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REPRESENTATION AND DERIVED REPRESENTATION RINGS OF NAKAYAMA TRUNCATED ALGEBRAS AND A VIEWPOINT UNDER MONOIDAL CATEGORIES*

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Abstract. The main aim of this study is to characterize representation rings and derived representation rings of a class of finite dimensional Hopf algebras constructed from the Nakayama truncated algebras KZ_n/J^d with certain constraints. For the representation ring $r(KZ_n/J^d)$, we completely determine its generators and the relations of generators via the method of the Pascal triangle. For the derived representation ring $dr(KZ_n/J^2)$(i.e., $d = 2$), we determine its generators and give the relations of generators. For these two aspects, the polynomial characterizations of the representation ring and the derived representation rings are both given.

Representation rings are well-known as Green rings from module categories over Hopf algebras. We have studied Green rings in the context of monoidal categories such that representations of Hopf algebras can be investigated through Green rings of various levels from module categories to derived categories from a unified viewpoint. Firstly, as analogues of representation rings of Hopf algebras, we set up so-called Green rings of monoidal categories, and then we list some such categories including module, complex, homotopy, derived and (derived) shift categories, and the relationship among their corresponding Green rings.

Key words. Representation ring, derived representation ring, shift ring, Nakayama truncated algebra, Pascal triangle, monoidal category.

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1. Introduction. Throughout this paper let K be an algebraically closed field and all modules left modules. The Grothendieck group $K_0(A)$ plays an important role for a $K-$algebra A. If A is not only an algebra, but also a Hopf algebra, we can define a ring structure on $K_0(A)$ using the comultiplication Δ, that is, the Grothendieck ring. However, the Grothendieck ring reflects only the relations among irreducible representations, but not all indecomposable representations. Therefore, in some studies (e.g. [6] [14]), representation rings of finite groups were introduced to study modular representation theory through the structure of representation rings such as semi-simplicity.

Let H be a K-Hopf algebra over K with comultiplication Δ and counit ε. It is well-known that ${}_H mod$ is a monoidal category with the tensor product $\otimes_K$. Usually, the representation ring of H, denoted by $r(H)$ (see [14]), is also called the Green ring, which we write as $gr({}_H mod)$. In this case, $r(H)$ is generated by all $[V]$ of indecomposable H-modules $V \in \text{ind}(H)$ with $[V] + [W] = [V \oplus W]$ and $[V][W] = [V \otimes_K W]$. Note that

(i) the module structure of $V \otimes_K W$ is obtained by $h(v \times w) = \sum_{(h)} (h'v \times h''w)$ for $h \in H, v \in V, w \in W$;

(ii) For the identity $[K]$, K is viewed as an H-module by $h \cdot k = \varepsilon(h)k$ for any $h \in H$, $k \in K$;

(iii) The distributive law of $r(H)$ is due to the bilinearity of $\otimes_K$;

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(iv) If H is cocommutative, then $r(H)$ is a commutative ring.

For a representation-finite K-Hopf algebra H, up to isomorphism, let $\mathcal{M} = \{M_1, \cdots, M_t\}$ be the set of all t indecomposable H-modules. In particular, say $M_1 = K$. The representation ring $r(H)$ can be realized as a quotient of a certain finitely generated free ring. That is, for the *free algebra* $K\langle \mathbf{X} \rangle$ on a set of indeterminates $\mathbf{X} = \{x_1, x_2, \cdots, x_t\}$, we can define an epimorphism of rings $\varphi : K\langle \mathbf{X} \rangle \to r(H)$ satisfying $\varphi(1) = [K]$, $\varphi(x_i) = [M_i]$ for $i = 1, \cdots, t$.

Then, for any $1 \leq i, j \leq t$, there are non-negative integers $k^l_{i,j} (i \leq l \leq t)$ appearing as the structure constants of the representation ring $r(H)$ satisfying

$$[M_i][M_j] = \sum_{1 \leq l \leq t} k^l_{i,j}[M_l]. \tag{1}$$

Define an ideal I of $K\langle \mathbf{X} \rangle$ generated by $\{x_i x_j - \sum_{1 \leq l \leq t} k^l_{i,j} x_l \mid 1 \leq i, j \leq t\}$, then the representation ring $r(H)$ can be realized as $r(H) \cong K\langle \mathbf{X} \rangle/(I, 1 - x_1)$.

Moreover, if H is cocommutative, then $r(H)$ becomes a commutative ring due to the definition of a representation ring. In this case, the free algebra $K\langle \mathbf{X} \rangle$ is replaced by the polynomial algebra $K[\mathbf{X}]$ through the relations $[M_i][M_j] = [M_j][M_i]$, that is, $r(H) \cong K[\mathbf{X}]/(I, 1 - x_1)$.

However, in general, using some relations among the t indecomposable modules of a representation-finite K-Hopf algebra H, it is possible to find a positive integer s with $s < t$ and s generators consisting of iso-classes of indecomposable modules to generate $r(H)$ with more relations. Then, we will be able to find $\mathbf{Y} = \{y_1, \cdots, y_s\}$ so as to obtain an epimorphism $K[Y] \to r(H)$. This will be described in Section 4 for some Nakayama truncated algebras $H = KZ_n/J^d$.

Most of this paper is devoted to characterizing representation rings and derived representation rings of a special class of finite dimensional Hopf algebras, i.e., the Nakayama truncated algebras KZ_n/J^d with $d = p^m \leq n$ over an algebraically closed field K of characteristic p. Nakayama algebras, also called generalized uniserial algebras, were introduced by Nakayama who characterized this kind of algebras as an artinian ring R in which each R-module is a direct sum of quasi-primitive modules (i.e. those factor modules of primitive one-sided ideals) (see [1],[11],[25]). The Nakayama truncated algebras, which were also studied in [2],[3], are always basic self-injective Nakayama algebras. Some well-known algebras can be realized as Nakayama truncated algebras, such as the Taft algebras and the generalized Taft algebras studied in [17],[26],[29]. Another reason why we consider this kind of Nakayama truncated algebras is that they have Hopf algebra structures, see Proposition 2.1.

The representation rings of certain Hopf algebras have been computed, such as finite dimensional semi-simple Hopf algebra, the enveloping algebra of a complex semi-simple Lie algebra, the polynomial Hopf algebra $K[x]$ and the Sweedler 4-dimensional Hopf algebra (see [9],[10],[12],[23],[31]). There have also been recent studies by Chen, Oystaeyen in [8] and Zhang for Taft algebras and by Li and Zhang for the generalized Taft algebras in [21].

Although all the algebraic structures of Taft algebras, generalized Taft algebras and Nakayama truncated algebras can be written as the form KZ_n/J^d for an oriented cycles Z_n in the cases of $d = n, d|n$ and $d = p^m \leq n$ respectively, their structures as coalgebras differ substantially.

The Nakayama truncated algebra KZ_n/J^d is of the finite representation type and becomes a cocommutative Hopf algebra in the case we discuss. We give the generators of its representation ring with cardinality less than the number of iso-classes of indecomposable modules and determine the relations of these generators. For clearer expression, we introduce the method of the Pascal triangle. Finally, we use linear recurrent sequence to obtain the polynomial characterization of the representation ring $r(KZ_n/J^d)$.

As an application of the polynomial characterization of representation rings, we give a preliminary discussion of the isomorphism problem on representation rings. The complexified representation ring $R(KZ_n/J^d)$ is defined from $r(KZ_n/J^d)$ and then using it, a sufficient condition for two such representation rings $R(KZ_{n_1}/J^{d_1})$ and $R(KZ_{n_2}/J^{d_2})$ with $n_1d_1 = n_2d_2$ to be isomorphic is given by that, in their polynomial characterizations, the ideals generated by the relations in the polynomial rings are both radical ideals.

In essence, representation rings are constructed by (indecomposable) objects from the module categories of Hopf algebras and then are used as a tool to reflect the properties of Hopf algebras. Hence, we can apply the same idea to more categories enjoying similarity, in which monoidal categories seem be the most natural. So, in this paper, we also introduce the concept of Green rings over monoidal categories which we regard as the de-categorification of monoidal categories. Conversely, monoidal categories can be thought of as the categorification of their Green rings.

The viewpoint of Green rings of monoidal categories is important, since under this we can unify a series of special monoidal categories via Green rings. Representation rings over module categories are most important. The others include those for a Hopf algebra H, the complex category $Ch(H)$, the homotopy category $K(H)$, the derived category $D^b(H)$ and their subcategories $Ch^{sh}(H)$ and $D^{sh}(H)$ with Green rings called complex rings, derived representation rings and (derived) shift rings, respectively. In this part, we also give the relations among the Green rings of these special monoidal categories.

In general, the Green rings mentioned above are non-commutative. However, when the Hopf algebras are cocommutative, even non-cocommutative for some special cases (e.g. (generalized) Taft algebras), Green rings are commutative. We can characterize these rings via free algebras. In particular, when the Hopf algebras are cocommutative, we can use polynomial algebras to characterize the Green rings. At the end of this section, we give the characterizations of shift rings and derived representation rings using free algebras and polynomial algebras.

Our other aim is to calculate the derived representation ring of the algebra KZ_n/J^d. In fact, the representation ring $r(KZ_n/J^d)$, which is characterized completely in Section 7, is a sub-ring of the derived ring $dr(KZ_n/J^d)$. Another sub-ring of $dr(KZ_n/J^d)$ is the shift ring $sh(KZ_n/J^d)$, which has a graded structure which is easy to describe (Proposition 5.1). However, the calculation of the structure coefficients of the derived representation ring $dr(KZ_n/J^d)$ is more difficult than that of $sh(KZ_n/J^d)$ and $r(KZ_n/J^d)$. Indeed, we have been able to investigate only the derived representation ring for the case $d = 2$, i.e. $H = KZ_n/J^2$, see Theorem 5.9 where the generators of $dr(KZ_n/J^2)$, the relations of generators and the polynomial characterization are given respectively in (i), (ii) and (iii). The remaining problem is that in Theorem 5.9, for the case $s > s' > 1$, the decomposition of the double complex $P^\bullet(j'+s'-1, j')\otimes P^\bullet(j+s-1, j)$ has three choices and we still cannot determinate actually which one is appropriate. For this problem, we conjecture that there can be only one choice of decomposition of the double complex $P^\bullet(j'+s'-1, j')\otimes P^\bullet(j+s-1, j)$ (see Conjecture 5.8).

This paper is organized as follows. In Section 2, we review some basic definitions of Nakayama truncated algebras KZ_n/J^d and list all the indecomposable modules, and then show the Hopf algebra structure of KZ_n/J^d according to some results from [1] and [13]. Moreover, we give an elementary lemma to simplify our proof in the process, and then for integers $0 \leq i, i', l \leq d-1$ and a sequence of integers $U = \{u_j\}_{0\leq j\leq l}$, we describe a combinatorial way to construct the indecomposable submodule $M(i, i', l, \{u_0, u_1, \cdots, u_l\})$ of the tensor product of indecomposable modules $M(i, \overline{0}) \otimes M(i', \overline{0})$ via the Pascal triangle. Furthermore, we describe the generators of the representation ring $r(KZ_n/J^d)$ of the Nakayama truncated algebras KZ_n/J^d. In Section 3, we show the relations between these generators and the representation ring $r(KZ_n/J^d)$ are characterized by the certain quotient ring of the corresponding polynomial ring. As an application of the polynomial characterization of representation rings, we then discuss the isomorphism problem of representation rings. In Section 4, we generalize our representation rings over module categories to the viewpoint of a Green ring from a monoidal category. In Section 5, we first give the structure of the shift ring $sh(KZ_n/J^d)$ and then calculate the derived representation ring $dr(KZ_n/J^2)$ through the generators and their relations, although the same discussion for general $dr(KZ_n/J^2)$ remains problematic.

2. The generators of the representation ring $r(KZ_n/J^d)$.

2.1. Hopf structure and indecomposable modules of Nakayama truncated algebra KZ_n/J^d. The Nakayama truncated algebras, or truncated cycle algebras, were first described by Bardzell *et al.* in [2]. Let Z_n be an oriented cycle of length n, which has n vertices $\{v_{\overline{0}}, \cdots, v_{\overline{n-1}}\}$ and n arrows $\{\alpha_{\overline{0}}, \cdots, \alpha_{\overline{n-1}}\}$ such that the origin vertex $s(\alpha_{\overline{i}})$ of the arrow $\alpha_{\overline{i}}$ equals the terminus vertex $t(\alpha_{\overline{i-1}})$ of $\alpha_{\overline{i-1}}$. Here, the subscript indices of vertices and arrows in Z_n are denoted by $\{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$, the set of elements of the residue class of the abelian group $\mathbb{Z}/n\mathbb{Z}$, since in the sequel we will correspond the set of the vertices of Z_n to the set of the elements of group $G = \mathbb{Z}/n\mathbb{Z}$, see (Proposition 2.1). Let J denote the two-sided ideal of the path algebra KZ_n generated by all arrows.

The *Nakayama truncated algebras* are defined as KZ_n/J^d for any positive integers d, which are called the *truncated quotients* of the path algebras KZ_n for any n.

For $A = KZ_n/J^d$, by Theorem V.3.5 in [1], for any indecomposable A-module M, there exists an indecomposable projective A-module P and an integer t with $1 \leq t \leq d$ such that $M \cong P/\text{rad}^t P$. Thus, there is a total of nd indecomposable A-modules

$$M(i, \overline{j}) = P_{\overline{j}}/\text{rad}^i P_{\overline{j}}, 1 \leq i \leq d$$

where $P_{\overline{j}}$ is the indecomposable projective A-module at the vertex $v_{\overline{j}}$. Denote by $S_{\overline{j}}$ the simple $A-$ module at $v_{\overline{j}}$ and notice that $M(1, \overline{j}) = S_{\overline{j}}$ and $M(d, \overline{j}) = P_{\overline{j}}$.

Green and Solberg(1998) showed in [1] that for a finite dimensional basic Hopf algebra H over K, there exists a finite group G and a *weight sequence* $W = (w_1, w_2, \cdots, w_r)$ of G (i.e., for each $g \in G$, the sequence W and $(gw_1g^{-1}, gw_2g^{-1}, \cdots, gw_rg^{-1})$ are the same up to a permutation), such that $H \cong K\Gamma_G(W)/I$ as Hopf algebras for an admissible ideal I, where the quiver $\Gamma_G(W)$ is defined as: the vertex set $\Gamma_G(W)_0 = \{v_g\}_{g\in G}$ and the arrow set $\Gamma_G(W)_1 = \{(a_i, g) = (v_{g^{-1}} \to v_{w_ig^{-1}}) | g \in G, w_i \in W\}$, which is simply the ordinary quiver of H. This quiver $\Gamma_G(W)$ is called the *covering quiver* of H with respect to the weight sequence W.

Now, we can give the condition for the Nakayama truncated algebra KZ_n/J^d to be a cocommutative Hopf algebra with the comultiplication Δ, the counit ε and the antipode S, which are defined as follows:

$$\Delta(v_h) = \sum_{g_1+g_2=h,\ g_1,g_2\in G} v_{g_1}\otimes v_{g_2}, \quad \varepsilon(v_h) = \begin{cases} 1, & h=0 \\ 0, & h\neq 0 \end{cases}, \quad S(v_h) = v_{-h} \tag{2}$$

$$\Delta(\alpha_h) = \sum_{g_1+g_2=h,\ g_1,g_2\in G} (\alpha_{g_1}\otimes v_{g_2} + v_{g_1}\otimes \alpha_{g_2}), \quad \varepsilon(\alpha_h)=0, \quad S(\alpha_h) = -\alpha_h \tag{3}$$

where $v_h \in (Z_n)_0$ the vertex set of Z_n and $\alpha_h \in (Z_n)_1$ the arrow set of Z_n.

Proposition 2.1. *For the Nakayama truncated algebra KZ_n/J^d over a field K of characteristic p, let $G = \mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$ be the residue class group modulo n with a weight sequence $W = \{\overline{1}\}$. Then,*

(i) *the covering quiver $\Gamma_G(W)$ is simply the n-th oriented cycle Z_n;*

(ii) *KZ_n/J^d is a cocommutative Hopf algebra with covering quiver Z_n and Hopf structure given in (2) and (3) if and only if $d = p^m \leq n$ for some $m > 0$.*

Proof. (i) It follows directly from the definition of $\Gamma_G(W)$.

(ii) **"Only if":** In the oriented cycle $\Gamma_G(W) = Z_n$, the vertex set $(Z_n)_0 = \{v_g\}_{g\in G}$ and the arrow set $(Z_n)_1 = \{\alpha_g = (v_g \to v_{\overline{1}+g}) \mid g \in G\}$. The conclusion follows from ([13], Lemma 5.3) and the fact that $\gcd(\binom{d}{1}, \binom{d}{2}, \cdots, \binom{d}{d-1}) = \begin{cases} p, & d \text{ is the power of prime } p; \\ 1, & \text{otherwise}, \end{cases}$ in [27].

"If": If $d = p^m \leq n$ holds for some $m > 0$, then J^d is a Hopf ideal in $K\Gamma_G(W)$ by the sufficient and necessary condition of Lemma 5.3 in [13], then KZ_n/J^d becomes a Hopf algebra due to the results of Corollary 5.4 in [13], whose cocommutativity follows directly from the definition of the comultiplication Δ. □

There are more results for criterion of KZ_n/J^d having Hopf structures which are various with that from (2) and (3), see [7] and [22]. One may discuss, under these Hopf structures, the same topics in this paper, e.g. derived representation rings in the sequel.

2.2. Elementary Lemma. First, we define some notations. Denote the basis of $M(i, \overline{j})$ as

$$\{v_{\overline{j}}, \alpha_{\overline{j}}, \alpha_{\overline{j+1}}\alpha_{\overline{j}}, \cdots, \alpha_{\overline{j+i-2}}\cdots\alpha_{\overline{j+1}}\alpha_{\overline{j}}\},$$

and abbreviate them as $v_{\overline{j}}^{\underline{0}} = v_{\overline{j}}$, $v_{\overline{j}}^{\underline{1}} = \alpha_{\overline{j}}, \cdots$, $v_{\overline{j}}^{\overline{i-1}} = \alpha_{\overline{j+i-2}}\cdots\alpha_{\overline{j+1}}\alpha_{\overline{j}}$, respectively. Fig. 1 illustrates the module $M(i, \overline{j})$.

