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UNIVERSAL COVERING CALABI-YAU MANIFOLDS OF THE HILBERT SCHEMES OF n POINTS OF ENRIQUES SURFACES*

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Abstract. The purpose of this paper is to investigate the Hilbert scheme of n points of an Enriques surface from the following three points of view: (i) the relationship between the small deformation of the Hilbert scheme of n points of an Enriques surface and that of its universal cover (Theorem 1.1), (ii) the natural automorphisms of the Hilbert scheme of n points of an Enriques surface (Theorem 1.4), and (iii) the number of distinct Hilbert schemes of n points of Enriques surfaces, which has the same universal covering space (Theorem 1.7).

Key words. Calabi-Yau manifold, Enriques surface, Hilbert scheme.

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1. Introduction. Throughout this paper, we work over $\mathbb{C}$, and n is an integer such that $n \geq 2$. A $K3$ surface K is a compact complex surface with $\omega_K \simeq \mathcal{O}_K$ and $\mathrm{H}^1(K, \mathcal{O}_K) = 0$. An Enriques surface E is a compact complex surface with $\mathrm{H}^1(E, \mathcal{O}_E) = 0$, $\mathrm{H}^2(E, \mathcal{O}_E) = 0$, and $\omega_E^{\otimes 2} \simeq \mathcal{O}_E$. A Calabi-Yau manifold X is an n-dimensional compact kähler manifold such that it is simply connected, there is no holomorphic k-form on X for $0 < k < n$, and there is a nowhere vanishing holomorphic n-form on X. By Oguiso and Schröer [11, Theorem 3.1], the Hilbert scheme of n points of an Enriques surface $E^{[n]}$ has a Calabi-Yau manifold X as the universal covering space of degree 2. Recall that when $n = 1$, $E^{[1]}$ is an Enriques surface E, and X is a $K3$ surface.

In this paper, we study the Hilbert scheme of n points of an Enriques surface $E^{[n]}$ from the relationship between $E^{[n]}$ and its universal covering space X (Theorem 1.1 and 1.7) and the natural automorphisms of $E^{[n]}$ (Theorem 1.4).

Section 2 is a preliminary section. We prepare and recall some basic facts on the Hilbert scheme of n points of a surface and show that for the universal covering space X of $E^{[n]}$, there is a quotient singular variety Z such that X is a resolution of Z (Theorem 2.7).

In Section 3, we investigate the relationship between the small deformation of $E^{[n]}$ and that of X. When $n = 1$, $E^{[1]}$ is an Enriques surface E, and X is a $K3$ surface. An Enriques surface has a 10-dimensional deformation space and a $K3$ surface has a 20-dimensional deformation space. Thus the small deformation of X is much bigger than that of E. For $n \geq 2$, by using the result of Göttsche and Soergel [7, Theorem 2] and the properties of the covering space $X \to E^{[n]}$, we compute the dimension of the deformation space of X. Consequently, we obtain Theorem 1.1 which is different from the case of $n = 1$:

THEOREM 1.1. *For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then every small deformation of X is induced by that of $E^{[n]}$.*

REMARK 1.2. By Fantechi [4, Theorems 0.1 and 0.3], every small deformation of $E^{[n]}$ is induced by that of E. Thus for $n \geq 2$, every small deformation of X is

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induced by that of E.

In Section 4, we study the natural automorphisms of $E^{[n]}$.

DEFINITION 1.3. For $n \geq 2$ and S a smooth compact surface, any automorphism $f \in \mathrm{Aut}(S)$ induces an automorphism $f^{[n]} \in \mathrm{Aut}(S^{[n]})$. An automorphism $g \in \mathrm{Aut}(S^{[n]})$ is called natural if there is an automorphism $f \in \mathrm{Aut}(S)$ such that $g = f^{[n]}$.

When S is a $K3$ surface, the natural automorphisms of $S^{[n]}$ were studied by Boissière and Sarti [3]. They showed that an automorphism of $S^{[n]}$ is natural if and only if it preserves the exceptional divisor of the Hilbert-Chow morphism [3, Theorem 1]. We obtain Theorem 1.4 which is similar to [3, Theorem 1]:

THEOREM 1.4. *For $n \geq 2$, let E be an Enriques surface, and D the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \to E^{(n)}$. An automorphism f of $E^{[n]}$ is natural if and only if $f(D) = D$.*

In Section 5, we compute the number of distinct Enriques surface type quotients of X for a fixed X.

DEFINITION 1.5. For $n \geq 1$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. A variety Y is called an Enriques surface type quotient of X if there is an Enriques surface E' and a free involution τ of X such that $Y \simeq E'^{[n]}$ and $E'^{[n]} \simeq X/\langle\tau\rangle$. Here we call two Enriques surface type quotients of X distinct if they are not isomorphic to each other.

Recall that when $n = 1$, $E^{[1]}$ is an Enriques surface E and X is a $K3$ surface. In [12, Theorem 0.1], Ohashi showed the following theorem:

THEOREM 1.6. *For any nonnegative integer l, there exists a $K3$ surface with exactly 2^{l+10} distinct Enriques quotients. In particular, there does not exist a universal bound for the number of distinct Enriques quotients of a $K3$ surface.*

We obtain Theorem 1.7 which is different from Theorem 1.6 in the sense of the Enriques surface type quotient:

THEOREM 1.7. *For $n \geq 3$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then the number of distinct Enriques surface type quotients of X is one.*

REMARK 1.8. When $n = 2$, we do not count the number of distinct Enriques surface type quotients of X. We compute the Hodge numbers of the universal covering space X of $E^{[2]}$ (Appendix A).

In Proposition 5.2, we show that for $n \geq 3$, the covering involution of $\pi : X \to E^{[n]}$ acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity. In Proposition 5.5, by using Theorem 2.7 and 1.4, we show that for $n \geq 2$, if an automorphism φ of X acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity, then φ is a lift of a natural automorphism of $E^{[n]}$. In Proposition 5.9, by using Proposition 5.5 and checking the action to $\mathrm{H}^1(X, \Omega_X^{2n-1})$, we classify involutions of X which act on $\mathrm{H}^2(X, \mathbb{C})$ as the identity. We prove Theorem 1.7 using those results.

In addition, let Y be a smooth compact Kähler surface. For a line bundle L on Y, by using the natural map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y^{[n]}))$, $L \mapsto L_n$, we put

$$h^{p,q}(Y^{[n]}, L_n) := \dim_{\mathbb{C}} \mathrm{H}^q(Y^{[n]}, \Omega^p_{Y^{[n]}} \otimes L_n),$$

$$h^{p,q}(Y,L) := \dim_{\mathbb{C}} \mathrm{H}^q(Y, \Omega_Y^p \otimes L),$$

$$A := \sum_{n,p,q=0}^{\infty} h^{p,q}(Y^{[n]}, L_n) x^p y^q t^n, \text{ and}$$

$$B := \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \Big(\frac{1}{1-(-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k} \Big)^{(-1)^{p+q} h^{p,q}(Y,L)}.$$

In [2, Conjecture 1], S. Boissière conjectured that

$$A = B.$$

In the proof of Theorem 1.1, we obtain the counterexample to this conjecture for Y an Enriques surface and $L = \Omega_Y^2$. See Appendix B for details.

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2. Preliminaries. Let S be a nonsingular surface, $S^{[n]}$ the Hilbert scheme of n points of S, $\pi_S : S^{[n]} \to S^{(n)}$ the Hilbert-Chow morphism, and $p_S : S^n \to S^{(n)}$ the natural projection. We denote the exceptional divisor of π_S by D. By Fogarty [5, Theorem 2.4], $S^{[n]}$ is smooth of $\dim_{\mathbb{C}} S^{[n]} = 2n$.

Let Δ^n be the set of n-uples $(x_1, \dots, x_n) \in S^n$ with at least two x_i's equal, S_*^n the set of n-uples $(x_1, \dots, x_n) \in S^n$ with at most two x_i's equal. We put

$$S_*^{(n)} := p_S(S_*^n),$$

$$\Delta^{(n)} := p_S(\Delta^n),$$

$$S_*^{[n]} := \pi_S^{-1}(S_*^{(n)}),$$

$$\Delta_*^n := \Delta^n \cap S_*^n,$$

$$\Delta_*^{(n)} := p_S(\Delta_*^n), \text{ and}$$

$$F := S^{[n]} \setminus S_*^{[n]}.$$

When $n = 2$, $S_*^2 = S^2$, $F = \emptyset$ and $\mathrm{Blow}_{\Delta^2} S^2/\mathcal{S}_2 \simeq S^{[2]}$. For $n \geq 3$, we have $\mathrm{Blow}_{\Delta_*^n} S_*^n/\mathcal{S}_n \simeq S_*^{[n]}$, and F is an analytic closed subset and its codimension is 2 in $S^{[n]}$ by Beauville [1, page 767-768]. Here $\mathcal{S}_n$ is the symmetric group of degree n which acts naturally on S^n by permuting of the factors.

Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space. Let $\mu : K \to E$ be the universal covering space of E where K is a $K3$ surface, and Λ the pullback of $\Delta^{(n)}$ by the morphism:

$$\mu^{(n)} : K^{(n)} \ni [(x_1, \dots, x_n)] \mapsto [(\mu(x_1), \dots, \mu(x_n))] \in E^{(n)}.$$

Then we get a 2^n-sheeted unramified covering space:

$$\mu^{(n)}|_{K^{(n)}\backslash\Lambda} : K^{(n)}\backslash\Lambda \to E^{(n)}\backslash\Delta^{(n)}.$$

Furthermore, let Γ be the pullback of Λ by the natural projection $p_K : K^n \to K^{(n)}$. Since Γ is an algebraic closed set with codimension 2, then

$$\mu^{(n)} \circ p_K : K^n\backslash\Gamma \to E^{(n)}\backslash\Delta^{(n)}$$

is the $2^n n!$-sheeted universal covering space. Since $E^{[n]}\backslash D = E^{(n)}\backslash\Delta^{(n)}$ where $D = \pi_E^{-1}(\Delta^{(n)})$, we regard the universal covering space

$$\mu^{(n)} \circ p_K : K^n\backslash\Gamma \to E^{(n)}\backslash\Delta^{(n)}$$

as the universal covering space of $E^{[n]} \setminus D$

$$\mu^{(n)} \circ p_K : K^n\backslash\Gamma \to E^{[n]}\backslash D.$$

Since $\pi : X \setminus \pi^{-1}(D) \to E^{[n]} \setminus D$ is a covering space, and $\mu^{(n)} \circ p_K : K^n \setminus \Gamma \to E^{[n]} \setminus D$ is the universal covering space, there is a morphism

$$\omega : K^n \setminus \Gamma \to X \setminus \pi^{-1}(D)$$

such that $\omega : K^n\backslash\Gamma \to X\backslash\pi^{-1}(D)$ is the universal covering space and $\mu^{(n)} \circ p_K = \pi\circ\omega$:

$$\begin{array}{ccc} K^n \setminus \Gamma & \xrightarrow{\ \omega\ } & X \setminus \pi^{-1}(D) \\ & \searrow^{\mu^{(n)}\circ p_K} & \downarrow \pi \\ & & E^{[n]} \setminus D. \end{array}$$

