Osserman [24] and Ding-Yuan [11], because the Lawson-Osserman cone [24] is a Lipschitz solution to the non-parametric minimal surface equations, and Ding-Yuan [11] then constructed a family of 4-dimensional entire minimal graphs in Im $\mathbf{O}$ whose tangent cones at infinity are Lawson-Osserman cones. Therefore, Moser's theorem does not fully extend to higher codimension.

Via Jordan angles, a new geometric property of the Lawson-Osserman cone was explored in the Appendix of [21]:

Proposition 1.2. *The Lawson-Osserman cone $M(a)$ is a 4-dimensional submanifold in Im $\mathbf{O}$ with CJA relative to Im $\mathbf{H}$, and $Arg^N = \{\arccos(2/3), \arccos(\sqrt{6}/6)\}$, $Arg^T = \{\arccos(2/3), \arccos(\sqrt{6}/6), 0\}$. Moreover, an arbitrary angle line with respect to $\arccos(2/3)$ is a ray of $M(a)$, and vise versa.*

In [24], Lawson-Osserman raised the following question: What is the largest constant C such that an entire minimal graph of arbitrary dimension and codimension with $v \leq C$ has to be affine linear? Up to now, the best positive answer to this question in a successive series of achievements by several mathematicians (see [17], [19], [29], [32], [20]) is gotten in [22], which says that for any entire minimal graph $M = \{(x, f(x)) : x \in \mathbf{R}^n\} \subset \mathbf{R}^{n+m}$ with $f : \mathbf{R}^n \to \mathbf{R}^m$, if $v \leq 3$, then M has to be an affine n-plane. The v-function of the Lawson-Osserman cone is constant; it equals 9 on $M(a)$. The v-functions of the Ding-Yuan examples take values in $[1, 9)$. Thus, there is still a large quantitative gap between 3 and 9, that is, between known Bernstein type theorems and the counterexamples.

The Lawson-Osserman problem can be viewed as the first gap problem of the v-function for entire minimal graphs of higher codimension. To study the gap phenomena of the v-function, it is natural to consider minimal graphs whose v-function is constant. Observing that the v-function is a function of all Jordan angles (see (1.25)), the v-function on any minimal graph with CJA relative to the coordinate plane (e.g. the Lawson-Osserman cone) is constant. So one can propose the following problem:

Problem 1.1. *Let M be a nonflat minimal graph with CJA relative to the coordinate plane, does the v-function takes values in a discrete set? Does any minimal graph in Euclidean space with constant v-function have to be a submanifold with CJA?*

1.3. Main results. This paper will be organized as follows.

In Section 2, the second fundamental form B of submanifolds with CJA in Euclidean space shall be studied. At first, differentiating the Jordan angle functions not only gives some nullity properties of B, but also reveals the relationship between the induced tangent (normal) connection and the second fundamental form. Taking the covariant derivative of the formulas obtained in the previous step, one can compute some components of ∇B in terms of B. With the aid of the Codazzi equations, we can derive a constraint equation for the second fundamental form (see Lemma 2.6), which is nontrivial when the multiplicity of a tangent Jordan angle $\theta \in (0, \pi/2)$, i.e. m_θ^T, is strictly larger than 1. This conclusion will play an important part in Section 3. Based on these formulas, it is easy to get some vanishing theorems for the second fundamental form B of submanfolds with CJA, including the following one.

THEOREM 1.1. *Let f be an $\mathbf{R}^m$-valued function on an open domain $D \subset \mathbf{R}^n$. If $M = \text{graph}\ f$ is a minimal submanifold with CJA relative to $\mathbf{R}^m$, and $g^N, g^T \leq 2$, then f has to be affine linear, i.e. M has to be an affine n-plane.*

Note that the example of Lawson-Osserman cone implies that the condition '$g^N, g^T \leq 2$' in Theorem 1.1 cannot be omitted.

Harvey-Lawson [16] introduced a new concept of coassociative submanifolds, as an important example of calibrated geometries, and showed that the Lawson-Osserman cone is a coassociative submanifold. Observing that coassociative submanifolds constitute an important class of 4-dimensional minimal submanifolds in $\mathbf{R}^7$, it is natural to study the structure of coassociative submanifolds with CJA, which is the main topic of Section 3. With the aid of the algebraic properties of octonions, one can obtain several interesting conclusions on the Jordan angles and the second fundamental form of coassociative submanifolds. In conjunction with Lemma 2.6, a structure theorem for coassociative submanifolds with CJA is deduced as follows.

THEOREM 1.2. *Let f be a smooth function from an open domain $D \subset \mathbf{H}$ into $Im\ \mathbf{H}$. If $M = \text{graph}\ f$ is a coassociative submanifold with CJA relative to $Im\ \mathbf{H}$, and $g^N \leq 2, g^T \leq 3$, then f is either an affine linear function or $f(x) = \eta(x - x_0) + y_0$, where $x_0 \in \mathbf{H}$, $y_0 \in Im\ \mathbf{H}$ and*

$$\eta(x) = \frac{\sqrt{5}}{2|x|}\bar{x}\varepsilon x$$

with ε an arbitrary unit element in $Im\ \mathbf{H}$. In other words, M is an affine 4-plane or a translate of an open subset of the Lawson-Osserman cone.

Theorem 1.1 and 1.2 give a partial positive answer to Problem 1.1.

2. On the second fundamental form of submanfolds with CJA. Let M be an n-dimensional submanifold in $\mathbf{R}^{n+m}$ with CJA relative to a fixed m-plane Q_0. We use the notations $\mathcal{P}_0$, $\mathcal{P}_0^\perp$, Arg^N, Arg^T, $N_\theta M$, $T_\theta M$, $R_\theta M$, m_θ^N, m_θ^T, g^N, g^T established in Section 1. For p in M, we put

$$N_{p,\theta}M := N_pM \cap N_\theta M, \qquad T_{p,\theta}M := T_pM \cap T_\theta M. \tag{2.1}$$

The second fundamental form B is a pointwise symmetric bilinear form on T_pM $(p \in M)$ with values in N_pM defined by

$$B_{XY} = (\overline{\nabla}_X Y)^N$$

with $\overline{\nabla}$ the Levi-Civita connection on $\mathbf{R}^{n+m}$. The induced connections on TM and NM are

$$\nabla_X Y = (\overline{\nabla}_X Y)^T, \qquad \nabla_X \nu = (\overline{\nabla}_X \nu)^N.$$

Here X, Y are smooth sections of TM and ν denotes a smooth section of NM. The second fundamental form, the curvature tensor of the submanifold, the curvature tensor of the normal bundle and the curvature tensor of the ambient manifold satisfy the Gauss, Codazzi and Ricci equations (see [31] for details).

Let A be the shape operator defined by

$$A^\nu(v) = (-\overline{\nabla}_v \nu)^T \qquad \forall \nu \in \Gamma(NM), v \in T_pM. \tag{2.2}$$

A^ν is a symmetric operator on T_pM and satisfies the Weingarten equations

$$\langle B_{XY}, \nu\rangle = \langle A^\nu(X), Y\rangle \qquad \forall X, Y \in \Gamma(TM). \tag{2.3}$$

The trace of the second fundamental form gives a normal vector field H on M, which is called the mean curvature vector field. If $\nabla H \equiv 0$, then we say that M has parallel mean curvature. Moreover if $H \equiv 0$, M is called a minimal submanifold.

2.1. Nullity lemmas. Let $\theta \in \mathrm{Arg}^N (\theta \neq 0, \pi/2)$ and $\Phi_\theta : R_\theta M \to R_\theta M$ denote the anti-involution associated to θ, then (1.16) gives

$$\begin{aligned} (\mathcal{P}_0^\perp v)^T &= \cos^2\theta \ v, \qquad & (\mathcal{P}_0^\perp v)^N &= -\cos\theta\sin\theta \ \Phi_\theta(v), \\ (\mathcal{P}_0 \mu)^N &= \cos^2\theta \ \mu, \qquad & (\mathcal{P}_0\mu)^T &= -\cos\theta\sin\theta \ \Phi_\theta(\mu). \end{aligned} \tag{2.4}$$

for any $v \in T_\theta M$ and $\mu \in N_\theta M$.

Based on the above formulas, one can easily deduce the following nullity lemmas for the second fundamental form of M.

LEMMA 2.1. *For each $\theta \in Arg^N$ which takes values in $(0, \pi/2)$,*

$$\langle B_{uv}, \Phi_\theta(w)\rangle + \langle B_{uw}, \Phi_\theta(v)\rangle = 0 \tag{2.5}$$

holds pointwise for any $u \in T_pM$ and $v, w \in T_{p,\theta}M$. In particular,

$$\langle B_{uv}, \Phi_\theta(v)\rangle = 0 \tag{2.6}$$

for every $v \in T_{p,\theta}M$.

Proof. By linearity, it suffices to prove (2.6) for any unit vector $v \in T_{p,\theta}M$.

Let X be a smooth local section of $T_\theta M$, such that $X_p = v$ and $|X| \equiv 1$, then

$$\langle \mathcal{P}_0^\perp X, \mathcal{P}_0^\perp X\rangle = |\mathcal{P}_0^\perp X|^2 \equiv \cos^2\theta. \tag{2.7}$$

Differentiating both sides with respect to u yields

$$\begin{aligned} 0 &= (1/2)\nabla_u \langle \mathcal{P}_0^\perp X, \mathcal{P}_0^\perp X\rangle = \langle \overline{\nabla}_u(\mathcal{P}_0^\perp X), \mathcal{P}_0^\perp v\rangle \\ &= \langle \mathcal{P}_0^\perp(\overline{\nabla}_u X), \mathcal{P}_0^\perp v\rangle = \langle \mathcal{P}_0^\perp(\nabla_u X), \mathcal{P}_0^\perp v\rangle + \langle \mathcal{P}_0^\perp B_{uv}, \mathcal{P}_0^\perp v\rangle \\ &= \langle \nabla_u X, (\mathcal{P}_0^\perp v)^T\rangle + \langle B_{uv}, (\mathcal{P}_0^\perp v)^N\rangle \\ &= \cos^2\theta \langle \nabla_u X, v\rangle - \cos\theta\sin\theta \langle B_{uv}, \Phi_\theta(v)\rangle \\ &= (1/2)\cos^2\theta \nabla_u |X|^2 - \cos\theta\sin\theta\langle B_{uv}, \Phi_\theta(v)\rangle \\ &= -\cos\theta\sin\theta\langle B_{uv}, \Phi_\theta(v)\rangle \end{aligned}$$

(where we have used (2.4)) and then we arrive at (2.6). □

Lemma 2.2. *For each $\theta \in Arg^N$ taking values in $(0, \pi/2)$,*

$$\langle B_{uv}, \nu \rangle = 0 \tag{2.8}$$

for any $u, v \in T_{p,\theta}M$ and $\nu \in N_{p,\theta}M$.

Proof. Let $w := -\Phi_\theta(\nu)$, then $w \in T_{p,\theta}M$ and $\Phi_\theta(w) = -\Phi_\theta^2(\nu) = \nu$. Applying Lemma 2.1 gives

$$\begin{aligned}\langle B_{uv}, \nu \rangle &= \langle B_{uv}, \Phi_\theta(w) \rangle = -\langle B_{uw}, \Phi_\theta(v) \rangle \\ &= -\langle B_{wu}, \Phi_\theta(v) \rangle = \langle B_{wv}, \Phi_\theta(u) \rangle \\ &= \langle B_{vw}, \Phi_\theta(u) \rangle = -\langle B_{vu}, \Phi_\theta(w) \rangle \\ &= -\langle B_{uv}, \Phi_\theta(w) \rangle = -\langle B_{uv}, \nu \rangle\end{aligned}$$

and (2.8) immediately follows from the above equation. □

Lemma 2.3. *If $\theta \in Arg^N \cap Arg^T$ and $\theta \equiv 0$ or $\pi/2$, then*

$$\langle B_{uv}, \nu \rangle = 0 \tag{2.9}$$

for any $u \in T_pM$, $v \in T_{p,\theta}M$ and $\nu \in N_{p,\theta}M$.

Proof. If $\theta \equiv 0$, let X be a smooth local section of $T_\theta M$ such that $X_p = v$, then $X_q \in Q_0^\perp$ for any q. Thus $(\overline{\nabla}_u X)_p \subset Q_0^\perp$. On the other hand, $\nu \in N_{p,\theta}M$ implies $\nu \in Q_0$, hence

$$\langle B_{uv}, \nu \rangle = \langle \overline{\nabla}_u X, \nu \rangle = 0.$$

The proof for $\theta \equiv \pi/2$ is similar. □

2.2. Connections. Let $\theta, \sigma \in \mathrm{Arg}^T$, $\theta \neq \sigma$, X a local section of TM, Y and Z local sections of $T_\theta M$ and $T_\sigma M$, respectively. Define

$$(S_{\theta\sigma})_{YZ}(X) := \langle \nabla_X Y, Z \rangle. \tag{2.10}$$

Then for any smooth function f defined on M, $(S_{\theta\sigma})_{YZ}(fX) = f(S_{\theta\sigma})_{YZ}(X)$, $(S_{\theta\sigma})_{Y,fZ}(X) = f(S_{\theta\sigma})_{YZ}(X)$ and

$$\begin{aligned}(S_{\theta\sigma})_{fY,Z}(X) &= \langle \nabla_X(fY), Z \rangle = f\langle \nabla_X Y, Z \rangle + (\nabla_X f)\langle Y, Z \rangle \\ &= f(S_{\theta\sigma})_{YZ}(X).\end{aligned}$$

This means $S_{\theta\sigma}$ is a smooth tensor field on M of type $(3,0)$. More precisely, $S_{\theta\sigma}$ is a smooth section of the tensor bundle $T^*M \otimes T_\theta^*M \otimes T_\sigma^*M$. Since ∇ is a Levi-Civita connection on M,

$$\begin{aligned}(S_{\theta\sigma})_{YZ}(X) &= \langle \nabla_X Y, Z \rangle = \nabla_X \langle Y, Z \rangle - \langle \nabla_X Z, Y \rangle \\ &= -\langle \nabla_X Z, Y \rangle = -(S_{\sigma\theta})_{ZY}(X).\end{aligned} \tag{2.11}$$

Now we additionally define

$$\Phi_\theta|_{R_\theta M} = 0 \qquad \text{whenever } \theta \equiv 0 \text{ or } \pi/2, \tag{2.12}$$

then (2.4) still holds when $\theta = 0$ or $\pi/2$. Let

$$\kappa_{\theta\sigma} := \frac{\sin 2\theta}{\cos 2\theta - \cos 2\sigma} \tag{2.13}$$

be a constant depending only on θ and σ. The following result reveals the relationship between $S_{\theta\sigma}$ and the second fundamental form.