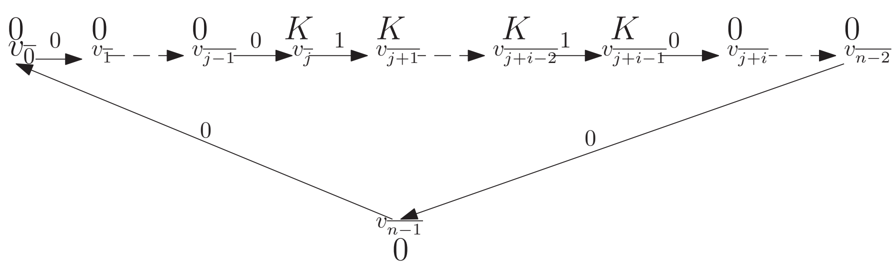

Fig. 1.

From now on, we assume that $d = p^m$ for some prime p, $m > 0$ and char$K = p$; dim,$\otimes$ and Hom stand for $\dim_K$, $\otimes_K$ and Hom_K, respectively. All modules stand for (KZ_n/J^d)-modules. For an A-module X, we denote $X_{\overline{l}}$ as its corresponding vector space at the vertex $v_{\overline{l}}$, $\mathbf{dim}X$ as its dimension vector and $(\mathbf{dim}X)_{\overline{l}}$ as the component of $\mathbf{dim}X$ at the vertex $v_{\overline{l}}$. The following two lemmas are easily proved from a straightforward verification.

Lemma 2.2. *For all* $1 \le i, i' \le d, 0 \le j, j' \le n-1$, $(\mathbf{dim}(M(i,\overline{j}) \otimes M(i',\overline{j'})))_{\overline{l}}$ *is equal to the number of partitions of* $\overline{l} = \overline{l_1} + \overline{l_2}$ *under the condition* $(\mathbf{dim}M(i,\overline{j}))_{\overline{l_1}} \ne 0$ *and* $(\mathbf{dim}M(i',\overline{j'}))_{\overline{l_2}} \ne 0$.

Proof. Choose the bases of $M(i,\overline{j})$ and $M(i',\overline{j'})$ as $\{v_{\overline{j}}^{\overline{0}}, v_{\overline{j}}^{\overline{1}}, \cdots, v_{\overline{j}}^{\overline{i-1}}\}$ and $\{v_{\overline{j'}}^{\overline{0}}, v_{\overline{j'}}^{\overline{1}}, \cdots, v_{\overline{j'}}^{\overline{i'-1}}\}$, respectively, then the basis of $M(i,\overline{j}) \otimes M(i',\overline{j'})$ is $\{v_{\overline{j}}^{\overline{k}} \otimes v_{\overline{j'}}^{\overline{k'}} | 0 \le k \le i-1, 0 \le k' \le i'-1\}$.

Additionally, $(\mathbf{dim}(M(i,\overline{j}) \otimes M(i',\overline{j'})))_{\overline{l}} = \dim(v_{\overline{l}} \cdot (M(i,\overline{j}) \otimes M(i',\overline{j'})))$, and

$$\begin{aligned} v_{\overline{l}} \cdot (v_{\overline{j}}^{\overline{k}} \otimes v_{\overline{j'}}^{\overline{k'}}) = \Delta(v_{\overline{l}})(v_{\overline{j}}^{\overline{k}} \otimes v_{\overline{j'}}^{\overline{k'}}) &= \sum_{g_1+g_2=\overline{l}, g_1,g_2\in G} (v_{g_1} \otimes v_{g_2})(v_{\overline{j}}^{\overline{k}} \otimes v_{\overline{j'}}^{\overline{k'}}) \\ &= \sum_{g_1+g_2=\overline{l}, g_1,g_2\in G} \delta_{g_1,\overline{j+k}}\delta_{g_2,\overline{j'+k'}}(v_{\overline{j}}^{\overline{k}} \otimes v_{\overline{j'}}^{\overline{k'}}) \\ &= \begin{cases} v_{\overline{j}}^{\overline{k}} \otimes v_{\overline{j'}}^{\overline{k'}}, & \text{if } g_1 = \overline{j+k}, g_2 = \overline{j'+k'} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker symbol. Thus, $(\mathbf{dim}M(i,\overline{j}) \otimes M(i',\overline{j'}))_{\overline{l}}$
$= \sharp\{$the partitions of $l = j + k + j' + k'$ such that $0 \le k \le i-1, 0 \le k' \le i'-1\}$.

Denote $l_1 = j + k$, $l_2 = j' + k'$. By the definition of $M(i,\overline{j})$, $(\mathbf{dim}M(i,\overline{j}))_{\overline{l_1}} \ne 0$ if and only if $j \le l_1 \le j + i - 1$, $(\mathbf{dim}M(i',\overline{j'}))_{\overline{l_2}} \ne 0$ if and only if $j' \le l_2 \le j' + i' - 1$. Then, we have $l = (j + k) + (j' + k') = l_1 + l_2$.

Thus, the number of the partitions of $l = j + k + j' + k'$ with $0 \le k \le i-1, 0 \le k' \le i'-1$ is equal to the number of the partitions of $l = l_1 + l_2$ with $j \le l_1 \le j + i - 1, j' \le k' \le j' + i' - 1$ and moreover, is equal to the number of partitions of $l = l_1 + l_2$ satisfying $(\mathbf{dim}M(i,\overline{j}))_{\overline{l_1}} \ne 0$ and $(\mathbf{dim}M(i',\overline{j'}))_{\overline{l}} \ne 0\}$. □

Following this result, in the case that $(\mathbf{dim}(M(i,\overline{j}) \otimes M(i',\overline{j'})))_{\overline{l}} \ne 0$, the basis of $(M(i,\overline{j}) \otimes M(i',\overline{j'}))_{\overline{l}}$ is that

$$\{v_{\overline{j}}^{\overline{l_1-j}} \otimes v_{\overline{j'}}^{\overline{l_2-j'}} \mid l_1 + l_2 = l, (\mathbf{dim}M(i,\overline{j}))_{\overline{l_1}} \ne 0, (\mathbf{dim}M(i',\overline{j'}))_{\overline{l_2}} \ne 0\}.$$

Lemma 2.3. *If* $(\mathbf{dim}M(i,\overline{j}))_{\overline{l_1}} \ne 0$, $(\mathbf{dim}M(i',\overline{j'}))_{\overline{l_2}} \ne 0$ *and* $l = l_1 + l_2$, *then in the module* $M(i,\overline{j}) \otimes M(i',\overline{j'})$, *it holds that*

$$\alpha_{\overline{l}} \cdot (v_{\overline{j}}^{\overline{l_1-j}} \otimes v_{\overline{j'}}^{\overline{l_2-j'}}) = \alpha_{\overline{l_1}}(v_{\overline{j}}^{\overline{l_1-j}}) \otimes v_{\overline{j'}}^{\overline{l_2-j'}} + v_{\overline{j}}^{\overline{l_1-j}} \otimes \alpha_{\overline{l_2}}(v_{\overline{j'}}^{\overline{l_2-j'}}).$$

Proof.

$$\begin{aligned}
&\alpha_{\overline{l}}\cdot(v_{\overline{j}}^{\overline{l_1-j}}\otimes v_{\overline{j'}}^{\overline{l_2-j'}})\\
&=\Delta(\alpha_{\overline{l}})(v_{\overline{j}}^{\overline{l_1-j}}\otimes v_{\overline{j'}}^{\overline{l_2-j'}})\\
&=\sum_{g_1+g_2=\overline{l},g_1,g_2\in G}(\alpha_{g_1}\otimes v_{g_2}+v_{g_1}\otimes\alpha_{g_2})(v_{\overline{j}}^{\overline{l_1-j}}\otimes v_{\overline{j'}}^{\overline{l_2-j'}})\\
&=\sum_{g_1+g_2=\overline{l},g_1,g_2\in G}\delta_{g_1,\overline{l_1}}\delta_{g_2,\overline{l_2}}(\alpha_{\overline{l_1}}(v_{\overline{j}}^{\overline{l_1-j}})\otimes v_{\overline{j'}}^{\overline{l_2-j'}}+v_{\overline{j}}^{\overline{l_1-j}}\otimes\alpha_{g_2}(v_{\overline{j'}}^{\overline{l_2-j'}}))\\
&=\alpha_{\overline{l_1}}(v_{\overline{j}}^{\overline{l_1-j}})\otimes v_{\overline{j'}}^{\overline{l_2-j'}}+v_{\overline{j}}^{\overline{l_1-j}}\otimes\alpha_{\overline{l_2}}(v_{\overline{j'}}^{\overline{l_2-j'}}).
\end{aligned}$$

□

The next lemma can express the module $M(i,\overline{j})$ as the tensor product of $M(i,\overline{0})$ and $M(1,\overline{1})$'s, which will be used to simplify calculations in the following discussion. We call this result the *elementary lemma*.

LEMMA 2.4 (Elementary lemma). *For all* $1\le i\le d$,

$$M(i,\overline{j})\otimes M(1,\overline{j'})\cong M(1,\overline{j'})\otimes M(i,\overline{j})\cong M(i,\overline{j+j'}).$$

Proof. By Lemma 2.2, we have $(\mathbf{dim}M(i,\overline{j})\otimes M(1,\overline{j'}))_{\overline{l}}=\begin{cases}1, & j+j'\le l\le j+j'+i-1\\ 0, & \text{otherwise}\end{cases}$ and for any $j+j'\le l\le j+j'+i-1$, the basis of $(M(i,\overline{j})\otimes M(1,\overline{j'}))_{\overline{l}}$ is $v_{\overline{j}}^{\overline{l-j-j'}}\otimes v_{\overline{j'}}^{\overline{0}}$.

It suffices to check the morphism coinciding with the morphism of $M(i,\overline{j+j'})$. By Lemma 2.3, for $j+j'\le l\le j+j'+i-1$, choose $l=l_1+l_2=(l-j')+j'$, then we have

$$\begin{aligned}
\alpha_{\overline{l}}\cdot(v_{\overline{j}}^{\overline{l-j'-j}}\otimes v_{\overline{j'}}^{\overline{0}})&=\alpha_{\overline{l-j'}}(v_{\overline{j}}^{\overline{l-j'-j}})\otimes v_{\overline{j'}}^{\overline{0}}+v_{\overline{j}}^{\overline{l-j'-j}}\otimes\alpha_{\overline{j'}}(v_{\overline{j'}}^{\overline{0}})\\
&=\alpha_{\overline{l-j'}}(v_{\overline{j}}^{\overline{l-j'-j}})\otimes v_{\overline{j'}}^{\overline{0}}\\
&=\begin{cases}v_{\overline{j}}^{\overline{l-j'+1-j}}\otimes v_{\overline{j'}}^{\overline{0}}, & j+j'\le l\le j+j'+i-2\\ 0, & \text{otherwise}\end{cases}
\end{aligned}$$

therefore $M(i,\overline{j})\otimes M(1,\overline{j'})\cong M(i,\overline{j+j'})$. The other one is similar. □

By the elementary lemma we can restrict our study to the form $M(i,\overline{0}),1\le i\le d$ and $M(1,\overline{1})$. Indeed, for any $1\le i,i'\le d,0\le j,j'\le n-1$, we have

$$M(i,\overline{j})\otimes M(i',\overline{j'})\cong M(i,\overline{0})\otimes M(i',\overline{0})\otimes M(1,\overline{1})^{\otimes(j+j')},\tag{4}$$

where $M(1,\overline{1})^{\otimes(j+j')}$ means $\underbrace{M(1,\overline{1})\otimes M(1,\overline{1})\otimes\cdots M(1,\overline{1})}_{(j+j')-\text{times}}$ for simplicity. In the sequel, the meaning of $M(i,\overline{j})^{\otimes k}$, in general, is the same.

Additionally, if $(M(i,\overline{0})\otimes M(i',\overline{0}))_{\overline{l}}\neq 0$, then the basis of $M(i,\overline{0})\otimes M(i',\overline{0})$ is

$$\{v_{\overline{0}}^{\overline{k}}\otimes v_{\overline{0}}^{\overline{k'}}|k+k'=l,0\le k\le i-1,0\le k'\le i'-1\}.$$

For simplicity, from now on, we abbreviate $v_{\overline{0}}^{\overline{k}}\otimes v_{\overline{0}}^{\overline{k'}}$ as $v^{\overline{k}}\otimes v^{\overline{k'}}$.

2.3. Indecomposable modules associated with the Pascal triangle . In this section, we give a combinatorial way to construct the indecomposable submodules M of $M(i,\overline{0}) \otimes M(i',\overline{0})$ via the Pascal triangle.

Step 1. First we introduce some notations. We label rows of the Pascal triangle as 0-th row, 1-st row, $\cdots$ from the top to the bottom, label the symmetry axis of the Pascal triangle as 0-th column; 1-st column, 2-nd column from this axis to the right; the (-1)-st column, (-2)-nd column from this axis to the left, and call the point in the i- row and j-th column the (i,j) position, (Fig. 2).

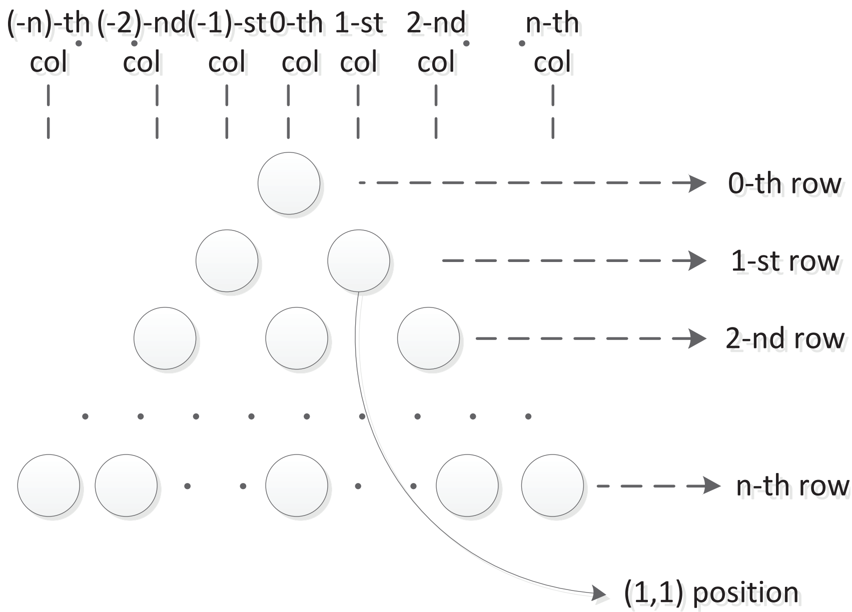

FIG. 2.

Step 2. Next, for the initial data $(i, i', l, \{u_j\}_{0\le j\le l})$, we complete the Pascal triangle by filling one number in each position. More precisely, for $0 \le i, i', l \le d-1$ and a sequence of integer coefficients $U = \{u_j\}_{0\le j\le l}$, we fill the Pascal triangle in the following way:

(i) For the entries in the (i,j) position with $0 \le i < l$, let $\diamondsuit(i,j) = 0$, where $\diamondsuit(i,j)$ denotes the number filling the (i,j) position.

(ii) For the entries in the (l,j) position, let $\diamondsuit(l,-l) = u_0, \diamondsuit(l,-l+2) = u_1, \cdots, \diamondsuit(l,l) = u_l$.

(iii) For the entries in the (i,j) position with $i > l$, finish the Pascal triangle via the Pascal triangle rules, that is, if $i+j = 0$, then let $\diamondsuit(i,j) = \diamondsuit(i-1,j+1)$. If $i = j$, then let $\diamondsuit(i,j) = \diamondsuit(i-1,j-1)$, otherwise let $\diamondsuit(i,j) = \diamondsuit(i-1,j-1)+\diamondsuit(i-1,j+1)$.

Step 3. Finally, we associate each vertex $v_{\overline{l+k}}$ a vector space $V_{\overline{l+k}}$ in the following way. It has basis $\diamondsuit(l+k,-l-k)\ v^{\overline{l+k}} \otimes v^{\overline{0}} + \cdots + \diamondsuit(l+k,l+k)\ v^{\overline{0}} \otimes v^{\overline{l+k}}$. Here are some remarks.

(i) We have $k \ge 0$ and $v_{\overline{n+s}} = v_{\overline{s}}, s \ge 0$.

(ii) If $v^{\overline{s}}$ is on the left side of the $\otimes$ symbol and $s \ge i$, then $v^{\overline{s}} = 0$.

(iii) If $v^{\overline{t}}$ is on the right side of the $\otimes$ symbol and $t \ge i'$, then $v^{\overline{t}} = 0$.

Now we associate a KZ_n/J^d-module with a given data $(i, i', l, \{u_j\}_{0\le j\le l})$ as follows.

Definition 2.5. Given initial data $(i, i', l, \{u_j\}_{0\le j\le l})$, define a KZ_n/J^d-module M as follows:

(i) $M = \bigoplus_{k\ge 0} V_{\overline{l+k}}$ as vector space;

(ii) For each vertex $v_{\overline{k}}$ and its associated vector space $V_{\overline{k}}$, the module action in M is defined as

$$\alpha_{\overline{k}} \cdot (v^{\overline{l_1}} \otimes v^{\overline{k-l_1}}) = \alpha_{\overline{l_1}}(v^{\overline{l_1}}) \otimes v^{\overline{k-l_1}} + v^{\overline{l_1}} \otimes \alpha_{\overline{k-l_1}}(v^{\overline{k-l_1}}),$$

which is just the simplified form of the result of Lemma 2.3.

From this definition, we denote this module by $M = M(i, i', l, \{u_j\}_{0\le j\le l})$.

The following proposition shows that M is indeed an indecomposable submodule of $M(i, \overline{0}) \otimes M(i', \overline{0})$.

Proposition 2.6. *For any $0 \le i, i', l \le d-1$ and a sequence of integer coefficients $U = \{u_j\}_{0\le j\le l}$. If $l \le \min\{i, i'\}$, then the module $M = M(i, i', l, \{u_j\}_{0\le j\le l})$ constructed above is an indecomposable submodule of $M(i, \overline{0}) \otimes M(i', \overline{0})$.*

Proof. Firstly, since $l \le \min\{i, i'\}$, $V_{\overline{l}}$ is a subspace of $(M(i, \overline{0}) \otimes M(i', \overline{0}))_{\overline{l}}$. Moreover, M inherits the module structure of $M(i, \overline{0}) \otimes M(i', \overline{0})$ and M is generated by $w_0 = u_0 v^{\overline{l}} \otimes v^{\overline{0}} + u_1 v^{\overline{l-1}} \otimes v^{\overline{1}} + \cdots + u_l v^{\overline{0}} \otimes v^{\overline{l}}$. Hence, M is a submodule of $M(i, \overline{0}) \otimes M(i', \overline{0})$.