We denote the covering transformation group of $\pi \circ \omega$ by

$$G := \{g \in \mathrm{Aut}(K^n \setminus \Gamma_K) : \pi \circ \omega \circ g = \pi \circ \omega\}.$$

Since $\deg(\mu^{(n)} \circ p_K) = 2^n n!$, the order of G is $2^n n!$. Let σ be the covering involution of $\mu : K \to E$. For

$$1 \le k \le n,\ 1 \le i_1 < \cdots < i_k \le n,$$

we define automorphisms $\sigma_{i_1 \dots i_k}$ of K^n in the following way: for $x = (x_i)_{i=1}^n \in K^n$,

$$\text{the j-th component of } \sigma_{i_1\dots i_k}(x) = \begin{cases} \sigma(x_j) & j \in \{i_1, \dots, i_k\} \\ x_j & j \notin \{i_1, \dots, i_k\}, \end{cases}$$

then $\mathcal{S}_n \subset G$, and $\{\sigma_{i_1\dots i_k}\}_{1\le k\le n,\ 1\le i_1<\cdots<i_k\le n} \subset G$. Let H be the subgroup of G generated by $\mathcal{S}_n$ and $\{\sigma_{ij}\}_{1\le i<j\le n}$.

Proposition 2.1. *G is generated by $\mathcal{S}_n$ and $\{\sigma_{i_1\dots i_k}\}_{1\le k\le n,\ 1\le i_1<\dots<i_k\le n}$.*

Proof. We assume that

$$s \circ t = s' \circ t'$$

for some $s, s' \in \mathcal{S}_n$ and $t, t' \in \{\sigma_{i_1 \dots i_k}\}_{1 \le k \le n,\ 1 \le i_1 < \cdots < i_k \le n}$. If $s \neq s'$, then $s'^{-1} \circ s \neq \mathrm{Id}_{K^n}$. We take an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in K^n$ with $\tilde{x}_i \neq \tilde{x}_j$ for $1 \le i < j \le n$ and $\sigma(\tilde{x}_i) \neq \tilde{x}_j$ for $1 \le i \le j \le n$. Since $s \circ t = s' \circ t'$, we have $s'^{-1} \circ s(x) = t' \circ t^{-1}(x)$. Thus for some i where $1 \le i \le n$,

$$\sigma(\tilde{x}_i) \in \{\tilde{x}_j\}_{j=1}^n.$$

This contradicts the definition of $\tilde{x}$. Therefore we get $s = s'$ and $t = t'$. Since $|\mathcal{S}_n| = n!$, $|\{\sigma_{i_1 \dots i_k}\}_{1 \le k \le n,\ 1 \le i_1 < \dots < i_k \le n}| = 2^n$, and $|G| = 2^n n!$, G is generated by $\mathcal{S}_n$ and $\{\sigma_{i_1 \dots i_k}\}_{1 \le k \le n,\ 1 \le i_1 < \cdots < i_k \le n}$. □

Proposition 2.2. $|H| = 2^{n-1} n!$.

Proof. For $s \in \mathcal{S}_n$ and $\sigma_{j_1 \dots j_l} \in \{\sigma_{i_1 \dots i_k}\}_{1 \le k \le n,\ 1 \le i_1 < \cdots < i_k \le n}$, there are positive numbers $u_1, \dots, u_k$ such that

$$\{u_1, \dots, u_k\} = \{s^{-1}(j_1), \dots, s^{-1}(j_l)\}, \text{ and } u_1 < \cdots < u_k.$$

Then we get $\sigma_{j_1 \dots j_l} \circ s = s \circ \sigma_{u_1 \dots u_k}$. For arbitrary j, $(i, j) \circ \sigma_i \circ (i, j) = \sigma_j$. Since H is generated by $\mathcal{S}_n$ and $\{\sigma_{ij}\}_{1 \le i < j \le n}$, from Proposition 2.1 we obtain $|G/H| = 2$, i.e. $|H| = 2^{n-1} n!$. □

Recall that $\mu : K \to E$ is the universal covering and σ is the covering involution of μ. We put

$$K^n_{*\mu} := (\mu^n)^{-1}(E^n_*),$$

where $\mu^n : K^n \ni (x_i)_{i=1}^n \mapsto (\mu(x_i))_{i=1}^n \in E^n$,

$$T_{ij} := \{(x_l)_{l=1}^n \in K^n_{*\mu} : \sigma(x_i) = x_j\},$$

$$U_{ij} := \{(x_l)_{l=1}^n \in K^n_{*\mu} : x_i = x_j\},$$

$$T := \bigcup_{1 \le i < j \le n} T_{i,j}, \text{ and}$$

$$U := \bigcup_{1 \le i < j \le n} U_{ij}.$$

When $n = 2$, $K^2_{*\mu} = K^2$, $U = \Delta^2$, and $T = \{(x, y) \in K^2 : \sigma(x) = y\}$. By the definition of $K^n_{*\mu}$, H acts on $K^n_{*\mu}$. For an element $\tilde{x} := (\tilde{x}_i)_{i=1}^n \in U \cap T$, some i, j, k, l with $k \neq l$ such that $\sigma(\tilde{x}_i) = \tilde{x}_j$ and $\tilde{x}_k = \tilde{x}_l$. Since σ does not have fixed points. Thus $\tilde{x}_i \neq \tilde{x}_l$. Therefore $\mu^n(\tilde{x}) \notin E^n_*$. This is a contradiction. We obtain $T \cap U = \emptyset$.

Lemma 2.3. *For $t \in H$ and $1 \le i < j \le n$, if $t \in H$ has a fixed point on U_{ij}, then $t = (i, j)$ or $t = \mathrm{id}_{K^n}$.*

Proof. Let $t \in H$ be an element of H where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in U_{ij}$ such that $t(\tilde{x}) = \tilde{x}$. By Proposition 2.1, for $t \in H$, there are $\sigma_{i_1 \dots i_k} \in \{\sigma_{i_1 \dots i_k}\}_{1 \le k \le n,\ 1 \le i_1 < \cdots < i_k \le n}$ and $(j_1, \dots, j_l) \in \mathcal{S}_n$ such that

$$t = (j_1, \dots, j_l) \circ \sigma_{i_1 \dots i_k}.$$

From the definition of U_{ij}, for $(x_l)_{l=1}^n \in U_{ij}$,

$$\{x_1, \dots, x_n\} \cap \{\sigma(x_1), \dots, \sigma(x_n)\} = \emptyset.$$

Suppose $\sigma_{i_1,\cdots,i_k} \neq \mathrm{id}_{K^n}$. Since $t(\tilde{x}) = \tilde{x}$, we have

$$\{\tilde{x}_1, \dots, \tilde{x}_n\} \cap \{\sigma(\tilde{x}_1), \dots, \sigma(\tilde{x}_n)\} \neq \emptyset.$$

This is a contradiction. Thus we have $t = (j_1, \dots, j_l)$. Similarly from the definition of U_{ij}, for $(x_l)_{l=1}^n \in U_{ij}$, if $x_s = x_t$ $(1 \le s < t \le n)$, then $s = i$ and $t = j$. Thus we have $t = (i, j)$ or $t = \mathrm{id}_{K^n}$. □

LEMMA 2.4. *For $t \in H$ and $1 \le i < j \le n$, if $t \in H$ has a fixed point on T_{ij}, then $t = \sigma_{i,j} \circ (i, j)$ or $t = \mathrm{id}_{K^n}$.*

Proof. Let $t \in H$ be an element of H where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in T_{ij}$ such that $t(\tilde{x}) = \tilde{x}$. By Proposition 2.1, for $t \in H$, there are $\sigma_{i_1\dots i_k} \in \{\sigma_{i_1\dots i_k}\}_{1\le k\le n,\ 1\le i_1<\cdots<i_k\le n}$ and $(j_1, \dots, j_l) \in \mathcal{S}_n$ such that

$$t = (j_1 \dots j_l) \circ \sigma_{i_1\dots i_k}.$$

Since $(j, j+1) \circ \sigma_{i,j} \circ (j, j+1) : U_{ij} \to T_{ij}$ is an isomorphism, and by Lemma 2.3, we have

$$(j, j+1) \circ \sigma_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \sigma_{i,j} \circ (j, j+1) = (i, j) \text{ or } \mathrm{id}_{K^n}.$$

If $(j, j+1) \circ \sigma_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \sigma_{i,j} \circ (j, j+1) = \mathrm{id}_{K^n}$, then $t = \mathrm{id}_{K^n}$. If $(j, j+1) \circ \sigma_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \sigma_{i,j} \circ (j, j+1) = (i, j)$, then

$$\begin{aligned} t &= (j, j+1) \circ \sigma_{i,j} \circ (j, j+1) \circ (i, j) \circ (j, j+1) \circ \sigma_{i,j} \circ (j, j+1) \\ &= (j, j+1) \circ \sigma_{i,j} \circ (i, j+1) \circ \sigma_{i,j} \circ (j, j+1) \\ &= (j, j+1) \circ \sigma_{i,j+1} \circ (i, j+1) \circ (j, j+1) \\ &= \sigma_{i,j} \circ (i, j). \end{aligned}$$

Thus we have $t = \sigma_{i,j} \circ (i, j)$. □

From Lemma 2.3 and Lemma 2.4, the universal covering map μ induces a local isomorphism

$$\mu_*^{[n]} : \mathrm{Blow}_{T\cup U} K_{*\mu}^n / H \to \mathrm{Blow}_{\Delta_{E_*}^n} E_*^n / \mathcal{S}_n = E_*^{[n]}.$$

Here $\mathrm{Blow}_A B$ is the blow up of B along $A \subset B$.