LEMMA 2.4. *Let $\theta, \sigma \in Arg^T$, $\theta \neq \sigma$, then for any $u \in T_pM$, $v \in T_{p,\theta}M$ and $w \in T_{p,\sigma}M$,*

$$(S_{\theta\sigma})_{vw}(u) = \kappa_{\sigma\theta}\langle B_{uv}, \Phi_\sigma(w)\rangle - \kappa_{\theta\sigma}\langle B_{uw}, \Phi_\theta(v)\rangle. \tag{2.14}$$

Proof. Let Y, Z be smooth local sections of $T_\theta M$ and $T_\sigma M$, respectively, such that $Y(p) = v$, $Z(p) = w$, then $(\mathcal{P}_0^\perp Y)^T = \cos^2\theta\, Y$, $(\mathcal{P}_0^\perp Z)^T = \cos^2\sigma\, Z$. Hence

$$\begin{aligned} 0 = \cos^2\theta\langle Y, Z\rangle &= \langle(\mathcal{P}_0^\perp Y)^T, Z\rangle \\ &= \langle \mathcal{P}_0^\perp Y, Z\rangle = \langle \mathcal{P}_0^\perp Y, \mathcal{P}_0^\perp Z\rangle. \end{aligned}$$

Differentiating both sides of the above equation with respect to $u \in T_pM$ yields

$$\begin{aligned} 0 &= \nabla_u\langle \mathcal{P}_0^\perp Y, \mathcal{P}_0^\perp Z\rangle = \langle \overline{\nabla}_u(\mathcal{P}_0^\perp Y), \mathcal{P}_0^\perp w\rangle + \langle \mathcal{P}_0^\perp v, \overline{\nabla}_u(\mathcal{P}_0^\perp Z)\rangle \\ &= \langle \mathcal{P}_0^\perp(\overline{\nabla}_u Y), \mathcal{P}_0^\perp w\rangle + \langle \mathcal{P}_0^\perp v, \mathcal{P}_0^\perp(\overline{\nabla}_u Z)\rangle \\ &= \langle \mathcal{P}_0^\perp(\nabla_u Y), \mathcal{P}_0^\perp w\rangle + \langle \mathcal{P}_0^\perp v, \mathcal{P}_0^\perp(\nabla_u Z)\rangle + \langle \mathcal{P}_0^\perp B_{uv}, \mathcal{P}_0^\perp w\rangle + \langle \mathcal{P}_0^\perp v, \mathcal{P}_0^\perp B_{uw}\rangle \\ &= \langle \nabla_u Y, (\mathcal{P}_0^\perp w)^T\rangle + \langle \nabla_u Z, (\mathcal{P}_0^\perp v)^T\rangle + \langle B_{uv}, (\mathcal{P}_0^\perp w)^N\rangle + \langle B_{uw}, (\mathcal{P}_0^\perp v)^N\rangle \\ &= \cos^2\sigma\langle \nabla_u Y, w\rangle + \cos^2\theta\langle \nabla_u Z, v\rangle - \cos\sigma\sin\sigma\langle B_{uv}, \Phi_\sigma(w)\rangle - \cos\theta\sin\theta\langle B_{uw}, \Phi_\theta(v)\rangle \\ &= (\cos^2\sigma - \cos^2\theta)(S_{\theta\sigma})_{vw}(u) - \cos\sigma\sin\sigma\langle B_{uv}, \Phi_\sigma(w)\rangle - \cos\theta\sin\theta\langle B_{uw}, \Phi_\theta(v)\rangle \\ &= (1/2)(\cos 2\sigma - \cos 2\theta)(S_{\theta\sigma})_{vw}(u) - (1/2)\sin 2\sigma\langle B_{uv}, \Phi_\sigma(w)\rangle - (1/2)\sin 2\theta\langle B_{uw}, \Phi_\theta(v)\rangle \end{aligned}$$

(we have used (2.4) and (2.11)), which is equivalent to (2.14). □

Similarly, given $u \in T_pM$, $\mu \in \Gamma(N_\theta M)$, $\nu \in \Gamma(N_\sigma M)$ with $\theta, \sigma \in \mathrm{Arg}^N$ and $\theta \neq \sigma$, one can define

$$(S^N_{\theta\sigma})_{\mu\nu}(u) := \langle \nabla_u \mu, \nu\rangle. \tag{2.15}$$

Then $S^N_{\theta\sigma}$ is a smooth section of $T^*M \otimes N^*_\theta M \otimes N^*_\sigma M$, and

$$\begin{aligned} (S^N_{\sigma\theta})_{\nu\mu}(u) &= \langle \nabla_u \nu, \mu\rangle = \nabla_u\langle \nu, \mu\rangle - \langle \nu, \nabla_u \mu\rangle \\ &= -\langle \nabla_u \mu, \nu\rangle = -(S^N_{\theta\sigma})_{\mu\nu}(u). \end{aligned} \tag{2.16}$$

Let μ, ν be local section of $N_\theta M$ and $N_\sigma M$ respectively, then

$$\begin{aligned} 0 = \cos^2\theta\langle \mu, \nu\rangle &= \langle(\mathcal{P}_0\mu)^N, \nu\rangle \\ &= \langle \mathcal{P}_0\mu, \nu\rangle = \langle \mathcal{P}_0\mu, \mathcal{P}_0\nu\rangle. \end{aligned} \tag{2.17}$$

Differentiating both sides of the above equality with respect to $u \in T_pM$, one can use (2.4) to get the following result, as in the proof of Lemma 2.4.

LEMMA 2.5. *Given $\theta, \sigma \in Arg^N$, $\theta \neq \sigma$,*

$$(S^N_{\theta\sigma})_{\mu\nu}(u) = \kappa_{\theta\sigma}\langle B_{u,\Phi_\theta(\mu)}, \nu\rangle - \kappa_{\sigma\theta}\langle B_{u,\Phi_\sigma(\nu)}, \mu\rangle \tag{2.18}$$

for any $u \in T_pM, \mu \in N_{p,\theta}M$ and $\nu \in N_{p,\sigma}M$.

2.3. Computation of ∇B and related results. Let $\theta \in \mathrm{Arg}^T$, $\sigma \in \mathrm{Arg}^N$, and $(\cdot)^\sigma$ be the orthogonal projection of N_pM onto $N_{p,\sigma}M$. Define

$$R_{\theta\sigma}(v_1,v_2,v_3,v_4) := \langle B^\sigma_{v_1v_3}, B^\sigma_{v_2v_4}\rangle - \langle B^\sigma_{v_1v_4}, B^\sigma_{v_2v_3}\rangle \tag{2.19}$$

for any $v_1,v_2,v_3,v_4 \in T_{p,\theta}M$. Then $R_{\theta\sigma}$ is a smooth section of the tensor bundle $T^*_\theta M \otimes T^*_\theta M \otimes T^*_\theta M \otimes T^*_\theta M$. Obviously $R_{\theta\sigma}(v_1,v_2,v_3,v_4) = -R_{\theta\sigma}(v_2,v_1,v_3,v_4) = -R_{\theta\sigma}(v_1,v_2,v_4,v_3) = R_{\theta\sigma}(v_3,v_4,v_1,v_2)$, and

$$\begin{aligned}
&R_{\theta\sigma}(v_1,v_2,v_3,v_4) + R_{\theta\sigma}(v_2,v_3,v_1,v_4) + R_{\theta\sigma}(v_3,v_1,v_2,v_4)\\
=&\langle B^\sigma_{v_1v_3}, B^\sigma_{v_2v_4}\rangle - \langle B^\sigma_{v_1v_4}, B^\sigma_{v_2v_3}\rangle + \langle B^\sigma_{v_2v_1}, B^\sigma_{v_3v_4}\rangle\\
&- \langle B^\sigma_{v_2v_4}, B^\sigma_{v_3v_1}\rangle + \langle B^\sigma_{v_3v_2}, B^\sigma_{v_1v_4}\rangle - \langle B^\sigma_{v_3v_4}, B^\sigma_{v_1v_2}\rangle\\
=&0.
\end{aligned} \tag{2.20}$$

Hence $R_{\theta\sigma}$ is a curvature type tensor. Note that $R_{\theta\sigma} = 0$ whenever $m^T_\theta \equiv 1$.

Let $\theta, \sigma \in \mathrm{Arg}^T$, and define

$$U_{\theta\sigma}(v_1,v_2,v_3,v_4) =: \big\langle (A^{\Phi_\theta(v_3)}v_1)_\sigma, (A^{\Phi_\theta(v_4)}v_2)_\sigma\big\rangle - \big\langle (A^{\Phi_\theta(v_4)}v_1)_\sigma, (A^{\Phi_\theta(v_3)}v_2)_\sigma\big\rangle \tag{2.21}$$

for any $v_1,v_2,v_3,v_4 \in T_{p,\theta}M$. Here $(\cdot)_\sigma$ denotes the orthogonal projection of T_pM onto $T_{p,\sigma}M$. Due to Lemma 2.1, $A^{\Phi_\theta(v)}w + A^{\Phi_\theta(w)}v = 0$ for any $v,w \in T_{p,\theta}M$, hence $U_{\theta\sigma}(v_1,v_2,v_3,v_4) = -U_{\theta\sigma}(v_2,v_1,v_3,v_4) = -U_{\theta\sigma}(v_1,v_2,v_4,v_3) = U_{\theta\sigma}(v_3,v_4,v_1,v_2)$ and $U_{\theta\sigma} = 0$ whenever $m^T_\theta \equiv 1$. Note, however, that $U_{\theta\sigma}$ does not satisfy a Bianchi type identity.

LEMMA 2.6. *Given $\theta \in Arg^T$ taking values in $(0,\pi/2)$,*

$$\sum_{\sigma\in Arg^N,\sigma\neq\theta} \kappa_{\theta\sigma}R_{\theta\sigma}(v,w,v,w) = 3\sum_{\sigma\in Arg^T,\sigma\neq\theta} \kappa_{\theta\sigma}U_{\theta\sigma}(v,w,v,w) \tag{2.22}$$

for any $v,w \in T_{p,\theta}M$, and moreover

$$\begin{aligned}
\langle(\nabla_vB)_{ww}, \Phi_\theta(v)\rangle =&(1/3)\sum_{\sigma\in Arg^N,\sigma\neq\theta} \kappa_{\theta\sigma}\left(\langle B^\sigma_{vv}, B^\sigma_{ww}\rangle + 2|B^\sigma_{vw}|^2\right)\\
&-2\sum_{\sigma\in Arg^T,\sigma\neq\theta}\kappa_{\sigma\theta}\langle B^\sigma_{vw}, \Phi_\sigma(A^{\Phi_\theta(v)}w)_\sigma\rangle.
\end{aligned} \tag{2.23}$$

Proof. Let

$$u_\sigma := (A^{\Phi_\theta(v)}w)_\sigma \tag{2.24}$$

for each $\sigma \in \mathrm{Arg}^T$, then Lemma 2.1 tells us

$$(A^{\Phi_\theta(w)}v)_\sigma = -(A^{\Phi_\theta(v)}w)_\sigma = -u_\sigma \tag{2.25}$$

and moreover

$$\begin{aligned}
U_{\theta\sigma}(v,w,v,w) &= \big\langle (A^{\Phi_\theta(v)}v)_\sigma, (A^{\Phi_\theta(w)}w)_\sigma\big\rangle - \big\langle (A^{\Phi_\theta(w)}v)_\sigma, (A^{\Phi_\theta(v)}w)_\sigma\big\rangle\\
&= |u_\sigma|^2.
\end{aligned} \tag{2.26}$$

In particular, combining the Weingarten equations and Lemma 2.2 gives

$$|u_\theta|^2 = \langle u_\theta, A^{\Phi_\theta(v)}w\rangle = \langle B_{u_\theta w}, \Phi_\theta(v)\rangle = 0,$$

i.e. $u_\theta = 0$.

Let Y, Z be local sections of $T_\theta M$ such that $Y_p = v$, $Z_p = w$. By Lemma 2.2, $\langle B_{ZZ}, \Phi_\theta(Y)\rangle \equiv 0$, hence

$$\begin{aligned}\langle(\nabla_v B)_{ww}, \Phi_\theta(v)\rangle &= \nabla_v\langle B_{ZZ}, \Phi_\theta(Y)\rangle - \langle B_{ww}, \nabla_v\Phi_\theta(Y)\rangle - 2\langle B_{\nabla_v Z, w}, \Phi_\theta(v)\rangle \\ &= -\langle B_{ww}, \nabla_v\Phi_\theta(Y)\rangle - 2\langle B_{\nabla_v Z, w}, \Phi_\theta(v)\rangle \\ &:= -I - 2II\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}I &= \sum_{\sigma\in\mathrm{Arg}^N}\langle B^\sigma_{ww}, \nabla_v\Phi_\theta(Y)\rangle = \sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}(S^N_{\theta\sigma})_{\Phi_\theta(v),B^\sigma_{ww}}(v) \\ &= \sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\Big(\kappa_{\theta\sigma}\langle B_{v,\Phi^2_\theta(v)}, B^\sigma_{ww}\rangle - \kappa_{\sigma\theta}\langle B_{v,\Phi_\sigma(B^\sigma_{ww})}, \Phi_\theta(v)\rangle\Big) \\ &= -\sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\kappa_{\theta\sigma}\langle B^\sigma_{vv}, B^\sigma_{ww}\rangle\end{aligned} \tag{2.28}$$

(Lemma 2.2, Lemma 2.1, Lemma 2.5 and $\Phi^2_\theta = -\mathbf{Id}$ have been used in this calculation) and

$$\begin{aligned}II &= \langle B_{\nabla_v Z, w}, \Phi_\theta(v)\rangle = \sum_{\sigma\in\mathrm{Arg}^T}\langle\nabla_v Z, u_\sigma\rangle = \sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}(S_{\theta\sigma})_{wu_\sigma}(v) \\ &= \sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}(\kappa_{\sigma\theta}\langle B_{vw}, \Phi_\sigma(u_\sigma)\rangle - \kappa_{\theta\sigma}\langle B_{vu_\sigma}, \Phi_\theta(w)\rangle) \\ &= \sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\left(\kappa_{\sigma\theta}\langle B_{vw}, \Phi_\sigma(u_\sigma)\rangle + \kappa_{\theta\sigma}|u_\sigma|^2\right).\end{aligned} \tag{2.29}$$

(Here we have used the Weingarten equations, (2.25) and Lemma 2.4.) Substituting (2.28) and (2.29) into (2.27) implies

$$\begin{aligned}\langle(\nabla_v B)_{ww}, \Phi_\theta(v)\rangle = &\sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\kappa_{\theta\sigma}\langle B^\sigma_{vv}, B^\sigma_{ww}\rangle - 2 \\ &\times\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\left(\kappa_{\sigma\theta}\langle B_{vw}, \Phi_\sigma(u_\sigma)\rangle + \kappa_{\theta\sigma}|u_\sigma|^2\right).\end{aligned} \tag{2.30}$$

Again applying Lemma 2.2 gives $\langle B_{ZY}, \Phi_\theta(Y)\rangle \equiv 0$, hence

$$\begin{aligned}\langle(\nabla_w B)_{wv}, \Phi_\theta(v)\rangle =&\nabla_w\langle B_{ZY}, \Phi_\theta(Y)\rangle - \langle B_{wv}, \nabla_w\Phi_\theta(Y)\rangle \\ &- \langle B_{\nabla_w Z, v}, \Phi_\theta(v)\rangle - \langle B_{w,\nabla_w Y}, \Phi_\theta(v)\rangle \\ =&-\langle B_{wv}, \nabla_w\Phi_\theta(Y)\rangle - \langle B_{w,\nabla_w Y}, \Phi_\theta(v)\rangle \\ :=&-I - II\end{aligned} \tag{2.31}$$

where

$$\begin{aligned}I &= \sum_{\sigma\in\mathrm{Arg}^N}\langle B^\sigma_{wv}, \nabla_w\Phi_\theta(Y)\rangle = \sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}(S^N_{\theta\sigma})_{\Phi_\theta(v),B^\sigma_{wv}}(w) \\ &= \sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\Big(\kappa_{\theta\sigma}\langle B_{w,\Phi^2_\theta(v)}, B^\sigma_{wv}\rangle - \kappa_{\sigma\theta}\langle B_{w,\Phi_\sigma(B^\sigma_{wv})}, \Phi_\theta(v)\rangle\Big) \\ &= \sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\left(-\kappa_{\theta\sigma}|B^\sigma_{wv}|^2 - \kappa_{\sigma\theta}\langle u_\sigma, \Phi_\sigma(B^\sigma_{wv})\rangle\right)\end{aligned} \tag{2.32}$$

and

$$\begin{aligned} II =&\langle B_{w,\nabla_w Y}, \Phi_\theta(v)\rangle = \sum_{\sigma\in \mathrm{Arg}^T}\langle \nabla_w Y, u_\sigma\rangle = \sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}(S_{\theta\sigma})_{vu_\sigma}(w)\\ =&\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}(\kappa_{\sigma\theta}\langle B_{wv},\Phi_\sigma(u_\sigma)\rangle-\kappa_{\theta\sigma}\langle B_{wu_\sigma},\Phi_\theta(v)\rangle)\\ =&\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\left(\kappa_{\sigma\theta}\langle B_{wv},\Phi_\sigma(u_\sigma)\rangle-\kappa_{\theta\sigma}|u_\sigma|^2\right). \end{aligned}\tag{2.33}$$

If $\sigma\neq 0,\pi/2$, then Φ_σ is isometric and $\Phi_\sigma^2=-\mathbf{Id}$. Hence

$$\langle u_\sigma,\Phi_\sigma(B^\sigma_{wv})\rangle=\langle\Phi_\sigma(u_\sigma),\Phi^2_\sigma(B^\sigma_{wv})\rangle=-\langle B^\sigma_{wv},\Phi_\sigma(u_\sigma)\rangle.$$