Moreover, the indecomposability of M can be obtained as a consequence of the actions of the morphisms $\alpha_{\overline{l}}, \alpha_{\overline{l+1}}, \cdots$ on the vector spaces $V_{\overline{l}}, V_{\overline{l+1}}, \cdots$, according to the construction of $M = \bigoplus_{k\ge 0} V_{\overline{l+k}}$. In fact, by the definition of M, it is generated by the element w_0, which is a basis of $V_{\overline{l}}$. If M is decomposable, write $M = M' \oplus M''$, then $w_0 \in M'$ or $w_0 \in M''$. Assume $w_0 \in M'$. Thus $M' \cap V_{\overline{l}} \ne \{0\}$. But $\dim V_{\overline{l}} = 1$, we obtain $V_{\overline{l}} \subseteq (M')_{\overline{l}}$. So, $M = \bigoplus_{k\ge 0} V_{\overline{l+k}}$ is a submodule of M' since M is generated by w_0. This means that $M = M'$ and $M'' = 0$, which implies that M is indecomposable. □

By Proposition 2.6, the coefficient of $v^{\overline{i}} \otimes v^{\overline{j}}$ is stored in the $(i+j, j-i)$ position. Therefore from now on, we write this phenomenon as $c(i, j) = \diamondsuit(i+j, j-i)$ for short.

We end this section with the following example for constructing module $M = M(i, i', l, \{u_j\}_{0\le j\le l})$.

Example 2.7. Let $i = i' = 2, l = 0, u_0 = 1, n = 6, d = p = 5$, then $M(2, 2, 0, \{1\}) = \oplus_{k=0}^{5} V_{\overline{k}}$ is an indecomposable submodule of $M(2, \overline{0}) \otimes M(2, \overline{0})$ by Proposition 2.6. Through the Pascal triangle we have the following processes.

(i) $V_{\overline{0}} = \mathrm{span}\{v^{\overline{0}} \otimes v^{\overline{0}}\} \cong K$;

(ii) $V_{\overline{1}} = \mathrm{span}\{v^{\overline{1}} \otimes v^{\overline{0}} + v^{\overline{0}} \otimes v^{\overline{1}}\} \cong K$, since $\alpha_{\overline{0}} \cdot (v^{\overline{0}} \otimes v^{\overline{0}}) = v^{\overline{1}} \otimes v^{\overline{0}} + v^{\overline{0}} \otimes v^{\overline{1}}$;

(iii) $V_{\overline{2}} = \mathrm{span}\{2v^{\overline{1}} \otimes v^{\overline{1}}\} \cong K$, since $\alpha_{\overline{1}} \cdot (v^{\overline{1}} \otimes v^{\overline{0}} + v^{\overline{0}} \otimes v^{\overline{1}}) = v^{\overline{2}} \otimes v^{\overline{0}} + 2v^{\overline{1}} \otimes v^{\overline{1}} + v^{\overline{0}} \otimes v^{\overline{2}} = 2v^{\overline{1}} \otimes v^{\overline{1}}$ with $v^{\overline{2}} = \alpha_{\overline{1}}\alpha_{\overline{0}} = 0$ in $M(2, \overline{0})$;

(iv) $V_{\overline{3}} = V_{\overline{4}} = V_{\overline{5}} = 0$, since $\alpha_{\overline{2}} \cdot (2v^{\overline{1}} \otimes v^{\overline{1}}) = 2v^{\overline{2}} \otimes v^{\overline{1}} + 2v^{\overline{1}} \otimes v^{\overline{2}} = 0$.

Thus, $M(2, 2, 0, \{1\}) = \mathrm{span}\{v^{\overline{0}} \otimes v^{\overline{0}}, v^{\overline{1}} \otimes v^{\overline{0}} + v^{\overline{0}} \otimes v^{\overline{1}}, 2v^{\overline{1}} \otimes v^{\overline{1}}\}$.

In Fig. 3, the figure on the right expresses $M(2, 2, 0, \{1\})$, in which the coefficients occurring in the basis of $M(2, 2, 0, \{1\})$ are surrounded by the dotted line; the figure on the left expresses $M(3, \overline{0})$. The figure on the right can be induced from the figure on the left under the action of arrows; conversely, the figure on the left is directly obtained from the figure on the right. Hence, from Fig. 3, we have shown that $M(2,2,0,\{1\}) \cong M(3,\overline{0})$, which is an indecomposable submodule of $M(2,\overline{0}) \otimes M(2,\overline{0})$.

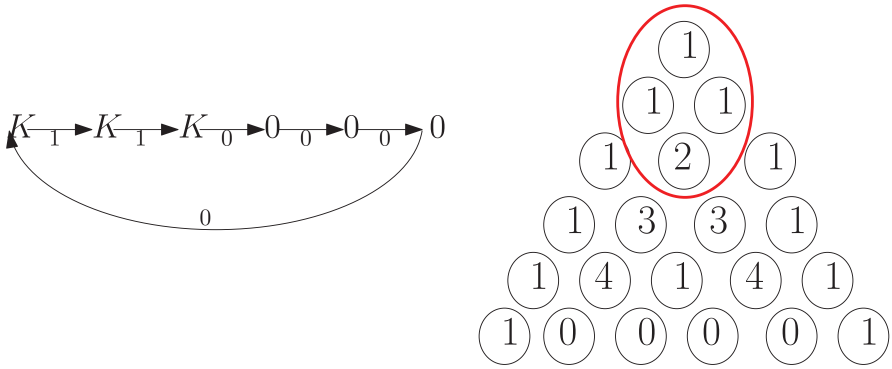

Fig. 3.

2.4. The generators of $r(KZ_n/J^d)$. This section is devoted to calculating the generators of representation ring $r(KZ_n/J^d)$.

First, we give the following Lemma which is similar to Lemma 3.8 given by Chen *et al.* in [8]. Here, we give another verification by the notation introduced in Section 5.

Lemma 2.8. (i) *For all* $1 \leq i \leq d, 0 \leq j \leq n-1$,

$$M(i,\overline{j}) \otimes M(1,\overline{0}) \cong M(1,\overline{0}) \otimes M(i,\overline{j}) \cong M(i,\overline{j}), \quad M(1,\overline{1})^{\otimes n} \cong M(1,\overline{0}).$$

(ii) $M(2,\overline{0}) \otimes M(t,\overline{0}) \cong M(t+1,\overline{0}) \oplus M(t-1,\overline{1})$ *for all* $t \geq 2, p \nmid t$.
(iii) $M(2,\overline{0}) \otimes M(t,\overline{0}) \cong M(t,\overline{0}) \oplus M(t,\overline{1})$ *for all* $t > 0, p|t$.

Proof. (i) follows directly from Lemma 2.4.

(ii) For $t \geq 2, p \nmid t$, by Proposition 2.6 we have two submodules $M_1 = M(2,t,0,\{1\})$ and $M_2 = M(2,t,1,\{1-t,1\})$ of $M(2,\overline{0}) \otimes M(t,\overline{0})$. The coefficients of the bases of $(M_1)_{\overline{l}}, (M_2)_{\overline{l}}$, $0 \leq l \leq n-1$ are given in the following table:

Rows in Pascal triangle	0	1	$\cdots$	l	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$n-1$
M_1	$(0,1)$	$(1,1)$	$\cdots$	$(l,1)$	$\cdots$	$(t-1,1)$	$(t,0)$	$(0,0)$	$\cdots$	$(0,0)$
M_2	$(0,0)$	$(1-t,1)$	$\cdots$	$(l-t,1)$	$\cdots$	$(-1,1)$	$(0,0)$	$(0,0)$	$\cdots$	$(0,0)$

where the coefficient pair (c_1,c_2) in the s-position means that $c_1 = c(1,s-1)$, $c_2 = c(0,s)$, which are stored in the upper-right two diagonals of the Pascal triangle.

Thus, for $p \nmid t$, we have $M_1 \cong M(t+1,\overline{0})$, $M_2 \cong M(t-1,\overline{1})$, dim $(M_1)_{\overline{l}} +$ dim $(M_2)_{\overline{l}} =$ dim $(M(2,\overline{0}) \otimes M(t,\overline{0}))_{\overline{l}}$ and $(M_1)_{\overline{l}} \cap (M_2)_{\overline{l}} = \{0\}$ for any $0 \leq l \leq n-1$. Therefore,

$$M(2,\overline{0}) \otimes M(t,\overline{0}) \cong M_1 \oplus M_2 \cong M(t+1,\overline{0}) \oplus M(t-1,\overline{1}).$$

(iii) Similarly with (ii), $M(t,\overline{0}) \cong M(2,t,0,\{1\})$, $M(t,\overline{1}) \cong M(2,t,1,\{1,0\})$. The basis of $M(t,\overline{0})_{\overline{l}}$ is $v^{\overline{0}} \otimes v^{\overline{l}} + l\, v^{\overline{1}} \otimes v^{\overline{l-1}}$ and the basis of $M(t,\overline{1})_{\overline{l}}$ is $v^{\overline{1}} \otimes v^{\overline{l-1}}$ for $1 \leq l \leq t$. □

To determine the generators of the Green ring $r(KZ_n/J^d)$, we need more observations.

Lemma 2.9. *For any* $1 \le u \le d$*, the isomorphism class* $[M(u,\overline{0})]$ *of the indecomposable module* $M(u,\overline{0})$ *can be expressed as a polynomial of* $[M(1,\overline{1})], [M(2,\overline{0})], [M(p^l+1,\overline{0})]$ *for* $1 \le l \le m-1$ *in* $r(KZ_n/J^d)$*.*

Proof. In the case that $u-1$ is a power of p, the conclusion follows at once.

Otherwise, in the case that $u-1$ is not a power of p, we prove this conclusion by induction on u.

When $u = 1$, the conclusion is trivial.

When $u > 1$ and $u-1$ is not a power of p, assuming that for any $1 \le v < u$, $M(v,\overline{0})$ can be expressed as a polynomial of $[M(1,\overline{1})], [M(2,\overline{0})], [M(p^l+1,\overline{0})], \forall 1 \le l \le m-1$ in $r(KZ_n/J^d)$. Because $\gcd(\binom{u-1}{1}, \binom{u-1}{2}, \cdots, \binom{u-1}{u-2}) = 1$ in [20], there exists $w, 1 \le w \le u-2$ such that $\binom{u-1}{w} \ne 0$ in K with $\mathrm{char}K = p$.

Now, we will prove the result on $M(u,\overline{0})$ by the induction hypotheses. For this purpose, consider the decomposition of $M(u-w,\overline{0}) \otimes M(w+1,\overline{0})$ into indecomposables.

Firstly, since $\binom{u-1}{w} \ne 0$ in K, $M(u-w, w+1, 0, \{1\})$ is a submodule of $M(u-w,\overline{0}) \otimes M(w+1,\overline{0})$, generated by $v^{\overline{0}} \otimes v^{\overline{0}}$ and the corresponding figure in the Pascal triangle is the rectangle with points at $(0,0)$, $(u-w-1,0)$, $(0,w)$, $(u-w-1,w)$ positions. Then, it is clear that $M(u,\overline{0}) \cong M(u-w, w+1, 0, \{1\})$.

Given the decomposition of $M(u-w,\overline{0}) \otimes M(w+1,\overline{0})$ into the direct sum of indecomposables as $M(u-w,\overline{0}) \otimes M(w+1,\overline{0}) = M_1 \oplus M_2 \cdots \oplus M_t$, since $\dim\,(M(u-w,\overline{0}) \otimes M(w+1,\overline{0}))_{\overline{0}} = 1$, there exists a unique s with $1 \le s \le t$ such that $\dim\,(M_s)_{\overline{0}} = 1$. Without loss of generality, say $s = 1$, then it follows that $v^{\overline{0}} \otimes v^{\overline{0}} \in M_1$. Then, $M(u,\overline{0})$ becomes a submodule of M_1 since $M(u,\overline{0})$ is generated by $v^{\overline{0}} \otimes v^{\overline{0}}$.

Because M_1 is a submodule of $M(u-w,\overline{0}) \otimes M(w+1,\overline{0})$, we have $\dim\,(M_1)_{\overline{l}} = 0$ for $u \le l \le n-1$. Thus, $\dim M_1 \le (u-1) - 0 + 1 = u = \dim M(u,\overline{0})$.

Therefore, $M_1 = M(u,\overline{0})$ and then we have $M(u-w,\overline{0}) \otimes M(w+1,\overline{0}) = M(u,\overline{0}) \oplus M_2 \cdots \oplus M_t$. From this, in $r(KZ_n/J^d)$, we find that

$$[M(u,\overline{0})] = [M(u-w,\overline{0})][M(w+1,\overline{0})] - [M_2] - \cdots - [M_t]. \tag{5}$$

For any $2 \le s \le t$, there are $1 \le i_s, j_s \le u-2$ such that $M_s \cong M(i_s, \overline{j_s})$ and by the elementary lemma, $M(i_s, \overline{j_s}) = M(i_s,\overline{0}) \otimes M(1,\overline{1})^{\otimes j_s}$. Thus $[M_s] = [M(i_s,\overline{j_s})] = [M(i_s,\overline{0})][M(1,\overline{1})]^{j_s}$.

Hence, from the induction hypotheses, all of $[M(u-w,\overline{0})], [M(w+1,\overline{0})], [M_2], \cdots, [M_t]$ can be expressed as a polynomial of $[M(1,\overline{1})], [M(2,\overline{0})], [M(p^l+1,\overline{0})], \forall 1 \le l \le m-1$ in $r(KZ_n/J^d)$. By (5), so can $[M(u,\overline{0})]$. □

By Lemma 2.4, we have $[M(u,\overline{w})] = [M(u,\overline{0})][M(1,\overline{1})]^w$ for $1 \le u \le d, 0 \le w \le n-1$. By this fact and Lemma 2.9, we obtain the following.

Theorem 2.10. *Let* $\mathrm{char}K = p$ *and* $n \ge d = p^m$*. Then, the representation ring* $r(KZ_n/J^d)$ *of the Nakayama truncated algebra* KZ_n/J^d *is generated by* $[M(1,\overline{1})], [M(2,\overline{0})]$ *and* $[M(p^l+1,\overline{0})], 1 \le l \le m-1$*.*

The following corollary is more precise.

Corollary 2.11. *For any* $1 \le u \le p^l, 1 \le l \le m, 0 \le r \le n-1$*, then the element* $[M(u,\overline{r})]$ *in* $r(KZ_n/J^d)$ *can be expressed as a polynomial of*

$$[M(1,\overline{1})],\ [M(2,\overline{0})],\ [M(p+1,\overline{0})],\ [M(p^2+1,\overline{0})],\ \cdots,\ [M(p^{l-1}+1,\overline{0})].$$

Proof. We may use the induction on l. When $l = 1$, by Lemma 2.4, we have

$$[M(i,\overline{j})] = [M(i,\overline{0})][M(1,\overline{j})] = [M(i,\overline{0})][M(1,\overline{1})]^j$$

and by Lemma 2.8, recursively, we have

$$[M(2,\overline{0})][M(2,\overline{0})] = [M(3,\overline{0})] + [M(1,\overline{1})],$$
$$[M(2,\overline{0})][M(3,\overline{0})] = [M(4,\overline{0})] + [M(2,\overline{1})], \cdots,$$
$$[M(2,\overline{0})][M(p-1,\overline{0})] = [M(p,\overline{0})] + [M(p-2,\overline{1})].$$

It follows that for any $1 \le u \le p, 0 \le r \le n-1$, $[M(u,\overline{r})]$ can be expressed as a polynomial of $[M(1,\overline{1})], [M(2,\overline{0})]$.

Assume the result is true for $l-1$. Now consider it for l. Let $1 \le u \le p^l, 0 \le r \le n-1$.

If $1 \le u \le p^{l-1}$, then the result follows by the induction hypothesis. So, we need only to consider the case $p^{l-1}+1 \le u \le p^l$ by using induction again on u.

Firstly, if $u = p^{l-1}+1$, then the result is trivial since we have $[M(u,\overline{r})] = [M(p^{l-1}+1,\overline{0})][M(1,\overline{1})]^r$ by Lemma 2.4. Now, suppose the result holds for all $u' < u$ and then consider $[M(u,\overline{0})]$.

In this case $p^{l-1}+2 \le u \le p^l$, $u-1$ is not a power of p. According to the proof of Lemma 2.9, there exists $1 \le w \le u-2$ such that $\binom{u-1}{w} \ne 0$ in K and the following holds:

$$[M(u,\overline{0})] = [M(u-w,\overline{0})][M(w+1,\overline{0})] - [M_2] - \cdots - [M_t]. \tag{6}$$

where for any $2 \le s \le t$, $M_s \cong M(i_s,\overline{j_s})$ for some i_s, j_s with $1 \le i_s, j_s \le u-2$.

Finally, the result follows by the induction hypotheses applied to the terms $[M(u-w,\overline{0})], [M(w+1,\overline{0})], [M_2], \cdots, [M_t]$. □

From this result, there is a unique ring epimorphism $\phi : \mathcal{Z}[y,z,w_1,\cdots,w_{m-1}] \longrightarrow r(KZ_n/J^d)$ with $\phi(1) = [M(1,\overline{0})], \phi(y) = [M(1,\overline{1})], \phi(z) = [M(2,\overline{0})]$ and $\phi(w_l) = [M(p^l+1,\overline{0})]$, for $1 \le l \le m-1$.

In the next section, we start to show the relations among the generators $[M(1,\overline{1})], [M(2,\overline{0})], [M(p^l+1,\overline{0})], \forall 1 \le l \le m-1$ of the representation ring $r(KZ_n/J^d)$ and their corresponding pre-images $y, z, w_1, \cdots, w_{m-1}$.

3. Polynomial characterization of $r(KZ_n/J^d)$ and its application.

3.1. Polynomial characterization . This section is devoted to a discussion of the relations among the generators of the representation ring $r(KZn/J^d)$. We will need the following facts.

LEMMA 3.1. *Let R be a unital ring and N_1, N_2 be two unital R-modules. If $N_1 \oplus N_2$ has a simple submodule S, then either N_1 or N_2 has a simple submodule isomorphic to S.*

Proof. Since S is a simple module, then $S = Rb$ for some $b \ne 0$ in $N_1 \oplus N_2$. Let $b = b_1 + b_2$ for $b_1 \in N_1$, $b_2 \in N_2$. Without loss of generality, say $b_1 \ne 0$. Then, the annihilator of b, that is, $\mathrm{Ann}(b) = \{r \in R \mid rb = 0\}$ is a maximal left ideal of R, since $R/\mathrm{Ann}(b) \cong Rb$ is simple as R-module.