LEMMA 2.5. *For every $x \in E_*^{[n]}$, $|(\mu_*^{[n]})^{-1}(x)| = 2$.*

Proof. For $(x_i)_{i=1}^n \in \Delta_*^n \subset E^n$ with $x_1 = x_2$, there are n elements $y_1, \dots, y_n$ of K such that $y_1 = y_2$ and $\mu(y_i) = x_i$ for $1 \le i \le n$. Then

$$(\mu^n)^{-1}((x_i)_{i=1}^n) = \{y_1, \sigma(y_1)\} \times \cdots \times \{y_n, \sigma(y_n)\}.$$

Since H is generated by $\mathcal{S}_n$ and $\{\sigma_{ij}\}_{1\le i<j\le n}$, for $(z_i)_{i=1}^n \in (\mu^n)^{-1}((x_i)_{i=1}^n)$ if the number of i with $z_i = y_i$ is even, then

$$(z_i)_{i=1}^n = \{\sigma(y_1), \sigma(y_2), y_3, \dots, y_n\} \text{ on } K_{*\mu}^n/H, \text{ and}$$

if the number of i with $z_i = y_i$ is odd, then

$$(z_i)_{i=1}^n = \{\sigma(y_1), y_2, y_3, \ldots, y_n\} \text{ on } K^n_{*\mu}/H.$$

Furthermore since $\sigma_i \notin H$ for $1 \leq i \leq n$,

$$\{\sigma(y_1), \sigma(y_2), y_3, \ldots, y_n\} \neq \{\sigma(y_1), y_2, y_3, \ldots, y_n\}, \text{ on } K^n_{*\mu}/H.$$

Thus for every $x \in E^{[n]}_*$, we get $|(\mu^{[n]}_*)^{-1}(x)| = 2$. □

PROPOSITION 2.6. *$\mu^{[n]}_* : \mathrm{Blow}_{T\cup U} K^n_{*\mu}/H \to \mathrm{Blow}_{\Delta^n_*} E^n_*/\mathcal{S}_n$ is the universal covering space, and $X \setminus \pi^{-1}(F) \simeq \mathrm{Blow}_{T\cup U} K^n_{*\mu}/H$. When $n = 2$, we have $X \simeq \mathrm{Blow}_{T\cup U} K^2/H$.*

Proof. Since $\mu^{[n]}_*$ is a local isomorphism, from Lemma 2.5 we get that $\mu^{[n]}_*$ is a covering map. Furthermore $\pi : X \setminus \pi^{-1}(F) \to E^{[n]}_*$ is the universal covering space of degree 2, $\mu^{[n]}_* : \mathrm{Blow}_{T\cup U} K^n_{*\mu}/H \to \mathrm{Blow}_{\Delta^n_*} E^n_*/\mathcal{S}_n$ is the universal covering space. By the uniqueness of the universal covering space, we have $X\setminus\pi^{-1}(F) \simeq \mathrm{Blow}_{T\cup U} K^n_{*\mu}/H$. When $n = 2$, since $E^2_* = E^2$, $K^2_{*\mu} = K^2$ and $\mathrm{Blow}_{\Delta^2} E^2/\mathcal{S}_2 \simeq E^{[2]}$, we have $X \simeq \mathrm{Blow}_{T\cup U} K^2/H$. □

THEOREM 2.7. *For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$. Then there is a birational morphism $\varphi_X : X \to K^n/H$ such that $\varphi_X^{-1}(\Gamma/H) = \pi^{-1}(D)$.*

Proof. When $n = 2$, this is proved by Proposition 2.6. From here we assume that $n \geq 3$. From Proposition 2.6, we have $X \setminus \pi^{-1}(F) \simeq \mathrm{Blow}_{T\cup U} K^n_{*\mu}/H$. Since the codimension of F is 2, there is a meromorphim f of X to K^n/H which satisfies the following commutative diagram:

$$\begin{array}{ccc} E^{[n]} \setminus F & \xrightarrow{\pi_E} & E^{(n)} \\ {\scriptstyle \pi}\uparrow & & \uparrow{\scriptstyle p_H} \\ X \setminus \pi^{-1}(F) & \xrightarrow{f} & K^n/H \end{array}$$

where $\pi_E : E^{[n]} \to E^{(n)}$ is the Hilbert-Chow morphism, and $p_H : K^n/H \to E^{(n)}$ is the natural projection. For an ample line bundle $\mathcal{L}$ on $E^{(n)}$, since the natural projection $p_H : K^n/H \to E^{(n)}$ is finite, $p_H^*\mathcal{L}$ is ample. From the above diagram, we have $\pi^*(\pi_E^*\mathcal{L}) \mid_{X\setminus\pi^{-1}(F)} = f^*(p_H^*\mathcal{L})$. Since $\pi^{-1}(F)$ is an analytic closed subset of codimension 2 in X and $p_H^*\mathcal{L}$ is ample, there is a holomorphism φ_X of X to K^n/H such that $\varphi_X \mid_{X\setminus\pi^{-1}(F)} = f \mid_{X\setminus\pi^{-1}(F)}$. Since $f : X \setminus \pi^{-1}(D) \cong (K^n \setminus \Gamma)/H$, this is a birational morphism. □

3. Proof of Theorem 1.1. Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$.

PROPOSITION 3.1. *For $n \geq 2$, we have* $\dim_{\mathbb{C}} \mathrm{H}^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) = 0$.

Proof. For a smooth projective manifold S, we put

$$h^{p,q}(S) := \dim_{\mathbb{C}} \mathrm{H}^q(S, \Omega^p_S) \text{ and}$$

$$h(S, x, y) := \sum_{p,q} h^{p,q}(S) x^p y^q.$$

By [7, Theorem 2] and [6, page 204], we have the equation (1):

$$\sum_{n=0}^{\infty}\sum_{p,q} h^{p,q}(E^{[n]})x^p y^q t^n = \prod_{k=1}^{\infty}\prod_{p,q=0}^{2}\Big(\frac{1}{1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k}\Big)^{(-1)^{p+q}h^{p,q}(E)}.$$

Since an Enriques surface E has Hodge numbers $h^{0,0}(E) = h^{2,2}(E) = 1$, $h^{1,0}(E) = h^{0,1}(E) = 0$, $h^{2,0}(E) = h^{0,2}(E) = 0$, and $h^{1,1}(E) = 10$, the equation (1) is

$$\sum_{n=0}^{\infty}\sum_{p,q} h^{p,q}(E^{[n]})x^p y^q t^n = \prod_{k=1}^{\infty}\Big(\frac{1}{1-x^{k-1}y^{k-1}t^k}\Big)\Big(\frac{1}{1-x^k y^k t^k}\Big)^{10}\Big(\frac{1}{1-x^{k+1}y^{k+1}t^k}\Big).$$

It follows that

$$h^{p,q}(E^{[n]}) = 0 \text{ for all } p,q \text{ with } p \neq q.$$

Thus we have $\dim_{\mathbb{C}} \mathrm{H}^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) = 0$ for $n \geq 2$. □

THEOREM 3.2. *For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then every small deformation of X is induced by that of $E^{[n]}$.*

Proof. In [4, Proposition 4.2 and Theorems 0.3], Fantechi showed that for a smooth projective surface with $\mathrm{H}^0(S, T_S) = 0$ or $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, and $\mathrm{H}^1(S, \mathcal{O}_S(-K_S)) = 0$, where K_S is the canonical divisor of S, then we get

$$\dim_{\mathbb{C}} \mathrm{H}^1(S, T_S) = \dim_{\mathbb{C}} \mathrm{H}^1(S^{[n]}, T_{S^{[n]}}).$$

Since an Enriques surface E satisfies $\mathrm{H}^0(E, T_E) = 0$ or $\mathrm{H}^1(E, \mathcal{O}_E) = 0$, and $\mathrm{H}^1(E, \mathcal{O}_E(-K_E)) = 0$, we have $\dim_{\mathbb{C}} \mathrm{H}^1(E^{[n]}, T_{E^{[n]}}) = 10$. Since $K_{E^{[n]}}$ is not trivial and $2K_{E^{[n]}}$ is trivial, we have

$$T_{E^{[n]}} \simeq \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}.$$

Therefore we have $\dim_{\mathbb{C}} \mathrm{H}^1(E^n, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}) = 10$. Since K_X is trivial, then we have $T_X \simeq \Omega^{2n-1}_X$. Since $\pi : X \to E^{[n]}$ is the covering map, we have

$$\mathrm{H}^k(X, \Omega^{2n-1}_X) \simeq \mathrm{H}^k(E^{[n]}, \pi_*\Omega^{2n-1}_X).$$

Since $X \simeq \mathcal{S}pec\, \mathcal{O}_{E^{[n]}} \oplus \mathcal{O}_{E^{[n]}}(K_{E^{[n]}})$ ([11, Theorem 3.1]), we have

$$\mathrm{H}^k(E^{[n]}, \pi_*\Omega^{2n-1}_X) \simeq \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \oplus (\Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}})).$$

Thus

$$\begin{aligned}\mathrm{H}^k(X, \Omega^{2n-1}_X) &\simeq \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \oplus (\Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}))\\ &\simeq \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) \oplus \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}).\end{aligned}$$

Combining this with Proposition 3.1, we obtain

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{H}^1(X, \Omega^{2n-1}_X) &= \dim_{\mathbb{C}} \mathrm{H}^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}})\\ &= 10.\end{aligned}$$

Let $p : \mathcal{Y} \to U$ be the universal family of $E^{[n]}$ and $f : \mathcal{X} \to \mathcal{Y}$ be the universal covering space. Then $q : \mathcal{X} \to U$ is a flat family of X where $q := p \circ f$. Then we have a commutative diagram:

$$\begin{array}{ccccc} T_{U,0} & \xrightarrow{\rho_p} & \mathrm{H}^1(\mathcal{Y}_0, T_{\mathcal{Y}_0}) & = & \mathrm{H}^1(E^{[n]}, T_{E^{[n]}}) \\ & \searrow^{\rho_q} & \downarrow \tau & & \downarrow \pi^* \\ & & \mathrm{H}^1(\mathcal{X}_0, T_{\mathcal{X}_0}) & = & \mathrm{H}^1(X, T_X). \end{array}$$

Since $\mathrm{H}^1(E^{[n]}, T_{E^{[n]}}) \simeq \mathrm{H}^1(X, T_X)$ by π^*, the vertical arrow τ is an isomorphism and

$$\dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{X}_u, T_{\mathcal{X}_u}) = \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{X}_u, \Omega^{2n-1}_{\mathcal{X}_u})$$

is a constant for some neighborhood of $0 \in U$, it follows that $q : \mathcal{X} \to U$ is the complete family of $\mathcal{X}_0 = X$, therefore $q : \mathcal{X} \to U$ is the versal family of $\mathcal{X}_0 = X$. Thus every small deformation of X is induced by that of $E^{[n]}$. ☐

4. Proof of Theorem 1.4**.** For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, and D the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \to E^{(n)}$. First we show that for an automorphism f of $E^{[n]}$, $f(D) = D$ if and only if $f^*(\mathcal{O}_{E^{[n]}}(D)) = \mathcal{O}_{E^{[n]}}(D)$ in $\mathrm{H}^2(E^{[n]}, \mathbb{C})$. Next, we show Theorem 1.4.