On the other hand, $\Phi_\sigma=0$ whenever $\sigma=0$ or $\pi/2$. Therefore

$$-\sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\kappa_{\sigma\theta}\langle u_\sigma,\Phi_\sigma(B^\sigma_{wv})\rangle=\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\kappa_{\sigma\theta}\langle B_{wv},\Phi_\sigma(u_\sigma)\rangle.\tag{2.34}$$

Substituting (2.32)-(2.34) into (2.31) yields

$$\begin{aligned}\langle(\nabla_w B)_{wv},\Phi_\theta(v)\rangle=&\sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\kappa_{\theta\sigma}|B^\sigma_{wv}|^2\\ &+\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\left(\kappa_{\theta\sigma}|u_\sigma|^2-2\kappa_{\sigma\theta}\langle B_{wv},\Phi_\sigma(u_\sigma)\rangle\right).\end{aligned}\tag{2.35}$$

The Codazzi equations imply $(\nabla_v B)_{ww}=(\nabla_w B)_{wv}$. Hence by comparing the right hand sides of (2.30) and (2.35) we arrive at

$$\sum_{\sigma\in\mathrm{Arg}^N,\sigma\neq\theta}\kappa_{\theta\sigma}\left(\langle B^\sigma_{vv},B^\sigma_{ww}\rangle-|B^\sigma_{vw}|^2\right)=3\sum_{\sigma\in\mathrm{Arg}^T,\sigma\neq\theta}\kappa_{\theta\sigma}|u_\sigma|^2\tag{2.36}$$

and then (2.22) immediately follows from the definition of $R_{\theta\sigma}$ and $U_{\theta\sigma}$. Finally (2.23) is obtained by substituting (2.36) into (2.35). □

LEMMA 2.7. *We consider $\theta\in Arg^T$ taking values in $(0,\pi/2)$ and $\sigma\in Arg^T$ such that $\theta\neq\sigma$. If $U_{\theta\sigma}(v_1,v_2,v_1,v_2)=0$ holds for any $v_1,v_2\in T_{p,\theta}M$, then*

$$\begin{aligned}\langle(\nabla_v B)_{ww},\Phi_\theta(v)\rangle=&-2\sum_{\tau\in Arg^T,\tau\neq\theta}\kappa_{\tau\theta}\langle B^\tau_{vw},\Phi_\tau(A^{\Phi_\theta(v)}w)_\tau\rangle\\ &+\sum_{\tau\in Arg^N,\tau\neq\theta}\kappa_{\theta\tau}|B^\tau_{vw}|^2+\sum_{\tau\in Arg^T,\tau\neq\theta}\kappa_{\theta\tau}|(A^{\Phi_\theta(v)}w)_\tau|^2\end{aligned}\tag{2.37}$$

for any $v\in T_{p,\theta}M$ and $w\in T_{p,\sigma}M$.

Proof. In the sequel we make use of the abbreviation $u_\tau:=(A^{\Phi_\theta(v)}w)_\tau$ for any $\tau\in\mathrm{Arg}^T$. By the definition of $U_{\theta\sigma}$,

$$\begin{aligned}0=&U_{\theta\sigma}(u_\theta,v,u_\theta,v)\\ =&\left\langle(A^{\Phi_\theta(u_\theta)}u_\theta)_\sigma,(A^{\Phi_\theta(v)}v)_\sigma\right\rangle-\left\langle(A^{\Phi_\theta(v)}u_\theta)_\sigma,(A^{\Phi_\theta(u_\theta)}v)_\sigma\right\rangle\\ =&|(A^{\Phi_\theta(v)}u_\theta)_\sigma|^2\end{aligned}$$

i.e. $(A^{\Phi_\theta(v)}u_\theta)_\sigma = 0$. Hence

$$0 = \langle A^{\Phi_\theta(v)}u_\theta, w\rangle = \langle A^{\Phi_\theta(v)}w, u_\theta\rangle = |u_\theta|^2$$

i.e. $u_\theta = 0$. Similarly, one can deduce that $B^\theta_{vw} = 0$.

Let Y be a local smooth section of $T_\theta M$ and Z be a local smooth section of $T_\sigma M$, such that $Y_p = v$, $Z_p = w$. Lemma 2.1 implies $\langle B_{YZ}, \Phi_\theta(Y)\rangle \equiv 0$, hence

$$\begin{aligned}\langle(\nabla_v B)_{ww}, \Phi_\theta(v)\rangle =&\langle(\nabla_w B)_{vw}, \Phi_\theta(v)\rangle\\ =&\langle \nabla_w\langle B_{YZ}, \Phi_\theta(Y)\rangle - \langle B_{vw}, \nabla_w \Phi_\theta(Y)\rangle\\ &-\langle B_{\nabla_w Y, w}, \Phi_\theta(v)\rangle - \langle B_{v,\nabla_w Z}, \Phi_\theta(v)\rangle\\ =&-\langle B_{vw}, \nabla_w\Phi_\theta(Y)\rangle - \langle B_{\nabla_w Y,w}, \Phi_\theta(v)\rangle\\ :=&-I-II\end{aligned} \tag{2.38}$$

where

$$\begin{aligned}I =& \sum_{\tau\in\mathrm{Arg}^N}\langle B^\tau_{vw}, \nabla_w\Phi_\theta(Y)\rangle = \sum_{\tau\in\mathrm{Arg}^N,\tau\neq\theta}(S^N_{\theta\tau})_{\Phi_\theta(v),B^\tau_{vw}}(w)\\ =&\sum_{\tau\in\mathrm{Arg}^N,\tau\neq\theta}\Big(\kappa_{\theta\tau}\langle B_{w,\Phi^2_\theta(v)}, B^\tau_{vw}\rangle - \kappa_{\tau\theta}\langle B_{w,\Phi_\tau(B^\tau_{vw})}, \Phi_\theta(v)\rangle\Big)\\ =&-\sum_{\tau\in\mathrm{Arg}^N,\tau\neq\theta}\big(\kappa_{\theta\tau}|B^\tau_{vw}|^2 + \kappa_{\tau\theta}\langle\Phi_\tau(B^\tau_{vw}), u_\tau\rangle\big)\\ =&\sum_{\tau\in\mathrm{Arg}^N,\tau\neq\theta}\kappa_{\tau\theta}\langle B^\tau_{vw}, \Phi_\tau(u_\tau)\rangle - \sum_{\tau\in\mathrm{Arg}^N,\tau\neq\theta}\kappa_{\theta\tau}|B^\tau_{vw}|^2\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}II =& \sum_{\tau\in\mathrm{Arg}^T}\langle\nabla_w Y, u_\tau\rangle = \sum_{\tau\in\mathrm{Arg}^T,\tau\neq\theta}(S_{\theta\tau})_{v,u_\tau}(w)\\ =&\sum_{\tau\in\mathrm{Arg}^T,\tau\neq\theta}(\kappa_{\tau\theta}\langle B_{vw}, \Phi_\tau(u_\tau)\rangle - \kappa_{\theta\tau}\langle B_{wu_\tau}, \Phi_\theta(v)\rangle)\\ =&\sum_{\tau\in\mathrm{Arg}^T,\tau\neq\theta}\kappa_{\tau\theta}\langle B^\tau_{vw}, \Phi_\tau(u_\tau)\rangle - \sum_{\tau\in\mathrm{Arg}^T,\tau\neq\theta}\kappa_{\theta\tau}|u_\tau|^2.\end{aligned} \tag{2.40}$$

Substituting (2.39) and (2.40) into (2.38) yields (2.37). □

LEMMA 2.8. *If $\theta \in Arg^T \cap Arg^N$ and $\theta \equiv 0$ or $\pi/2$, then for any $v \in T_{p,\theta}M$, $\nu \in \Gamma(N_\theta M)$ and $w \in T_pM$,*

$$\langle(\nabla_v B)_{ww}, \nu\rangle = -2\sum_{\sigma\in Arg^T}\kappa_{\sigma\theta}\langle B^\sigma_{vw}, \Phi_\sigma(A^\nu w)_\sigma\rangle. \tag{2.41}$$

Proof. Let Y be a local section of $T_\theta M$ and Z be a local section of TM, such that $Y_p = v$ and $Z_p = w$, then Lemma 2.3 tells us $\langle B_{YZ}, \nu\rangle \equiv 0$. Therefore

$$\begin{aligned}\langle(\nabla_v B)_{ww}, \nu\rangle =&\langle(\nabla_w B)_{vw}, \nu\rangle\\ =&\nabla_w\langle B_{YZ}, \nu\rangle - \langle B_{vw}, \nabla_w\nu\rangle - \langle B_{\nabla_w Y,w}, \nu\rangle - \langle B_{v,\nabla_w Z}, \nu\rangle\\ =&-\langle B_{vw}, \nabla_w\nu\rangle - \langle B_{\nabla_w Y,w}, \nu\rangle\\ :=&-I-II\end{aligned} \tag{2.42}$$

where

$$
\begin{aligned}
I &= \sum_{\sigma\in \mathrm{Arg}^N,\sigma\neq\theta} \langle B^\sigma_{vw}, \nabla_w \nu\rangle = \sum_{\sigma\in \mathrm{Arg}^N,\sigma\neq\theta} (S^N_{\theta\sigma})_{\nu, B^\sigma_{vw}}(w) \\
&= -\sum_{\sigma\in \mathrm{Arg}^N,\sigma\neq\theta} \kappa_{\sigma\theta}\langle B_{w,\Phi_\sigma(B^\sigma_{vw})}, \nu\rangle = -\sum_{\sigma\in \mathrm{Arg}^N} \kappa_{\sigma\theta}\langle \Phi_\sigma(B^\sigma_{vw}), u_\sigma\rangle \\
&= \sum_{\sigma\in \mathrm{Arg}^T} \langle B^\sigma_{vw}, \Phi_\sigma(u_\sigma)\rangle
\end{aligned}
\tag{2.43}
$$

and

$$
\begin{aligned}
II &= \sum_{\sigma\in \mathrm{Arg}^T,\sigma\neq\theta} \langle \nabla_w Y, u_\sigma\rangle = \sum_{\sigma\in \mathrm{Arg}^T,\sigma\neq\theta} (S_{\theta\sigma})_{vu_\sigma}(w) \\
&= \sum_{\sigma\in \mathrm{Arg}^T,\sigma\neq\theta} \kappa_{\sigma\theta}\langle B_{wv}, \Phi_\sigma(u_\sigma)\rangle = \sum_{\sigma\in \mathrm{Arg}^T} \kappa_{\sigma\theta}\langle B^\sigma_{vw}, \Phi_\sigma(u_\sigma)\rangle.
\end{aligned}
\tag{2.44}
$$

Here $u_\sigma := (A^\nu w)_\sigma$, and $u_\theta = B^\theta_{vw} = 0$ is a direct corollary of Lemma 2.3. Substituting (2.43) and (2.44) into (2.42), we arrive at at (2.41). □

2.4. Vanishing theorems. With the above lemmas, we can now derive vanishing theorems for the second fundamental form of submanifolds with CJA.

THEOREM 2.1. *Let M^n be a submanifold of $\mathbf{R}^{n+m}$ with CJA relative to a fixed m-plane Q_0 (M need not be complete), then*

(i) *If $g^T = g^N = 1$, then M has to be an affine linear subspace;*

(ii) *If $g^T = 1, g^N = 2, \pi/2 \notin Arg^T$ and M has parallel mean curvature, then M is affine linear;*

(iii) *If $g^T = 2, g^N = 1, \pi/2 \notin Arg^N$, and M has parallel mean curvature, then M is affine linear;*

(iv) *If $g^T = g^N = 2$, $Arg^N \neq \{0, \pi/2\}$, and M is minimal, then M is affine linear.*

REMARKS.

- Let $S^1 := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0\}$ be a circle whose tangent vectors are all orthogonal to the x_3-axis, then S^1 has CJA and $\mathrm{Arg}^T = \{\pi/2\}$, $\mathrm{Arg}^N = \{\pi/2, 0\}$. It is easy to check that S^1 has parallel mean curvature. Hence the condition '$\pi/2 \notin \mathrm{Arg}^T$' cannot be dropped in (ii).
- Let $S := S^1 \times \mathbf{R}$ be a circular cylinder, whose normal vectors are all orthogonal to the x_3-axis, then S has CJA and $\mathrm{Arg}^N = \{\pi/2\}$, $\mathrm{Arg}^T = \{\pi/2, 0\}$. Its mean curvature vector field is parallel along S. Hence the condition '$\pi/2 \notin \mathrm{Arg}^N$' cannot be dropped in (iii).
- Let S be a nonflat minimal surface in $\mathbf{R}^3$, then $M := S \times \mathbf{R}$ is a minimal submanifold in $\mathbf{R}^3 \times \mathbf{R}^2 = \mathbf{R}^5$. It is easily-seen that M has CJA relative to $Q_0 := \mathbf{R}^2$ (the second factor in the product $\mathbf{R}^3 \times \mathbf{R}^2$), and $\mathrm{Arg}^N = \mathrm{Arg}^T = \{0, \pi/2\}$. Hence the condition '$\mathrm{Arg}^N \neq \{0, \pi/2\}$' cannot be dropped in (iv).

Proof. (i) Denote $g^T = g^N = \{\theta\}$, then Lemma 2.2 and 2.3 tell us

$$\langle B_{vw}, \nu\rangle = 0$$

for any $v, w \in T_{p,\theta}M = T_pM$ and $\nu \in N_{p,\theta}M = N_pM$. Hence M is totally geodesic.

(ii) As shown in Section 1, there exists $\theta_0 \neq 0, \pi/2$, such that $\mathrm{Arg}^T = \{\theta_0\}$, $\mathrm{Arg}^N = \{0, \theta_0\}$.

By Lemma 2.6,

$$\begin{aligned}&\kappa_{\theta_0 0}\big(\langle B^0_{vv}, B^0_{ww}\rangle - |B^0_{vw}|^2\big) = \kappa_{\theta_0 0} R_{\theta_0 0}(v,w,v,w)\\ =&\sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta_0} \kappa_{\theta_0\sigma} R_{\theta_0\sigma}(v,w,v,w) = 3\sum_{\sigma\in \mathrm{Arg}^T, \sigma\neq\theta_0} \kappa_{\theta_0\sigma} U_{\theta_0\sigma}(v,w,v,w)\\ =&0\end{aligned} \tag{2.45}$$

for any $v, w \in T_{p,\theta_0}M = T_pM$. In conjunction with $\kappa_{\theta_0 0} = \frac{\sin 2\theta_0}{\cos 2\theta_0 - 1} \neq 0$, we have $\langle B^0_{vv}, B^0_{ww}\rangle = |B^0_{vw}|^2$. Substituting it into (2.23) implies

$$\begin{aligned}\langle (\nabla_v B)_{ww}, \Phi_{\theta_0}(v)\rangle &= (1/3)\kappa_{\theta_0 0}\big(\langle B^0_{vv}, B^0_{ww}\rangle + 2|B^0_{vw}|^2\big)\\ &= \kappa_{\theta_0 0}|B^0_{vw}|^2.\end{aligned} \tag{2.46}$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_{p,\theta_0}M = T_pM$. Since M has parallel mean curvature,

$$0 = \sum_{i=1}^n \langle \nabla_v H, \Phi_{\theta_0}(v)\rangle = \sum_{i=1}^n \langle (\nabla_v B)_{e_i e_i}, \Phi_{\theta_0}(v)\rangle = \kappa_{\theta_0 0}\sum_{i=1}^n |B^0_{ve_i}|^2 \tag{2.47}$$

which forces $|B^0_{ve_i}| = 0$ for any $1 \leq i \leq n$. Thus $B^0_{vw} = 0$ for any $v, w \in T_pM$. On the other hand, Lemma 2.2 implies $B^{\theta_0}_{vw} = 0$. Therefore $B \equiv 0$ on M.

