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ON LOWER BOUNDS FOR SLOPES OF TOTALLY RAMIFIED TRIPLE COVER FIBRATIONS*

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Abstract. Let $f : S \to C$ be a totally ramified triple cover fibration of type (g, γ). We prove that the slope of f has a sharp lower bound $\frac{24(g-1)}{5g-6\gamma+1}$ given that $g > \frac{15}{2}\gamma + \frac{5}{2}$. We also characterize fibrations that achieve the bound.

Key words. Triple cover, slope, index theorem, singularity.

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1. Introduction. In this paper, we always work over $\mathbb{C}$.

A fibration, denoted by $f : S \to C$, is a smooth projective surface S with a surjective morphism f onto a smooth curve C whose fibers are connected. A fibration $f : S \to C$ is relatively minimal if there is no (-1)-curve on S contained in any fiber of f. A fibration is of genus g if the genus of a general fiber is g. Let $b = g(C)$ and $g = g(F)$ be genera of the base curve C and a general fiber F respectively.

We have the following very important relative numerical invariants:

$$K_f^2 \overset{\text{def}}{=} K_{S/C}^2 = K_S^2 - 8(g-1)(b-1), \tag{1}$$

$$\chi_f \overset{\text{def}}{=} \deg f_*\mathcal{O}_S(K_f) = \chi(\mathcal{O}_S) - (g-1)(b-1), \tag{2}$$

$$e_f \overset{\text{def}}{=} \chi_{\text{top}}(S) - \chi_{\text{top}}(C)\chi_{\text{top}}(F) = \chi_{\text{top}}(S) - 4(g-1)(b-1). \tag{3}$$

It is well known that they are all non-negative and related by Noether's formula (see for instance [BHPV04])

$$12\chi_f = K_f^2 + e_f.$$

Whenever $\chi_f \neq 0$, the slope of the fibration f is defined to be $\lambda_f = K_f^2/\chi_f$. Noether's formula implies that $\lambda_f \leq 12$. The equality holds if and only if f is a Kodaira fibration.

The slope λ_f and its bounds play important roles in the study of geography of algebraic surfaces. Under different conditions, lower bounds as well as upper bounds for slopes of fibrations have been studied by many authors (For lower bounds, see for instance [Xia87], [CH88], [Hor91], [Che93], [Kon91], [Kon93], [Kon96], [SF00], [Mat90], [Per81] and [Tan96]. For upper bounds, see for instance [CT06], [Tan96], and [Xia85]). It needs to be pointed out that searching for sharp lower bounds is more difficult in general and still active. The paper [AK02] provides a very good survey in this direction.

In this paper, we focus on lower bounds for slopes of triple cover fibrations. We are motivated by the work [Bar01], [BZ01] and [CS08] on double cover fibrations.

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A fibration $f : S \to C$ of genus g is called a double cover (resp. triple cover) fibration of type (g, γ) if it is relatively minimal and there exists a relatively minimal fibration $\varphi : P_0 \to C$ of genus γ and a generic double cover (resp. triple cover) $\phi : S \dashrightarrow P_0$ such that the following diagram commutes,

$$\begin{array}{ccc} S & \overset{\phi}{\dashrightarrow} & P_0 \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} \\ & & C \end{array} \tag{4}$$

For a double cover fibration of type (g, γ), Barja [Bar01] (for $\gamma = 1$), Barja-Zucconi [BZ01] and Cornalba-Stoppino [CS08] proved that

$$\lambda_f \geq \frac{4(g-1)}{(g-\gamma)}$$

for $g \geq 4\gamma + 2$ by using double cover trick and Hodge index theorem. Moreover, there are examples showing that the bound is sharp.

For triple cover fibrations, the same idea works but the calculation is much more difficult.

Assume that f is a triple cover fibration. One can lift ϕ to a general finite morphism $\widetilde{S} \to P_0$ by a sequence of blow-ups $\rho : \widetilde{S} \to S$. Furthermore, by Stein factorization, we have

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & S_0 \\ {\scriptstyle \rho}\downarrow & \searrow & \downarrow{\scriptstyle \pi} \\ S & \overset{\phi}{\dashrightarrow} & P_0 \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} \\ & & C \end{array} \tag{5}$$

where $\pi : S_0 \to P_0$ is a triple cover. We call f a totally ramified triple cover fibration, if π is a totally ramified triple cover.

When $\gamma = 0$, to find lower bounds for the slope λ_f, we may and, unless otherwise stated, will assume that π is a *normalized triple cover*(see Lemma 2.1).