However, $\mathrm{Ann}(b_1)$ is a proper left ideal of R and $\mathrm{Ann}(b_1) \supseteq \mathrm{Ann}(b)$. Therefore, $\mathrm{Ann}(b_1) = \mathrm{Ann}(b)$ due to the maximum of $\mathrm{Ann}(b)$. Then, as a submodule of N_1, we have $Rb_1 \cong R/\mathrm{Ann}(b_1) = R/\mathrm{Ann}(b) \cong S$. □

LEMMA 3.2. $M(p^l+1,\overline{0}) \otimes M(kp^l+1,\overline{0}) \cong \begin{cases} W_1, & \text{if } k \equiv -1 (\mathrm{mod}\ p) \\ W_2, & \text{if } k \equiv 0 (\mathrm{mod}\ p) \\ W_3, & \text{otherwise} \end{cases}$ *where*

$W_1 = M(kp^l+p^l,\overline{0}) \oplus M(kp^l+p^l,\overline{1}) \oplus (\oplus_{j=2}^{p^l-1} M(kp^l,\overline{j})) \oplus M(kp^l-p^l+1,\overline{p^l})$,

$W_2 = M(kp^l+p^l+1,\overline{0}) \oplus (\oplus_{j=1}^{p^l} M(kp^l,\overline{j}))$,

$W_3 = M(kp^l+p^l+1,\overline{0}) \oplus M(kp^l+p^l-1,\overline{1}) \oplus (\oplus_{j=2}^{p^l-1} M(kp^l,\overline{j})) \oplus M(kp^l-p^l+1,\overline{p^l})$.

Proof. Here, we give the precise proof only for the W_3 case with $l=1, p>2$. For the remaining cases, the proofs are similar.

Firstly, we construct a sequence of submodules $\{M_i\}_{0\le i\le p}$ of $M(p+1,\overline{0}) \otimes M(kp+1,\overline{0})$ via the Pascal triangle, and then prove that

$$M_0 \cong M(kp+p+1,\overline{0}), M_1 \cong M(kp+p-1,\overline{1}), M_p \cong M(kp-p+1,\overline{p}), \quad (7)$$

$$M_2 \cong M(kp,\overline{2}), M_3 \cong M(kp,\overline{3}), \cdots, M_{p-1} \cong M(kp,\overline{p-1}). \quad (8)$$

Lastly, we show that $M(p+1,\overline{0}) \otimes M(kp+1,\overline{0}) \cong \bigoplus_{i=0}^{p} M_i$.

Step 1. We define the submodules $\{M_i\}_{0\le i\le p}$ as follows:

$$M_l = \begin{cases} M(p+1,kp+1,l,\{0,\cdots,0,1,0,\cdots,0\}), & \text{if } l \text{ is even;} \\ M(p+1,kp+1,l,\{0,\cdots,0,1,-k,0,\cdots,0\}), & \text{if } l \text{ is odd} \end{cases}$$

where the numbers of 0 on the left-side and the right-side of 1 are the same, as are those on the left-side and the right-side of $\{1,-k\}$.

Step 2. Claim that $M_0 \cong M(kp+p+1,\overline{0})$ and for even $l=2,4,\cdots,p-1$, $M_l \cong M(kp,\overline{l})$. For odd l, the proof is similar.

Indeed, the Pascal triangle associated with M_0 is given on the left of Fig. 4, where the coefficients are surrounded by a red dotted rectangle.

For example, since $\binom{p}{0} = \binom{p}{p} = 1$ and $\binom{p}{i} = 0$ in K for $1 \le i \le p-1$, we see that the p-row of the Pascal triangle associated with M_0 is $\{1,0,\cdots,0,1\}$. By the definition of $M(kp+p+1,\overline{0})$, it follows that $M_0 \cong M(kp+p+1,\overline{0})$ at once.

Furthermore, operating M_0's Pascal triangle by going down 1 unit, we obtain the Pascal triangle associated with M_2, shown on the right-side of Fig. 4, where the coefficients are again surrounded by a red dotted rectangle. It is clear that $M_2 \cong M(kp,2)$ by the definition of $M(kp,\overline{2})$.

Continuing this process, through applying M_0's Pascal triangle by going down $2,3,\cdots,\frac{p-1}{2}$ units, it follows that $M_l \cong M(kp,\overline{l})$ for even $l=4,\cdots,p-1$.

Step 3. It remains to be shown that $M(p+1,\overline{0}) \otimes M(kp+1,\overline{0}) \cong \bigoplus_{l=0}^{p} M_l$.

By Proposition 2.6, all M_l, as well as $\sum_{l=0}^{p} M_l$, are submodules of $M(p+1,\overline{0}) \otimes M(kp+1,\overline{0})$. Next, we claim by induction that $\sum_{l=0}^{p} M_l$ is indeed a direct sum.

For any i with $0 \le i \le n-1$, we denote by $\beta_{l,i}$ the basis of the vector space $\{(M_l)_{\overline{i}}\}$ when $(M_l)_{\overline{i}} \ne 0$ and denote by P_l the Pascal triangle corresponding to the module M_l.

It is easy to see that $T_0 = S_{\overline{kp+p+1}}$, $T_1 = S_{\overline{kp+p-1}}$, $T_i = S_{\overline{kp+i-1}}, \forall 2 \le i \le p-1$, $T_p = S_{\overline{kp}}$ is respectively the unique simple submodule of M_0, M_1, $M_i, \forall 2 \le i \le p-1$, M_p.

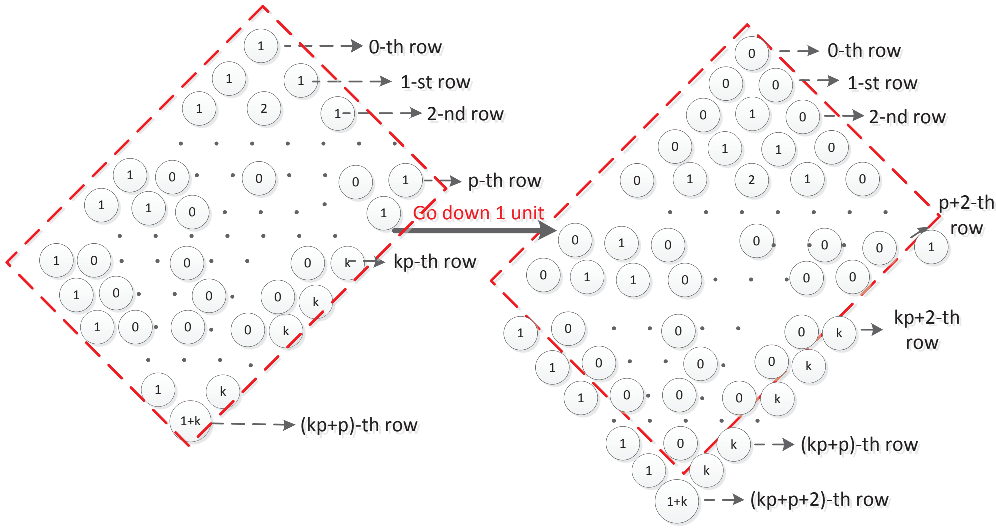

FIG. 4.

When $p=1$, we have $M_0 \cap M_1 = \{0\}$ since $T_0 \ncong T_1$. Hence $M_0 + M_1$ is a direct sum.

Assume that for $p-1$ the result holds, that is, $\sum_{j=0}^{p-1} M_{l_j}$ is a direct sum for any subset $\{l_0, l_1, \cdots, l_{p-1}\}$ of $\{0, 1, \cdots, p\}$. Now consider the case of p for the sum $\sum_{l=0}^{p} M_l$.

By Lemma 3.1 and the induction assumption, the direct sum $\sum_{j=0}^{p-1} M_{l_j} = \bigoplus_{j=0}^{p-1} M_{l_j}$ has all p simple submodules $T_{l_0}, \cdots, T_{l_{p-1}}$, all of which are not isomorphic to T_{l_p}, the unique simple submodule of M_{l_p} for l_p. Thus, $\bigoplus_{j=0}^{p-1} M_{l_j}$ has no any simple submodule isomorphic to T_{l_p}. It follows that for any $l_p \in \{0, 1, \cdots, p\}$,

$$(\bigoplus_{j=0}^{p-1} M_{l_j}) \cap M_{l_p} = \{0\}.$$

Therefore, $\sum\limits_{l=0}^{p} M_l = \sum\limits_{j=0}^{p} M_{l_j} = \bigoplus_{j=0}^{p} M_{l_j}$ is a direct sum.

Finally, let us show that the dimension vectors of $M(p+1, \overline{0}) \otimes M(kp+1, \overline{0})$ and $\bigoplus_{l=0}^{p} M_l$ are the same. It follows from the equation below, that for each i with $0 \le i \le p$,

$$\begin{aligned}\dim\ (M(p+1,\overline{0}) \otimes M(kp+1,\overline{0}))_{\overline{i}} &= \begin{cases} i+1, & \text{if } 0 \le i \le p-1; \\ p+1, & \text{if } p \le i \le kp; \\ kp+p+1-i, & \text{if } kp+1 \le i \le kp+p; \\ 0, & \text{otherwise.} \end{cases} \\ &= \sum_{l=0}^{p} \dim\ (M_l)_{\overline{i}}. \end{aligned} \tag{9}$$

In fact, the basis of $M(p+1, \overline{0}) \otimes M(kp+1, \overline{0})$ is $\{v^{\overline{s}} \otimes v^{\overline{t}} | 0 \le s \le p, 0 \le t \le kp\}$ and for each case on i, $\dim\ (M(p+1, \overline{0}) \otimes M(kp+1, \overline{0}))_{\overline{i}}$ can be easily calculated to obtain the value in (9) by partition of the rectangle $(0,0),(0,p),(kp,0),(kp,p)$ in the x-y-coordinate system via the diagonals $x+y=l$. On the other hand, dim $(\sum_{l=0}^{p} M_l)_{\bar{i}} = \sum_{l=0}^{p} \dim(M_l)_{\bar{i}}$, which is equal to the middle value in (9) for each case on i according to the isomorphisms

$$M_0 \cong M(kp+p+1,\overline{0}),\quad M_1 \cong M(kp+p-1,\overline{1}),\quad M_p \cong M(kp-p+1,\overline{p})$$

$$M_2 \cong M(kp,\overline{2}),\quad M_3 \cong M(kp,\overline{3}),\quad \cdots,\quad M_{p-1} \cong M(kp,\overline{p-1}).$$

□

Now, we can realize the representation ring $r(KZ_n/J^d)$ as the quotient of a polynomial ring by using Lemma 3.2. In preparation, we first give the formula of a linear recurrent sequence. Given two initial values a_1, a_2 and the recursion relation

$$a_n = za_{n-1} - ya_{n-2},\ n \geq 2$$

with fixed y, z satisfying $z^2 - 4y \neq 0$, by the well-known result on linear recurrence sequences, we have

$$a_n = \frac{2a_2 - (z-\sqrt{\Delta})a_1}{2\sqrt{\Delta}}\left(\frac{z+\sqrt{\Delta}}{2}\right)^{n-1} + \frac{(z+\sqrt{\Delta})a_1 - 2a_2}{2\sqrt{\Delta}}\left(\frac{z-\sqrt{\Delta}}{2}\right)^{n-1}. \tag{10}$$

and further write the right side of (10) as a polynomial g of a_1, a_2, denoted a_n by $g(n, a_1, a_2)$.

For the following discussion, we first define the order

$$y < z < w_1 < w_2 < \cdots < w_{m-1} \tag{11}$$

in the polynomial ring $\mathbb{Z}[y, z, w_1, \cdots, w_{m-1}]$ and then consider the order of the monomials according to the dictionary order.

For a monomial $az_1^{i_1}\cdots z_n^{i_n}$ of a polynomial $f(z_1,\cdots,z_n)$, denote by $I(ax_1^{i_1}\cdots x_n^{i_n}) = (i_1,\cdots,i_n)$, which is called the index series of the monomial $az_1^{i_1}\cdots z_n^{i_n}$. Arranging its monomials in descending order, the leading term of the polynomial f is the largest monomial, denoted by $lt(f)$. Meanwhile, we denote $I(f)$ as the index series of the leading term $lt(f)$ of the polynomial f under the descending order, i.e. $I(f) = I(lt(f))$.

Now we are in the position of determining the relations among the generators $[M(1,\overline{1})], [M(2,\overline{0})]$ and $[M(p^l+1,\overline{0})] (\forall 1 \leq l \leq m-1)$ of the Nakayama truncated algebra KZ_n/J^d.

Step 1. Determine the relations among $[M(1,\overline{1})]$ and $[M(2,\overline{0})]$.

By Lemma 2.8 (i), the relation $[M(1,\overline{1})]^n = [M(1,\overline{0})]$ holds, which corresponds to $y^n - 1 = 0$.

By Lemma 2.8 (iii), the relation $([M(2,\overline{0})] - [M(1,\overline{1})] - [M(1,\overline{0})])[M(p,\overline{0})] = 0$ holds, which corresponds to $(z-y-1)g(p,1,z) = 0$.

Now define

$$g_0(y,z,w_1,\cdots,w_{m-1}) = y^n - 1, \qquad g_1(y,z,w_1,\cdots,w_{m-1}) = (z-y-1)g(p,1,z).$$

Note the leading terms of $g_0(y,z,w_1,\cdots,w_{m-1})$ and $g_1(y,z,w_1,\cdots,w_{m-1})$ are y^n and z^p, respectively.

Step 2. Determine the relations among $[M(p^l+1,\overline{0})]$ for $1 \le l \le m-1$.

Firstly, choosing $k = 1, 2, \cdots, p-1$ in Lemma 3.2 successively, we obtain the following equalities:

$[M(p^l+1,\overline{0})]^2 = [M(2p^l+1,\overline{0})] + \cdots$

$[M(p^l+1,\overline{0})][M(2p^l+1,\overline{0})] = [M(3p^l+1,\overline{0})] + \cdots$

....

$[M(p^l+1,\overline{0})][M((p-2)p^l+1,\overline{0})] = [M((p-1)p^l+1,\overline{0})] + \cdots$

$[M(p^l+1,\overline{0})][M((p-1)p^l+1,\overline{0})] = [M(p^{l+1},\overline{0})] + [M(p^{l+1},\overline{1})] + \cdots$

Then, from the first formula, we have $[M(2p^l+1,\overline{0})] = [M(p^l+1,\overline{0})]^2 - \cdots$. By substituting it into the second formula, we find that $[M(3p^l+1,\overline{0})] = [M(p^l+1,\overline{0})]^3 - \cdots$, then by substituting $[M(3p^l+1,\overline{0})]$ into the third formula, $\cdots\cdots$, by continuing this process, we obtain that $[M((p-1)p^l+1,\overline{0})] = [M(p^l+1,\overline{0})]^{p-1} - \cdots$. By substituting it into the last formula, we finally obtain a polynomial g_{l+1} such that the formula

$$g_{l+1}([M(1,\overline{1})], [M(2,\overline{0})], [M(p+1,\overline{0})], [M(p^2+1,\overline{0})], \cdots, [M(p^l+1,\overline{0})]) = 0.$$

holds.

Then after substituting $[M(1,\overline{1})]$ by y, $[M(2,\overline{0})]$ by z and $[M(p^s+1,\overline{0})]$ by w_s, $1 \le s \le l$ respectively, we have the polynomials $g_{l+1}(y, z, w_1, \cdots, w_l)$ for $1 \le l \le m-1$.

Step 3. Finally, we construct the ideal I of $\mathbb{Z}[y, z, w_1, \cdots, w_{m-1}]$ generated by the polynomials $g_i(y, z, w_1, \cdots, w_{m-1}), 0 \le i \le m$.

The following observations are crucial.

Lemma 3.3. *Given l, s satisfying $1 \le l \le m, 2 \le s \le p$, let $p \le u \le sp^l, 0 \le r \le n-1$ and denote by $f_{u,\overline{r}}$ the corresponding polynomial of $[M(u,\overline{r})]$ generated by $[M(1,\overline{1})], [M(2,\overline{0})]$ and $[M(p^l+1,\overline{0})], 1 \le l \le m-1$ in Theorem 2.10. Then $f_{u,\overline{r}}$ is a polynomial in $\mathbb{Z}[y, z, w_1, \cdots, w_l]$, however, no w_l^t $(t \ge s)$ appears in any monomials of $f_{u,\overline{r}}$.*

Proof. It is obvious that the polynomial $f_{u,\overline{r}}$ is a polynomial in $\mathbb{Z}[y, z, w_1, \cdots, w_l]$ according to Corollary 2.11 and the correspondence among $[M(1,\overline{1})], [M(2,\overline{0})]$, $[M(p^l+1,\overline{0})]$ and y, z, w_l for $1 \le l \le m-1$.

Now we need only to prove that w_l^t $(t \ge s)$ will not appear in any monomials of $f_{u,\overline{r}}$. It is sufficient to prove the result for $[M(u,\overline{0})]$ instead of $[M(u,\overline{r})]$ since $I(f_{u,\overline{r}}) = I(f_{u,\overline{0}}) + (0, 0, \cdots, 0, r)$ for $u \le p$. We will prove the result by induction on s satisfying $2 \le s \le p$.

When $s = 2$, we have $p \le u \le 2p^l$. We will claim that $I(f_{u,\overline{0}}) < I(w_l^2)$.

In the case $p \le u \le p^l$, the result follows from Corollary 2.11 since w_l does not appear in the polynomial $f_{u,\overline{0}}$.

In the case $p^l+1 \le u \le 2p^l$, we will use the induction on u. Firstly if $u = p^l+1$, then $I(f_{u,0}) = I(w_l) < I(w_l^2)$. Now we assume the result holds for u', $p^l+1 \le u' < u \le 2p^l$. Since in this case $u-1$ is not a power of p, we have $[M(u,\overline{0})] = [M(u-w,\overline{0})][M(w+1,\overline{0})] - [M_2] - \cdots - [M_t]$ for some w such that $1 \le w \le u-2$ and $\binom{u-1}{w} \ne 0$ in K, due to the proof of Lemma 2.9. Moreover, $M_k \cong M(i_k, \overline{j_k})$ for any $2 \le k \le t$ with $1 \le i_k, j_k \le u-2$.

By the induction hypotheses on u, we have $I(f_{i_k,\overline{j_k}}) < I(w_l^2)$ for $2 \le k \le t$. Thus, it remains to prove that $I(f_{u-w,\overline{0}} f_{w+1,\overline{0}}) < I(w_l^2)$. Since $(u-w)+(w+1) = u+1 \le 2p^l+1$, we find that either $u-w < p^l+1$ or $w+1 < p^l+1$. Assume the former case is true, then by Corollary 2.11, we have $I(f_{u-w,\overline{0}}) < I(w_l)$, i.e., w_l does not appear in $f_{u-w,0}$. Again by the induction hypotheses on u, since $w+1<u$, we have $I(f_{w+1,\overline{0}}) < I(w_l^2)$. Therefore,

$$I(f_{u-w,\overline{0}}f_{w+1,\overline{0}}) = I(f_{u-w,\overline{0}}) + I(f_{w+1,\overline{0}}) < I(w_l^2).$$

In the latter case, i.e. $w+1 < p^l+1$, a similar discussion can be given.