Proposition 4.1. *For any positive integer $l \in \mathbb{N}$ we have*

$$\dim_{\mathbb{C}} \mathrm{H}^0(E^{[n]}, \mathcal{O}_{E^{[n]}}(lD)) = 1.$$

Proof. Since D is effective, we obtain $\dim_{\mathbb{C}} \mathrm{H}^0(E^{[n]}, \mathcal{O}_{E^{[n]}}(lD)) \geq 1$. Since $E^{[n]} \backslash D \simeq E^{(n)} \backslash \Delta^{(n)}$, and $\mathcal{O}_{E^{[n]}}(lD) \simeq \mathcal{O}_{E^{[n]}}$ on $E^{[n]} \setminus D$, we have

$$(\pi_E)_*(\mathcal{O}_{E^{[n]}}(lD)) \simeq \mathcal{O}_{E^{(n)}} \text{ on } E^{(n)} \backslash \Delta^{(n)},$$

where $\pi_E : E^{[n]} \to E^{(n)}$ is the Hilbert-Chow morphism. Since the codimension of $\Delta^{(n)}$ is 2, and $E^{(n)}$ is normal, we have $\Gamma(E^{(n)} \backslash \Delta^{(n)}, \mathcal{O}_{E^{(n)}}) = \Gamma(E^{(n)}, \mathcal{O}_{E^{(n)}})$. Since $\mathcal{O}_{E^{[n]}}(lD))$ is a local free sheaf, the restriction map:

$$\Gamma(E^{[n]}, \mathcal{O}_{E^{[n]}}(lD))) \to \Gamma(E^{[n]} \backslash D, \mathcal{O}_{E^{[n]}}(lD)))$$

is injective. Thus we obtain $\dim_{\mathbb{C}} \mathrm{H}^0(E^{[n]}, \mathcal{O}_{E^{[n]}}(lD)) = 1$. ☐

Remark 4.2. Since $\mathrm{H}^1(E^{[n]}, \mathcal{O}_{E^{[n]}}) = 0$, the map $\mathrm{Pic}(E^{[n]}) \to \mathrm{H}^2(E^{[n]}, \mathbb{C})$ is injective. By Proposition 4.1, and D is effective, we have that for an automorphism $\varphi \in \mathrm{Aut}(E^{[n]})$, the condition $\varphi^*(\mathcal{O}_{E^{[n]}}(D)) = \mathcal{O}_{E^{[n]}}(D)$ in $\mathrm{H}^2(E^{[n]}, \mathbb{C})$ is equivalent to the condition $\varphi(D) = D$.

Recall that σ is the covering involution of $\mu : K \to E$, $\pi \circ \omega : K^n \setminus \Gamma \to E^{[n]} \setminus D$ is the universal covering space, and $G := \{g \in \mathrm{Aut}(K^n \setminus \Gamma) : \pi \circ \omega \circ g = \pi \circ \omega\}$ is the covering transformation group of $\pi \circ \omega$.

Proposition 4.3. *Let f be an automorphism of $E^{[n]} \setminus D$, and $g_1, \ldots, g_n$ automorphisms of K such that $(\pi \circ \omega) \circ (g_1 \times \cdots \times g_n) = f \circ (\pi \circ \omega)$, where $(g_1 \times \cdots \times g_n)$ is the automorphism of K^n. Then we have $g_i = g_1$ or $g_i = g_1 \circ \sigma$ for each $1 \leq i \leq n$. Moreover $g_1 \circ \sigma = \sigma \circ g_1$.*

Proof. We show the first assertion by contradiction. Without loss of generality, we may assume that $g_2 \neq g_1$ and $g_2 \neq g_1 \circ \sigma$. Let h_1 and h_2 be two morphisms of K where $g_i \circ h_i = \mathrm{id}_K$ and $h_i \circ g_i = \mathrm{id}_K$ for $i = 1$, 2. We define two morphisms $H_{1,2}$ and $H_{1,2,\sigma}$ from K to K^2 by

$$H_{1,2} : K \ni x \mapsto (h_1(x), h_2(x)) \in K^2$$

$$H_{1,2,\sigma} : K \ni x \mapsto (h_1(x), \sigma \circ h_2(x)) \in K^2.$$

Let $S_\sigma := \{(x, y) : y = \sigma(x)\}$ be the subset of K^2. Since $h_1 \neq h_2$ and $h_1 \neq \sigma \circ h_2$, $H_{1,2}^{-1}(\Delta^2) \cup H_{1,2,\sigma}^{-1}(S_\sigma)$ do not coincide with K. Thus there is $x' \in K$ such that $H_{1,2}(x') \notin \Delta^2$ and $H_{1,2,\sigma}(x') \notin S_\sigma$. For $x' \in K$, we put $x_i := h_i(x') \in K$ for $i = 1$, 2. Then there are some elements $x_3, \ldots, x_n \in K$ such that $(x_1, \ldots, x_n) \in K^n \setminus \Gamma$. We have $g((x_1, \ldots, x_n)) \notin K^n \setminus \Gamma$ by the assumption of x_1 and x_2. It is contradiction, because g is an automorphism of $K^n \setminus \Gamma$. Thus we have $g_i = g_1$ or $g_i = g_1 \circ \sigma$ for $1 \leq i \leq n$.

We show the second assertion. Since the covering transformation group of $\pi \circ \omega$ is G, the liftings of f are given by

$$\{g \circ u : u \in G\} = \{u \circ g : u \in G\}.$$

Thus for $\sigma_1 \circ g$, there is an element $\sigma_{i_1 \cdots i_k} \circ s$ of G where $s \in \mathcal{S}_n$ and $t \in \{\sigma_{i_1 \ldots i_k}\}_{1 \leq k \leq n,\ 1 \leq i_1 < \ldots < i_k \leq n}$ such that $\sigma_1 \circ g = g \circ \sigma_{i_1 \ldots i_k} \circ s$. If we think about the first component of $\sigma_1 \circ g$, we have $s = \mathrm{id}$ and $t = \sigma_1$. Therefore $g \circ \sigma_1 \circ g^{-1} = \sigma_1$, we have $\sigma \circ g_1 = g_1 \circ \sigma$. □

THEOREM 4.4. *For $n \geq 2$, let E be an Enriques surface, D the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \to E^{(n)}$. An automorphism f of $E^{[n]}$ is natural if and only if $f(D) = D$, i.e. $f^*(\mathcal{O}_{E^{[n]}}(D)) = \mathcal{O}_{E^{[n]}}(D)$ in $H^2(E^{[n]}, \mathbb{C})$.*

Proof. Let f be an automorphism of $E^{[n]}$ with $f(D) = D$. Then f induces an automorphism of $E^{[n]} \setminus D$. Since the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma$ such that $\pi \circ \omega \circ g = f \circ \pi \circ \omega$:

$$\begin{array}{ccc} E^{[n]} \setminus D & \xrightarrow{f} & E^{[n]} \setminus D \\ {\scriptstyle \pi\circ\omega}\uparrow & & \uparrow{\scriptstyle \pi\circ\omega} \\ K^n \setminus \Gamma & \xrightarrow{g} & K^n \setminus \Gamma. \end{array}$$

Since Γ is an analytic set of codimension 2, and K^n is projective, g can be extended to a birational automorphism of K^n. By Oguiso [10, Theorem 4.1], g is an automorphism of K^n, and there are some automorphisms $g_1, \ldots, g_n \in \mathrm{Aut}(K)$ and $s \in \mathcal{S}_n$ such that $g = s \circ g_1 \times \cdots \times g_n$. Since $\mathcal{S}_n \subset G$, we can assume that $g = g_1 \times \cdots \times g_n$. By Proposition 4.3, we have $g_i = g_1$ or $g_1 \circ \sigma$ for $1 \leq i \leq n$ and $g_1 \circ \sigma = \sigma \circ g_1$. We denote $g_1^{[n]}$ the induced automorphism of $E^{[n]}$ given by g_1. Then ${g_1}^{[n]}|_{E^{[n]} \setminus D} = f|_{E^{[n]} \setminus D}$. Thus $g_1^{[n]} = f$, i.e. f is natural. The other implication is obvious. □

5. Proof of Theorem 1.7. Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$.

In Proposition 5.2, we shall show that for $n \geq 3$, the covering involution of $\pi : X \to E^{[n]}$ acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity. In Proposition 5.5, by using Theorem

2.7 and 1.4, we shall show that for $n \geq 2$, if an automorphism φ of X acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity, then φ is a lift of a natural automorphism of $E^{[n]}$. In Proposition 5.9, by using Proposition 5.5 and checking the action to $\mathrm{H}^1(X, \Omega_X^{2n-1}) \cong \mathrm{H}^{2n-1,1}(X)$, we classify involutions of X which act on $\mathrm{H}^2(X, \mathbb{C})$ as the identity. We prove Theorem 1.7 using those results.