We say that a fibration $f : S \to C$ is isotrivial if all smooth fibers are isomorphic. We say that a fibration is locally trivial if it is isotrivial and all fibers are smooth.

By carefully studying local invariants of triple cover fibrations, we obtain

THEOREM 1.1. *Let $f : S \to C$ be a totally ramified triple cover fibration of type (g, γ). If $g > \frac{15}{2}\gamma + \frac{5}{2}$, then*

$$\lambda_f \geq \frac{24(g-1)}{5g - 6\gamma + 1}.$$

The bound is sharp. When $\gamma \geq 1$ or that $\gamma = 0$, $g > 3$ and $\pi : S_0 \to P_0$ (see diagram (5)) is normalized, the equality holds if and only if the following conditions hold:

(1) the morphism $\varphi : P_0 \to C$ (see diagram (4)) is locally trivial;

(2) the branch locus R of π is numerically equivalent to a $\mathbb{Q}$-linear combination of the relative canonical divisor $K_{P_0/C}$ and a fiber of $\varphi : P_0 \to C$ whenever $\gamma \neq 1$;

(3) *the singularities of the triple cover* $\pi : S_0 \to P_0$ *(see diagram* (5)*) are at most rational double points.*

REMARK 1.2. *In the case that* $(g, \gamma) = (3, 0)$ *and* $\lambda_f = 3$*, the singularities of* S_0 *are not necessarily rational double points (See Example 3.3).*

The proof of this theorem involves computations on invariants of triple cover fibrations which are much more difficult than those of double cover fibrations. We will first study the singularities of branch loci of triple covers and describe how to compute the invariants in section 2. By using the invariants of triple covers and the Hodge index theorem, we will prove Theorem 1.1 in section 3 and give an example to show that our bound is sharp.

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2. Preliminaries on the triple covers. In this section we recall some facts about triple covers from [Tan01] and [Tan02].

2.1. Triple cover data. Let P_0 be a smooth algebraic surface and $\pi : S_0 \to P_0$ be a normal triple cover. It is known that a normal triple cover is always determined by some triple cover data $(s, t, \mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on P_0, $s \in H^0(P_0, \mathcal{L}^2)$ and $0 \neq t \in H^0(P_0, \mathcal{L}^3)$. Note that S_0 is the normalization of the surface defined by $z^3 + sz + t = 0$ in the line bundle of $\mathcal{L}$ on P_0.

If $s = 0$, then the triple cover is cyclic and everything is known (see for instance [Tan02, Section 1.4]). Assume that $s \neq 0$. Let

$$a = \frac{4s^3}{\gcd(s^3, t^2)}, \; b = \frac{27t^2}{\gcd(s^3, t^2)}, \; c = \frac{4s^3 + 27t^2}{\gcd(s^3, t^2)}.$$

Then a, b and c are coprime sections of an invertible sheaf such that $a + b = c$. Conversely, any coprime triples (a, b, c) with $a + b = c$ determines a triple cover over P_0(in fact, one can recover s and t by using (a, b, c)). It is well known that the following decompositions hold:

$$a = 4a_1a_2^2a_0^3, \quad b = 27b_1b_0^2, \quad c = c_1c_0^2,$$

where a_1, a_2, b_1, c_1 are square-free and $\gcd(a_1, a_2) = 1$. The corresponding divisors of those sections are denoted by

$$A_i = \mathrm{Div}(a_i), \quad B_i = \mathrm{Div}(b_i), \quad C_i = \mathrm{Div}(c_i).$$

Recall that the branch locus of the triple cover π is $R = 2D_2 + D_1$, where $D_2 = A_1 + A_2$ and $D_1 = B_1 + C_1$ are the totally ramified branch locus and simply ramified branch locus respectively. We remark that the triple cover $\pi : S_0 \to P_0$ is totally ramified if and only if the divisor $D_1 = 0$. A Galois triple cover is clearly totally ramified. However, the converse may be not true (see [Tok92]).

2.2. Canonical resolution. For a triple cover $\pi_0 : S_0 \to P_0$, a canonical resolution $\tau : \tilde{S} \to S_0$ of singularities of S_0 is a commutative diagram.

$$\begin{array}{ccccccccc} \tilde{S} = S_k & \xrightarrow{\tau_k} & \cdots & \longrightarrow & S_2 & \xrightarrow{\tau_2} & S_1 & \xrightarrow{\tau_1} & S_0 \\ {\scriptstyle \tilde{\pi}=\pi_k}\downarrow & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0=\pi} \\ \tilde{P} = P_k & \xrightarrow{\sigma_k} & \cdots & \longrightarrow & P_2 & \xrightarrow{\sigma_2} & P_1 & \xrightarrow{\sigma_1} & P_0 \end{array} \tag{6}$$