(iii) Denote $\mathrm{Arg}^N = \{\theta_0\}$, $\mathrm{Arg}^T = \{0, \theta_0\}$ with $\theta_0 \neq 0, \pi/2$. Again applying Lemma 2.6 gives

$$\kappa_{\theta_0 0} U_{\theta_0 0}(v,w,v,w) = (1/3)\sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta_0} \kappa_{\theta_0\sigma} R_{\theta_0\sigma}(v,w,v,w) = 0 \tag{2.48}$$

i.e. $U_{\theta 0}(v,w,v,w) = 0$ for any $v, w \in T_{p,\theta_0}M$. This means

$$\begin{aligned}0 &= \langle (A^{\Phi_{\theta_0}(v)}v)_0, (A^{\Phi_{\theta_0}(w)}w)_0\rangle - \langle (A^{\Phi_{\theta_0}(w)}v)_0, (A^{\Phi_{\theta_0}(v)}w)_0\rangle\\ &= \big|(A^{\Phi_{\theta_0}(v)}w)_0\big|^2.\end{aligned} \tag{2.49}$$

Since $\Phi_{\theta_0} : T_{p,\theta_0}M \to N_{p,\theta_0}M = N_pM$ is an isomorphism, $(A^\nu w)_0 = 0$ holds for every $\nu \in N_pM$. On the other hand, $(A^\nu w)_{\theta_0} = 0$ is a direct corollary of Lemma 2.2. Thus $A^\nu w = 0$ for every $w \in T_{p,\theta_0}M$.

Let $\{e_1, \cdots, e_{m_{\theta_0}}\}$ be an orthonormal basis of $T_{p,\theta_0}M$, and $\{e_{m_{\theta_0}+1}, \cdots, e_n\}$ be an orthonormal basis of $T_{p,0}M$. For any $v \in T_{p,\theta_0}M$, by (2.23) and (2.37),

$$\langle (\nabla_v B)_{e_i e_i}, \Phi_{\theta_0}(v)\rangle = \begin{cases} 0 & \text{if } 1 \leq i \leq m_{\theta_0},\\ \kappa_{\theta_0 0}\big|(A^{\Phi_{\theta_0}(v)}e_i)_0\big|^2 & \text{if } m_{\theta_0}+1 \leq i \leq n.\end{cases} \tag{2.50}$$

Hence

$$\begin{aligned}0 = \langle \nabla_v H, \Phi_{\theta_0}(v)\rangle &= \sum_{i=1}^n \langle (\nabla_v B)_{e_i e_i}, \Phi_{\theta_0}(v)\rangle\\ &= \sum_{i=m_{\theta_0}+1}^n \kappa_{\theta_0 0}\big|(A^{\Phi_{\theta_0}(v)}e_i)_0\big|^2\end{aligned} \tag{2.51}$$

and then $(A^\nu e_i)_0 = 0$ for any $\nu \in N_pM$. On the other hand, $\langle A^\nu e_i, v\rangle = \langle A^\nu v, e_i\rangle = 0$ holds for any $v \in T_{p,\theta_0}M$. Therefore $A^\nu e_i = 0$ for each $m_{\theta_0} + 1 \le i \le n$.

In summary, $A^\nu \equiv 0$ for any smooth section ν of NM and then M has to be affine linear.

(iv) Denote $\mathrm{Arg}^N = \mathrm{Arg}^T = \{\theta_1, \theta_2\}$. Without loss of generality one can assume $\theta_1 \in (0, \pi/2)$. Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of $T_{p,\theta_1}M$ and $\{e_{m+1}, \cdots, e_n\}$ be an orthonormal basis of $T_{p,\theta_2}M$. By Lemma 2.6, for any $1 \le i, j \le m$,

$$\begin{aligned}&\langle B^{\theta_2}_{e_ie_i}, B^{\theta_2}_{e_je_j}\rangle - |B^{\theta_2}_{e_ie_j}|^2 = R_{\theta_1\theta_2}(e_i, e_j, e_i, e_j)\\ =&3\ U_{\theta_1\theta_2}(e_i, e_j, e_i, e_j) = 3\big|(A^{\Phi_{\theta_1}(e_j)}e_i)\big|^2\end{aligned}$$

i.e.

$$\langle B^{\theta_2}_{e_ie_i}, B^{\theta_2}_{e_je_j}\rangle = 3\big|(A^{\Phi_{\theta_1}(e_j)}e_i)\big|^2 + |B^{\theta_2}_{e_ie_j}|^2. \tag{2.52}$$

On the other hand, Lemma 2.2 and Lemma 2.3 tell us $B^{\theta_2}_{e_ie_j} = 0$ for every $m + 1 \le i, j \le n$. Since M is a minimal submanifold,

$$\begin{aligned}0 = |H^{\theta_2}|^2 &= \big|\sum_{i=1}^n B^{\theta_2}_{e_ie_i}\big|^2\\ &= \big|\sum_{i=1}^m B^{\theta_2}_{e_ie_i}\big|^2 = \sum_{i,j=1}^m \langle B^{\theta_2}_{e_ie_i}, B^{\theta_2}_{e_je_j}\rangle\\ &= \sum_{i,j=1}^m \big(3|(A^{\Phi_{\theta_1}(e_j)}e_i)_{\theta_2}|^2 + |B^{\theta_2}_{e_ie_j}|^2\big).\end{aligned} \tag{2.53}$$

Hence $(A^{\Phi_{\theta_1}(e_j)}e_i)_{\theta_2} = B^{\theta_2}_{e_ie_j} = 0$ for all $1 \le i, j \le m$. In other words, $B^{\theta_2}_{v_1v_2} = 0$ for any $v_1, v_2 \in T_{p,\theta_1}M$, and $B^{\theta_1}_{vw} = 0$ for any $v \in T_{p,\theta_1}M$ and $w \in T_{p,\theta_2}M$, which follows from the Weingarten equations.

If $\theta_2 \in (0, \pi/2)$, then similarly one can deduce that $B^{\theta_1}_{w_1w_2} = 0$ for any $w_1, w_2 \in T_{p,\theta_2}M$ and $B^{\theta_2}_{vw} = 0$ for any $v \in T_{p,\theta_1}M$ and $w \in T_{p,\theta_2}M$. In conjunction with $B^{\theta_1}_{v_1v_2} = 0$ for any $v_1, v_2 \in T_{p,\theta_1}M$ and $B^{\theta_2}_{w_1w_2} = 0$ for any $w_1, w_2 \in T_{p,\theta_2}M$, we have $B \equiv 0$ on M and M has to be totally geodesic.

If $\theta_2 = 0$ or $\pi/2$, then (2.23) implies

$$\langle(\nabla_v B)_{e_ie_i}, \Phi_{\theta_1}(v)\rangle = (1/3)\kappa_{\theta_1\theta_2}\big(\langle B^{\theta_2}_{e_ie_i}, B^{\theta_2}_{vv}\rangle + 2|B^{\theta_2}_{e_iv}|^2\big) = 0 \tag{2.54}$$

for any $v \in T_{p,\theta_1}M$ and each $1 \le i \le m$. Since $U_{\theta_1\theta_2}(v_1, v_2, v_1, v_2) = 0$ for any $v_1, v_2 \in T_{p,\theta_1}M$, (2.37) tells us

$$\langle(\nabla_v B)_{e_ie_i}, \Phi_{\theta_1}(v)\rangle = \kappa_{\theta_1\theta_2}|B^{\theta_2}_{ve_i}|^2 + \kappa_{\theta_1\theta_2}|(A^{\Phi_{\theta_1}(v)}e_i)_{\theta_2}|^2, \tag{2.55}$$

for each $m + 1 \le i \le n$. Thus

$$\begin{aligned}0 = \langle\nabla_v H, \Phi_\theta(v)\rangle &= \sum_{i=1}^n \langle(\nabla_v B)_{e_ie_i}, \Phi_\theta(v)\rangle\\ &= \sum_{i=m+1}^n \kappa_{\theta_1\theta_2}\left(|B^{\theta_2}_{ve_i}|^2 + |(A^{\Phi_{\theta_1}(v)}e_i)_{\theta_2}|^2\right),\end{aligned} \tag{2.56}$$

which forces $B^{\theta_2}_{ve_i} = (A^{\Phi_{\theta_1}(v)}e_i)_{\theta_2} = 0$ for each $m+1 \leq i \leq n$. In other words, $B^{\theta_2}_{vw} = 0$ for any $v \in T_{p,\theta_1}M$ and $w \in T_{p,\theta_2}M$, and $B^{\theta_1}_{w_1w_2} = 0$ for any $w_1, w_2 \in T_{p,\theta_2}M$. Therefore $B \equiv 0$ on M and M has to be affine linear. □

Let $f : D \subset \mathbf{R}^n \to \mathbf{R}^m$ be a smooth vector-valued function, then for any $p \in M :=$ graph f, any Jordan angle between N_pM and the coordinate m-plane takes values in $[0, \pi/2)$ (see [32]). Hence Theorem 2.1 implies Theorem 1.1 mentioned in §1.3.

3. Coassociative submanifolds with CJA.

3.1. Associative subspace of Im O. Let $\mathbf{O}$ denote the octonions, which is an 8-dimensional normed algebra over $\mathbf{R}$ with multiplicative unit 1. More precisely, $\mathbf{O}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, whose associated norm $| \cdot |$ satisfies

$$|xy| = |x||y| \tag{3.1}$$

for any $x, y \in \mathbf{O}$. Denote by Re $\mathbf{O}$ the 1-dimensional subspace spanned by 1, and by Im $\mathbf{O}$ the orthogonal complement of Re $\mathbf{O}$. Then every $x \in \mathbf{O}$ has a unique decomposition

$$x = \text{Re } x + \text{Im } x$$

with Re $x \in$ Re $\mathbf{O}$, Im $x \in$ Im $\mathbf{O}$. The conjugation of x is defined by

$$\bar{x} = \text{Re } x - \text{Im } x. \tag{3.2}$$

For $w \in \mathbf{O}$, let R_w (L_w) denote the linear operator of right (left) multiplication by w, respectively. With the aid of (3.1) and (3.2), one can easily deduce the following fundamental formulas (see Appendix IV.A of [16]):

$$\langle R_w x, R_w y \rangle = \langle x, y \rangle |w|^2, \qquad \langle L_w x, L_w y \rangle = \langle x, y \rangle |w|^2, \tag{3.3}$$

$$\langle x, R_w y \rangle = \langle R_{\bar{w}} x, y \rangle, \qquad \langle x, L_w y \rangle = \langle L_{\bar{w}} x, y \rangle, \tag{3.4}$$

$$\bar{\bar{x}} = x, \ \overline{xy} = \bar{y}\bar{x}, \qquad x\bar{x} = |x|^2, \ \langle x, y \rangle = \text{Re } x\bar{y}. \tag{3.5}$$

Let P be a 3-dimensional real subspace of Im $\mathbf{O}$, if $A :=$ Re $\mathbf{O} \oplus P$ is a quarternion subalgebra of $\mathbf{O}$ (i.e. A is isomorphic to $\mathbf{H}$), then P is said to be *associative*.

Lemma 3.1. *Let P be an associative subspace of Im $\mathbf{O}$ and x, y be unit elements in P that are orthogonal to each other, then $\{x, y, z := xy\}$ is an orthonormal basis of P, and*

$$xy = -yx = z, \quad yz = -zy = x, \quad zx = -xz = y. \tag{3.6}$$

Conversely, if $\{x, y, z\}$ is an orthonormal basis of an associative subspace P, then $z = xy$ or $-xy$.

Proof. Since Re $\mathbf{O} \oplus P$ is a subalgebra of $\mathbf{O}$, $xy \in$ Re $\mathbf{O} \oplus P$. By (3.2) and (3.5),

$$\text{Re } (xy) = -\text{Re } (x\bar{y}) = -\langle x, y \rangle = 0,$$

i.e. $xy \in P$. Applying (3.3) and (3.1) gives

$$\begin{aligned}\langle xy, x\rangle &= \langle L_x y, L_x 1\rangle = \langle y, 1\rangle |x|^2 = 0,\\ \langle xy, y\rangle &= \langle R_y x, R_y 1\rangle = \langle x, 1\rangle |y|^2 = 0,\\ |xy| &= |x||y| = 1.\end{aligned}$$

Hence $\{x, y, z := xy\}$ is an orthonormal basis of P.

Similarly, one can show yx is also a unit element in P orthogonal to span$\{x, y\}$, hence $yx = z$ or $-z$. If $yx = z$, then

$$\begin{aligned}&(x+y)(x-y) = x^2 - y^2 + yx - xy\\ =& -x\bar{x} + y\bar{y} + z - z = -|x|^2 + |y|^2 \\ =&0.\end{aligned} \tag{3.7}$$

On the other hand, since x and y are linearly independent, $x+y, x-y \neq 0$ and it follows from (3.1) that $|(x+y)(x-y)| = |x+y||x-y| \neq 0$, which contradicts (3.7). Hence $yx = -z$ and it follows that

$$\begin{aligned}yz =& y(-yx) = -y^2 x\\ =& y\bar{y}x = |y|^2 x = x.\end{aligned}$$

Similarly one can prove $zy = -x$ and $zx = -xz = y$.

Conversely, if $\{x, y, z\}$ is an orthonormal basis of P, then z and xy are both unit elements orthogonal to span$\{x, y\}$, which implies $z = xy$ or $-xy$. □

LEMMA 3.2. *Let A be a quarternion subalgebra of $\mathbf{O}$, $\varepsilon \in A^\perp$ with $|\varepsilon| = 1$, then $A\varepsilon \perp A$, $\mathbf{O} = A \oplus A\varepsilon$ and*

$$(x + y\varepsilon)(v + w\varepsilon) = (xv - \bar{w}y) + (wx + y\bar{v})\varepsilon \tag{3.8}$$

for any $x, y, v, w \in A$.

Proof. The lemma immediately follows from Lemma A.8 in [16]. □

3.2. Jordan angles between associative subspaces. Now we explore the Jordan angles between an associative subspace P and Im $\mathbf{H}$.

Case I. $0 \in \text{Arg}(P, \text{Im}\,\mathbf{H})$ and $m_0 \geq 2$. This means there exist 2 unit elements $a, b \in P \cap \text{Im}\,\mathbf{H}$ that are orthogonal to each other, then it follows from Lemma 3.1 that $\{a, b, ab\}$ is an orthonormal basis of $P \cap \text{Im}\,\mathbf{H}$. Hence $P = \text{Im}\,\mathbf{H}$ and $\text{Arg}(P, \text{Im}\,\mathbf{H}) = \{0\}$.

Case II. $\pi/2 \in \text{Arg}(P, \text{Im}\,\mathbf{H})$ and $m_{\pi/2} \geq 2$. Then there exists 2 unit elements $ae, be \in P \cap (\text{Im}\,\mathbf{H})^\perp = P \cap \mathbf{H}e$ that are orthogonal to each other. By Lemma 3.1, $(ae)(be) = -\bar{b}a$ is a unit vector in P, and $-\bar{b}a \in \mathbf{H} \cap \text{Im}\,\mathbf{O} = \text{Im}\,\mathbf{H}$. Hence $\text{Arg}(P, \text{Im}\,\mathbf{H}) = \{0, \pi/2\}$, $m_0 = 1$, $m_{\pi/2} = 2$, and P is spanned by ae, be and $-\bar{b}a$, which are the angle directions of P relative to Im $\mathbf{H}$.

Case III. $m_0 \leq 1$ and $m_{\pi/2} \leq 1$. (Note that $m_0 = 0$ ($m_{\pi/2} = 0$) means $0 \notin \text{Arg}(P, \text{Im}\,\mathbf{H})$ ($\pi/2 \notin \text{Arg}(P, \text{Im}\,\mathbf{H})$), respectively.) First, we claim $m_0 + m_{\pi/2} \leq 1$. If not, there exist unit elements $a \in P \cap \text{Im}\,\mathbf{H}$ and $be \in P \cap \mathbf{H}e$; by Lemma 3.1, P is spanned by a, be and $a(be) = (ba)e \in \mathbf{H}e$; hence $m_{\pi/2} = 2$, contradicting $m_{\pi/2} \leq 1$.