LEMMA 5.1. *Let X be a smooth complex manifold, $Z \subset X$ a closed submanifold whose codimension is 2, $\tau : X_Z \to X$ the blow up of X along Z, $E = \tau^{-1}(Z)$ the exceptional divisor, and h the first Chern class of the line bundle $\mathcal{O}_{X_Z}(E)$.*
Then $\tau^* : \mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X_Z, \mathbb{C})$ is injective, and

$$\mathrm{H}^2(X_Z, \mathbb{C}) \simeq \mathrm{H}^2(X, \mathbb{C}) \oplus \mathbb{C}h.$$

Proof. Let $U := X \setminus Z$ be an open set of X. Then U is isomorphic to an open set $U' = X_Z \setminus E$ of X_Z. As τ gives a morphism between the pair (X_Z, U') and the pair (X, U), we have a morphism τ^* between the long exact sequence of cohomology relative to these pairs:

$$\begin{array}{ccccccc}
\mathrm{H}^k(X,U,\mathbb{C}) & \longrightarrow & \mathrm{H}^k(X,\mathbb{C}) & \longrightarrow & \mathrm{H}^k(U,\mathbb{C}) & \longrightarrow & \mathrm{H}^{k+1}(X,U,\mathbb{C}) \\
\downarrow{\scriptstyle \tau^*_{X,U}} & & \downarrow{\scriptstyle \tau^*_X} & & \downarrow{\scriptstyle \tau^*_U} & & \downarrow{\scriptstyle \tau^*_{X,U}} \\
\mathrm{H}^k(X_Z,U',\mathbb{C}) & \longrightarrow & \mathrm{H}^k(X_Z,\mathbb{C}) & \longrightarrow & \mathrm{H}^k(U',\mathbb{C}) & \longrightarrow & \mathrm{H}^{k+1}(X_Z,U',\mathbb{C}).
\end{array}$$

By Thom isomorphism, the tubular neighborhood Theorem, and Excision theorem, we have

$$\mathrm{H}^q(Z, \mathbb{C}) \simeq \mathrm{H}^{q+4}(X, U, \mathbb{C}), \text{ and}$$

$$\mathrm{H}^q(E, \mathbb{C}) \simeq \mathrm{H}^{q+2}(X_Z, U', \mathbb{C}).$$

In particular, we have

$$\mathrm{H}^l(X, U, \mathbb{C}) = 0 \text{ for } l = 0, 1, 2, 3, \text{ and}$$

$$\mathrm{H}^j(X_Z, U', \mathbb{C}) = 0 \text{ for } l = 0, 1.$$

Thus we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^1(X,\mathbb{C}) & \longrightarrow & \mathrm{H}^1(U,\mathbb{C}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \tau^*_{X,U}} & & \downarrow{\scriptstyle \tau^*_X} & & \downarrow{\scriptstyle \tau^*_U} & & \downarrow{\scriptstyle \tau^*_{X,U}} \\
0 & \longrightarrow & \mathrm{H}^1(X_Z,\mathbb{C}) & \longrightarrow & \mathrm{H}^1(U',\mathbb{C}) & \longrightarrow & \mathrm{H}^0(E,\mathbb{C}),
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^2(X,\mathbb{C}) & \longrightarrow & \mathrm{H}^2(U,\mathbb{C}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \tau^*_{X,U}} & & \downarrow{\scriptstyle \tau^*_X} & & \downarrow{\scriptstyle \tau^*_U} & & \downarrow{\scriptstyle \tau^*_{X,U}} \\
\mathrm{H}^0(E,\mathbb{C}) & \longrightarrow & \mathrm{H}^2(X_Z,\mathbb{C}) & \longrightarrow & \mathrm{H}^2(U',\mathbb{C}) & \longrightarrow & \mathrm{H}^3(X_Z,U',\mathbb{C}).
\end{array}$$

Since $\tau\,|_{U'}\colon U' \xrightarrow{\sim} U$, we have isomorphisms $\tau_U^* : \mathrm{H}^k(U,\mathbb{C}) \simeq \mathrm{H}^k(U',\mathbb{C})$. Thus we have

$$\dim_{\mathbb{C}}\mathrm{H}^2(X_Z,\mathbb{C}) = \dim_{\mathbb{C}}\mathrm{H}^2(X,\mathbb{C}) + 1, \text{ and}$$

$$\tau^* : \mathrm{H}^2(X,\mathbb{C}) \to \mathrm{H}^2(X_Z,\mathbb{C}) \text{ is injective,}$$

and therefore we obtain

$$\mathrm{H}^2(X_Z,\mathbb{C}) \simeq \mathrm{H}^2(X,\mathbb{C}) \oplus \mathbb{C}h.$$

□

PROPOSITION 5.2. *Suppose $n \geq 3$. For the covering involution ρ of the universal covering space $\pi : X \to E^{[n]}$, the induced map $\rho^* : \mathrm{H}^2(X,\mathbb{C}) \to \mathrm{H}^2(X,\mathbb{C})$ is the identity.*

Proof. Since the codimension of $\pi^{-1}(F)$ is 2, we get

$$\mathrm{H}^2(X,\mathbb{C}) \cong \mathrm{H}^2(X \setminus \pi^{-1}(F),\mathbb{C}).$$

By Proposition 2.6, $X \setminus \pi^{-1}(F) \simeq \mathrm{Blow}_{T\cup U} K^n_{*\mu}/H$.

Let $\tau : \mathrm{Blow}_{T\cup U} K^n_{*\mu} \to K^n_{*\mu}$ be the blow up of $K^n_{*\mu}$ along $T \cup U$,

$$h_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\mathrm{Blow}_{T\cup U} K^n_{*\mu}}(\tau^{-1}(U_{ij})),$$

and

$$k_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\mathrm{Blow}_{T\cup U} K^n_{*\mu}}(\tau^{-1}(T_{ij})).$$

By Lemma 5.1, we have

$$\mathrm{H}^2(\mathrm{Blow}_{T\cup U} K^n_{*\mu},\mathbb{C}) \cong \mathrm{H}^2(K^n,\mathbb{C}) \oplus \left(\bigoplus_{1\leq i<j\leq n} \mathbb{C}h_{ij}\right) \oplus \left(\bigoplus_{1\leq i<j\leq n} \mathbb{C}k_{ij}\right).$$

Since $n \geq 3$, there is an isomorphism

$$(j, j+1) \circ \sigma_{ij} \circ (j, j+1) : U_{ij} \xrightarrow{\sim} T_{ij}.$$

Thus we have $\dim_{\mathbb{C}}\mathrm{H}^2(\mathrm{Blow}_{T\cup U} K^n_{*\mu}/H,\mathbb{C}) = 11$, i.e. $\dim_{\mathbb{C}}\mathrm{H}^2(X,\mathbb{C}) = 11$. Since $\mathrm{H}^2(E^{[n]},\mathbb{C}) = \mathrm{H}^2(X,\mathbb{C})^{\rho^*}$, ρ^* is the identity. □

PROPOSITION 5.3. *For any positive integer $l \in \mathbb{N}$ we have*

$$\dim_{\mathbb{C}}\mathrm{H}^0(X,\mathcal{O}_X(l\pi^*(D))) = 1.$$

Proof. From Propositin 4.1 we obtain $\dim_{\mathbb{C}}\mathrm{H}^0(X,\mathcal{O}_X(l\pi^*(D))) \geq 1$. Like the proof of Proposition 4.1, we have $\dim_{\mathbb{C}}\mathrm{H}^0(X,\mathcal{O}_X(l\pi^*(D))) = 1$. from Theorem 2.7. □

REMARK 5.4. Since $\mathrm{H}^1(X,\mathcal{O}_X) = 0$, the map $\mathrm{Pic}(X) \to \mathrm{H}^2(X,\mathbb{C})$ is injective. By Proposition 5.3, and $\pi^{-1}(D)$ is effective, for an automorphism $\varphi \in \mathrm{Aut}(X)$, the condition $\varphi^*(\mathcal{O}_X(\pi^*D)) = \mathcal{O}_X(\pi^*D)$ in $\mathrm{H}^2(X,\mathbb{C})$ is equivalent to the condition $\varphi(\pi^{-1}(D)) = \pi^{-1}(D)$.

Recall that $\omega : K^n \setminus \Gamma \to X \setminus \pi^{-1}(D)$ is the universal covering space.

PROPOSITION 5.5. *For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, and D the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \to E^{(n)}$.*

For an automorphism φ of X with $f^*(\mathcal{O}(\pi^* D)) = \mathcal{O}(\pi^* D)$ in $\mathrm{H}^2(X, \mathbb{C})$, there is an automorphism ϕ of E such that φ is a lift of $\phi^{[n]}$ where $\phi^{[n]}$ is the natural automorphism of $E^{[n]}$ induced by ϕ. Furthermore, if the order of φ is 2, then the order of ϕ is at most 2.

Proof. Let φ be an automorphism of X with $\varphi^*(\mathcal{O}(\pi^* D)) = \mathcal{O}(\pi^* D)$ in $\mathrm{H}^2(X, \mathbb{C})$. By Remark 5.4, $\varphi|_{X \setminus \pi^{-1}(D)}$ is automorphism of $X \setminus \pi^{-1}(D)$. By the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma$ such that $\varphi \circ \omega = \omega \circ g$:

$$\begin{array}{ccc} X \setminus \pi^{-1}(D) & \xrightarrow{\varphi} & X \setminus \pi^{-1}(D) \\ \omega \uparrow & & \uparrow \omega \\ K^n \setminus \Gamma & \xrightarrow{g} & K^n \setminus \Gamma. \end{array}$$

Like the proof of Proposition 4.3, we can assume that there are some automorphisms g_i of K such that $g = g_1 \times \cdots \times g_n$, for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, and $g_1 \circ \sigma = \sigma \circ g_1$. Since $g_1 \circ \sigma = \sigma \circ g_1$, g_1 induces an automorphism ϕ of E. Let $\phi^{[n]}$ is the natural automorphism of $E^{[n]}$ induced by ϕ. Then we have $\pi \circ \omega \circ g = \phi^{[n]} \circ \pi \circ \omega$:

$$\begin{array}{ccc} E^{[n]} \setminus D & \xrightarrow{\phi^{[n]}} & E^{[n]} \setminus D \\ \pi \circ \omega \uparrow & & \uparrow \pi \circ \omega \\ K^n \setminus \Gamma & \xrightarrow{g} & K^n \setminus \Gamma. \end{array}$$

Since $\varphi \circ \omega = \omega \circ g$, we have $\phi^{[n]} \circ \pi = \pi \circ \varphi$:

$$\begin{array}{ccc} E^{[n]} \setminus D & \xrightarrow{\phi^{[n]}} & E^{[n]} \setminus D \\ \pi \uparrow & & \uparrow \pi \\ X \setminus \pi^{-1}(D) & \xrightarrow{\varphi} & X \setminus \pi^{-1}(D). \end{array}$$

We assume that the order of φ is 2. Since $\omega = \varphi^2 \circ \omega = \omega \circ g^2$, we get $g^2 \in H$. Now $g = g_1 \times \cdots \times g_n$, for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, and $g_1 \circ \sigma = \sigma \circ g_1$. Thus we have $g_1^2 = \mathrm{id}_K$ or σ. By [9, Lemma 1.2], we have $g_1^2 = \mathrm{id}_K$. Therefore the order of ϕ is at most 2. □

DEFINITION 5.6. Let S be a smooth surface. An automorphism φ of S is numerically trivial if the induced automorphism φ^* of the cohomology ring over $\mathbb{Q}$, $\mathrm{H}^*(S, \mathbb{Q})$ is the identity.

We suppose that an Enriques surface E has numerically trivial involutions. By [9, Proposition 1.1], there is just one numerically trivial involution of E, denoted υ. For υ, there are just two involutions of K which are liftings of υ, one acts on $\mathrm{H}^0(K, \Omega_K^2)$ as the identity, and another acts on $\mathrm{H}^0(K, \Omega_K^2)$ as $-\mathrm{id}_{\mathrm{H}^0(K, \Omega_K^2)}$, we denote by υ_+ and υ_-, respectively. Then they satisfies $\upsilon_+ = \upsilon_- \circ \sigma$.