In the diagram (6),

(1) the morphism σ_{i+1} is the blowing-up of P_i at a singular point p_i of the branch locus of π_i, the surface S_{i+1} is the normalization the fiber product of $P_{i+1} \times_{P_i} S_i$, the morphism $\tilde{\pi} = \pi_k$ has a smooth branch locus, and hence $\tilde{S} = S_k$ is smooth;

(2) the corresponding data $(a^{(i)}, b^{(i)}, c^{(i)})$ of π_i is obtained from

$$(\sigma_i^* a^{(i-1)}, \sigma_i^* b^{(i-1)}, \sigma_i^* c^{(i-1)})$$

by eliminating the common factors.

The above canonical resolution works for any triple cover [Tan91]. We remark that a canonical resolution for a cyclic triple cover was also given in [AK91]. Also another method to deal with canonical resolutions of triple covers was given by Ashikaga [Ash92].

Unless otherwise stated, we will assume that our triple covers are totally ramified triple covers in what follows. In this case, the branch locus $R = 2(A_1 + A_2)$. Such a triple cover is smooth if and only if $A_1 + A_2$ is a smooth divisor.

We put

$$d_i = \min\{m_{p_i}(A^{(i)}), m_{p_i}(B^{(i)}), m_{p_i}(C^{(i)})\},$$

where $m_p(D)$ is the multiplicity of a divisor D at p. We note that each triple cover $\pi_i : S_i \to P_i$ in the canonical resolution is still totally ramified, which can be seen by induction on i. For $i = 0$, $\pi_0 = \pi$ is totally ramified by our assumption. Assume that π_i is totally ramified. Since $m_{p_i}(B^{(i)})$ and $m_{p_i}(C^{(i)})$ are both even, d_i is also even. Therefore, after blowing up, there is no new square free component contained in $B^{(i+1)}$ or $C^{(i+1)}$. Thus $B_1^{(i+1)} + C_1^{(i+1)} = 0$, i.e., π_{i+1} is also totally ramified.

Let E_i be the exceptional curve of σ_i, $\mathcal{E}_i$ (resp., $\overline{E}_i$) be the total (resp., strict) transform of E_i in $\tilde{P}$, and $\sigma = \sigma_1 \cdots \sigma_k : \tilde{P} \to P_0$ be the compositions of the maps σ_i. Then the branch locus of $\tilde{\pi}$ is

$$\widetilde{D}_2 = \sigma^*(D_2) - \sum_{i=0}^{k-1} n_i \mathcal{E}_{i+1},$$

where the numbers n_i are given by (see [Tan02, page 158] or [CT06, page 4])

$$n_i = \begin{cases} m_{p_i}(D_2^{(i)}), & \text{if } d_i \equiv m_{p_i}(A^{(i)}) \pmod 3; \\ m_{p_i}(D_2^{(i)}) - 1, & \text{otherwise.} \end{cases} \tag{7}$$

It is not difficult to see that (also see [CT06, Lemma 2.1])

$$n_i = \begin{cases} m_{p_i}(D_2^{(i)}) - 1, & E_{i+1} \text{ in the support of } D_2^{(i+1)}; \\ m_{p_i}(D_2^{(i)}), & \text{otherwise.} \end{cases}$$

Now we have the following formulae, which will be used in the proof of Theorem 1.1, for a totally ramified triple cover (see [Tan02, Section 6] or [CT06, Section 2]).

$$\chi(\mathcal{O}_{\tilde{S}}) = 3\chi(\mathcal{O}_{P_0}) + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_{P_0} - \sum_{i=0}^{k-1}\frac{n_i(5n_i-9)}{18}, \tag{8}$$

$$K_{\tilde{S}}^2 = 3K_{P_0}^2 + \frac{4}{3}D_2^2 + 4D_2K_{P_0} - \sum_{i=0}^{k-1}\frac{4n_i(n_i-3)}{3} - 3k. \tag{9}$$

Noting that $R = 2D_2$ in our case, one can restate Lemma 5.2 in [CT06] as follows.

LEMMA 2.1. *If $\gamma = 0$, then the surface $\widetilde{P}$ in the diagram (6) can be contracted to a relatively minimal model P with a ruling $\varphi : P \to C$ satisfying the following conditions.*

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{\psi}} & P \\ & {\scriptstyle\tilde{\varphi}}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & C & \end{array}$$

(1) Let $\widetilde{D}_2$ be the totally ramified branch locus of $\tilde{\pi}$, and D_2 be the image of $\widetilde{D}_2$. Then $\tilde{\psi} : \widetilde{P} \to P$ is the canonical resolution of D_2.