Hence there exist mutually orthogonal elements $x_1, x_2 \in P$ that are unit angle directions of P relative to Im $\mathbf{H}$ associated to $\theta_1, \theta_2 \in \text{Arg}(P, \text{Im}\,\mathbf{H}) \cap (0, \pi/2)$, respectively. More precisely,

$$(\mathcal{P} \circ \mathcal{P}_0) x_\alpha = \cos^2 \theta_\alpha x_\alpha \qquad \forall \alpha = 1, 2. \tag{3.9}$$

Here $\mathcal{P}_0$ denotes the orthogonal projection of Im $\mathbf{O}$ onto Im $\mathbf{H}$ and $\mathcal{P}$ denotes the orthogonal projection of Im $\mathbf{O}$ onto P. As in Section 1, we denote by $\mathcal{P}_0^\perp$ the orthogonal projection of Im $\mathbf{O}$ onto $\mathbf{H}e = (\text{Im }\mathbf{H})^\perp$, then

$$\begin{aligned} x_\alpha &= \mathcal{P}_0 x_\alpha + \mathcal{P}_0^\perp x_\alpha \\ &= \cos\theta_\alpha a_\alpha + \sin\theta_\alpha y_\alpha \end{aligned} \tag{3.10}$$

with $a_\alpha := \sec\theta_\alpha\ \mathcal{P}_0 x_\alpha \in \text{Im }\mathbf{H}$ and $y_\alpha := \csc\theta_\alpha\ \mathcal{P}^\perp x_\alpha \in \mathbf{H}e$, satisfying $|a_\alpha| = |y_\alpha| = 1$ for each $\alpha = 1, 2$. Let ε be the unique element in $\mathbf{O}$ satisfying $y_1 = a_1\varepsilon$, then for every $c \in \mathbf{H}$,

$$\begin{aligned} \langle \varepsilon, c\rangle &= \langle L_{a_1}\varepsilon, L_{a_1}c\rangle = \langle a_1\varepsilon, a_1 c\rangle \\ &= \langle y_1, a_1 c\rangle = 0, \end{aligned}$$

which implies $\varepsilon \in \mathbf{H}e$. And $|\varepsilon| = 1$ directly follows from $y_1 = a_1\varepsilon$ and $|y_1| = |a_1| = 1$. Similarly, one can prove that there exists a unique $b \in \mathbf{H}$ which satisfies $y_2 = b\varepsilon$, and moreover $|b| = 1$.

Let $x_3 := x_1 x_2$, then Lemma 3.2 enables us to obtain

$$\begin{aligned} x_3 =&(\cos\theta_1 a_1 + \sin\theta_1 a_1\varepsilon)(\cos\theta_2 a_2 + \sin\theta_2 b\varepsilon) \\ =&(\cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2 \bar{b} a_1) \\ &+ (\cos\theta_1\sin\theta_2 b a_1 + \sin\theta_1\cos\theta_2 a_1\bar{a}_2)\varepsilon. \end{aligned} \tag{3.11}$$

By Lemma 3.1, $\{x_1, x_2, x_3\}$ is an orthonormal basis of P, thus for each $\alpha = 1, 2$,

$$\begin{aligned} 0 &= \cos^2\theta_\alpha\langle x_\alpha, x_3\rangle = \langle(\mathcal{P}\circ\mathcal{P}_0)x_\alpha, x_3\rangle \\ &= \langle \mathcal{P}_0 x_\alpha, x_3\rangle = \langle \mathcal{P}_0 x_\alpha, \mathcal{P}_0 x_3\rangle. \end{aligned} \tag{3.12}$$

When $\alpha = 1$, the above equation gives

$$\begin{aligned} 0 &= \langle \mathcal{P}_0 x_1, \mathcal{P}_0 x_3\rangle = \langle \cos\theta_1 a_1, \cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2\bar{b}a_1\rangle \\ &= \cos^2\theta_1\cos\theta_2\langle a_1, a_1 a_2\rangle - \cos\theta_1\sin\theta_1\sin\theta_2\langle a_1, \bar{b}a_1\rangle \\ &= \cos^2\theta_1\cos\theta_2\langle 1, a_2\rangle - \cos\theta_1\sin\theta_1\sin\theta_2\langle 1, \bar{b}\rangle \\ &= -\cos\theta_1\sin\theta_1\sin\theta_2\langle \bar{b}, 1\rangle. \end{aligned}$$

In conjunction with $\theta_1, \theta_2 \in (0, \pi/2)$ we have $\langle \bar{b}, 1\rangle = 0$, therefore $b \in \text{Im }\mathbf{H}$. Letting $\alpha = 2$ in (3.12) yields

$$\begin{aligned} 0 &= \langle \mathcal{P}_0 x_2, \mathcal{P}_0 x_3\rangle = \langle \cos\theta_2 a_2, \cos\theta_1\cos\theta_2 a_1 a_2 - \sin\theta_1\sin\theta_2\bar{b}a_1\rangle \\ &= \cos\theta_1\cos^2\theta_2\langle a_2, a_1 a_2\rangle - \cos\theta_2\sin\theta_1\sin\theta_2\langle a_2, \bar{b}a_1\rangle \\ &= -\cos\theta_2\sin\theta_1\sin\theta_2\langle a_2, \bar{b}a_1\rangle \end{aligned}$$

and moreover

$$\begin{aligned} 0 &= \langle a_2, \bar{b}a_1\rangle = \langle a_2, R_{a_1}\bar{b}\rangle = \langle R_{\bar{a}_1}a_2, \bar{b}\rangle \\ &= \langle a_2\bar{a}_1, \bar{b}\rangle = \langle -a_2 a_1, -b\rangle = \langle a_2 a_1, b\rangle. \end{aligned}$$

Observing that a_1, a_2 and $a_2 a_1$ form an orthonormal basis of Im $\mathbf{H}$, we have $b \in \text{span}\{a_1, a_2\}$.

By the definition of angle directions,

$$\begin{aligned}&\langle \mathcal{P}_0^\perp x_1, \mathcal{P}_0^\perp x_2\rangle = \langle \mathcal{P}_0^\perp x_1, x_2\rangle = \langle (\mathcal{P}\circ\mathcal{P}_0^\perp)x_1, x_2\rangle\\ =&\langle \mathcal{P}(x_1-\mathcal{P}_0x_1), x_2\rangle = \langle x_1,x_2\rangle - \langle(\mathcal{P}\circ\mathcal{P}_0)x_1,x_2\rangle\\ =&\langle x_1,x_2\rangle - \cos^2\theta_1\langle x_1,x_2\rangle = 0,\end{aligned} \tag{3.13}$$

which implies

$$\begin{aligned}0 &= \langle \sin\theta_1 y_1, \sin\theta_2 y_2\rangle = \sin\theta_1\sin\theta_2\langle a_1\varepsilon, b\varepsilon\rangle\\ &= \sin\theta_1\sin\theta_2\langle a_1, b\rangle\end{aligned}$$

i.e. $\langle a_1, b\rangle = 0$. Therefore $b = a_2$ or $-a_2$.

If $b = a_2$, then (3.11) shows

$$\begin{aligned}x_3 &= (\cos\theta_1\cos\theta_2 a_1a_2 - \sin\theta_1\sin\theta_2\bar{a}_2a_1) + (\cos\theta_1\sin\theta_2 a_2a_1 + \sin\theta_1\cos\theta_2 a_1\bar{a}_2)\varepsilon\\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)a_3 - (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)a_3\varepsilon\\ &= \cos(\theta_1+\theta_2)a_3 - \sin(\theta_1+\theta_2)a_3\varepsilon.\end{aligned} \tag{3.14}$$

Noting that x_3 is also an angle direction of P relative to $\text{Im}\,\mathbf{H}$, $\theta_3 := \arccos|\cos(\theta_1 + \theta_2)| \in \text{Arg}(P, \text{Im}\,\mathbf{H})$. In other words,

$$\theta_3 = \begin{cases}\theta_1+\theta_2 & \text{if } \theta_1+\theta_2 \le \pi/2,\\ \pi-(\theta_1+\theta_2) & \text{if } \theta_1+\theta_2 > \pi/2.\end{cases}$$

Otherwise, $b = -a_2$ and (3.11) gives

$$\begin{aligned}x_3 =&(\cos\theta_1\cos\theta_2 a_1a_2 - \sin\theta_1\sin\theta_2(-\bar{a}_2)a_1)\\ &+ (\cos\theta_1\sin\theta_2(-a_2a_1) + \sin\theta_1\cos\theta_2 a_1\bar{a}_2)\varepsilon\\ =&(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)a_3 - (-\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)a_3\varepsilon\\ =&\cos(\theta_1-\theta_2)a_3 - \sin(\theta_1-\theta_2)a_3\varepsilon,\end{aligned} \tag{3.15}$$

which implies $\theta_3 := \arccos|\cos(\theta_1-\theta_2)| = |\theta_1-\theta_2| \in \text{Arg}(P, \text{Im}\,\mathbf{H})$. Without loss of generality, one can assume $\theta_1 \ge \theta_2$, then $\theta_3 = \theta_1 - \theta_2$. Now we put

$$\begin{aligned}\theta_1' &:= \theta_2, & \theta_2' &:= \theta_3, & \theta_3' &:= \theta_1,\\ a_1' &:= a_2, & a_2' &:= a_3, & a_3' &:= a_1,\\ x_1' &:= x_2, & x_2' &:= x_3, & x_3' &:= x_1\end{aligned}$$

and $\varepsilon' := -\varepsilon$, then

$$\begin{aligned}x_1' &= \cos\theta_1' a_1' + \sin\theta_1' a_1'\varepsilon',\\ x_2' &= \cos\theta_2' a_2' + \sin\theta_2' a_2'\varepsilon',\\ x_3' &= \cos\theta_3' a_3' - \sin\theta_3' a_3'\varepsilon',\end{aligned} \tag{3.16}$$

which satisfy $\theta_3' = \theta_1' + \theta_2'$, $a_3' = a_1'a_2'$ and $x_3' = x_1'x_2'$.

Altogether, we have shown

PROPOSITION 3.1. *Let P be an associative subspace of Im $\mathbf{O}$, and $0 \le \theta_1 \le \theta_2 \le \theta_3 \le \pi/2$ be the Jordan angles between P and Im $\mathbf{H}$, then*

$$\theta_3 = \begin{cases}\theta_1+\theta_2 & \text{if } \theta_1+\theta_2 \le \pi/2,\\ \pi-(\theta_1+\theta_2) & \text{if } \theta_1+\theta_2 > \pi/2.\end{cases} \tag{3.17}$$

Moreover, there exist an orthonormal basis $\{a_1, a_2, a_3\}$ *of Im* $\mathbf{H}$ *satisfying* $a_3 = a_1 a_2$, *and a unit element* $\varepsilon \in \mathbf{H}e$, *such that*

$$\begin{aligned} x_1 &:= \cos\theta_1 a_1 + \sin\theta_1 a_1 \varepsilon, \\ x_2 &:= \cos\theta_2 a_2 + \sin\theta_2 a_2 \varepsilon, \\ x_3 &:= \cos(\theta_1+\theta_2) a_3 - \sin(\theta_1+\theta_2) a_3 \varepsilon \end{aligned} \tag{3.18}$$

are unit angle directions of P *relative to Im* $\mathbf{H}$, *and* $x_3 = x_1 x_2$.

3.3. On the second fundamental form of coassociative submanifolds. Let M be a 4-dimensional submanifold in Im $\mathbf{O}$. If the normal space at every point of M is associative, then we call M a *coassociative submanifold* (see [16]). Let p be an arbitrary point of M, denote by $0 \le \theta_1 \le \theta_2 \le \theta_3 \le \pi/2$ the Jordan angles between $N_p M$ (an associative subspace) and Im $\mathbf{H}$, then by Proposition 3.1,

$$\theta_3 = \begin{cases} \theta_1 + \theta_2 & \text{if } \theta_1 + \theta_2 \le \pi/2, \\ \pi - (\theta_1 + \theta_2) & \text{if } \theta_1 + \theta_2 > \pi/2. \end{cases} \tag{3.19}$$

Denote $\{a_1, a_2, a_3\}$ to be the orthonormal basis of Im $\mathbf{H}$ satisfying $a_3 = a_1 a_2$ and ε to be the unit element in $\mathbf{H}e$, such that

$$\begin{aligned} \nu_1 &:= \cos\theta_1 a_1 + \sin\theta_1 a_1 \varepsilon, \\ \nu_2 &:= \cos\theta_2 a_2 + \sin\theta_2 a_2 \varepsilon, \\ \nu_3 &:= \cos(\theta_1+\theta_2) a_3 - \sin(\theta_1+\theta_2) a_3 \varepsilon \end{aligned} \tag{3.20}$$

are all unit angle directions of $N_p M$ relative to Im $\mathbf{H}$, and $\nu_3 = \nu_1 \nu_2$. Denote

$$\begin{aligned} e_1 &:= -\nu_1 \varepsilon = \sin\theta_1 a_1 - \cos\theta_1 a_1 \varepsilon, \\ e_2 &:= -\nu_2 \varepsilon = \sin\theta_2 a_2 - \cos\theta_2 a_2 \varepsilon, \\ e_3 &:= -\nu_3 \varepsilon = -\sin(\theta_1+\theta_2) a_3 - \cos(\theta_1+\theta_2) a_3 \varepsilon, \\ e_4 &:= \varepsilon, \end{aligned} \tag{3.21}$$

then it is easy to check that $\langle e_i, \nu_\alpha \rangle = 0$ and $\langle e_i, e_j \rangle = \delta_{ij}$ for each $1 \le i, j \le 4$ and $1 \le \alpha \le 3$. Hence $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $T_p M$. Whenever $\theta_\alpha \in (0, \pi/2)$, let Φ_{p,θ_α} denote the isometric automorphism of $R_{p,\theta_\alpha} M := N_{p,\theta_\alpha} M \oplus T_{p,\theta_\alpha} M$ as in §1.1, then it follows from (1.9) that

$$\sec\theta_1 \mathcal{P}_0^\perp e_1 = \cos\theta_1 e_1 - \sin\theta_1 \Phi_{p,\theta_1}(e_1).$$

Hence

$$\begin{aligned} \Phi_{p,\theta_1}(e_1) &= \cot\theta_1 e_1 - \sec\theta_1 \csc\theta_1 \mathcal{P}_0^\perp e_1 \\ &= \cot\theta_1 (\sin\theta_1 a_1 - \cos\theta_1 a_1 \varepsilon) - \sec\theta_1 \csc\theta_1 (-\cos\theta_1 a_1 \varepsilon) \\ &= \cos\theta_1 a_1 + \sin\theta_1 a_1 \varepsilon \\ &= \nu_1 \end{aligned}$$

and similarly $\Phi_{p,\theta_2}(e_2) = \nu_2$; in conjunction with (3.19),

$$\begin{aligned} \Phi_{p,\theta_3}(e_3) &= \cot\theta_3 e_3 - \sec\theta_3 \csc\theta_3 \mathcal{P}_0^\perp e_3 \\ &= -\cos\theta_3 a_3 + \operatorname{sgn}\big(\cos(\theta_1+\theta_2)\big) \sin\theta_3 a_3 \varepsilon \\ &= \begin{cases} -\nu_3 & \text{if } \theta_1 + \theta_2 < \pi/2, \\ \nu_3 & \text{if } \theta_1 + \theta_2 > \pi/2. \end{cases} \end{aligned}$$

In summary we get a proposition as follows.

PROPOSITION 3.2. *Let M be a coassociative submanifold in Im* $\mathbf{O}$, $p \in M$ *and* $0 \le \theta_1 \le \theta_2 \le \theta_3 \le \pi/2$ *be the Jordan angles between* N_pM *and Im* $\mathbf{H}$, *then there exist an orthonormal basis* $\{\nu_1, \nu_2, \nu_3\}$ *of* N_pM *and an orthonormal basis* $\{e_1, e_2, e_3, e_4\}$ *of* T_pM, *such that*

(i) *For each* $1 \le \alpha \le 3$, ν_α (e_α) *is an angle direction of* N_pM (T_pM) *relative to Im* $\mathbf{H}$ *(or* $\mathbf{H}e$*), corresponding to the Jordan angle* θ_α;

(ii) $e_4 \in \mathbf{H}e$;

(iii) $e_\alpha = -\nu_\alpha e_4$ *for each* $1 \le \alpha \le 3$;

(iv) $\Phi_{p,\theta_\alpha}(e_\alpha) = \nu_\alpha$ *for any* $1 \le \alpha \le 2$ *satisfying* $\theta_\alpha \in (0, \pi/2)$;

(v) $\Phi_{p,\theta_3}(e_3) = \begin{cases} -\nu_3 & \text{if } \theta_1 + \theta_2 < \pi/2, \\ \nu_3 & \text{if } \theta_1 + \theta_2 > \pi/2, \\ 0 & \text{if } \theta_1 + \theta_2 = \pi/2. \end{cases}$

REMARK. Here we additionally define $\Phi_{p,0} = \Phi_{p,\pi/2} = 0$, as in §2.2.