Let $\upsilon^{[n]}$ be the automorphism of $E^{[n]}$ which is induced by υ. For $\upsilon^{[n]}$, there are just two automorphisms of X which are liftings of $\upsilon^{[n]}$, denoted ς and ς', respectively:

$$\begin{array}{ccc} E^{[n]} & \xrightarrow{\upsilon^{[n]}} & E^{[n]} \\ \pi\uparrow & & \uparrow\pi \\ X & \xrightarrow{\varsigma\,(\varsigma')} & X. \end{array}$$

Then they satisfies $\varsigma = \varsigma' \circ \sigma$. As the proof of Proposition 5.5, each order of ς and ς' is 2. From here, we classify involutions acting on $\mathrm{H}^2(X, \mathbb{C})$ as the identity by checking the action to $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$.

LEMMA 5.7. $\dim_{\mathbb{C}} \mathrm{H}^{2n-1,1}(K^n/H, \mathbb{C}) = 10$.

Proof. Let σ be the covering involution of $\mu : K \to E$. Put

$$\mathrm{H}^{p,q}_{\pm}(K, \mathbb{C}) := \{\alpha \in \mathrm{H}^{p,q}(K, \mathbb{C}) : \sigma^*(\alpha) = \pm\alpha\} \text{ and}$$

$$h^{p,q}_{\pm}(K) := \dim_{\mathbb{C}} \mathrm{H}^{p,q}_{\pm}(K, \mathbb{C}).$$

Since K is a $K3$ surface, we have

$$h^{0,0}(K) = 1,\ h^{1,0}(K) = 0,\ h^{2,0}(K) = 1,\ h^{1,1}(K) = 20,$$

$$h^{0,0}_{+}(K) = 1,\ h^{1,0}_{+}(K) = 0,\ h^{2,0}_{+}(K) = 0\ h^{1,1}_{+}(K) = 10,$$

$$h^{0,0}_{-}(K) = 0,\ h^{1,0}_{-}(K) = 0,\ h^{2,0}_{-}(K) = 1, \text{ and } h^{2,0}_{-}(K) = 10.$$

Let

$$\Lambda := \{(s_1, \cdots, s_n, t_1, \cdots, t_n) \in \mathbb{Z}^{2n}_{\geq 0} : \Sigma^n_{i=1} s_i = 2n-1,\ \Sigma^n_{j=1} t_j = 1\}.$$

From the Künneth Theorem, we have

$$\mathrm{H}^{2n-1,1}(K^n, \mathbb{C}) \simeq \bigoplus_{(s_1,\cdots,s_n,t_1,\cdots,t_n)\in\Lambda} \left(\bigotimes_{i=1}^{n} \mathrm{H}^{s_i,t_i}(K, \mathbb{C}) \right).$$

We take a base α of $\mathrm{H}^{2,0}(K, \mathbb{C})$ and a base $\{\beta_i\}_{i=1}^{20}$ of $\mathrm{H}^{1,1}(K, \mathbb{C})$ such that $\{\beta_i\}_{i=1}^{10}$ is a base of $\mathrm{H}^{1,1}_{-}(K, \mathbb{C})$ and $\{\beta_i\}_{i=11}^{20}$ is a base of $\mathrm{H}^{1,1}_{+}(K, \mathbb{C})$. Let

$$\tilde{\beta}_i := \bigotimes_{j=1}^{n} \epsilon_j$$

where $\epsilon_j = \alpha$ for $j \neq i$ and $\epsilon_j = \beta_i$ for $j = i$, and

$$\gamma_i := \bigoplus_{j=1}^{n} \tilde{\beta}_j.$$

Then $\{\gamma_i\}_{i=1}^{20}$ is a base of $\mathrm{H}^{2n-1,1}(K^n, \mathbb{C})^{\mathcal{S}_n}$. Since $\sigma^*\alpha = -\alpha$, $\sigma^*\beta_i = -\beta_i$ for $1 \leq i \leq 10$, and $\sigma^*\beta_i = \beta_i$ for $11 \leq i \leq 20$, we obtain

$$\sigma^*_{ij}\gamma_i = \gamma_i \text{ for } 1 \leq i \leq 10, \text{ and}$$

$$\sigma_{ij}^* \gamma_i = -\gamma_i \text{ for } 11 \leq i \leq 20.$$

Since $\mathrm{H}^{2n-1,1}(K^n/H, \mathbb{C}) \simeq \mathrm{H}^{2n-1,1}(K^n, \mathbb{C})^H$ and $H = \langle \mathcal{S}_n, \{\sigma_{ij}\}_{1 \leq i < j \leq n} \tau \rangle$, we obtain

$$\mathrm{H}^{2n-1,1}(K^n/H, \mathbb{C}) = \bigoplus_{i=1}^{10} \mathbb{C}\gamma_i.$$

Thus we get $\dim_{\mathbb{C}} \mathrm{H}^{2n-1,1}(K^n/H, \mathbb{C}) = 10$. □

REMARK 5.8. By Theorem 2.7, there is a resolution $\varphi_X : X \to K^n/H$. Then $\varphi_X^* : \mathrm{H}^{p,q}(K^n/H, \mathbb{C}) \to \mathrm{H}^{p,q}(X, \mathbb{C})$ is an injective (see [13]). By Lemma 5.7, $\varphi_X^* : \mathrm{H}^{2n-1,1}(K^n/H, \mathbb{C}) \to \mathrm{H}^{2n-1,1}(X, \mathbb{C})$ is an isomorphism.

Recall that $\pi \circ \omega : K^n \setminus \Gamma \to E^{[n]} \setminus D$ is the universal covering space.

PROPOSITION 5.9. *We suppose that E has a numerically trivial involution, denoted υ. Let $\upsilon^{[n]}$ be the natural automorphism of $E^{[n]}$ which is induced by υ. Since the degree of $\pi : X \to E^{[n]}$ is 2, there are just two involutions ζ and ζ' of X which are lifts of $\upsilon^{[n]}$. Then ς and ς' do not act on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$.*

Proof. Since $\upsilon^{[n]}(D) = D$, $\upsilon^{[n]}|_{E^{[n]} \setminus D}$ is an automorphism of $E^{[n]} \setminus D$. By the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma$ such that $\upsilon^{[n]} \circ \pi \circ \omega = \pi \circ \omega \circ g$:

$$\begin{array}{ccc} E^{[n]} \setminus D & \xrightarrow{\upsilon^{[n]}} & E^{[n]} \setminus D \\ {\scriptstyle \pi\circ\omega}\uparrow & & \uparrow{\scriptstyle \pi\circ\omega} \\ K^n \setminus \Gamma & \xrightarrow{g} & K^n \setminus \Gamma. \end{array}$$

By Proposition 4.3, there are some automorphisms g_i of K such that $g = g_1 \times \cdots \times g_n$ for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, and $g_1 \circ \sigma = \sigma \circ g_1$. By Theorem 2.7, we get $K^n \setminus \Gamma/H \simeq X \setminus \pi^{-1}(D)$. Put

$$\upsilon_{+,even} := u_1 \times \cdots \times u_n$$

where

$$u_i = \upsilon_+ \text{ or } u_i = \upsilon_- \text{ and the number of } i \text{ with } u_i = \upsilon_+ \text{ is even.}$$

$\upsilon_{+,even}$ is an automorphism of K^n and induces an automorphism $\widetilde{\upsilon_{+,even}}$ of $K^n \setminus \Gamma/H$. We define automorphisms $\widetilde{\upsilon_{+,odd}}$, $\widetilde{\upsilon_{-,even}}$, and $\widetilde{\upsilon_{-,odd}}$ of $K^n \setminus \Gamma/H$ in the same way. Since $\sigma_{ij} \in H$ for $1 \leq i < j \leq n$, and $\upsilon_+ = \upsilon_- \circ \sigma$, if n is odd,

$$\widetilde{\upsilon_{+,odd}} = \widetilde{\upsilon_{-,even}}, \ \widetilde{\upsilon_{+,even}} = \widetilde{\upsilon_{-,odd}}, \text{ and } \widetilde{\upsilon_{+,odd}} \neq \widetilde{\upsilon_{+,even}},$$

and if n is even,

$$\widetilde{\upsilon_{+,odd}} = \widetilde{\upsilon_{-,odd}}, \ \widetilde{\upsilon_{+,even}} = \widetilde{\upsilon_{-,even}}, \text{ and } \widetilde{\upsilon_{+,odd}} \neq \widetilde{\upsilon_{+,even}}.$$

Since $\upsilon^{(n)} \circ \pi_E = \pi_E \circ \upsilon^{[n]}$ and $K^n \setminus \Gamma/H \simeq X \setminus \pi^{-1}(D)$, we have $\upsilon^{[n]} \circ \pi = \pi \circ \widetilde{\upsilon_{+,odd}}$ and $\upsilon^{[n]} \circ \pi = \pi \circ \widetilde{\upsilon_{+,even}}$ where $\pi_E : E^{[n]} \to E^{(n)}$ is the Hilbert-Chow morphism, and $\upsilon^{(n)}$ is the automorphism of $E^{(n)}$ induced by υ. Since the degree of π is 2, we have

$\{\varsigma, \varsigma'\} = \{\widetilde{\upsilon_{+,odd}}, \widetilde{\upsilon_{+,even}}\}$. By [9, page 386-389], there is an element $\alpha_{\pm} \in \mathrm{H}^{1,1}_{-}(K, \mathbb{C})$ such that $\upsilon_{+}^{*}(\alpha_{\pm}) = \pm\alpha_{\pm}$. We fix a basis α of $\mathrm{H}^{2,0}(K, \mathbb{C})$, and let

$$\widetilde{\alpha_{\pm}}_i := \bigotimes_{j=1}^{n} \epsilon_j$$

where $\epsilon_j = \alpha$ for $j \neq i$ and $\epsilon_j = \alpha_{\pm}$ for $j = i$, and

$$\widetilde{\alpha_{\pm}} := \bigoplus_{j=1}^{n} \widetilde{\alpha_{\pm}}_i.$$

Since $K^n \setminus \Gamma/H \simeq X \setminus \pi^{-1}(D)$, and by the definition of $\widetilde{\upsilon_{+,odd}}$ and $\widetilde{\upsilon_{+,even}}$, we have

$$\widetilde{\upsilon_{+,odd}}^{*}(\varphi_X^{*}(\widetilde{\alpha_{+}})) = \varphi_X^{*}(\widetilde{\alpha_{+}}) \text{ and } \widetilde{\upsilon_{+,even}}^{*}(\varphi_X^{*}(\widetilde{\alpha_{-}})) = \varphi_X^{*}(\widetilde{\alpha_{-}}).$$

Thus ς and ς' do not act on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$. □

Definition 5.10. For $n \geq 1$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. A variety Y is called an Enriques surface type quotient of X if there is an Enriques surface E' and a free involution τ of X such that $Y \simeq E'^{[n]}$ and $E'^{[n]} \simeq X/\langle\tau\rangle$. Here we call two Enriques surface type quotients of X distinct if they are not isomorphic to each other.