(2) Let $D_{2,h}$ be the horizontal part of D_2 (i.e., $D_{2,h}$ does not contain any fibers of φ and $D_{2,v} = D_2 - D_{2,h}$ is the sum of some fibers), then the multiplicities of the singular points of $D_{2,h}$ (resp. D_2) are less or equal to $(g+2)/2$ (resp. $(g+4)/2$).

The triple cover over P totally ramified over D_2 (induced by the triple cover $\tilde{S} \to \tilde{P}$) is called normalized.

Note that the invariants $K_{\tilde{S}}$ and $\chi(\mathcal{O}_{\tilde{S}})$ can also be obtained via the formulae (8) and (9) from the induced normalized triple cover over P. So, in what follows, whenever $\gamma = 0$, we will assume that the triple cover $\pi_0 : S_0 \to P_0$ is the normalized triple cover induced by $\tilde{S} \to \tilde{P}$.

2.3. Singular points of branch locus. Let p_i be a singular point of the branch locus of the totally ramified triple cover $\pi_i : S_i \to P_i$ occurring in the canonical resolution diagram (6).

As a special case in [LT14, Theorem 1.3], we have

LEMMA 2.2. *A singularity in S_0 corresponding to p_i is a rational double (resp. triple) point if and only if the following conditions hold.*

(1) $n_i = 1$ (resp. $n_i = 2$);

(2) each infinitely close point of p_i, says p_j, satisfies $n_j \leq 2$.

From [LT14, Corollary 5.2, Corollary 6.1], one also has

LEMMA 2.3. *If $n_i = 1$, then $\overline{E}_{i+1}$ is a (-3)-curve and lies in the totally ramified branch locus. Thus the pull-back of $\overline{E}_{i+1}$ is a (-1)-curve.*

To prove Theorem 1.1, we need classify the singular points p_i with $n_i = 1$. From the definition (7) of n_i, we see that $n_i = 1$ if and only if $m_{p_i}(D_2^{(i)}) = 2$ and the exceptional curve E_{i+1} lies in the totally ramified branch locus. Moreover, the local equation of $D_2^{(i)}$ at p_i can be written as

$$x^2 + y^m = 0, \quad p_i = (0,0). \tag{10}$$

In a local neighborhood of p_i, such a triple cover is a Galois cover totally ramified over D_2 (See [Tan02, Theorem 2.1] and its proof).

From the above discussion and Lemma 2.3, if $n_i = 1$, the local triple cover over a neighborhood of p_i can be defined by a local equation

$$z^3 = x^2 + y^m.$$

So one can easily write down the canonical resolution of the corresponding singularities (see [Tan02, Section 1.2]).

In Figures 1-5, a thick line represents a component contained in the triple ramification divisor D_2, and a dash line represents a component not contained in the branch locus. The self-intersection number is marked near the component. The number under an arrow $\underset{n_i=?}{\longleftarrow}$ represents the invariant n_i associated to this blowing-up, and $\overset{1:3}{\longleftarrow}$ means a triple cover.

Based on the above discussion and Lemma 2.3, by a straightforward resolution, we see that a singular point p_i with $n_i = 1$ is one of the following types:

Type 1: $n_i = 1$ and D_2 is locally defined as $x^2 + y^2 = 0$ at $p_i = (0,0)$. The corresponding singularity in S_i is a rational double point of type A_2.

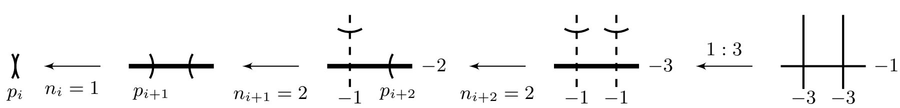

FIG. 1. *Resolution graph for type 1 singularity*

p_i has two distinct infinitely close points p_{i+1} and p_{i+2} in E_{i+1} with $n_{i+1} = n_{i+2} = 2$.

Type 2: $n_i = 1$ and D_2 is locally defined as $x^2 + y^3 = 0$ at $p_i = (0,0)$. The corresponding singularity in S_i is a rational double point of type D_4.

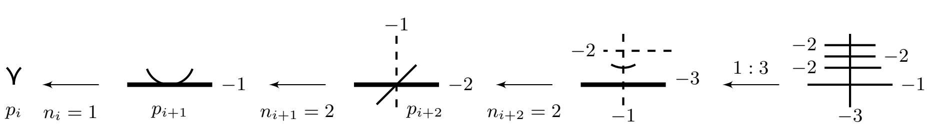

FIG. 2. *Resolution graph for type 2 singularity*

Type 3: $n_i = 1$ and D_2 is locally defined as $x^2 + y^4 = 0$ at $p_i = (0,0)$. The corresponding singularity in S_i is a rational double point of type E_6.