Now we extend $\{\nu_1, \nu_2, \nu_3\}$ as an orthonormal normal frame field on U, a neighborhood of p, such that $\nabla_v \nu_\alpha = 0$ for every $v \in T_pM$. Lemma 3.1 implies $\nu_3(q) = \nu_1(q)\nu_2(q)$ or $-\nu_1(q)\nu_2(q)$ for an arbitrary $q \in U$. Due to the continuity, $\nu_3 = \nu_1\nu_2$ on U and differentiating both sides with respect to $e_i \in T_pM$ gives

$$\begin{aligned} -h_{3,ij}e_j &= \overline{\nabla}_{e_i}\nu_3 = \overline{\nabla}_{e_i}(\nu_1\nu_2) \\ &= (\overline{\nabla}_{e_i}\nu_1)\nu_2 + \nu_1(\overline{\nabla}_{e_i}\nu_2) \\ &= -h_{1,ij}e_j\nu_2 - h_{2,ij}\nu_1 e_j, \end{aligned}$$

i.e.

$$h_{3,ij}e_j = h_{1,ij}e_j\nu_2 + h_{2,ij}\nu_1 e_j. \tag{3.22}$$

With the aid of Lemma 3.1, Lemma 3.2 and Proposition 3.2, a straightforward calculation shows

$$\begin{aligned} \text{LHS of (3.22)} &= h_{3,i\alpha}e_\alpha + h_{3,i4}e_4 \\ &= -h_{3,i\alpha}\nu_\alpha e_4 + h_{3,i4}e_4 \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} \text{RHS of (3.22)} =& h_{1,i\alpha}e_\alpha\nu_2 + h_{1,i4}e_4\nu_2 + h_{2,i\alpha}\nu_1 e_\alpha + h_{2,i4}\nu_1 e_4 \\ =& - h_{1,i\alpha}(\nu_\alpha e_4)\nu_2 - h_{1,i4}\nu_2 e_4 - h_{2,i\alpha}\nu_1(\nu_\alpha e_4) + h_{2,i4}\nu_1 e_4 \\ =& h_{1,i\alpha}(\nu_\alpha\nu_2)e_4 - h_{1,i4}\nu_2 e_4 - h_{2,i\alpha}(\nu_\alpha\nu_1)e_4 + h_{2,i4}\nu_1 e_4 \\ =& h_{1,i1}\nu_3 e_4 - h_{1,i2}e_4 - h_{1,i3}\nu_1 e_4 - h_{1,i4}\nu_2 e_4 \\ &+ h_{2,i1}e_4 + h_{2,i2}\nu_3 e_4 - h_{2,i3}\nu_2 e_4 + h_{2,i4}\nu_1 e_4 \\ =& (-h_{1,i2} + h_{2,i1})e_4 + (-h_{1,i3} + h_{2,i4})\nu_1 e_4 \\ &+ (-h_{1,i4} - h_{2,i3})\nu_2 e_4 + (h_{1,i1} + h_{2,i2})\nu_3 e_4. \end{aligned} \tag{3.24}$$

Comparing with (3.23) and (3.24), we arrive at the following conclusion.

PROPOSITION 3.3. *Let M be a coassociative submanifold in Im* $\mathbf{O}$, $p \in M$. *Let* $\{e_i : 1 \le i \le 4\}$ *and* $\{\nu_\alpha : 1 \le \alpha \le 3\}$ *be as in Proposition 3.2. Then for each* $1 \le i \le 4$,

$$h_{3,i1} = h_{1,i3} - h_{2,i4}, \tag{3.25}$$

$$h_{3,i2} = h_{1,i4} + h_{2,i3}, \tag{3.26}$$
$$h_{3,i3} = -h_{1,i1} - h_{2,i2}, \tag{3.27}$$
$$h_{3,i4} = -h_{1,i2} + h_{2,i1}. \tag{3.28}$$

Here $\{h_{ij}^{\alpha} := \langle B_{e_ie_j}, \nu_\alpha\rangle(p) : 1 \le i, j \le 4, 1 \le \alpha \le 3\}$ *are the coefficients of the second fundamental form at* p.

3.4. The characterization of the Lawson-Osserman cone. Now we additionaly assume M has CJA relative to Im $\mathbf{H}$. Let $p_0 \in M$, $0 \le \theta_1 \le \theta_2 \le \theta_3 \le \pi/2$ be the Jordan angles between $N_{p_0}M$ and Im $\mathbf{H}$, $\{\nu_1, \nu_2, \nu_3\}$ be the orthonormal basis of $N_{p_0}M$ and $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis of $T_{p_0}M$, satisfying the properties in Proposition 3.2. Then (2.6) and Lemma 2.3 implies

$$h_{\alpha,\alpha i} = 0 \qquad \forall 1 \le \alpha \le 3, 1 \le i \le 4 \tag{3.29}$$

and substituting it into (3.25)-(3.28) gives

$$h_{1,23} = h_{2,31} = h_{3,12}; \tag{3.30}$$
$$h_{1,22} = -h_{3,24}, \quad h_{1,33} = h_{2,34}, \quad h_{1,44} = -h_{2,34} + h_{3,24}; \tag{3.31}$$
$$h_{2,33} = -h_{1,34}, \quad h_{2,11} = h_{3,14}, \quad h_{2,44} = -h_{3,14} + h_{1,34}; \tag{3.32}$$
$$h_{3,11} = -h_{2,14}, \quad h_{3,22} = h_{1,24}, \quad h_{3,44} = -h_{1,24} + h_{2,14}. \tag{3.33}$$

Furthermore, applying Lemma 2.6 and 2.7 yields the following propositions.

PROPOSITION 3.4. *Let* M *be a coassociative submanifold in Im* $\mathbf{O}$*, with CJA relative to Im* $\mathbf{H}$*. If* $g^N \le 2$, $\pi/2 \notin Arg^N$ *and* $Arg^N \neq \{\arccos(\sqrt{6}/6), \arccos(2/3)\}$*, then* M *has to be affine linear.*

Proof. Let p_0 be an arbitrary point in M, and the notations θ_α, ν_α, e_i, $h_{\alpha,ij}$ are same as above.

Case I. $\theta_1 = 0$ and $\theta_2 = \theta_3 < \pi/2$. Then $g^T = g^N \le 2$ and the equality holds if and only if $\theta_2 \neq 0$. It is well-known that coassociative submanfolds are absolutely area-minimizing (see [16] §IV.2.B). By Theorem 2.1, M has to be an open set of an affine 4-plane.

Case II. $\theta_1 = \theta_2 \in (0, \pi/4) \cup (\pi/4, \pi/3)$ and $\theta_3 = \begin{cases} 2\theta_1 & \text{if } \theta_1 < \pi/4, \\ \pi - 2\theta_1 & \text{if } \theta_1 > \pi/4. \end{cases}$
Denote $\theta := \theta_1$, then $\text{Arg}^N = \{\theta, \theta_3\}$, $\text{Arg}^T = \{0, \theta, \theta_3\}$; $T_{p_0,\theta}M = \text{span}\{e_1, e_2\}$ and $N_{p_0,\theta}M = \text{span}\{\nu_1, \nu_2\}$ with $\nu_\alpha = \Phi_\theta(e_\alpha)$ for each $1 \le \alpha \le 2$; $T_{p_0,\theta_3}M = \text{span}\{e_3\}$, $N_{p_0,\theta_3}M = \text{span}\{\nu_3\}$ and

$$\Phi_{\theta_3}(e_3) = \begin{cases} \nu_3 & \text{if } \theta_1 > \pi/4, \\ -\nu_3 & \text{if } \theta_1 < \pi/4; \end{cases}$$

$T_{p_0,0} = \text{span}\{e_4\}$. Lemma 2.2 implies

$$h_{1,22} = h_{2,11} = 0. \tag{3.34}$$

Substituting the above equation into (3.31) and (3.32), we get

$$h_{3,24} = h_{3,14} = 0. \tag{3.35}$$

Applying Lemma 2.1 gives

$$0 = h_{1,23} + h_{2,13} = h_{1,24} + h_{2,14}. \tag{3.36}$$

In conjunction with (3.30), we have

$$h_{1,23} = h_{2,31} = h_{3,12} = 0. \tag{3.37}$$

Let $R_{\theta\sigma}$ and $U_{\theta\sigma}$ be tensors of type $(4,0)$, defined in (2.19) and (2.21), respectively. Then

$$\begin{aligned} R_{\theta\theta_3}(e_1,e_2,e_1,e_2) &= \langle B^{\theta_3}_{e_1e_1}, B^{\theta_3}_{e_2e_2}\rangle - \langle B^{\theta_3}_{e_1e_2}, B^{\theta_3}_{e_2e_1}\rangle \\ &= h_{3,11}h_{3,22} - h_{3,12}h_{3,21} \\ &= -h_{2,14}h_{1,24} = h^2_{1,24}, \end{aligned} \tag{3.38}$$

$$\begin{aligned} &U_{\theta\theta_3}(e_1,e_2,e_1,e_2) \\ =&\langle (A^{\Phi_\theta(e_1)}e_1)_{\theta_3}, (A^{\Phi_\theta(e_2)}e_2)_{\theta_3}\rangle - \langle (A^{\Phi_\theta(e_2)}e_1)_{\theta_3}, (A^{\Phi_\theta(e_1)}e_2)_{\theta_3}\rangle \\ =&\langle A^{\nu_1}e_1, e_3\rangle\langle A^{\nu_2}e_2, e_3\rangle - \langle A^{\nu_2}e_1, e_3\rangle\langle A^{\nu_1}e_2, e_3\rangle \\ =&h_{1,13}h_{2,23} - h_{2,13}h_{1,23} = 0 \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} &U_{\theta 0}(e_1,e_2,e_1,e_2) \\ =&\langle (A^{\Phi_\theta(e_1)}e_1)_0, (A^{\Phi_\theta(e_2)}e_2)_0\rangle - \langle (A^{\Phi_\theta(e_2)}e_1)_0, (A^{\Phi_\theta(e_1)}e_2)_0\rangle \\ =&\langle A^{\nu_1}e_1, e_4\rangle\langle A^{\nu_2}e_2, e_4\rangle - \langle A^{\nu_2}e_1, e_4\rangle\langle A^{\nu_1}e_2, e_4\rangle \\ =&h_{1,14}h_{2,24} - h_{2,14}h_{1,24} = h^2_{1,24}. \end{aligned} \tag{3.40}$$

By (2.22),

$$\begin{aligned} 0 &= \sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq\theta} \kappa_{\theta\sigma}R_{\theta\sigma}(e_1,e_2,e_1,e_2) - 3\sum_{\sigma\in \mathrm{Arg}^T, \sigma\neq\theta} \kappa_{\theta\sigma}U_{\theta\sigma}(e_1,e_2,e_1,e_2) \\ &= \kappa_{\theta\theta_3}R_{\theta\theta_3}(e_1,e_2,e_1,e_2) - 3\kappa_{\theta\theta_3}U_{\theta\theta_3}(e_1,e_2,e_1,e_2) - 3\kappa_{\theta 0}U_{\theta 0}(e_1,e_2,e_1,e_2) \\ &= (\kappa_{\theta\theta_3} - 3\kappa_{\theta 0})h^2_{1,24} \end{aligned}$$

where

$$\begin{aligned} &\kappa_{\theta\theta_3} - 3\kappa_{\theta 0} = \frac{\sin 2\theta}{\cos 2\theta - \cos 2\theta_3} - \frac{3\sin 2\theta}{\cos 2\theta - 1} \\ =&\frac{\sin 2\theta}{\cos 2\theta - \cos 4\theta} + \frac{3\sin 2\theta}{1-\cos 2\theta} = \frac{2\cos\theta\sin\theta}{2\sin 3\theta\sin\theta} + \frac{6\cos\theta\sin\theta}{2\sin^2\theta} \\ =&\frac{\cos\theta(\sin\theta + 3\sin 3\theta)}{\sin 3\theta\sin\theta} > 0. \end{aligned}$$

Hence $h_{1,24} = 0$ and moreover

$$h_{3,22} = h_{1,24} = 0, \quad h_{3,11} = -h_{2,14} = h_{1,24} = 0, \quad h_{3,44} = -h_{1,24} + h_{2,14} = 0. \tag{3.41}$$

In conjunction with (3.29), (3.35) and (3.37), we obtain

$$A^{\nu_3} = 0. \tag{3.42}$$

Putting $v = w = e_3$ in (2.23) gives

$$\begin{aligned}\langle(\nabla_{e_3}B)_{e_3e_3}, \Phi_{\theta_3}(e_3)\rangle =&(1/3)\kappa_{\theta_3\theta}\left(\langle B^\theta_{e_3e_3}, B^\theta_{e_3e_3}\rangle + 2|B^\theta_{e_3e_3}|^2\right)\\ &- 2\kappa_{\theta\theta_3}\langle B^\theta_{e_3e_3}, \Phi_\theta(A^{\Phi_{\theta_3}(e_3)}e_3)\rangle\\ =&\kappa_{\theta_3\theta}|B^\theta_{e_3e_3}|^2 = \kappa_{\theta_3\theta}(h^2_{1,33} + h^2_{2,33})\\ =&\kappa_{\theta_3\theta}(h^2_{2,34} + h^2_{1,34})\end{aligned} \tag{3.43}$$

(where we have used (3.42), (3.31) and (3.32)). By Lemma 2.7,

$$\begin{aligned}&\langle(\nabla_{e_3}B)_{e_4e_4}, \Phi_{\theta_3}(e_3)\rangle\\ =&- 2\kappa_{\theta\theta_3}\langle B^\theta_{e_3e_4}, \Phi_\theta(A^{\Phi_{\theta_3}(e_3)}e_4)_\theta\rangle\\ &+ \kappa_{\theta_3\theta}|B^\theta_{e_3e_4}|^2 + \kappa_{\theta_3\theta}|(A^{\Phi_{\theta_3}(e_3)}e_4)_\theta|^2 + \kappa_{\theta_30}|(A^{\Phi_{\theta_3}(e_3)}e_4)_0|^2\\ =&\kappa_{\theta_3\theta}(h^2_{1,34} + h^2_{2,34}),\end{aligned} \tag{3.44}$$

$$\begin{aligned}&\langle(\nabla_{e_3}B)_{e_1e_1}, \Phi_{\theta_3}(e_3)\rangle\\ =&- 2\kappa_{\theta\theta_3}\langle B^\theta_{e_3e_1}, \Phi_\theta(A^{\Phi_{\theta_3}(e_3)}e_1)_\theta\rangle\\ &+ \kappa_{\theta_3\theta}|B^\theta_{e_3e_1}|^2 + \kappa_{\theta_3\theta}|(A^{\Phi_{\theta_3}(e_3)}e_1)_\theta|^2 + \kappa_{\theta_30}|(A^{\Phi_{\theta_3}(e_3)}e_1)_0|^2\\ =&\kappa_{\theta_3\theta}(h^2_{1,31} + h^2_{2,31}) = 0\end{aligned} \tag{3.45}$$

and similarly

$$\langle(\nabla_{e_3}B)_{e_2e_2}, \Phi_{\theta_3}(e_3)\rangle = 0. \tag{3.46}$$

Combining (3.43)-(3.46) gives

$$\begin{aligned}0 = \langle\nabla_{e_3}H, \Phi_{\theta_3}(e_3)\rangle &= \sum_{i=1}^4\langle(\nabla_{e_3}B)_{e_ie_i}, \Phi_{\theta_3}(e_3)\rangle\\ &= 2\kappa_{\theta_3\theta}(h^2_{1,34} + h^2_{2,34}),\end{aligned}$$

which forces $h_{1,34} = h_{2,34} = 0$ (since $\kappa_{\theta_3\theta} = \frac{\sin 2\theta_3}{\cos 2\theta_3 - \cos 2\theta} \neq 0$) and moreover

$$\begin{aligned}h_{1,33} = h_{2,34} = 0, &\quad h_{1,44} = -h_{2,34} + h_{3,24} = 0;\\ h_{2,33} = -h_{1,34} = 0, &\quad h_{2,44} = -h_{3,14} + h_{1,34} = 0.\end{aligned} \tag{3.47}$$

In conjunction with (3.29), (3.34), (3.37), (3.41) and (3.42), we have $B(p_0) = 0$. The arbitrariness of p_0 implies $B \equiv 0$, i.e. M is totally geodesic.