Hence, the result is proved when $s=2$.

Now we assume for all integers s' with $2 \leq s' < s$, the result is true. Then, if $p \leq u \leq (s-1)p^l$, by the induction hypotheses on s, we have $I(f_{u,\overline{0}}) < I(w_l^{s-1}) < I(w_l^s)$, which means the result is true in this case.

Now we consider the result in the case $(s-1)p^l+1 \leq u \leq sp^l$, using the induction on u simultaneously.

Actually, for any $u, (s-1)p^l+1 \leq u \leq sp^l$, since $u-1$ is not a power of p, we again have the equation $[M(u,\overline{0})] = [M(u-w,\overline{0})][M(w+1,\overline{0})] - [M_2] - \cdots - [M_t]$ for some w satisfying $1 \leq w \leq u-2$ and $\binom{u-1}{w} \neq 0$ in K. In this case the remaining proof is similar to that of the case $s=2$ by replacing $I(w_l^2)$ with $I(w_l^s)$. Indeed, we will use the fact that $(u-w)+(w+1) = u+1 \leq sp^l+1$ and the induction hypotheses on u. □

Lemma 3.4. *Under the order $y < z < w_1 < w_2 < \cdots < w_{m-1}$, the leading terms of the polynomials $g_0, g_1, g_2, \cdots, g_m$ constructed above are $y^n, z^p, w_1^p, \cdots, w_{m-1}^p$ respectively.*

Proof. The leading terms of g_0, g_1 are y^n, z^p due to the definitions of g_0, g_1.

Due to **Step** 2 in this section, for $2 \leq s \leq m$, we have a series of equalities in the form $[M(p^{s-1}+1,\overline{0})]^p = \sum[M(r,\overline{t})]$ with $r \leq p^s$, which correspond respectively to the equalities $w_{s-1}^p = \sum f_{r,\overline{t}}$ satisfying $r \leq p^s$. Then according to the construction of g_s in **Step** 2, the polynomial $g_s = w_{s-1}^p - \sum f_{r,\overline{t}}$. By Theorem 2.10 and Lemma 3.3, all the terms $f_{r,\overline{t}}$ with $r \leq p^s$ on the right side of the equations can be represented by a polynomial with variables $y, z, w_1, \cdots, w_{s-1}$ and $I(f_{r,\overline{t}}) < I(w_{s-1}^p)$. Thus, the result follows at once. □

With the above constructions, we have the following main theorem.

Theorem 3.5. *Let* char$K = p$ *and $n \geq d = p^m$, then the representation ring $r(KZ_n/J^d)$ of the Nakayama truncated algebra KZ_n/J^d is isomorphic to the quotient ring $\mathbb{Z}[y,z,w_1,\cdots,w_{m-1}]/I$, induced by the ring epimorphism*

$$\phi : \mathbb{Z}[y,z,w_1,\cdots,w_{m-1}] \longrightarrow r(KZ_n/J^d)$$

satisfying $\phi(1) = [M(1,\overline{0})], \phi(y) = [M(1,\overline{1})], \phi(z) = [M(2,\overline{0})]$ and $\phi(w_l) = [M(p^l+1,\overline{0})], \forall\ 1 \leq l \leq m-1$, where the ideal I is generated by the polynomials $g_i(y,z,w_1,\cdots,w_{m-1}), 0 \leq i \leq m$ defined as above.

Proof. Firstly, by the construction of the ideal I, the ring epimorphism ϕ can induce a ring epimorphism

$$\bar{\phi} : \mathbb{Z}[y,z,w_1,\cdots,w_{m-1}]/I \to r(KZ_n/J^d) \tag{12}$$

such that $\bar{\phi}(\bar{v}) = \phi(v)$ for all $v \in \mathbb{Z}[y,z,w_1,\cdots,w_{m-1}]$.

To prove that $\bar{\phi}$ is a ring isomorphism, it is enough to claim that the ranks of two sides of the epimorphism $\bar{\phi}$ in (12) are equal.

By Lemma 3.4, the leading terms of the polynomials $g_0, g_1, g_2, \cdots, g_m$ are $y^n, z^p, w_1^p, \cdots, w_{m-1}^p$, respectively. Thus it is easy to see that

$$B = \{y^i z^j w_1^{l_1} \cdots w_{m-1}^{l_{m-1}} \mid 0 \leq i < n, 0 \leq j < p, 0 \leq l_k < p, k = 1, \cdots, m-1\}$$

forms a $\mathbb{Z}$-basis of $\mathbb{Z}[y, z, w_1, \cdots, w_{n-1}]/I$.

Since the ranks of two sides of the epimorphism $\bar{\phi}$ in (12) are both $np^m = nd$, the conclusion follows from the fact that two free abelian groups are isomorphic if and only if both have the same rank. □

Finally, we will give the following example to illustrate our result precisely.

Example 3.6. Let $n = 10$, then the representation ring $r(KZ_{10}/J^d)$ is shown as follows:

Case 1. When char $K_1 = 2$, then
$r(K_1Z_{10}/J^2) \cong \mathbb{Z}[y, z]/(y^{10} - 1, (z - y - 1)z)$.
$r(K_1Z_{10}/J^4) \cong \mathbb{Z}[y, z, w_1]/(y^{10} - 1, (z - y - 1)z, (w_1 - y)(w_1 + y - z - yz))$.
$r(K_1Z_{10}/J^8) \cong \mathbb{Z}[y, z, w_1, w_2]/(y^{10} - 1, (z - y - 1)z, (w_1 - y)(w_1 + y - z - yz), w_2^2 - y^4 - z(w_1 - y)(1 + y)(w_2 + 2y^2 - y^2z - yz))$.

Case 2. When char $K_2 = 3$, then
$r(K_2Z_{10}/J^3) \cong \mathbb{Z}[y, z]/(y^{10} - 1, (z - y - 1)(z^2 - y))$.
$r(K_2Z_{10}/J^9) \cong \mathbb{Z}[y, z, w_1]/(y^{10} - 1, (z - y - 1)(z^2 - y), (w_1 + y + y^2 + yz - z^2 - yz^2)(w_1^2 - y^3 - 2yzw_1 + y^2z^2))$.

Case 3. When char $K_3 = 5$, then
$r(K_3Z_{10}/J^5) \cong \mathbb{Z}[y, z]/(y^{10} - 1, (z - y - 1)(z^4 - 3yz^2 + y^2))$.

Case 4. When char $K_4 = 7$, then
$r(K_4Z_{10}/J^7) \cong \mathbb{Z}[y, z]/(y^{10} - 1, (z - y - 1)(z^6 - 5yz^4 + 6y^2z^2 - y^3))$.

3.2. Application to isomorphism problem. At the end of this section, we apply the polynomial characterization in Theorem 3.5 and discuss the isomorphic problem on representation rings, that is, what conditions should be satisfied for two Nakayama truncated algebras KZ_{n_1}/J^{d_1} and KZ_{n_2}/J^{d_2} such that $r(KZ_{n_1}/J^{d_1}) \cong r(KZ_{n_2}/J^{d_2})$ as rings?

In fact, we have to replace $r(KZ_n/J^d)$ by its *complexified representation algebra* $R(KZ_n/J^d) = \mathbb{C} \otimes_{\mathbb{Z}} r(KZ_n/J^d)$ and then discuss the conditions in which $R(KZ_{n_1}/J^{d_1}) \cong R(KZ_{n_2}/J^{d_2})$ holds.

Recall that for a field K, a positive integer t and the polynomial algebra $K[x_1, \cdots, x_t]$, let V be a subset of K^t, then denote $I(V) = \{f \in K[x_1, \cdots, x_t] | f(v) = 0 \text{ for all } v \in V\}$ which is an ideal of $K[x_1, \cdots, x_t]$, known as the *corresponding ideal* of V; conversely for an ideal I of $K[x_1, \cdots, x_t]$, denote $V(I) = \{v \in K^t | f(v) = 0 \text{ for all } f \in I\} \subset K^t$, which is called an *algebraic set* of K^t corresponding to I.

A subset $V \subset K^t$ is called an *affine variety* if it is an irreducible closed subset of K^t in terms of Zariski topology; and an ideal I of $K[x_1, \cdots, x_t]$ is called a *radical ideal* if $\sqrt{I} = I$ for $\sqrt{I} = \{f | f^s \in I \text{ for some integer } s > 0\}$.

It is well-known that there exists a one-to-one correspondence between affine varieties V of K^t and radical ideals I of $K[x_1, \cdots, x_t]$.

Furthermore, we will denote $K[V]$ as the *homogeneous coordinate ring* of the affine variety V. For details, see [16],[28].

Lemma 3.7 (Corollary 4.5, [28]). *For two affine varieties V and W in K^t, a polynomial map $f : V \to W$ is an isomorphism if and only if the dual map $f^* : K[W] \to K[V]$ is an isomorphism.*

We know $K[V] \overset{\text{def.}}{=} \{f : V \to K | f \text{ is a polynomial function}\}$; for a polynomial function $g \in K[W]$, f^* is defined by $f^*(g) = g \circ f$. For two finite sets V, W in K^t, there exists a polynomial isomorphism between V and W if and only if $|V| = |W|$. Then, we have the following:

Corollary 3.8. *Let* $\text{char}K = p$ *and* $n_i \geq d_i = p^{m_i}, i = 1, 2$ *satisfying* $n_1d_1 = n_2d_2$*, and* I, I' *be the ideals of* $\mathbb{Z}[y, z, w_1, \cdots, w_{m_1-1}]$ *and* $\mathbb{Z}[y, z, w_1, \cdots, w_{m_2-1}]$ *respectively, given in Theorem 3.5 in the process of polynomial characterization of the representation rings* $r(KZ_{n_1}/J^{d_1})$, $r(KZ_{n_2}/J^{d_2})$*. If both* I *and* I' *are radical ideals, then their complexified representation algebras are isomorphic, that is,* $R(KZ_{n_1}/J^{d_1}) \cong R(KZ_{n_2}/J^{d_2})$.

Proof. Denote $N = n_1d_1 = n_2d_2$, then the algebraic sets $V(I), V(I')$ both have N points, i.e. $N = |V(I)| = |V(I')|$. In fact, by the Grobner basis theory, since the degree of $g_0, g_1, g_2, \cdots, g_{m_1}$ in Theorem 3.5 are $n_1, p, p, \cdots, p$ respectively, in the order $y < z < w_1 < w_2 < \cdots < w_{m_1-1}$, it follows that the algebraic set $V(I)$ has $n_1 \cdot p \cdot p \cdots p = n_1 \cdot d_1 = N$ points. So does $V(I')$.

It follows that $N = |V(\sqrt{I})| = |V(I)| = |V(I')| = |V(\sqrt{I'})| < \infty$, and then there exists a polynomial isomorphism f from $V(\sqrt{I})$ to $V(\sqrt{I'})$, which means by Lemma 3.7 there exists a K-algebra isomorphism f^*, as the dual of f, from the homogeneous coordinate ring $\mathbb{C}[V(\sqrt{I'})]$ to $\mathbb{C}[V(\sqrt{I})]$.

Since both I and I' are radical ideals, we have $\mathbb{C}[V(\sqrt{I})] \cong \mathbb{C}[y, z, w_1, \cdots, w_{m_1-1}]/I(V(\sqrt{I})) = \mathbb{C}[y, z, w_1, \cdots, w_{m_1-1}]/\sqrt{I} = \mathbb{C}[y, z, w_1, \cdots, w_{m_1-1}]/I \cong R(KZ_{n_1}/J^{d_1})$, similarly the isomorphism $\mathbb{C}[V(\sqrt{I'})] \cong R(KZ_{n_2}/J^{d_2})$ holds. Hence, we find that $R(KZ_{n_1}/J^{d_1}) \cong R(KZ_{n_2}/J^{d_2})$, which completes the proof. □

From this result, it is possible to find some different Nakayama truncated algebras whose complexified representation algebras are isomorphic.

However, in general, the ideal I in Theorem 3.5 is not always a radical ideal. For instance, if we choose $n = 10, d = p = 2$, then in Example 3.6 we find that $r(K_1Z_{10}/J^2) \cong \mathbb{Z}[y, z]/(y^{10} - 1, (z - y - 1)z)$. That is, the ideal $I = (y^{10} - 1, (z - y - 1)z)$, but, by calculation using Maple 17 software, it is easy to see that $\sqrt{I} = (y^{10} - 1, (z - y - 1)z, \frac{y^{10}-1}{y+1}z) \supsetneqq I$, which shows that I is not a radical ideal.

Moreover, although two complexified representation algebras $R(KZ_{n_1}/J^{d_1}) \cong R(KZ_{n_2}/J^{d_2})$, their original representation rings can still be not isomorphic, in general. For example, let $A = \mathbb{Z}[x]/(x^2 - 1)$ and $B = \mathbb{Z}[x]/(x^2 + 1)$. First, note that A and B are not isomorphic as $\mathbb{Z}$-algebras. Indeed, for any homomorphism $f : A \to B$, we have $(f(\overline{x}))^2 = (\overline{ax + b})^2 = 1$ in B, hence $a = 0, b = \pm 1$, so $f(\overline{x}) = \overline{ax + b} = \pm 1$, which can not be an isomorphism. However, it is easy to see that $f : \mathbb{C}[x]/(x^2 - 1) \to \mathbb{C}[x]/(x^2 + 1)$ which shows that $\overline{x}$ to $i\overline{x}$ is an isomorphism. Therefore, A and B are not isomorphic as $\mathbb{Z}$-algebras themselves, but are isomorphic as $\mathbb{C}$-algebras after complexification.

4. Green rings from monoidal categories and some special cases.

4.1. General theory. In this section, we describe the theory of representation rings (i.e. Green rings) of module categories in the context of monoidal categories.

Monoidal categories (also called tensor categories) were introduced by Bènabou [5] in 1963 and are used to define the concept of a monoid object and an associated action on the objects of the category. They are also used in enriched categories. The theory of monoidal categories has numerous applications, for example, to define models for the multiplicative fragment of intuitionistic linear logic, and to form the mathematical foundation for the topological order in condensed matter. Braided monoidal categories have applications in quantum field theory and string theory.

We will review the method of Green rings in the context of monoidal categories and then introduce analogue methods via module, complex, homotopy, derived and (derived) shift categories, where derived shift categories can be viewed as full subcategories of complex categories (bounded derived categories).

A *monoidal category* $(\mathcal{C}, \otimes, I, a, l, r)$ is a category $\mathcal{C}$ equipped with a *tensor product* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object I, called the *unit* of $\mathcal{C}$, an associativity constraint a, a *left unit constraint* l and a *right unit constraint* r with respect to I such that the Pentagon Axiom and the Triangle Axiom are satisfied, see [20].

Throughout this paper, we assume the monoidal category $\mathcal{C}$ to be an additive Krull-Schmidt category and its tensor product $\otimes$ to be an additive bifunctor.

Definition 4.1. Assume a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ is an additive Krull-Schmidt category with $\otimes$ as an additive bifunctor. Denote by $[C]$ the isomorphism class of an object C in $\mathcal{C}$. Then the *Green ring* $gr(\mathcal{C})$ of $\mathcal{C}$ is defined as the ring with identity $[I]$ which is an additive abelian group generated by all $[C]$ modulo the relations $[C] + [D] = [C \oplus D]$ with multiplication given by $[C][D] = [C \otimes D]$ for any $C, D \in Ob(\mathcal{C})$.

There have been some references, e.g. [19][18], where Green rings were introduced and studied for some special monoidal categories.

Indeed, the Green ring $gr(\mathcal{C})$ can be viewed as the de-categorification of the monoidal category $\mathcal{C}$ in categorification theory, as we will show below.

Recall that an *additive 2-category* is a category enriched over the category of additive categories, i.e. for any two objects i, j, the morphism set between i and j is endowed with an additive category structure which satisfies some axioms. For details, refer to [24].

The *split Grothendieck category* $[\mathscr{C}]_\oplus$ of a 2-category $\mathscr{C}$ is the category which has the same objects as that of $\mathscr{C}$ and for any $i, j \in [\mathscr{C}]$, we have $\mathrm{Hom}_{[\mathscr{C}]_\oplus}(i, j) = [\mathscr{C}(i, j)]_\oplus$, where $[\mathscr{C}(i, j)]_\oplus$ denotes the split Grothendieck group of $\mathscr{C}(i, j)$. That is, the quotient of the free abelian group generated by the isomorphic classes of $\mathscr{C}(i, j)$ modulo the relations $[Y] - [X] - [Z]$ when $Y \cong X \oplus Z$, with the multiplication of morphisms given by $[M] \circ [N] = [M \circ N]$.

An additive 2-category $\mathscr{C}$ is called a *categorification* of a K-linear category $\mathscr{D}$ if $K \otimes_{\mathbb{Z}} [\mathscr{C}]$ is isomorphic to $\mathscr{D}$. In this case, the category $\mathscr{D}$ is called a *K-decategorification* of $\mathscr{C}$.

Clearly, an additive Krull-Schmidt monoidal category $\mathcal{C}$ with an additive bifunctor $\otimes$ can be regarded as an additive 2-category $\mathscr{C}$ which has only one object i and the morphism class $\mathscr{C}(i, i) = \mathcal{C}$ with the multiplication of morphisms as $M \circ N = M \otimes N$ for any $M, N \in \mathcal{C}$. We denote this 2-category as $\mathscr{C}_\mathcal{C}$. Moreover, a K-algebra A can be regarded as a linear K-category $\mathcal{A}$ which has only one object i and the morphism class $\mathcal{A}(i, i) = A$ with the multiplication of morphisms as $a \circ b = ab$ for any $a, b \in A$.

From these viewpoints, for a monoidal category $\mathcal{C}$ in Definition 4.1 and its corresponding 2-category $\mathscr{C}_\mathcal{C}$, we have the ring isomorphism $gr(\mathcal{C}) \cong [\mathscr{C}_\mathcal{C}]_\oplus$. After expanding $gr(\mathcal{C})$ to be over K, denoted by $gr_K(\mathcal{C}) = K \otimes_{\mathbb{Z}} gr(\mathcal{C})$ (called the *K-Green ring*) as a K-linear category, we find that $gr_K(\mathcal{C}) \cong K \otimes_{\mathbb{Z}} [\mathscr{C}_\mathcal{C}]_\oplus$. In summary, we have:

Theorem 4.2. *For a monoidal category $\mathcal{C}$, its K-green ring $gr_K(\mathcal{C})$ is a K-de-categorification of $\mathcal{C}$; this is equivalent to saying that $\mathcal{C}$ is a categorification of the*

K-Green ring $gr_K(\mathcal{C})$.