Theorem 5.11. *For $n \geq 3$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then the number of distinct Enriques surface type quotients of X is one.*

Proof. Let ρ be the covering involution of $\pi : X \to E^{[n]}$ for $n \geq 3$. Since for $n \geq 3$ $\dim_{\mathbb{C}}\mathrm{H}^2(E^{[n]}, \mathbb{C}) = \dim_{\mathbb{C}}\mathrm{H}^2(X, \mathbb{C}) = 11$, $\dim_{\mathbb{C}}\mathrm{H}^{2n-1,1}(E'^{[n]}, \mathbb{C}) = 0$, and $\dim_{\mathbb{C}}\mathrm{H}^{2n-1,1}(X, \mathbb{C}) = 10$, we obtain that ρ^* acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity, and $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$.

Let φ be an involution of X, which acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity and on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$. By Proposition 5.5, for φ, there is an automorphism ϕ of E such that φ is a lift of $\phi^{[n]}$ where $\phi^{[n]}$ is the natural automorphism of $E^{[n]}$ induced by ϕ. Furthermore since the order of ϕ is at most 2, the order of φ is 2. Since $\phi^{[n]} \circ \pi = \pi \circ \varphi$, $\phi^{[n]*}$ acts on $\mathrm{H}^2(E^{[n]}, \mathbb{C})$ as the identity. Thus ϕ^* acts on $\mathrm{H}^2(E, \mathbb{C})$ as the identity. If E does not have numerically trivial automorphisms, then $\phi = \mathrm{id}_E$. Thus $\varphi = \rho$.

We assume that ϕ does not the identity map. Then ϕ is numerically trivial. Then $\phi = \upsilon$ and $\varphi \in \{\zeta, \zeta'\}$. By Proposition 5.9, we obtain that φ does not act on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$. This is a contradiction. Thus $\phi = \mathrm{id}_E$, and we get $\varphi = \rho$. This proves the theorem. □

Theorem 5.12. *For $n \geq 2$, let $\pi : X \to E^{[n]}$ be the universal covering space. For any automorphism φ of X, if φ^* is acts on $\mathrm{H}^*(X, \mathbb{C}) := \bigoplus_{i=0}^{2n} \mathrm{H}^i(X, \mathbb{C})$ as the identity, then $\varphi = \mathrm{id}_X$.*

Proof. By Proposition 5.5, for φ, there is an automorphism ϕ of E such that φ is a lift of $\phi^{[n]}$ where $\phi^{[n]}$ is the natural automorphism of $E^{[n]}$ induced by ϕ. Since φ^* acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity, ϕ^* acts on $\mathrm{H}^2(E, \mathbb{C})$ as the identity. From [9, page 386-389] the order of ϕ is at most 4.

If the order of ϕ is 2, by Proposition 5.9 φ does not act on $\mathrm{H}^{2n-1,1}(X,\mathbb{C})$ as the identity. This is a contradiction.

If the order of ϕ is 4, there is an element $\alpha' \in \mathrm{H}^{1,1}_{-}(K,\mathbb{C})$ such that $g_1^*(\alpha'_{\pm}) = \pm\sqrt{-1}\alpha'$ from [9, page 390-391]. Like the proof of Proposition 5.9, φ does not act on $\mathrm{H}^{2n-1,1}(X,\mathbb{C})$ as the identity. This is a contradiction. Thus we have $\phi = \mathrm{id}_E$ and $\varphi \in \{\mathrm{id}_X, \rho\}$. Since ρ does not act on $\mathrm{H}^{2n-1,1}(X,\mathbb{C})$ as the identity, we have $\varphi = \mathrm{id}_X$. □

Corollary 5.13. *For $n \geq 2$, let $\pi : X \to E^{[n]}$ be the universal covering space. For any two automorphisms f and g of X, if $f^* = g^*$ on $\mathrm{H}^*(X,\mathbb{C})$, then $f = g$.*

Theorem 5.14. *For $n \geq 3$, let E be an Enriques surfaces, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space. Then there is an exact sequence:*

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(X) \to \mathrm{Aut}(E^{[n]}) \to 0.$$

Proof. Let f be an automorphism f of X. We put $g = f^{-1} \circ \rho \circ f$. Since for $n \geq 3$ ρ^* acts on $H^2(X,\mathbb{C})$ as the identity and on $H^{2n-1,1}(X)$ as $-\mathrm{id}_{H^{2n-1,1}(X)}$, we get that $g^* = \rho^*$ as automorphisms of $H^2(X,\mathbb{C}) \oplus H^{2n-1,1}(X)$. Like the proof of Theorem 5.12, we have $g = \rho$, i.e. $f \circ \rho = \rho \circ f$. Thus f induces a automorphism of $E^{[n]}$, and we have an exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(X) \to \mathrm{Aut}(E^{[n]}) \to 0.$$

□

6. Appendix A. We compute the Hodge number of the universal covering space X of $E^{[2]}$. Let σ be the covering involution of $\mu : K \to E$, and $\tau : \mathrm{Blow}_{\Delta\cup T}K^2 \to K^2$ the natural map, where $T = \{(x,y) \in K^2 : y = \sigma(x)\}$ and $\Delta = \{(x,x) \in K^2\}$. By Proposition 2.6, we have

$$X \simeq \mathrm{Blow}_{\Delta\cup T}K^2/H.$$

We put

$$D_\Delta := \tau^{-1}(\Delta) \text{ and}$$

$$D_T := \tau^{-1}(T).$$

For two inclusions

$$j_{D_\Delta} : D_\Delta \hookrightarrow \mathrm{Blow}_{\Delta\cup T}K^2, \text{ and}$$

$$j_{D_T} : D_T \hookrightarrow \mathrm{Blow}_{\Delta\cup T}K^2,$$

let j_{*D_Δ} be the Gysin morphism

$$j_{*D_\Delta} : \mathrm{H}^p(D_\Delta,\mathbb{C}) \to \mathrm{H}^{p+2}(\mathrm{Blow}_{\Delta\cup T}K^2,\mathbb{C}),$$

j_{*D_T} the Gysin morphism

$$j_{*D_T} : \mathrm{H}^p(D_T,\mathbb{C}) \to \mathrm{H}^{p+2}(\mathrm{Blow}_{\Delta\cup T}K^2,\mathbb{C}), \text{ and}$$

$$\psi := \tau^* + j_{*D_\Delta} \circ \tau|_{D_\Delta}^* + j_{*D_T} \circ \tau|_{D_T}^*$$

the morphism from $\mathrm{H}^p(K^2,\mathbb{C}) \oplus \mathrm{H}^{p-2}(\Delta,\mathbb{C}) \oplus \mathrm{H}^{p-2}(T,\mathbb{C})$ to $\mathrm{H}^p(\mathrm{Blow}_{\Delta\cup T}K^2,\mathbb{C})$. From [14, Theorem 7.31], we have isomorphisms of Hodge structures by ψ:

$$\mathrm{H}^k(K^2,\mathbb{C}) \oplus \mathrm{H}^{k-2}(\Delta,\mathbb{C}) \oplus \mathrm{H}^{k-2}(T,\mathbb{C}) \simeq \mathrm{H}^k(\mathrm{Blow}_{\Delta\cup T}K^2,\mathbb{C}).$$

Furthermore, for automorphism f of K, let $\bar{f}$ (resp. $\bar{f}_\sigma$) be the automorphism of $\mathrm{Blow}_{\Delta\cup T}K^2$ which is induced by $f \times f$ (resp. $f \times (f \circ \sigma)$, f_Δ the automorphism of Δ which is induced by $f \times f$, f_T the automorphism of T which is induced by $f \times f$, and $\tilde{f}$ the isomorphism from T to Δ which is induced by $f \times (f \circ \sigma)$. For $\alpha \in \mathrm{H}^*(K^2,\mathbb{C})$, $\beta \in \mathrm{H}^*(\Delta,\mathbb{C})$, and $\gamma \in \mathrm{H}^*(T,\mathbb{C})$, we obtain

$$\bar{f}^*(\tau^*\alpha) = \tau^*((f \times f)^*\alpha),$$

$$\bar{f}^*(j_{*D_\Delta} \circ \tau|_{D_\Delta}^*\beta) = j_{*D_\Delta} \circ \tau|_{D_\Delta}^*(f_\Delta^*\beta),$$

$$\bar{f}^*(j_{*D_T} \circ \tau|_{D_T}^*\gamma) = j_{*D_T} \circ \tau|_{D_T}^*(f_T^*\gamma),$$

$$\bar{f}_\sigma^*(\tau^*\alpha) = \tau^*((f \times (f \circ \sigma)^*\alpha),$$

$$\bar{f}_\sigma^*(j_{*D_\Delta} \circ \tau|_{D_\Delta}^*\beta) = j_{*D_T} \circ \tau|_{D_\Delta}^*(\tilde{f}^*\beta),$$

in $\mathrm{H}^*(\mathrm{Blow}_{\Delta\cup T}K^2,\mathbb{C})$.