Case III. $\theta_1 = \theta_2 = \theta_3 = \pi/3$. Then $g^N = 1$, $g^T = 2$ and Theorem 2.1 implies M is affine linear.

Case IV. $\theta_2 = \theta_3 \in (\pi/3, \arccos(\sqrt{6}/6)) \cup (\arccos(\sqrt{6}/6), \pi/2)$ and $\theta_1 = \pi - 2\theta_2$. Denote $\theta := \theta_2$, then $\mathrm{Arg}^N = \{\theta, \theta_1\}$, $\mathrm{Arg}^T = \{0, \theta, \theta_1\}$; $T_{p_0,\theta}M = \mathrm{span}\{e_2, e_3\}$ and $N_{p_0,\theta}M = \mathrm{span}\{\nu_2, \nu_3\}$ with $\nu_\alpha = \Phi_\theta(e_\alpha)$ for each $2 \leq \alpha \leq 3$; $T_{p_0,\theta_1}M = \mathrm{span}\{e_1\}$ and $N_{p_0,\theta_1}M = \mathrm{span}\{\nu_1\}$ with $\nu_1 = \Phi_{\theta_1}(e_1)$; $T_{p_0,0}M = \mathrm{span}\{e_4\}$. Applying Lemma 2.1 and 2.2 gives

$$h_{2,33} = h_{3,22} = 0, \quad 0 = h_{2,31} + h_{3,21} = h_{2,34} + h_{3,24}. \tag{3.48}$$

Substituting the above equations into (3.30)-(3.33) yields

$$h_{1,23} = h_{2,31} = h_{3,12} = 0; \tag{3.49}$$

$$h_{1,22} = -h_{3,24} = h_{2,34} = h_{1,33}, \quad h_{1,44} = -2h_{2,34}; \tag{3.50}$$
$$h_{1,34} = 0, \quad h_{2,11} = h_{3,14} = -h_{2,44}; \tag{3.51}$$
$$h_{1,24} = 0, \quad h_{3,11} = -h_{2,14} = -h_{3,44}. \tag{3.52}$$

A straightforward calculation shows

$$\begin{aligned} R_{\theta\theta_1}(e_2, e_3, e_2, e_3) &= \langle B^{\theta_1}_{e_2e_2}, B^{\theta_1}_{e_3e_3}\rangle - \langle B^{\theta_1}_{e_2e_3}, B^{\theta_1}_{e_3e_2}\rangle \\ &= h_{1,22}h_{1,33} - h_{1,23}h_{1,32} = h^2_{2,34}, \end{aligned} \tag{3.53}$$

$$\begin{aligned} &U_{\theta\theta_1}(e_2, e_3, e_2, e_3) \\ =&\langle (A^{\Phi_\theta(e_2)}e_2)_{\theta_1}, A^{\Phi_\theta(e_3)}e_3)_{\theta_1}\rangle - \langle A^{\Phi_\theta(e_3)}e_2)_{\theta_1}, A^{\Phi_\theta(e_2)}e_3)_{\theta_1}\rangle \\ =&h_{2,21}h_{3,31} - h_{3,21}h_{2,31} = 0, \end{aligned} \tag{3.54}$$

$$\begin{aligned} &U_{\theta 0}(e_2, e_3, e_2, e_3) \\ =&\langle A^{\Phi_\theta(e_2)}e_2)_0, A^{\Phi_\theta(e_3)}e_3)_0\rangle - \langle A^{\Phi_\theta(e_3)}e_2)_0, A^{\Phi_\theta(e_2)}e_3)_0\rangle \\ =&h_{2,24}h_{3,34} - h_{3,24}h_{2,34} = h^2_{2,34}, \end{aligned} \tag{3.55}$$

and then Lemma 2.6 implies

$$\begin{aligned} 0 =& \sum_{\sigma\in \mathrm{Arg}^N, \sigma\neq 0} \kappa_{\theta\sigma}R_{\theta\sigma}(e_2, e_3, e_2, e_3) - 3\sum_{\sigma\in \mathrm{Arg}^T, \sigma\neq\theta} \kappa_{\theta\sigma}U_{\theta\sigma}(e_2, e_3, e_2, e_3) \\ =& \kappa_{\theta\theta_1}R_{\theta\theta_1}(e_2, e_3, e_2, e_3) - 3\kappa_{\theta\theta_1}U_{\theta\theta_1}(e_2, e_3, e_2, e_3) - 3\kappa_{\theta 0}U_{\theta 0}(e_2, e_3, e_2, e_3) \\ =& (\kappa_{\theta\theta_1} - 3\kappa_{\theta 0})h^2_{2,34}, \end{aligned} \tag{3.56}$$

where

$$\begin{aligned} \kappa_{\theta\theta_1} - 3\kappa_{\theta 0} =& \frac{\sin 2\theta}{\cos 2\theta - \cos 2\theta_1} - \frac{3\sin 2\theta}{\cos 2\theta - 1} \\ =&\frac{\sin 2\theta}{\cos 2\theta - \cos 4\theta} + \frac{3\sin 2\theta}{1 - \cos 2\theta} = \frac{\cos\theta(\sin\theta + 3\sin 3\theta)}{\sin 3\theta\sin\theta} \\ =&\frac{2\cos\theta(5 - 6\sin^2\theta)}{\sin 3\theta}. \end{aligned} \tag{3.57}$$

Since $\theta \neq \arccos(\sqrt{6}/6)$, $\sin^2\theta \neq 5/6$ and then $\kappa_{\theta\theta_1} - 3\kappa_{\theta 0} \neq 0$. Hence $h_{2,34} = 0$ and moreover

$$h_{1,22} = h_{1,33} = h_{1,44} = h_{3,24} = h_{2,34} = 0. \tag{3.58}$$

In conjunction with (3.29), (3.49), (3.51) and (3.52), we have $A^{\nu_1} = 0$. With the aid of Lemma 2.7, one can then proceed as in Case II to deduce that $B(p_0) = 0$. Since p_0 is arbitrary, M has to be affine linear. □

Proposition 3.5. *Let M be a coassociative submanifold of Im* $\mathbf{O}$. *Assume M has CJA relative to Im* $\mathbf{H}$, *and* $Arg^N = \{\arccos(\sqrt{6}/6), \arccos(2/3)\}$, *then either M is affine linear, or there exists $a_0 \in Sp_1$, such that M is a translate of an open subset of $M(a_0)$. Here $M(a_0)$ denotes the Lawson-Osserman cone, see (1.2).*

Proof. Let $\theta_1 := \arccos(2/3)$, $\theta_2 = \theta_3 := \arccos(\sqrt{6}/6)$ and $\theta := \theta_2$. For an arbitrary point $p_0 \in M$, let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal tangent frame field and

$\{\nu_1, \nu_2, \nu_3\}$ be an orthonormal normal frame field on U, a neighborhood of p_0, such that for any $p \in M$, $e_i(p)$ and $\nu_\alpha(p)$ satisfy the properties in Proposition 3.2. In particular, $\nu_\alpha = \Phi_\alpha(e_\alpha)$ for each $1 \leq \alpha \leq 3$. With the aid of Lemma 2.1, Lemma 2.2 and Proposition 3.3, one can proceed as above to get some pointwise relations between the coefficients of the second fundamental form, see (3.29), (3.48)-(3.52). Denote

$$h := h_{1,22}, \tag{3.59}$$

then h can be seen as a smooth function on U, and

$$h_{1,33} = h_{2,34} = h, \quad h_{3,24} = -h, \quad h_{1,44} = -2h. \tag{3.60}$$

Step I. Prove

$$h_{2,11} = h_{3,11} = h_{2,14} = h_{3,14} = h_{2,44} = h_{3,44} = 0. \tag{3.61}$$

By (2.23),

$$\begin{aligned}\langle(\nabla_{e_1}B)_{e_1e_1}, \Phi_{\theta_1}(e_1)\rangle =&(1/3)\kappa_{\theta_1\theta}\left(\langle B^\theta_{e_1e_1}, B^\theta_{e_1e_1}\rangle + 2|B^\theta_{e_1e_1}|^2\right)\\ &- 2\kappa_{\theta\theta_1}\langle B^\theta_{e_1e_1}, \Phi_\theta(A^{\Phi_{\theta_1}(e_1)}e_1)\rangle\\ =&\kappa_{\theta_1\theta}(h^2_{2,11} + h^2_{3,11}).\end{aligned} \tag{3.62}$$

Applying Lemma 2.7, we have

$$\begin{aligned}&\langle(\nabla_{e_1}B)_{e_4e_4}, \Phi_{\theta_1}(e_1)\rangle\\ =&-2\kappa_{\theta\theta_1}\langle B^\theta_{e_1e_4}, \Phi_\theta(A^{\Phi_{\theta_1}}e_4)_\theta\rangle\\ &+\kappa_{\theta_1\theta}|B^\theta_{e_1e_4}|^2 + \kappa_{\theta_1\theta}|(A^{\Phi_{\theta_1}(e_1)}e_4)_\theta|^2 + \kappa_{\theta_10}|(A^{\Phi_{\theta_1}(e_1)}e_4)_0|^2\\ =&-2\kappa_{\theta\theta_1}(h_{2,14}h_{1,42} + h_{3,14}h_{1,43})\\ &+\kappa_{\theta_1\theta}(h^2_{2,14} + h^2_{3,14}) + \kappa_{\theta_1\theta}(h^2_{1,42} + h^2_{1,43}) + \kappa_{\theta_10}h^2_{1,44}\\ =&\kappa_{\theta_1\theta}(h^2_{2,14} + h^2_{3,14}) + 4\kappa_{\theta_10}h^2,\end{aligned} \tag{3.63}$$

$$\begin{aligned}&\langle(\nabla_{e_1}B)_{e_2e_2}, \Phi_{\theta_1}(e_1)\rangle\\ =&-2\kappa_{\theta\theta_1}\langle B^\theta_{e_1e_2}, \Phi_\theta(A^{\Phi_{\theta_1}}e_2)_\theta\rangle\\ &+\kappa_{\theta_1\theta}|B^\theta_{e_1e_2}|^2 + \kappa_{\theta_1\theta}|(A^{\Phi_{\theta_1}(e_1)}e_2)_\theta|^2 + \kappa_{\theta_10}|(A^{\Phi_{\theta_1}(e_1)}e_2)_0|^2\\ =&-2\kappa_{\theta\theta_1}(h_{2,12}h_{1,22} + h_{3,12}h_{1,23})\\ &+\kappa_{\theta_1\theta}(h^2_{2,12} + h^2_{3,12}) + \kappa_{\theta_1\theta}(h^2_{1,22} + h^2_{1,23}) + \kappa_{\theta_10}h^2_{1,24}\\ =&\kappa_{\theta_1\theta}h^2_{1,22} = \kappa_{\theta_1\theta}h^2\end{aligned} \tag{3.64}$$

and similarly

$$\langle(\nabla_{e_1}B)_{e_3e_3}, \Phi_{\theta_1}(e_1)\rangle = \kappa_{\theta_1\theta}h^2_{1,33} = \kappa_{\theta_1\theta}h^2. \tag{3.65}$$

Adding (3.62)-(3.65) gives

$$\begin{aligned}0 = \langle\nabla_{e_1}H, \Phi_{\theta_1}(e_1)\rangle &= \sum_{i=1}^4\langle(\nabla_{e_1}B)_{e_ie_i}, \Phi_{\theta_1}(e_1)\rangle\\ &= \kappa_{\theta_1\theta}(h^2_{2,11} + h^2_{3,11} + h^2_{2,14} + h^2_{3,14}) + 2(2\kappa_{\theta_10} + \kappa_{\theta_1\theta})h^2,\end{aligned}$$

where

$$
\begin{aligned}
2\kappa_{\theta_1 0}+\kappa_{\theta_1\theta} &= \frac{2\sin 2\theta_1}{\cos 2\theta_1-1}+\frac{\sin 2\theta_1}{\cos 2\theta_1-\cos 2\theta}\\
=&\frac{2\sin 2\theta_1}{\cos 2\theta_1-1}+\frac{\sin 2\theta_1}{\cos 2\theta_1+\cos\theta_1}=\frac{\sin 2\theta_1\big[2(\cos 2\theta_1+\cos\theta_1)+\cos 2\theta_1-1\big]}{(\cos 2\theta_1-1)(\cos 2\theta_1+\cos\theta_1)}\\
=&\frac{2\sin 2\theta_1(3\cos\theta_1-2)(\cos\theta_1+1)}{(\cos 2\theta_1-1)(\cos 2\theta_1+\cos\theta_1)}=0.
\end{aligned}
\tag{3.66}
$$

Hence $h_{2,11}=h_{3,11}=h_{2,14}=h_{3,14}=0$ and substituting it into (3.51)-(3.52) implies $h_{2,44}=h_{3,44}=0$.

Step II. Calculation of the connection coefficients.

Denote

$$
\Gamma_{ij}^k:=\langle\nabla_{e_i}e_j,e_k\rangle,\quad \bar\Gamma_{i\alpha}^\beta:=\langle\nabla_{e_i}\nu_\alpha,\nu_\beta\rangle. \tag{3.67}
$$

Then differentiating both sides of $\langle e_i,e_k\rangle=\delta_{jk}$ with respect to e_i gives $\Gamma_{ij}^k+\Gamma_{ik}^j=0$. In particular, $\Gamma_{ij}^j=0$. Similarly $\bar\Gamma_{i\alpha}^\beta+\bar\Gamma_{i\beta}^\alpha=0$ and especially $\bar\Gamma_{i\alpha}^\alpha=0$.

Based on Lemma 2.4, a direct calculation shows

$$
\begin{aligned}
\Gamma_{i1}^4 &= (S_{\theta_1 0})_{e_1e_4}(e_i)=\kappa_{0\theta_1}\langle B_{e_ie_1},\Phi_0(e_4)\rangle-\kappa_{\theta_1 0}\langle B_{e_ie_4},\Phi_{\theta_1}(e_1)\rangle\\
&= -\kappa_{\theta_1 0}h_{1,i4},
\end{aligned}
\tag{3.68}
$$

$$
\begin{aligned}
\Gamma_{i1}^2 &= (S_{\theta_1\theta})_{e_1e_2}(e_i)=\kappa_{\theta\theta_1}\langle B_{e_ie_1},\Phi_\theta(e_2)\rangle-\kappa_{\theta_1\theta}\langle B_{e_ie_2},\Phi_{\theta_1}(e_1)\rangle\\
&= \kappa_{\theta\theta_1}h_{2,i1}-\kappa_{\theta_1\theta}h_{1,i2},
\end{aligned}
\tag{3.69}
$$

$$
\begin{aligned}
\Gamma_{i2}^4 &= (S_{\theta 0})_{e_2e_4}(e_i)=\kappa_{0\theta}\langle B_{e_ie_2},\Phi_0(e_4)\rangle-\kappa_{\theta 0}\langle B_{e_ie_4},\Phi_\theta(e_2)\rangle\\
&= -\kappa_{\theta 0}h_{2,i4}
\end{aligned}
\tag{3.70}
$$

and similarly

$$
\Gamma_{i1}^3=(S_{\theta_1\theta})_{e_1e_3}(e_i)=\kappa_{\theta\theta_1}h_{3,i1}-\kappa_{\theta_1\theta}h_{1,i3}, \tag{3.71}
$$

$$
\Gamma_{i3}^4=(S_{\theta 0})_{e_3e_4}(e_i)=-\kappa_{\theta 0}h_{3,i4}. \tag{3.72}
$$

By Lemma 2.5,

$$
\begin{aligned}
\bar\Gamma_{21}^3 &= (S^N_{\theta_1\theta})_{\nu_1\nu_3}(e_2)=\kappa_{\theta_1\theta}\langle B_{e_2,\Phi_{\theta_1}(\nu_1)},\nu_3\rangle-\kappa_{\theta\theta_1}\langle B_{e_2,\Phi_\theta(\nu_3)},\nu_1\rangle\\
&= \kappa_{\theta\theta_1}h_{1,23}-\kappa_{\theta_1\theta}h_{3,21}=0.
\end{aligned}
\tag{3.73}
$$

Step III. Proof that the angle lines with respect to θ_1, i.e. integral curves of the vector field e_1, must be straight lines in Euclidean space.