Some properties of the Green ring $gr(\mathcal{C})$ of a monoidal category $\mathcal{C}$ are listed as follows:

(i) $gr(\mathcal{C})$ is an associative ring with identity $[I]$ for the unit object I in $\mathcal{C}$. $[I]$ is the identity of $gr(\mathcal{C})$ since I is the unit object of $\mathcal{C}$. The associativity of $gr(\mathcal{C})$ follows the associativity constraint a of $\mathcal{C}$; its distributive law is true because $\otimes$ is an additive bifunctor.

(ii) Under the addition, $gr(\mathcal{C})$ is a free abelian group with a $\mathbb{Z}$-basis $\{[D]: D \in \mathrm{ind}\mathcal{C}\}$ where $\mathrm{ind}\mathcal{C}$ denotes the subcategory consisting of indecomposable objects in $\mathcal{C}$.

For an arbitrary K-Hopf algebra H, the finitely generated module category ${}_H mod$, the complex category $Ch(H)$, the homotopy category $K(H)$, and the bounded derived category $D^b(H)$ are all monoidal categories. Meanwhile, we will construct the new monoidal categories, that is, two full subcategories $Ch^{sh}(H)$ and $D^{sh}(H)$ of $Ch(H)$ and $D^b(H)$ respectively, where $Ob(Ch^{sh}(H)) = Ob(D^{sh}(H)) = M^\bullet[s], s \in \mathbb{Z}$, $M \in \mathrm{ind}({}_H mod)$ and their Green rings are the same, known as the *shift ring* $sh(H)$.

Besides the module category ${}_H mod$ and its representation ring $r(H)$, in the sequel, we will also study the categories $D^b(H)$, $Ch^{sh}(H)$ and $D^{sh}(H)$ and their special Green rings. That is, the derived representation ring $dr(H)$ and its sub-ring, and the shift ring $sh(H)$.

4.2. On derived representation rings of bounded derived categories. Recall that in general, for a K-algebra A, $Ch(A)$ denotes the complex category of A, $K(A)$ the homotopy category of A, $D(A)$ the derived category of A and $D^b(A)$ the bounded derived category of A. For details of derived categories, see [15],[30].

For a complex $M^\bullet = \{M_i, d_i\}$ in these categories, we call $M^\bullet$ an *A-complex* since all M_i are A-modules.

For two indecomposable complexes $M^\bullet = \{M_i, d_i\}, N^\bullet = \{N_i, d'_i\}$ in $D^b(A)$, the tensor product of chain complexes $M^\bullet \otimes N^\bullet$ is given by [30] as follows:

Constructing the double complex $C_{M^\bullet,N^\bullet} = \{M_i \otimes N_j\}$ together with the maps

$$d^h_i = d_i \otimes \mathrm{id}_{N_j} : M_i \otimes N_j \to M_{i-1} \otimes N_j; \quad d^v_j = (-1)^i \, \mathrm{id}_{\mathrm{M_i}} \otimes d'_j : M_i \otimes N_j \to M_i \otimes N_{j-1},$$

the total complex $\mathrm{Tot}^\oplus(M^\bullet, N^\bullet)$ is defined by $(\mathrm{Tot}^\oplus(M^\bullet, N^\bullet))_n = \bigoplus_{i+j=n} M_i \otimes N_j$ with the maps $\widehat{d_n} = \bigoplus_{i+j=n} (d^h_i + d^v_j)$.

We call $\mathrm{Tot}^\oplus(M^\bullet, N^\bullet)$ the *tensor product of chain complexes* $M^\bullet$ and $N^\bullet$. That is, define $M^\bullet \otimes N^\bullet = \mathrm{Tot}^\oplus(M^\bullet, N^\bullet)$, where $\widehat{d_n}$ is said to be the *differentials* of $M^\bullet \otimes N^\bullet$.

For an arbitrary algebra A, we can only define $M_i \otimes N_j$ as a left A-module via the left A-action on M_i, which is not related to the module structure of N_j. So, we do not think of $M^\bullet \otimes N^\bullet$ as a complex in $D^b(A)$, in general.

We always assume that $A = H$ is a K-Hopf algebra with comultiplication Δ. Then, for two complexes $M^\bullet = \{M_i, d_i\}, N^\bullet = \{N_j, d'_j\} \in D^b(H)$, based on the H-module structures of M_i and N_j, we can define canonically an H-module structure on all terms. That is, for any $a \in H$, $m_i \in M_i$, $n_j \in N_j$, define $a(m_i \otimes n_j) = \sum_{(a)} a' m_i \otimes a'' n_j$. From this, we have the following:

LEMMA 4.3. *Let $M^\bullet = \{M_i, d_i\},\ N^\bullet = \{N_i, d'_i\}$ be two H-complexes in $D^b(H)$. Then the tensor product $M^\bullet \otimes N^\bullet$ is also an H-complex in $D^b(H)$.*

Proof. We know all $M_i \otimes N_j$ are H-modules given the above. So, we need only to claim all differentials $\widehat{d_n} = \bigoplus_{i+j=n} (d_i^h + d_j^v)$ are H-module morphisms among $\{M_i \otimes N_j\}$. In particular, it suffices to prove $d_i^h + d_j^v$ are H-module morphisms. In fact, for any $a \in H$, $m_i \in M_i$, $n_j \in N_j$,

$$\begin{aligned}(d_i^h + d_j^v)(a(m_i \otimes n_j)) &= \sum_{(a)} d_i(a'm_i) \otimes a''n_j + \sum_{(a)} a'm_i \otimes d_j'(a''n_j)\\ &= \sum_{(a)} a'd_i(m_i) \otimes a''n_j + \sum_{(a)} a'm_i \otimes a''d_j'n_j\\ &= a((d_i^h + d_j^v)(m_i \otimes n_j)).\end{aligned}$$

□

Now let us recall that a morphism $f : M^\bullet \to N^\bullet$ is called a *quasi-isomorphism* if it induces group isomorphisms between all homology groups of $M^\bullet$ and $N^\bullet$, written as $M^\bullet \overset{f}{\cong} N^\bullet$. We denote by $[M^\bullet]$ the isomorphism class of a chain complex $M^\bullet$ in $D^b(H)$.

The *mapping cone* of f, denoted as Cone(f), is defined as a complex $M^\bullet[1] \oplus N^\bullet$, with the differentials

$$\begin{pmatrix} -d_{i-1} & 0 \\ f_i & d_i' \end{pmatrix}, \quad \text{for all} \quad i \in \mathbb{Z}.$$

Lemma 4.4. *Let $f : M^\bullet \to M'^\bullet$ and $g : N^\bullet \to N'^\bullet$ be two quasi-isomorphisms in $Ch(H)$, then $f \otimes g : M^\bullet \otimes N^\bullet \to M'^\bullet \otimes N'^\bullet$ is also a quasi-isomorphism.*

Proof. We know from [30] that a morphism between two complexes is a quasi-isomorphism if and only if its mapping cone is acyclic. So, it is enough to prove that the complex Cone$(f \otimes g)$ is acyclic. Due to symmetry, we consider only the exactness of Cone$(f \otimes \mathrm{id})$. But, we have Cone$(f \otimes \mathrm{id})$ = Cone$(f) \otimes N^\bullet$. Since the double complexes are row and column bounded, we need only to prove that each row or column is exact by the Acyclic Assembly Lemma of [30]. The row exactness of the double complex Cone$(f) \otimes N^\bullet$ follows from the exactness of Cone(f). □

Now by Lemma 4.3 and 4.4, the bounded derived category $D^b(H)$ is a monoidal category.

Definition 4.5. Let H be a K-Hopf algebra. The *derived representation ring* $dr(H)$ of a K-Hopf algebra H is defined as the Green ring $gr(D^b(H))$ of the monoidal category $D^b(H)$. Concretely, for a bounded H-complex $M^\bullet$ in $D^b(H)$, we denote by $[M^\bullet]$ the isomorphism class of $M^\bullet$.

(1) $dr(H)$ is the abelian group generated by all isomorphism classes of bounded complexes in $D^b(H)$ with addition defined by the relations $[M^\bullet] + [N^\bullet] = [M^\bullet \oplus N^\bullet]$ for any bounded complexes $M^\bullet, N^\bullet$ in $D^b(H)$;

(2) In $dr(H)$, the multiplication is defined by the relations $[M^\bullet][N^\bullet] = [M^\bullet \otimes N^\bullet]$.

Note that

(i) The multiplication of $dr(H)$ is well-defined due to Lemma 4.4;

(ii) $[K^\bullet]$ is the identity of $dr(H)$, since $K^\bullet \otimes M^\bullet \cong M^\bullet \otimes K^\bullet \cong M^\bullet$ for any $M^\bullet \in D^b(H)$, where $K^\bullet$ is the stalk complex of the trivial H-module K at the 0-position;

(iii) The associativity of $dr(H)$ holds since for any $L^\bullet, M^\bullet, N^\bullet$ in $D^b(H)$,

$$([L^\bullet][M^\bullet])[N^\bullet] = ([L^\bullet \otimes M^\bullet])[N^\bullet] = [L^\bullet \otimes M^\bullet \otimes N^\bullet] = [L^\bullet]([M^\bullet \otimes N^\bullet]) = [L^\bullet]([M^\bullet][N^\bullet]).$$

(iv) The distributive law of $dr(H)$ holds since $(L^\bullet \oplus M^\bullet) \otimes N^\bullet \cong (L^\bullet \otimes N^\bullet) \oplus (M^\bullet \otimes N^\bullet)$;

(v) The abelian group $dr(H)$ is free with a $\mathbb{Z}$-basis $\{[M^\bullet] \mid M^\bullet \in \mathrm{ind}(D^b(H))\}$ by the Krull-Schmidt property of $D^b(H)$;

(vi) If H is cocommutative, then $dr(H)$ is a commutative ring.

4.3. On shift rings of (derived) shift categories. For any indecomposable module $M \in \mathrm{ind}({}_H mod)$, its stalk complex $M^\bullet$ and shift objects $M^\bullet[n], n \in \mathbb{Z}$ are indecomposable complexes in $Ch(H)$ and $D^b(H)$, where [1] means the shift functor. It is clear that for all $M, N \in \mathrm{ind}({}_H mod), i, j \in \mathbb{Z}$, in $Ch(H)$ and $D^b(H)$, we have

$$M^\bullet[i] \otimes N^\bullet[j] \cong (M^\bullet \otimes N^\bullet)[i+j] = (M \otimes N)^\bullet[i+j]. \tag{13}$$

The isomorphism in (13) on stalk complexes can be extended to any indecomposable objects in $D^b(H)$. That is, for any $M^\bullet, N^\bullet \in \mathrm{ind}D^b(H)$, $m, n \in \mathbb{Z}$, the isomorphism holds:

$$M^\bullet[m] \otimes N^\bullet[n] \cong (M^\bullet \otimes N^\bullet)[m+n]. \tag{14}$$

Define $Ch^{sh}(H)$ as the full subcategory of $Ch(H)$ whose objects are all $M^\bullet[n], \forall M \in {}_H mod, n \in \mathbb{Z}$. $M^\bullet[n]$ is indecomposable in $Ch^{sh}(H)$ if and only if $M \in \mathrm{ind}({}_H mod)$. Because $M^\bullet \otimes N^\bullet \cong (M \otimes N)^\bullet$ in $Ch(H)$ and $Ch^{sh}(H)$ is closed under $\otimes$ by (13), we see that $Ch^{sh}(H)$ is a monoidal subcategory of $Ch(H)$ with the same tensor product $\otimes$.

We call the monoidal category $Ch^{sh}(H)$ the *shift category* of H, whose Green ring is said to be the *shift ring* of H, denoted as $sh(H)$.

Analogously, we define $D^{sh}(H)$ as the full subcategory of $D^b(H)$ whose objects are $M^\bullet[n]$ for any $M \in {}_H mod, n \in \mathbb{Z}$, which is known as the *derived shift category* of H. Also, $D^{sh}(H)$ is a monoidal category.

$\mathrm{Hom}_{D^b(H)}(M^\bullet[n], M^\bullet[n])$ is local when M is an indecomposable H-module, since

$$\mathrm{Hom}_{D^b(H)}(M^\bullet[n], M^\bullet[n]) \cong \mathrm{Hom}_H(M, M).$$

Therefore, the indecomposable objects in $D^{sh}(H)$ are all $M^\bullet[n]$ for $M \in \mathrm{ind}({}_H mod), n \in \mathbb{Z}$.

Hence, from the above, indecomposable objects of $Ch^{sh}(H)$ are the same as those of $D^{sh}(H)$. That is, all $M^\bullet[n]$ for $M \in \mathrm{ind}({}_H mod), n \in \mathbb{Z}$. It follows that the Green ring of $D^{sh}(H)$ is the same as that of $Ch^{sh}(H)$, that is, the shift ring $sh(H)$.

According to the definition, the following facts are obtained:

FACT 4.6. Let H be a K-Hopf algebra. Then,

(i) The shift ring $sh(H)$ is a sub-ring of the derived representation ring $dr(H)$ generated by $[M^\bullet[n]], \forall M \in \mathrm{ind}({}_H mod),\ n \in \mathbb{Z}$;

(ii) $sh(H)$ is a free abelian group with basis $[M^\bullet[n]]$ for $M \in \mathrm{ind}({}_H mod),\ n \in \mathbb{Z}$;

(iii) The shift ring $sh(H)$ has a $\mathbb{Z}$-graded structure, more precisely, for $n \in \mathbb{Z}$, whose component of n-degree $sh(H)_{(n)}$ is the free abelian subgroup generated by $[M^\bullet[n]]$ for all $M \in \mathrm{ind}({}_H mod)$;

(iv) $sh(H)_{(i)} sh(H)_{(j)} = sh(H)_{(i+j)}$ for any $i, j \in \mathbb{Z}$;

(v) The representation ring $r(H)$ is indeed the 0-component of $sh(H)$, that is, $r(H) \cong sh(H)_{(0)}$.

Among of these facts, (iv) is from the isomorphism (13) and (v) can be easily seen from the fact that $[M^\bullet] = [M^\bullet[0]]$ for all $M \in \mathrm{ind}({}_H mod)$.

Now we can give the connection among $r(H)$, $sh(H)$, $dr(H)$ as visualised in Fig. 5.

FIG. 5.

4.4. Polynomial characterizations of shift rings and derived representation rings. Throughout this part, H refers to a representation-finite K-Hopf algebra. Following the above discussion, as the analogue of polynomial characterizations of representation rings in Section 1, we can draw the following conclusions about shift rings and derived representation rings.

THEOREM 4.7. *For a representation-finite K-Hopf algebra H, assume the number of the indecomposable modules of H is t up to isomorphism. Then, the following statements hold:*

(i) *The shift ring $sh(H) \cong K\langle \mathbf{X}' \rangle/(I', 1 - X_1)$, where*

$$\mathbf{X}' = \{X_1[i_1], X_2[i_2], \cdots, X_t[i_t] \mid i_1, \cdots, i_t \in \mathbb{Z}\}$$

is the set of the indeterminates, and in particular, denote $X_k \equiv X_k[0], \forall 1 \le k \le t$, the ideal I' is generated by

$$\{X_i[r]X_j[s] - \sum_{1\le l\le t} k_{i,j}^l X_l[r+s] \mid 1 \le i, j \le t, r, s \in \mathbb{Z}\}$$

with $k_{i,j}^l (1 \le i, j, l \le t)$ the structure constants of $r(H)$ in (1).

(ii) *Let $\{N_\lambda, \lambda \in \Lambda\}$ be the set of representatives of the orbits of indecomposable complexes in $D^b(H)$ under the shift functor* [1]. *Then the derived representation ring $dr(H) \cong K\langle \mathbf{Y}' \rangle/(J', 1 - Y_1)$, where $\mathbf{Y}' = \{Y_\lambda[s] \mid \lambda \in \Lambda, s \in \mathbb{Z}\}$ is the set of the indeterminates, and in particular, Y_1 a fixed indeterminate with $1 \in \Lambda$, denote $Y_\lambda \equiv Y_\lambda[0], \forall \lambda \in \Lambda$, the ideal J' is generated by*

$$\{Y_\lambda[s]Y_\mu[t] - \sum_{\eta\in\Lambda, r\in\mathbb{Z}} k_{\lambda,\mu}^{\eta,r} Y_\eta[r+s+t] \mid \lambda, \mu \in \Lambda, s, t \in \mathbb{Z}\},$$

and $k_{\lambda,\mu}^{\eta,r}$ is the multiplicity of $N_\eta[r]$ in the decomposition of $N_\lambda \otimes N_\mu$ for $\lambda, \mu, \eta \in \Lambda, r \in \mathbb{Z}$.

(iii) *Furthermore, when H is cocommutative, then $sh(H) \cong K[\mathbf{X}']/(I', 1 - X_1)$ and $dr(H) \cong K[\mathbf{Y}']/(J', 1 - Y_1)$.*

Proof. For (i), we define a ring homomorphism $\varphi : K\langle X' \rangle \to sh(H)$ such that $\varphi(X_i[s]) = [M_i^\bullet[s]]$ for any $1 \le i \le t$, $s \in \mathbb{Z}$ and $\varphi(1) = [M_1^\bullet[0]]$. It is clear that $\ker\varphi$ is an ideal of $K\langle X' \rangle$ generated by I' and $1 - X_1[0]$, then (i) follows at once, from (13).

The proof of (ii) is similar to that of (i) from (14), only note that to define a ring homomorphism $\psi : K\langle \mathbf{Y}' \rangle \to dr(H)$ such that $\psi(Y_\lambda[s]) = [N_\lambda[s]]$ for any $\lambda \in \Lambda, s \in \mathbb{Z}$.

Now (iii) follows from (i) and (ii), and the fact that the cocommutativity of H implies the commutativity of $dr(H)$ and $sh(H)$ directly. □

Remark 4.8. In Theorem 4.7 (i), the structure constants of the shift ring $sh(H)$ are the same as those of the representation ring $r(H)$. Similarly, for the derived representation ring $dr(H)$ in Theorem 4.7 (ii), the structure constants $k^{\eta,r}_{\lambda,\mu}$ in $Y_\lambda[s]Y_\mu[t] = \sum_{\eta\in\Lambda,r\in\mathbb{Z}} k^{\eta,r}_{\lambda,\mu} Y_\eta[r+s+t]$ are the same as those in $Y_\lambda Y_\mu = \sum_{\eta\in\Lambda,r\in\mathbb{Z}} k^{\eta,r}_{\lambda,\mu} Y_\eta[r]$.

From the next section, we will embed the representation rings of the Nakayama truncated algebras into the corresponding shift rings and derived representation rings. For these Nakayama truncated algebras as cocommutative Hopf algebras, the polynomial characterizations of shift rings and derived representation rings will be given by fewer numbers of indeterminates.