THEOREM 6.1. *For the universal covering space $\pi : X \to E^{[2]}$, we have $h^{0,0}(X) = 1$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 0$, $h^{1,1}(X) = 12$, $h^{3,0}(X) = 0$, $h^{2,1}(X) = 0$, $h^{4,0}(X) = 1$, $h^{3,1}(X) = 10$, and $h^{2,2}(X) = 131$.*

Proof. Since $X \simeq \mathrm{Blow}_{\Delta\cup T}K^2/H$, we have

$$h^{p,q}(X) = \dim_{\mathbb{C}}\{\alpha \in \mathrm{H}^{p,q}(\mathrm{Blow}_{\Delta\cup T}K^2,\mathbb{C}) : \ h^*\alpha = \alpha \text{ for } h \in H\}.$$

Let σ be the covering involution of $\mu : K \to E$. We put

$$\mathrm{H}_{\pm}^{p,q}(K,\mathbb{C}) := \{\alpha \in \mathrm{H}^{p,q}(K,\mathbb{C}) : \sigma^*(\alpha) = \pm\alpha\} \text{ and}$$

$$h_{\pm}^{p,q}(K) := \dim_{\mathbb{C}}\mathrm{H}_{\pm}^{p,q}(K,\mathbb{C}).$$

From $E = K/\langle\sigma\rangle$, we have

$$\mathrm{H}^{p,q}(E,\mathbb{C}) \simeq \mathrm{H}_{+}^{p,q}(K,\mathbb{C}).$$

Since K is a $K3$ surface, we have

$$h^{0,0}(K) = 1,\ h^{1,0}(K) = 0,\ h^{2,0}(K) = 1, \text{ and } h^{1,1}(K) = 20, \text{ and}$$

$$h_{+}^{0,0}(K) = 1,\ h_{+}^{1,0}(K) = 0,\ h_{+}^{2,0}(K) = 0, \text{ and } h_{+}^{1,1}(K) = 10, \text{ and}$$

$$h_{-}^{0,0}(K) = 0,\ h_{-}^{1,0}(K) = 0,\ h_{-}^{2,0}(K) = 1, \text{ and } h_{-}^{2,0}(K) = 10.$$

Recall that H is generated by $\mathcal{S}_2$ and $\sigma_{1,2}$. Since $\sigma \times \sigma(\Delta) = \Delta$ and $\sigma \times \sigma(T) = T$, from $E = K/\langle\sigma\rangle$ we have $\Delta/H \simeq E$ and $T/H \simeq E$. Thus we have

$$h^{0,0}(\Delta/H) = 1,\ h^{1,0}(\Delta/H) = 0,\ h^{2,0}(\Delta/H) = 0,\ h^{1,1}(\Delta/H) = 10,$$

$$h^{0,0}(T/H) = 1,\ h^{1,0}(T/H) = 0,\ h^{2,0}(T/H) = 0, \text{ and } h^{1,1}(T/H) = 10.$$

From the Künneth Theorem, we have

$$\mathrm{H}^{p,q}(K^2, \mathbb{C}) \simeq \bigoplus_{s+u=p,t+v=q} \mathrm{H}^{s,t}(K, \mathbb{C}) \otimes \mathrm{H}^{u,v}(K, \mathbb{C}), \text{ and}$$

$$\mathrm{H}^{p,q}(K^2/H, \mathbb{C}) \simeq \{\alpha \in \mathrm{H}^{p,q}(K^2, \mathbb{C}) : s^*(\alpha) = \alpha \text{ for } s \in \mathcal{S}_2 \text{ and } \sigma^*_{1,2}(\alpha) = \alpha\}.$$

Thus we obtain

$$h^{0,0}(K^2/H) = 1,\ h^{1,0}(K^2/H) = 0,\ h^{2,0}(K^2/H) = 0,\ h^{1,1}(K^2/H) = 10,$$

$$h^{3,0}(K^2/H) = 0,\ h^{2,1}(K^2/H) = 0,\ h^{4,0}(K^2/H) = 1,$$

$$h^{3,1}(K^2/H) = 10, \text{ and } h^{2,2}(K^2/H) = 111.$$

We fix a basis β of $\mathrm{H}^{2,0}(K, \mathbb{C})$ and a basis $\{\gamma_i\}_{i=1}^{10}$ of $\mathrm{H}^{1,1}_{-}(K, \mathbb{C})$, then we have

$$\mathrm{H}^{3,1}(K^2/H, \mathbb{C}) \simeq \bigoplus_{i=1}^{10} \mathbb{C}(\beta \otimes \gamma_i + \gamma_i \otimes \beta).$$

By the above equation, we have

$$h^{0,0}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 1,\ h^{1,0}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 0,$$

$$h^{2,0}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 0,\ h^{1,1}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 12,$$

$$h^{3,0}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 0,\ h^{2,1}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 0,$$

$$h^{4,0}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 1,\ h^{3,1}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 10, \text{ and}$$

$$h^{2,2}(\mathrm{Blow}_{\Delta\cup T} K^2/H) = 131.$$

Thus we obtain $h^{0,0}(X) = 1$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 0$, $h^{1,1}(X) = 12$, $h^{3,0}(X) = 0$, $h^{2,1}(X) = 0$, $h^{4,0}(X) = 1$, $h^{3,1}(X) = 10$, and $h^{2,2}(X) = 131$. □

7. Appendix B. Now we show that the conjecture in [2, Conjecture 1] is not established for Y an Enriques surface and $L=\Omega_Y^2$.

Let Y be a smooth compact Kähler surface. Recall that $Y^{[n]}$ is the Hilbert scheme of n points of Y, $\pi_Y : Y^{[n]} \to Y^{(n)}$ the Hilbert-Chow morphism, and $p_Y : Y^n \to Y^{(n)}$ the natural projection. For a line bundle L on Y, there is a unique line bundle $\mathcal{L}$ on $Y^{(n)}$ such that $p_Y^*\mathcal{L} = \bigotimes_{i=1}^n p^{i*}L$. By using pull back we have the natural map

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(Y^{[n]}),\ L \mapsto L_n := \pi_Y^*\mathcal{L}.$$

we put

$$h^{p,q}(Y^{[n]}, L_n) := \dim_{\mathbb{C}} \mathrm{H}^q(Y^{[n]}, \Omega^p_{Y^{[n]}} \otimes L_n),$$

$$h^{p,q}(Y, L) := \dim_{\mathbb{C}} \mathrm{H}^q(Y, \Omega^p_Y \otimes L),$$

$$A := \sum_{n,p,q=0}^{\infty} h^{p,q}(Y^{[n]}, L_n) x^p y^q t^n, \text{ and}$$

$$B := \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \Big(\frac{1}{1-(-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k)} \Big)^{(-1)^{p+q} h^{p,q}(Y,L)}.$$

Then in [2, Conjecture 1] S. Boissière conjectured that

$$A = B.$$

For Y an Enriques surface and $L = \Omega_Y^2$, as in the proof on Theorem 3.2 and the Serre duality, we have

$$\begin{aligned} h^{2n-1,1}(Y^{[n]}, (\Omega_Y^2)_n) &= \dim_{\mathbb{C}} \mathrm{H}^1(Y^{[n]}, \Omega^{2n-1}_{Y^{[n]}} \otimes \Omega^{2n}_{Y^{[n]}}) \\ &= \dim_{\mathbb{C}} \mathrm{H}^1(Y^{[n]}, T_{Y^{[n]}}) \\ &= 10. \end{aligned}$$

for $n \geq 2$. It follows that the coefficient of x^3yt^2 of A is 10.

We show that the coefficient of x^3yt^2 of B is not 10.

$$h^{0,0}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \mathcal{O}_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \Omega_Y^2) = 0.$$

$$h^{0,1}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, \mathcal{O}_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, \Omega_Y^2) = 0.$$

$$h^{0,2}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, \mathcal{O}_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, \Omega_Y^2) = 1.$$

By Serre duality, we get

$$\Omega_Y \otimes \Omega_Y^2 \simeq T_Y.$$

Since Y is an Enriques surface, we have

$$h^{1,0}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \Omega_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, T_Y) = 0.$$

$$h^{1,1}(Y,\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^1(Y,\Omega_Y\otimes\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^1(Y,T_Y)=10.$$

$$h^{1,2}(Y,\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^2(Y,\Omega_Y\otimes\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^2(Y,T_Y)=0.$$

Since Y is an Enriques surface, we obtain

$$\Omega_Y^2\otimes\Omega_Y^2\simeq\mathcal{O}_Y.$$

$$h^{2,0}(Y,\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^0(Y,\Omega_Y^2\otimes\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^0(Y,\mathcal{O}_Y)=1.$$

$$h^{2,1}(Y,\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^1(Y,\Omega_Y^2\otimes\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^1(Y,\mathcal{O}_Y)=0.$$

$$h^{2,2}(Y,\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^2(Y,\Omega_Y^2\otimes\Omega_Y^2)=\dim_{\mathbb{C}}\mathrm{H}^2(Y,\mathcal{O}_Y)=0.$$

Thus we obtain

$$\begin{aligned}B&=\prod_{k=1}^{\infty}\prod_{p,q=0}^{2}\Big(\frac{1}{1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k)}\Big)^{(-1)^{p+q}h^{p,q}(E,\Omega_E^2)}\\&=\prod_{k=1}^{\infty}\Big(\frac{1}{1-x^{k-1}y^{k+1}t^k)}\Big)\Big(\frac{1}{1-x^ky^kt^k)}\Big)^{10}\Big(\frac{1}{1-x^{k+1}y^{k-1}t^k)}\Big)\\&=\prod_{k=1}^{\infty}\Big(\sum_{a=0}^{\infty}(x^{k-1}y^{k+1}t^k)^a\Big)\Big(\sum_{b=0}^{\infty}(x^ky^kt^k)^b\Big)^{10}\Big(\sum_{c=0}^{\infty}(x^{k+1}y^{k-1}t^k)^c\Big).\end{aligned}$$

Thus we have

$$\begin{aligned}B&\equiv\prod_{k=1}^{2}(1+x^{k-1}y^{k+1}t^k+x^{2k-2}y^{2k+2}t^{2k})\times(1+x^ky^kt^k+x^{2k}y^{2k}t^{2k})^{10}\times\\&\quad(1+x^{k+1}y^{k-1}t^k+x^{2k+2}y^{2k-2}t^{2k})\ (mod\ t^3)\\&\equiv\Big((1+y^2t+y^4t^2)\times(1+xy^3t^2)\Big)\times\\&\quad\Big((1+10(xyt+x^2y^2t^2)+45(xyt+x^2y^2t^2)^2)\times(1+x^2y^2t^2)\Big)\times\\&\quad\Big((1+x^2t+x^4t^2)\times(1+x^3yt^2)\Big)\ (mod\ t^3)\\&\equiv\Big(1+y^2t+(xy^3+y^4)t^2\Big)\times\Big(1+10xyt+56x^2y^2t^2\Big)\times\\&\quad\Big(1+x^2t+(x^3y+x^4)t^2\Big)\ (mod\ t^3)\\&\equiv\Big(1+(10xy+y^2)t+(56x^2y^2+11xy^3+y^4)t^2\Big)\times\\&\quad\Big(1+x^2t+(x^3y+x^4)t^2\Big)\ (mod\ t^3)\\&\equiv1+(x^2+10xy+y^2)t+(x^4+11x^3y+56x^2y^2+11xy^3+y^4)t^2\ (mod\ t^3).\end{aligned}$$

Therefore the coefficient of x^3yt^2 of B is 11. The conjecture in [2, Conjecture 1] is not established for Y an Enriques surface and $L=\Omega_Y^2$.

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