This is equivalent to $\overline{\nabla}_{e_1}e_1=0$ holding everywhere, which follows from the following straightforward calculation.

$$
\begin{aligned}
\overline{\nabla}_{e_1}e_1 &= B_{e_1e_1}+\nabla_{e_1}e_1=h_{\alpha,11}\nu_\alpha+\Gamma_{11}^ie_i=\sum_{i=2}^3\Gamma_{11}^ie_i+\Gamma_{11}^4e_4\\
&= \sum_{i=2}^3(\kappa_{\theta\theta_1}h_{i,11}-\kappa_{\theta_1\theta}h_{1,1i})e_i-\kappa_{\theta_1 0}h_{1,14}e_4=0.
\end{aligned}
$$

Step IV. Proof that there exists a hypersurface N of U, such that $p_0 \in N$ and $e_1(p) \perp T_pN$ for every $p \in N$.

By the Frobenius theorem, it suffices to prove that the subbundle $e_1^\perp$ of TU is integrable; more precisely, given arbitrary smooth sections X, Y of $e_1^\perp$, $[X, Y]$ takes values in $e_1^\perp$ as well.

Now we write $X = \sum_{i=2}^4 X^i e_i$ and $Y = \sum_{j=2}^4 Y^j e_j$, then

$$[X, Y] = X^i Y^j [e_i, e_j] + X^i (\nabla_{e_i} Y^j) e_j - Y^j (\nabla_{e_j} X^i) e_i$$

and hence

$$\langle [X, Y], e_1 \rangle = X^i Y^j \langle [e_i, e_j], e_1 \rangle.$$

Hence it is necessary and sufficient for us to show $\langle [e_i, e_j], e_1 \rangle = 0$ for any $2 \le i < j \le 4$.

Since ∇ is torsion-free,

$$\begin{aligned}\langle [e_2, e_3], e_1 \rangle =& \langle \nabla_{e_2} e_3, e_1 \rangle - \langle \nabla_{e_3} e_2, e_1 \rangle = \Gamma_{23}^1 - \Gamma_{32}^1 = -\Gamma_{21}^3 + \Gamma_{31}^2 \\ =& -(\kappa_{\theta\theta_1} h_{3,21} - \kappa_{\theta_1\theta} h_{1,23}) + (\kappa_{\theta\theta_1} h_{2,31} - \kappa_{\theta_1\theta} h_{1,32}) \\ =& 0,\end{aligned}$$

$$\begin{aligned}\langle [e_2, e_4], e_1 \rangle =& \langle \nabla_{e_2} e_4, e_1 \rangle - \langle \nabla_{e_4} e_2, e_1 \rangle = \Gamma_{24}^1 - \Gamma_{42}^1 = -\Gamma_{21}^4 + \Gamma_{41}^2 \\ =& \kappa_{\theta_1 0} h_{1,24} + (\kappa_{\theta\theta_1} h_{2,41} - \kappa_{\theta_1\theta} h_{1,42}) \\ =& 0\end{aligned}$$

and similarly

$$\langle [e_3, e_4], e_1 \rangle = \kappa_{\theta_1 0} h_{1,34} + (\kappa_{\theta\theta_1} h_{3,41} - \kappa_{\theta_1\theta} h_{1,43}) = 0.$$

Then the claim is proved.

Without loss of generality, we can assume that the closure of N is contained in U. Then there exists $\delta > 0$, such that $\mathbf{X}(p) + te_1 \in U$ for every $p \in N$ and any $t \in (-\delta, \delta)$, where $\mathbf{X}(p)$ denotes the position vector of p in Im $\mathbf{O}$. Define $\phi : N \times (-\delta, \delta) \to U$

$$(p, t) \mapsto \mathbf{X}(p) + te_1, \tag{3.74}$$

then ϕ is a diffeomorphism between $N \times (-\delta, \delta)$ and a neighborhood of p_0 in M, which is denoted by W.

Step V. The function h defined in (3.59) is constant on N.

Applying the Codazzi equations,

$$\begin{aligned}\nabla_{e_4} h &= \nabla_{e_4} h_{1,22} = \nabla_{e_4} \langle B_{e_2 e_2}, \nu_1 \rangle \\ &= \langle (\nabla_{e_4} B)_{e_2 e_2}, \nu_1 \rangle + 2\Gamma_{42}^i h_{1,2i} + \Gamma_{41}^\alpha h_{\alpha,22} \\ &= \langle (\nabla_{e_4} B)_{e_2 e_2}, \nu_1 \rangle = \langle (\nabla_{e_2} B)_{e_2 e_4}, \nu_1 \rangle \\ &= \nabla_{e_2} h_{1,24} - \Gamma_{22}^i h_{1,i4} - \Gamma_{24}^i h_{1,2i} - \bar{\Gamma}_{21}^\alpha h_{\alpha,24} \\ &= 2\Gamma_{22}^4 h - \Gamma_{24}^2 h + \bar{\Gamma}_{21}^3 h = 3\Gamma_{22}^4 h \\ &= -3\kappa_{\theta 0} h_{2,24} h = 0,\end{aligned}$$

$$\begin{aligned}
\nabla_{e_2} h &= \nabla_{e_2} h_{1,33} = \nabla_{e_2}\langle B_{e_3e_3}, \nu_1\rangle \\
&= \langle(\nabla_{e_2} B)_{e_3e_3}, \nu_1\rangle + 2\Gamma_{23}^i h_{1,3i} + \bar{\Gamma}_{21}^\alpha h_{\alpha,33} \\
&= \langle(\nabla_{e_2} B)_{e_3e_3}, \nu_1\rangle = \langle(\nabla_{e_3} B)_{e_2e_3}, \nu_1\rangle \\
&= \nabla_{e_3} h_{1,23} - \Gamma_{32}^i h_{1,i3} - \Gamma_{33}^i h_{1,2i} - \bar{\Gamma}_{31}^\alpha h_{\alpha,23} \\
&= -\Gamma_{32}^3 h - \Gamma_{33}^2 h = 0
\end{aligned}$$

and similarly

$$\nabla_{e_3} h = \nabla_{e_3} h_{1,22} = -\Gamma_{23}^2 h - \Gamma_{22}^3 h = 0.$$

Hence $\nabla h \equiv 0$ on N. Without loss of generality, we can assume $h|_N \equiv h_0$, with h_0 a nonnegative constant.

Step VI. W is a cone whenever $h_0 > 0$.

Define $\psi : N \to \mathrm{Im}\ \mathbf{O}$

$$\psi(p) = \mathbf{X}(p) + R_0 e_1(p),$$

where R_0 is a constant to be chosen. Then

$$\begin{aligned}
\psi_* e_i &= e_i + R_0 \overline{\nabla}_{e_i} e_1 \\
&= e_i + R_0 \big(B_{e_ie_1} + \nabla_{e_i} e_1\big) \\
&= e_i + R_0 \Gamma_{i1}^j e_j \\
&= e_i + R_0 \Big[\sum_{j=2}^3 (\kappa_{\theta\theta_1} h_{j,i1} - \kappa_{\theta_1\theta} h_{1,ij}) e_j - \kappa_{\theta_1 0} h_{1,i4} e_4\Big]
\end{aligned}$$

for each $2 \le i \le 4$. More precisely,

$$\begin{aligned}
\psi_* e_2 &= (1 - R_0 \kappa_{\theta_1\theta} h_{1,22}) e_2 = (1 - R_0 \kappa_{\theta_1\theta} h_0) e_2, \\
\psi_* e_3 &= (1 - R_0 \kappa_{\theta_1\theta} h_{1,33}) e_3 = (1 - R_0 \kappa_{\theta_1\theta} h_0) e_3, \\
\psi_* e_4 &= (1 - R_0 \kappa_{\theta_1 0} h_{1,44}) e_4 = (1 + 2R_0 \kappa_{\theta_1 0} h_0) e_4.
\end{aligned} \tag{3.75}$$

Now we put

$$R_0 := (\kappa_{\theta_1\theta} h_0)^{-1},$$

then combining (3.75) and (3.66) implies $\psi_* e_i = 0$ for each $2 \le i \le 4$. Hence ψ is a constant map on N. Without loss of generality, we can assume $\psi \equiv 0$, i.e. $F(p) = -R_0 e_1(p)$ for every $p \in N$. In other words, N lies in the Euclidean sphere centered at 0 and of radius R_0, and an arbitrary normal line of N, i.e. $\{F(p) + te_1 : t \in \mathbf{R}\}$ with $p \in N$, must go through the origin. Therefore W is a cone.

Step VII. M is an open subset of $M(a_0)$ provided that $h_0 > 0$ and $\psi \equiv 0$.

Let

$$S := \{x \in \mathrm{Im}\ \mathbf{O} : |x| = R_0, |\mathcal{P}_0^\perp x| = \cos\theta_1 R_0\} \tag{3.76}$$

be a submanifold of Im $\mathbf{O}$. For any $x \in S$, there exist a unit element $b \in \mathrm{Im}\ \mathbf{H}$ and a unit element $\varepsilon \in \mathbf{H}e$, such that

$$x = R_0(-\sin\theta_1 b + \cos\theta_1 b\varepsilon).$$

Define

$$E_x = \mathbf{R}\varepsilon \oplus \{\sin\theta c - \cos\theta c\varepsilon : c \in \mathrm{Im}\ \mathbf{H}, \langle b, c\rangle = 0\}, \tag{3.77}$$

then E_x is a 3-dimensional subspace of $T_x S$. Furthermore

$$E := \{E_x : x \in S\} \tag{3.78}$$

is a 3-dimensional distribution on S.

For any $p \in N$, $e_1(p)$ is a unit tangent angle direction associated to θ_1. Hence there exist $b \in \mathrm{Im}\ \mathbf{H}$ and $\varepsilon \in \mathbf{H}e$ satisfying $|b| = |\varepsilon| = 1$, such that

$$e_1(p) = \sin\theta_1 b - \cos\theta_1 b\varepsilon.$$

Moreover,

$$\begin{aligned}\mathbf{X}(p) &= \psi(p) - R_0 e_1(p) = -R_0 e_1(p)\\ &= R_0(-\sin\theta_1 b + \cos\theta_1 b\varepsilon).\end{aligned}$$

Therefore $N \subset S$.

Denote

$$\begin{aligned}\nu_1 :=& \Phi_{\theta_1}(e_1) = (-\tan\theta_1 \mathcal{P}_0^{\perp} + \cot\theta_1 \mathcal{P}_0)e_1\\ =& \cos\theta_1 b + \sin\theta_1 b\varepsilon,\end{aligned}$$

then ν_1 is a unit angle direction of $N_p M$ with respect to Im $\mathbf{H}$. On the other hand, Proposition 3.1 implies the existence of an orthonormal basis $\{b_1, b_2, b_3\}$ of Im $\mathbf{H}$ satisfying $b_3 = b_1 b_2$ and a unit element $\varepsilon' \in \mathbf{H}e$, such that

$$\nu'_\alpha := \cos\theta_\alpha b_\alpha + \sin\theta_\alpha b_\alpha \varepsilon' \qquad \forall 1 \le \alpha \le 3$$

are all unit angle directions of $N_p M$ relative to Im $\mathbf{H}$. Since $m_{\theta_1} = 1$, $\nu'_1 = \pm\nu_1$, and then one can assume $b_1 = b$, $\varepsilon' = \varepsilon$ without loss of generality, which implies

$$N_p M = \mathbf{R}\nu_1 \oplus \{\cos\theta c + \sin\theta c\varepsilon : c \in \mathrm{Im}\ H, \langle b, c\rangle = 0\}.$$

Noting that $T_p N \perp N_p M$ and $T_p N \perp e_1$, it is easy to deduce that $T_p N = E_p$, i.e. N is an integral manifold of E.

For any $a \in \mathrm{Sp}_1$, $M(a)$ is a coassociative cone, which has CJA with $\mathrm{Arg}^T = \{\theta_1, \theta, 0\}$, and each ray is an angle line with respect to θ_1 (see 1.2). As above, one can show that $M(a) \cap B(R_0) \subset S$ and that it is also an integral manifold of E.

Now we write

$$\mathbf{X}(p_0) = R_0(\sin\theta_1 b_0 + \cos\theta_1 c_0 e) = (2/3)R_0\big[(\sqrt{5}/2)b_0 + c_0 e\big] \tag{3.79}$$

with $b_0 \in \mathrm{Im}\ \mathbf{H}, c_0 \in \mathbf{H}$ satisfying $|b_0| = |c_0| = 1$. Then choosing

$$a_0 := c_0 b_0 \bar{c}_0, \quad q_0 := \bar{c}_0 \tag{3.80}$$

gives

$$\mathbf{X}(p_0) = (2/3)R_0\big[(\sqrt{5}/2)q_0 a_0 \bar{q}_0 + \bar{q}_0 e\big] \in M(a_0).$$

Therefore N and $M(a_0) \cap B(R_0)$ are both integral manifolds of E. Since $M(a_0) \cap B(R_0)$ is complete, applying the Frobenius theorem implies $N \subset M(a_0) \cap B(R_0)$, and hence

$W \subset M(a_0)$. Finally, because minimal submanfolds in Euclidean space are analytic manifolds, M has to be an open subset of $M(a_0)$.

Step VIII. M is affine linear whenever $h_0 = 0$.

First, $h_0 = 0$ implies $B \equiv 0$ on N. Denote by $\tilde{B}$ the second fundamental form of N in Im $\mathbf{O}$, then

$$\langle \tilde{B}_{e_i e_j}, \nu_\alpha \rangle = \langle \overline{\nabla}_{e_i} e_j, \nu_\alpha \rangle = \langle B_{e_i e_j}, \nu_\alpha \rangle = 0$$

for any $2 \le i, j \le 4$ and $1 \le \alpha \le 3$,

$$\begin{aligned} \langle \tilde{B}_{e_i e_j}, e_1 \rangle &= \langle \overline{\nabla}_{e_i} e_j, e_1 \rangle = \langle \nabla_{e_i} e_j, e_1 \rangle = \Gamma^1_{ij} \\ &= -\Gamma^j_{i1} = -(\kappa_{\theta\theta_1} h_{j,i1} - \kappa_{\theta_1\theta} h_{1,ij}) = 0 \end{aligned}$$

for any $2 \le j \le 3$ and

$$\langle \tilde{B}_{e_i e_4}, e_1 \rangle = \Gamma^1_{i4} = -\Gamma^4_{i1} = \kappa_{\theta_1 0} h_{1,i4} = 0.$$

Thus $\tilde{B} \equiv 0$, i.e. N is totally geodesic.

Since

$$\begin{aligned} \overline{\nabla}_{e_i} e_1 &= \sum_{j=2}^{4} \langle \nabla_{e_i} e_1, e_j \rangle e_j + B_{e_i e_1} \\ &= \sum_{j=2}^{4} \Gamma^j_{i1} e_j + B_{e_i e_1} = 0 \end{aligned}$$

for each $2 \le i \le 4$, e_1 is parallel along N. Therefore W is an open subset of an affine linear subspace of Im $\mathbf{O}$. Due to the analyticity of minimal submanifolds, M has to be affine linear. And the proof is completed. ∎

Proposition 3.4 and Proposition 3.5 together imply the following theorem.

THEOREM 3.1. *Let M be a coassociative submanifold in Im $\mathbf{O}$. Assume M has CJA relative to Im $\mathbf{H}$. If $g^N \le 2$ and $\pi/2 \notin Arg^N$, then either M is affine linear, or there exists $a_0 \in Sp_1$ and $w_0 \in$ Im $\mathbf{O}$, such that M is an open subset of*

$$M(a_0, w_0) := \{ r\big[(\sqrt{5}/2) q a_0 \bar{q} + \bar{q} e\big] + w_0 : q \in Sp_1, r \in \mathbf{R}^+ \}.$$

In other words, M is a translate of a portion of the Lawson-Osserman cone.

Theorem 1.2 in §1.3 is a direct corollary of Theorem 3.1.

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