5. Shift rings and derived representation rings of Nakayama truncated algebra KZ_n/J^d.

5.1. The shift ring $sh(KZ_n/J^d)$. For the Nakayama truncated algebra $H = KZ_n/J^d$ with char$K = p, n \geq d = p^m, m > 0$, recall that the indecomposable modules $M(i,\overline{j}) = P_{\overline{j}}/\mathrm{rad}^i P_{\overline{j}}, 0 \leq i \leq n-1, 1 \leq j \leq d$, $s \in \mathbb{Z}$, where $P_{\overline{j}}$ is the indecomposable projective H-module at the vertex $\overline{j}$. To calculate the shift ring $sh(KZ_n/J^d)$, it suffices to compute the structure constants $k^{i'',j''}_{i,j,i',j'}$ in the decomposition

$$M(i,\overline{j})^\bullet[s] \otimes M(i',\overline{j'})^\bullet[s'] \cong \sum_{0\leq i''\leq n-1, 1\leq j''\leq d} k^{i'',j''}_{i,j,i',j'} M(i'',\overline{j''})^\bullet[s+s']$$

for $0 \leq i, i', i'' \leq n-1, 1 \leq j, j', j'' \leq d, s, s' \in \mathbb{Z}$. However, by the formula (13) in Section 2, these $k^{i'',j''}_{i,j,i',j'}$ are just the structure constants in the decomposition

$$M(i,\overline{j}) \otimes M(i',\overline{j'}) \cong \sum_{0\leq i''\leq n-1, 1\leq j''\leq d} k^{i'',j''}_{i,j,i',j'} M(i'',\overline{j''})$$

for $0 \leq i, i', i'' \leq n-1, 1 \leq j, j', j'' \leq d$ in the module category, which we have already proved by Lemma 2.8 and Lemma 3.2. Hence we can give the generators and relations of the shift ring $r^{sh}(KZ_n/J^d)$ as follows.

Proposition 5.1. *Let* char$K = p, n \geq d = p^m, m > 0$. *Then*

(i) *The shift ring $sh(KZ_n/J^d)$ is generated by $[M(1,\overline{1})^\bullet[r]], [M(2,\overline{0})^\bullet[r]]$ and $[M(p^l+1,\overline{0})^\bullet[r]]$, where $r \in \mathbb{Z}, 1 \leq l \leq m-1$.*

(ii) *The relations in shift ring $sh(KZ_n/J^d)$ can be determined using the following formulas:*

$$\begin{cases} M(i,\overline{j})^\bullet[r] \otimes M(1,\overline{0})^\bullet[s] \cong M(1,\overline{0})^\bullet[s] \otimes M(i,\overline{j})^\bullet[r] \cong M(i,\overline{j})^\bullet[r+s], \\ M(1,\overline{1})^\bullet[r]^{\otimes n} \cong M(1,\overline{0})^\bullet[nr], \\ M(2,\overline{0})^\bullet[r] \otimes M(t,\overline{0})^\bullet[s] \cong M(t+1,\overline{0})^\bullet[r+s] \oplus M(t-1,\overline{1})^\bullet[r+s] \quad \forall t \geq 2, p \nmid t, \\ M(2,\overline{0})^\bullet[r] \otimes M(t,\overline{0})^\bullet[s] \cong M(t,\overline{0})^\bullet[r+s] \oplus M(t,\overline{1})^\bullet[r+s] \ \ \textit{for all } t > 0, p | t, \end{cases}$$

$$M(p^l+1,\overline{0})^\bullet[r] \otimes M(kp^l+1,\overline{0})^\bullet[s] \cong \begin{cases} W_1[r+s], & \textit{if } \ k \equiv -1 (\mathrm{mod}\ p) \\ W_2[r+s], & \textit{if } \ k \equiv 0 (\mathrm{mod}\ p) \\ W_3[r+s], & \textit{otherwise} \end{cases}$$

where $r, s \in \mathbb{Z}$ and W_1, W_2, W_3 are defined in Lemma 3.2.

(iii) *Let I' be the ideal of $\mathbb{Z}[y[r], z[r], w_l[r]]_{r\in\mathbb{Z},1\le l\le m-1}$ generated by the relations which are given from (ii) through replacing $[M(1,\overline{1})^\bullet[r]], [M(2,\overline{0})^\bullet[r]], [M(p^l+1,\overline{0})^\bullet[r]]$ by the indeterminates $y[r], z[r], w_l[r], r\in\mathbb{Z}, 1\le l\le m-1$, respectively. Then there is the ring isomorphism:*

$$sh(KZ_n/J^d) \cong \mathbb{Z}[y[r], z[r], w_l[r]]_{r\in\mathbb{Z},1\le l\le m-1}/I'.$$

Proof. (i) follows from Theorem 2.10 and (ii) from Lemma 2.8 and Lemma 3.2. Additionally, (iii) follows from Theorem 3.5. Concretely, we can define a ring homomorphism

$$\phi : \mathbb{Z}[y[r], z[r], w_l[r]]_{r\in\mathbb{Z},1\le l\le m-1} \to sh(KZ_n/J^d)$$

by $\phi(y[r]) = [M(1,\overline{1})^\bullet[r]]$, $\phi(z[r]) = [M(2,\overline{0})^\bullet[r]]$ and $\phi(w_l[r]) = [M(p^l+1,\overline{0})^\bullet[r]]$, the remaining is similar. □

5.2. The derived representation ring $dr(KZ_n/J^2)$ **.** So far, it is difficult to obtain all indecomposable objects in the bounded derived category $D^b(H)$, in general, for an arbitrary representation-finite K-Hopf algebra H, even for the Nakayama truncated algebra $H = KZ_n/J^d$.

When $d = 2$, however, we can list all the indecomposable objects of $D^b(H)$ using the method introduced by Bautista and Liu in [3] since $H = KZ_n/J^2$ is a gentle algebra, or more generally, an elementary algebra with a 2-nilpotent radical. All the notations and terminologies can be found in [3].

5.2.1. Construction of the indecomposable objects in $D^b(KZ_n/J^2)$**.**

LEMMA 5.2. *The set of indecomposable objects of $D^b(KZ_n/J^2)$ is given by*

$$\mathrm{ind}(D^b(KZ_n/J^2)) = \{P^\bullet(i,j)[l] \mid -\infty \le j \le i < +\infty, l\in\mathbb{Z}\},$$

where the complex

$$P^\bullet(i,j): \qquad \cdots \to 0 \to P_{\overline{i}} \to P_{\overline{i-1}} \to \cdots \to P_{\overline{j}} \to 0 \to \cdots,$$

with the morphisms $P_{\overline{a+1}} \to P_{\overline{a}}$ defined by the action of arrows for all $j \le a < i$.

Proof. Our proof is dependent on Theorem 3.11 in [3], where the composition of arrows is from left to right, which differs from the composition in this paper. Here, we will use the dual conclusion for the converse composition, that naturally follows from that in [3].

Firstly, it is easy to see that the minimal grading covering $\widetilde{Q}$ of Q is just the infinite Dynkin graph A_∞ with the universal covering map sends m to m(under the relation of modulo n), see Fig. 6. We may further denote the vertices of A_∞ as $\cdots - 2, -1, 0, 1, 2, \cdots$.

Since KQ/J^2 is an elementary algebra with a 2-nilpotent radical, by Theorem 3.11 in [3] there is a functor $\mathscr{F} : \mathrm{rep}^{-,p}(\widetilde{Q}) \to D^b(KQ/J^2)$ which preserves isomorphism classes and indecomposability, where $\mathrm{rep}^{-,p}(\widetilde{Q})$ denotes the fully subcategory of $\mathrm{rep}(\widetilde{Q})$ consisting of the bounded-above truncated injective representations. Moreover, by ([3], Lemma 3.4), there exists an indecomposable object N in $\mathrm{rep}^{-,p}(\widetilde{Q})$ and an integer n such that $M \cong \mathscr{F}(N)[n]$ for any indecomposable object in $D^b(KZ_n/J^2)$.

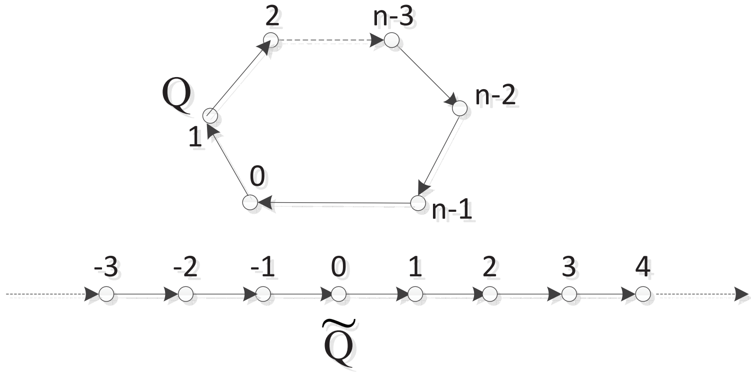

FIG. 6.

Hence, to characterize the indecomposable objects of $D^b(KZ_n/J^2)$, it suffices to compute the indecomposable objects of $\mathrm{rep}^{-,p}(\widetilde{Q})$.

Secondly, as the quiver $\widetilde{Q} = A_\infty$ with linear orientation, all the indecomposable bounded-above objects in $\mathrm{rep}^{-,p}(\widetilde{Q})$ are $\{L(i,j)\}_{-\infty\leq j\leq i<+\infty}$, where $L(i,j)$ for any i,j satisfies

$$L(i,j)_a = \begin{cases} K, & (j \leq a \leq i), \\ 0, & \text{otherwise}, \end{cases} \quad L(i,j)_{a\to a+1} = \begin{cases} id_K, & (j \leq a \leq i-1), \\ 0, & \text{otherwise}. \end{cases}$$

Finally, we consider the images of $L(i,j)$ under the action of the functor $\mathscr{F}$. By the definition of $L(i,j)$ above, we find that $\mathscr{F}(L(i+1,j+1)) = P^\bullet(i,j)$, where $P^\bullet(i,j)$ are defined in this lemma. Because $P^\bullet(i,j)$ has bounded homology for any $-\infty \leq j \leq i < +\infty$, each such $L(i,j)$ is included in $\mathrm{rep}^{-,p}(\widetilde{Q})$ by ([3], Proposition 3.8). Therefore, the set of all indecomposable objects of $D^b(KZ_n/J^2)$ is just the set $\{P^\bullet(i,j)[l]\}_{-\infty\leq j\leq i<+\infty, l\in\mathbb{Z}}$ by Proposition 3.1 and Theorem 3.11 in [3]. ◻

In particular, we have $P^\bullet(i,-\infty) = \cdots \to 0 \to P_{\bar{i}} \to P_{\overline{i-1}} \to \cdots$. since $S^\bullet_{\overline{i+1}}[i]$ is quasi-isomorphic to $P^\bullet(i,-\infty)$ in the complex category $Ch(H)$. Hence,

$$P^\bullet(i,-\infty) \cong S^\bullet_{\overline{i+1}}[i] \quad \text{in} \quad D^b(H). \tag{15}$$

Note that if we choose $j = i$, then $\mathscr{F}(L(i,i))$ is the stalk complex $P_{\bar{i}}$ in the i-th position.

5.2.2. Structure constants of the derived representation ring $dr(KZ_n/J^2)$**.** The derived representation ring $dr(KZ_n/J^2)$ is generated by all objects in $\mathrm{ind}(D^b(H)) = \{(P^\bullet(i,j))[l] \mid -\infty \leq j \leq i < +\infty, l \in \mathbb{Z}\}$. From (15), we have

$$\begin{aligned} P^\bullet(i',-\infty)[l'] \otimes P^\bullet(i,-\infty)[l] &\cong S^\bullet_{\overline{i'+1}}[i'+l'] \otimes S^\bullet_{\overline{i+1}}[i+l] \\ &\cong S^\bullet_{\overline{i'+i+2}}[i'+i+l'+l] \\ &\cong P^\bullet(i'+i+1,-\infty)[l'+l-1], \end{aligned} \tag{16}$$

So, it remains to consider the structure constants associated with the tensor product $P^\bullet(i',j')[l'] \otimes P^\bullet(i,j)[l]$ where at least one of j' and j is not equal to $-\infty$. For simplicity, we may further assume that $l = l' = 0$.

To calculate the decomposition of $P^\bullet(i',j') \otimes P^\bullet(i,j)$ in the case that one of j' and j is equal to $-\infty$ and the other is not $-\infty$, we need the following lemma about $P_{\overline{i'}} \otimes S_{\overline{i}}$.

LEMMA 5.3. *There is a complex isomorphism from*

$$\cdots \to 0 \to P_{\overline{i'+1}} \otimes S_{\overline{i}} \xrightarrow{\alpha_{i'}\otimes id} P_{\overline{i'}} \otimes S_{\overline{i}} \to 0 \to \cdots$$

to

$$\cdots \to 0 \to P_{\overline{i+i'+1}} \xrightarrow{\alpha_{i+i'}} P_{\overline{i+i'}} \to 0 \to \cdots$$

where α_j is defined by the action of the corresponding arrows for $j = i', i+i'$.

Proof. Note that $P_{\overline{j}} = M(2,\overline{j}), S_{\overline{j}} = M(1,\overline{j})$ for any $j \in \mathbb{Z}$. By Lemma 2.4, we have $P_{\overline{j}} \otimes S_{\overline{i}} \cong P_{\overline{i+j}}$. Denote this isomorphism by f_j for $j = i', i'+1$. Hence $f_j(e_{\overline{j}} \otimes e_{\overline{i}}) = e_{\overline{j+i}}$, $f_j(\alpha_{\overline{j}} \otimes e_{\overline{i}}) = \alpha_{\overline{j+i}}$ for $j = i', i'+1$, where $e_{\overline{j}}$ denotes the primitive idempotent element corresponding to vertices $j = i', i'+i$, respectively. Now by calculation, we obtain the following commutative diagram.

$$\begin{array}{ccc}
P_{\overline{i'+1}} \otimes S_{\overline{i}} & \xrightarrow{\alpha_{i'}\otimes id} & P_{\overline{i'}} \otimes S_{\overline{i}} \\
{\scriptstyle f_{i'+1}}\downarrow & & {\scriptstyle f_{i'}}\downarrow \\
P_{\overline{i+i'+1}} & \xrightarrow{\alpha_{i+i'}} & P_{\overline{i+i'}}
\end{array},$$

then the complex isomorphism follows at once. □

PROPOSITION 5.4.

$$P^\bullet(i',j')[l'] \otimes P^\bullet(i,-\infty)[l] \cong P^\bullet(i'+i+1, j'+i+1)[l'+l-1] \tag{17}$$

for any $-\infty \leq j' \leq i' < \infty$, $i, l', l \in \mathbb{Z}$.

Proof. Firstly, in the case for $j' = -\infty$, the result follows from (16).

Next, in the case for $-\infty < j' \leq i'$, by Lemma 5.3 we have the following complex isomorphism

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P_{\overline{i'}} \otimes S_{\overline{i}} & \longrightarrow & P_{\overline{i'-1}} \otimes S_{\overline{i}} & \longrightarrow & \cdots & \longrightarrow & P_{\overline{j'}} \otimes S_{\overline{i}} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle f_{i'}}\downarrow & & {\scriptstyle f_{i'-1}}\downarrow & & & & {\scriptstyle f_{j'}}\downarrow & & {\scriptstyle 0}\downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P_{\overline{i'+i}} & \longrightarrow & P_{\overline{i'+i-1}} & \longrightarrow & \cdots & \longrightarrow & P_{\overline{j'+i}} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

i.e., $P^\bullet(i',j') \otimes S^\bullet_{\overline{i}}[i] \cong P^\bullet(i'+i, j'+i)$. From (15), we have the isomorphisms

$$P^\bullet(i',j') \otimes P^\bullet(i,-\infty) \cong P^\bullet(i',j') \otimes S^\bullet_{\overline{i+1}}[i] \cong P^\bullet(i'+i+1, j'+i+1)[-1].$$

Finally the result follows by $M^\bullet[l'] \otimes N^\bullet[l] \cong M^\bullet \otimes N^\bullet[l'+l]$ for any $M^\bullet, N^\bullet \in D^b(KZ_n/J^2)$, $l', l \in \mathbb{Z}$. □

Now we consider the decomposition of $P^\bullet(i',j')[l] \otimes P^\bullet(i,j))[l']$ in the case j, j' are both not $-\infty$. Firstly, we calculate all of its homological groups.

PROPOSITION 5.5. *For $j, j', s, s' \in \mathbb{Z}$ and $-\infty < s' \leq s$, let*

$$\mathcal{L} = P^\bullet(j'+s'-1, j') \otimes P^\bullet(j+s-1, j).$$

Then the homological groups $H_m(\mathcal{L})$ of $\mathcal{L}$ for all $m \in \mathbb{Z}$ are listed as follows:

$$H_m(\mathcal{L}) = \begin{cases} S_{\overline{j+j'+s+s'}}, & \text{if } m = j+j'+s+s'-2, s' \neq 1; \\ S_{\overline{j+j'+s}}, & \text{if } m = j+j'+s-1, s \neq s', s' \neq 1; \\ S_{\overline{j+j'+s'}}, & \text{if } m = j+j'+s'-1, s \neq s', s' \neq 1; \\ S_{\overline{j+j'+s}} \oplus S_{\overline{j+j'+s'}}, & \text{if } m = j+j'+s-1, s = s' \neq 1; \\ P_{\overline{j+j'+s}}, & \text{if } m = j+j'+s-1, s \neq s' = 1; \\ S_{\overline{j+j'}}, & \text{if } m = j+j', s' \neq 1; \\ P_{\overline{j+j'}}, & \text{if } m = j+j', s \neq s' = 1; \\ P_{\overline{j+j'+1}} \oplus P_{\overline{j+j'}}, & \text{if } m = 0, s = s' = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Proof. For the notations in relation to spectral sequence, refer to ([30], §5.6). As there exists a double complex C_{**} such that $\mathcal{L} = \mathrm{Tot}^{\oplus}(C_{**})$, we can calculate the homological groups $H_m(\mathcal{L})$ through C_{**} using the method of spectral sequences.

When $s \geq s' > 1$, since C_{**} is bounded, we obtain a spectral sequence $\{{}^I E^r_{pq}\}$, which converges to $H_{p+q}(\mathcal{L})$ via the natural filtration by columns. More precisely, the spectral sequence starts with $\{{}^I E^0_{p,q} = C_{p,q}\}$ and the differentials $d^0_{pq} = d^v_{pq}$, where d^v_{pq} ($\forall p, q \in \mathbb{Z}$) are the vertical differentials of C_{**} and denote $d^v = \{d^v_{pq}\}_{p,q\in\mathbb{Z}}$, $d^0 = \{d^0_{pq}\}_{p,q\in\mathbb{Z}}$. Now we find that

$${}^I E^1_{p,q} = H^v_q(C_{p,*}) = \begin{cases} P_{\overline{j'+p}} \otimes S_{\overline{j+s}}, & \text{if } j' \leq p \leq j'+s'-1,\ q = j+s-1; \\ P_{\overline{j'+p}} \otimes S_{\overline{j}}, & \text{if } j' \leq p \leq j'+s'-1,\ q = j; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, the maps $d^1_{pq} : {}^I E^1_{p,q} \to {}^I E^1_{p-1,q}$ are induced by the horizontal differentials d^h_{pq} of C_{**}. Hence we find that the 2-piece ${}^I E^2_{p,q}$ is as follows:

$${}^I E^2_{p,q} = \begin{cases} S_{\overline{j'+s'}} \otimes S_{\overline{j+s}} = S_{\overline{j+j'+s+s'}}, & \text{if } p = j'+s'-1, q = j+s-1; \\ S_{\overline{j'+s'}} \otimes S_{\overline{j}} = S_{\overline{j+j'+s'}}, & \text{if } p = j'+s'-1, q = j; \\ S_{\overline{j'}} \otimes S_{\overline{j+s}} = S_{\overline{j+j'+s}}, & \text{if } p = j', q = j+s-1; \\ S_{\overline{j'}} \otimes S_{\overline{j}} = S_{\overline{j+j'}}, & \text{if } p = j', q = j; \\ 0, & \text{otherwise.} \end{cases}$$

Since $s \geq s'$, there exist no p, q such that both ${}^I E^2_{p,q}$ and ${}^I E^2_{p-2,q+1}$ are not zero. Thus the maps $d^2_{pq} : {}^I E^2_{p,q} \to {}^I E^2_{p-2,q+1}$ are zero maps, which implies that ${}^I E^2_{p,q} = {}^I E^3_{p,q} = \cdots {}^I E^m_{p,q} = {}^I E^\infty_{p,q}$. Since the spectral sequence $\{{}^I E^r_{pq}\}$ converges to $H_{p+q}(\mathcal{L})$, it follows that $H_m(\mathcal{L})$ has a filtration $\{F_p H_m(\mathcal{L})\}$ such that $F_p H_m(\mathcal{L})/F_{p-1} H_m(\mathcal{L}) \cong {}^I E^\infty_{p,m-p}$ for any $m \in \mathbb{N}$.

In case $s \neq s'$, for any m, there is at most one p_0 such that ${}^I E^\infty_{p,m-p} \neq 0$. Hence by the filtration of $H_m(\mathcal{L})$, we have $H_m(\mathcal{L}) ={}^I E^\infty_{p_0,m-p_0}$.

In case $s = s'$, by the filtration of $H_{s-1}(\mathcal{L})$, there is an exact sequence

$$0 \to S_{\overline{j+j'+s}} \to H_{j+j'+s-1}(\mathcal{L}) \to S_{\overline{j+j'+s}} \to 0.$$

Moreover, since $\mathrm{Ext}^1_H(S_{\overline{j+j'+s}}, S_{\overline{j+j'+s}}) = 0$, we have $H_{j+j'+s-1}(\mathcal{L}) = S_{\overline{j+j'+s}} \oplus S_{\overline{j+j'+s}}$. Additionally, for $m \neq s-1$, there is at most one p_0 such that ${}^I E^\infty_{p_0,m-p_0} \neq 0$. Hence by the filtration of $H_m(\mathcal{L})$, we obtain $H_m(\mathcal{L}) ={}^I E^\infty_{p_0,m-p_0}$.

In a word, when $s \neq s'$, we have

$$H_m(\mathcal{L}) = \begin{cases} S_{\overline{j+j'+s+s'}}, & \text{if } m = j+j'+s+s'-2; \\ S_{\overline{j+j'+s}}, & \text{if } m = j+j'+s-1; \\ S_{\overline{j+j'+s'}}, & \text{if } m = j+j'+s'-1; \\ S_{\overline{j+j'}}, & \text{if } m = j+j'; \\ 0, & \text{otherwise;} \end{cases} \tag{19}$$

when $s = s'$, we have

$$H_m(\mathcal{L}) = \begin{cases} S_{\overline{j+j'+s+s'}}, & \text{if } m = j+j'+s+s'-2; \\ S_{\overline{j+j'+s}} \oplus S_{\overline{j+j'+s'}}, & \text{if } m = j+j'+s-1; \\ S_{\overline{j+j'}}, & \text{if } m = j+j'; \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

The proof in the case $s' = 1$ is similar following the explanation below.

When $s > s' = 1$, the first piece of the spectral sequence $\{{}^I E^r_{p,q}\}$ can be calculated as:

$$^I E^1_{p,q} = \begin{cases} P_{\overline{j'}} \otimes S_{\overline{j+s}} \cong P_{\overline{j+j'+s}}, & \text{if } p = j', q = j+s-1; \\ P_{\overline{j'}} \otimes S_{\overline{j}} \cong P_{\overline{j+j'}}, & \text{if } p = j', q = j; \\ 0, & \text{otherwise.} \end{cases}$$

Since there exists no p, q such that both ${}^I E^1_{p,q}$ and ${}^I E^1_{p-1,q}$ are non-zeros, the maps $d^1_{pq} : {}^I E^1_{p,q} \to {}^I E^1_{p-1,q}$ are zero maps, which implies that ${}^I E^1_{p,q} = {}^I E^2_{p,q} = \cdots {}^I E^m_{p,q} = {}^I E^\infty_{p,q}$. Thus,

$$H_m(\mathcal{L}) = \begin{cases} P_{\overline{j+j'+s}}, & \text{if } m = j+j'+s-1; \\ P_{\overline{j+j'}}, & \text{if } m = j+j'; \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Finally, when $s = s' = 1$, $\mathcal{L} = P_{\overline{j'}}[j'] \otimes P_{\overline{j}}[j'] \cong P_{\overline{j+j'+1}}[j+j'] \oplus P_{\overline{j+j'}}[j+j']$. $\square$

Through Proposition 5.5 and some calculations, we can characterize objects in $D^b(H)$ whose homological groups are isomorphic to $H_m(\mathcal{L})$ in two cases.

(I) When $s' = 1$, an object $\mathcal{R}$ in $D^b(H)$ satisfies $H_m(\mathcal{R}) \cong H_m(\mathcal{L})$ for all $m \in \mathbb{Z}$ if and only if $\mathcal{R} \cong P^\bullet_{\overline{j+j'}}[j+j'] \oplus P^\bullet_{\overline{j+j'+s}}[j+j'+s-1]$.

(II) When $s' > 1$, an object $\mathcal{R}$ in $D^b(H)$ satisfies $H_m(\mathcal{R}) \cong H_m(\mathcal{L})$ for all $m \in \mathbb{Z}$ if and only if one of the following four cases holds:

Case 1: $\mathcal{R}$ has a direct summand with the form $S_{\overline{k}}[m]$ for some $0 \leq k \leq n-1$ and $m \in \mathbb{Z}$;

Case 2: $\mathcal{R} \cong \mathcal{R}_2 = P^\bullet(j+j'+s'-1, j+j') \oplus P^\bullet(j+j'+s+s'-1, j+j'+s)[-1]$;

Case 3: $\mathcal{R} \cong \mathcal{R}_3 = P^\bullet(j+j'+s-1, j+j') \oplus P^\bullet(j+j'+s+s'-1, j+j'+s')[-1]$;

Case 4: $\mathcal{R} \cong \mathcal{R}_4 = P^\bullet(j+j'+s+s'-1, j+j') \oplus P^\bullet(j+j'+s-1, j+j'+s')[-1]$. This case applies only if $s > s'$.

Obviously the isomorphic objects in $D^b(H)$ must have isomorphic homological groups. Therefore, from (I), when $s' = 1$, we have the decomposition

$$\mathcal{L} = P^\bullet(j'+s'-1, j') \otimes P^\bullet(j+s-1, j) = P^\bullet_{j+j'}[j+j'] \oplus P^\bullet_{j+j'+s}[j+j'+s-1]; \tag{22}$$

when $s' > 1$, the decomposition of $\mathcal{L}$ must be equal to one of $\mathcal{R}$'s in the above four cases. However, the following Proposition 5.7 tells us that the decomposition of $\mathcal{L}$ is impossible when it takes the form in Case 1 above in (II).

LEMMA 5.6. *A chain map $f : P^\bullet(i, -\infty) \to P^\bullet(i, -\infty)$ is null homotopic if and only if $f = 0$.*

Proof. If $f = (f_j)_{j\in\mathbb{Z}}$ is null homotopic, then there exist $s_j : P_{\overline{j}} \to P_{\overline{j+1}}$ such that $f_j = s_{j-1}d_j + d_{j+1}s_j$ for all $j \in \mathbb{Z}$, where d_j denote the differentials of $P^\bullet(i, -\infty)$. By the definition of H, it is easy to see that $\mathrm{Hom_H}(\mathrm{P}_{\bar{\mathrm{i}}}, \mathrm{P}_{\overline{\mathrm{i+1}}}) = 0$. Therefore $s_j = 0$ for all $j \in \mathbb{Z}$. Hence $f_j = 0$ for all $j \in \mathbb{Z}$. The converse is trivial. □

PROPOSITION 5.7. *$\mathcal{L} = P^\bullet(j' + s' - 1, j') \otimes P^\bullet(j + s - 1, j)$ contains no direct summand as the form $S^\bullet_{\overline{k}}[m]$ for $0 \le k \le n - 1$ and $m \in \mathbb{Z}$.*

Proof. Otherwise, there exists $m \in \mathbb{Z}$, $0 \le k \le n - 1$ such that $S^\bullet_{\overline{k}}[m]$ is a direct summand of $\mathcal{L}$. Since $K^-(\mathcal{I}) \cong D^b(H)$, $P^\bullet(k - 1, -\infty)$ is the minimal injective resolution of $S_{\overline{k}}$ and $K^-(\mathcal{I})$ is the bounded-above homotopy category of injective complexes, which is a subcategory of a homotopy category consisting of bounded-above injective complexes, thus $P^\bullet(k - 1, -\infty)[m]$ is a direct summand of $\mathcal{L}$ in $K^-(\mathcal{I})$. Therefore, there exists $\overline{f} \in \mathrm{Hom}_{K^-(\mathcal{I})}(\mathcal{L}, \mathrm{P}^\bullet(\mathrm{k} - 1, -\infty)[\mathrm{m}])$ and $\overline{g} \in \mathrm{Hom}_{K^-(\mathcal{I})}(\mathrm{P}^\bullet(\mathrm{k} - 1, -\infty)[\mathrm{m}], \mathcal{L})$ such that $\overline{f}\overline{g} = id$ in $K^-(\mathcal{I})$. Equivalently, there exists $f \in \mathrm{Hom_{Ch(H)}}(\mathcal{L}, \mathrm{P}^\bullet(\mathrm{k} - 1, -\infty)[\mathrm{m}])$ and $g \in \mathrm{Hom_{Ch(H)}}(\mathrm{P}^\bullet(\mathrm{k} - 1, -\infty)[\mathrm{m}], \mathcal{L})$ such that $fg - id$ is null homotopic in $Ch(H)$. By Lemma 5.6, $fg - id = 0$, so $fg = id$. Then we deduce that $P^\bullet(k - 1, -\infty)[m]$ is a direct summand of $\mathcal{L}$ in $Ch(H)$. But this is impossible because $P^\bullet(j + s - 1, j) \otimes P^\bullet(j' + s' - 1, j')$ is a bounded complex. □

From Proposition 5.7, we know that the decomposition of $\mathcal{L}$ cannot be in Case 1 when $s' > 1$.

Additionally, when $s = s' > 1$, Case 2 coincides with Case 3 and Case 4 does not appear. Hence in this condition, we have the decomposition:

$$\mathcal{L} = P^\bullet(j + j' + s' - 1, j + j') \oplus P^\bullet(j + j' + s + s' - 1, j + j' + s)[-1]. \tag{23}$$

Therefore, we need only to consider the decomposition of $\mathcal{L}$ when $s > s' > 1$. Unfortunately, so far we cannot determine the decomposition of $\mathcal{L}$ in the form of Cases 2, 3 and 4 exactly in general. We conjecture that only Case 2 can occur in the decomposition of $\mathcal{L}$, according to several decompositions we have already calculated. We can summarize the above discussion in the following conjecture.

CONJECTURE 5.8. *Using the above notations in $D^b(KZ_n/J^2)$ for all $j, j', s, s' \in \mathbb{Z}$ and $s' \le s$, we always have the following decomposition of indecomposable complexes:*

$$\begin{aligned}\mathcal{L} &= P^\bullet(j' + s' - 1, j') \otimes P^\bullet(j + s - 1, j)\\ &\cong P^\bullet(j + j' + s' - 1, j + j') \oplus P^\bullet(j + j' + s + s' - 1, j + j' + s)[-1].\end{aligned} \tag{24}$$

From the above discussion, we know that this conjecture has been affirmed except for the case $s > s' > 1$.

Lastly, we give a summary of the derived representation ring $dr(KZ_n/J^2)$ as follows.

THEOREM 5.9. *For* $\mathrm{char}K = p = 2, n \ge 2$*, the following statements hold:*

(i) *The derived representation ring $dr(KZ_n/J^2)$ is generated by $[P^\bullet(i, j)[l]]$, where $l \in \mathbb{Z}, -\infty \le j \le i < +\infty$.*

(ii) *The relations of the generators of $dr(KZ_n/J^2)$ in (i) can be determined by:*
(a) $P^\bullet(i', j')[l'] \otimes P^\bullet(i, -\infty)[l] \cong P^\bullet(i, -\infty)[l] \otimes P^\bullet(i', j')[l'] \cong P^\bullet(i' + i + 1, j' + i +$

$1)[l'+l-1]$ *for all* $-\infty \leq j' \leq i' < \infty$, $i, l, l' \in \mathbb{Z}$;
(b) *for all* $j, j', s, s' \in \mathbb{Z}$ *and* $s' \leq s$:
(1) *when* $s' = 1$, *the decomposition*

$$\begin{aligned} & P^{\bullet}(j'+s'-1, j')[l'] \otimes P^{\bullet}(j+s-1, j)[l] \\ = & P^{\bullet}_{\overline{j+j'}}[j+j'+l+l'] \oplus P^{\bullet}_{\overline{j+j'+s}}[j+j'+s+l+l'-1]; \end{aligned}$$

(2) *when* $s = s' > 1$, *the decomposition:*

$$\begin{aligned} & P^{\bullet}(j'+s'-1, j')[l'] \otimes P^{\bullet}(j+s-1, j)[l] \\ = & P^{\bullet}(j+j'+s'-1, j+j')[l+l'] \oplus P^{\bullet}(j+j'+2s-1, j+j'+s)[l+l'-1]; \end{aligned}$$

(3) *when* $s > s' > 1$, *one of the three possible decompositions:*

$$\begin{aligned} & P^{\bullet}(j'+s'-1, j')[l'] \otimes P^{\bullet}(j+s-1, j)[l] \\ = & P^{\bullet}(j+j'+s'-1, j+j')[l+l'] \oplus P^{\bullet}(j+j'+s+s'-1, j+j'+s)[l+l'-1]; \\ & P^{\bullet}(j'+s'-1, j')[l'] \otimes P^{\bullet}(j+s-1, j)[l] \\ = & P^{\bullet}(j+j'+s-1, j+j')[l+l'] \oplus P^{\bullet}(j+j'+s+s'-1, j+j'+s')[l+l'-1]; \\ & P^{\bullet}(j'+s'-1, j')[l'] \otimes P^{\bullet}(j+s-1, j)[l] \\ = & P^{\bullet}(j+j'+s+s'-1, j+j')[l+l'] \oplus P^{\bullet}(j+j'+s-1, j+j'+s')[l+l'-1]. \end{aligned}$$

(iii) *For the indeterminates* $\nu(i,j)[l]$, *let* I'' *be the ideal of the polynomial ring* $\mathbb{Z}[\nu(i,j)[l]]_{l\in\mathbb{Z}, -\infty\leq j\leq i<+\infty}$ *generated by the relations given in (ii) in this theorem through replacing* $[P^{\bullet}(i,j)[l]]$ *by* $\nu(i,j)[l]$, *for* $l \in \mathbb{Z}, -\infty \leq j \leq i < +\infty$. *Then there is the ring isomorphism*

$$dr(KZ_n/J^2) \cong \mathbb{Z}[\nu(i,j)[l]]_{l\in\mathbb{Z}, -\infty\leq j\leq i<+\infty}/I''.$$

Proof. (i) follows by Lemma 5.2. For the proof of (iii), in detail, it suffices to note that all the indecomposable objects of $D^b(KZ_n/J^2)$ are $P^{\bullet}(i,j)[l]_{-\infty\leq j\leq i<+\infty, l\in\mathbb{Z}}$ and the ring homomorphism $\psi : \mathbb{Z}[\nu(i,j)[l]]_{l\in\mathbb{Z}, -\infty\leq j\leq i<+\infty} \to dr(KZ_n/J^2)$ can be defined by $\psi(\nu(i,j)[l]) = [P^{\bullet}(i,j)[l]]$.

For (ii), (a) follows from Proposition 5.4; and (b) is according to the above discussion with (22), (23) and the cases 2,3 and 4 in (II). □

Furthermore, if Conjecture 5.8 is confirmed, we can unify (1), (2) and (3) of Theorem 5.9 (ii) with the decomposition (24) in Conjecture 5.8.

Remark 5.10. In Proposition 5.1, when $d = 2 = p$, then $m = 1$ and thus the shift ring $sh(KZ_n/J^2)$ is generated by $[M(1,\overline{1})^{\bullet}[r]], [M(2,\overline{0})^{\bullet}[r]]$ for $r \in \mathbb{Z}$. As we know, $sh(KZ_n/J^2)$ is a sub-ring of $dr(KZ_n/J^2)$. And, $M(1,\overline{1})^{\bullet}[r] = S^{\bullet}_{\overline{1}}[r] \cong P^{\bullet}(0,-\infty)[r]$ and $M(2,\overline{0})^{\bullet}[r] \cong P^{\bullet}_{\overline{0}}[r] \cong P^{\bullet}(0,0)[r]$. So, the generators $[M(1,\overline{1})^{\bullet}[r]]$ and $[M(2,\overline{0})^{\bullet}[r]]$ of $sh(KZ_n/J^2)$ in Proposition 5.1 are simply those $P^{\bullet}(0,-\infty)[r]$ and $P^{\bullet}(0,0)[r]$ of $dr(KZ_n/J^2)$ for $r \in \mathbb{Z}$ in Theorem 5.9.

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