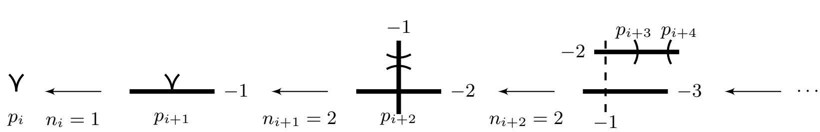

FIG. 3. *Resolution graph for type 3 singularity*

Both infinitely close points p_{i+3} and p_{i+4} are of type 1 again. We omit the rest of the resolution.

Type 4**:** $n_i = 1$ and D_2 is locally defined as $x^2 + y^5 = 0$ at $p_i = (0,0)$. The corresponding singularity in S_i is a rational double point of type E_8.

Here the infinitely close point p_{i+3} is of type 3. We omit the rest of the resolution again.

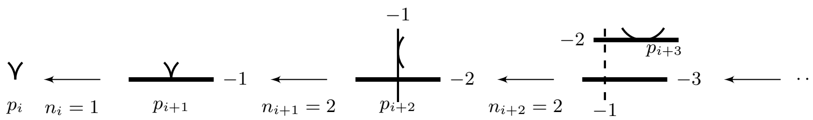

FIG. 4. *Resolution graph for type 4 singularity*

Type 5**:** $n_i = 1$ and D_2 is locally defined as $x^2 + y^m = 0$ at $p_i = (0,0)$, $m \geq 6$. However the corresponding singularity in S_i is not a rational double point by Lemma 2.2.

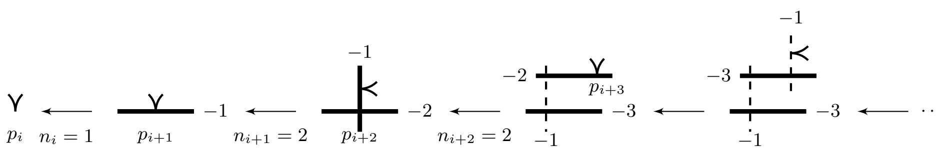

FIG. 5. *Resolution graph for type 5 singularity*

If p_{i+3} admits an infinitely close point, say p_{i+4}, after a blowing-up, then p_{i+4} is one of type 1-5 again and its local equation is $x^2 + y^{m-6} = 0$.

Summarizing the above classification, we get

LEMMA 2.4. *Let p_i be a singular point of the branch locus of the triple cover $\pi_i : S_i \to P_i$ in the commutative diagram of the canonical resolution. Assume that the invariant $n_i = 1$. Then*

(1) The point p_i has at least two infinitely close points in E_{i+1}, say p_{i+1} and p_{i+2}, with invariants $n_{i+1} = n_{i+2} = 2$.

(2) The exceptional curve E_{i+1} lies in totally ramified branch locus, and its pulling-back in $\tilde{S} = S_k$ is a (-1)-curve.

(3) The singularity in S_i corresponding to p_i is a rational double point if and only if p_i is one of types 1-4.

3. Proof of Theorem 1.1. This section is devoted to prove Theorem 1.1. We use notations introduced in the previous sections. Let $f : S \to C$ be a totally ramified triple cover fibration of type (g, γ). Then the canonical resolution $\tilde{\pi} : \tilde{S} \to \tilde{P}$ (Figure 6) induces a fibration $\tilde{f} : \tilde{S} \to C$. Since $f : S \to C$ is relatively minimal, by the uniqueness of relatively minimal model, after contracting all (-1)-curves in fibers of

$\tilde{f}$, we get $f : S \to C$ again. Namely, we have the following commutative diagram

$$\begin{array}{ccccc} & & S & & \\ & \overset{\rho}{\nearrow} & \downarrow & \overset{\phi}{\searrow} & \\ \tilde{S} & \xrightarrow{\tau} & S_0 & \longrightarrow & P_0 \\ \downarrow{\scriptstyle\tilde{\pi}} & & \downarrow{\scriptstyle\pi} & // & \downarrow{\scriptstyle\varphi} \\ \tilde{P} & \xrightarrow{\sigma} & P_0 & \longrightarrow & C \end{array} \tag{11}$$

where $\rho : \tilde{S} \to S$ is the morphism contracting all (-1)-curves in fibers of $\tilde{f}$. We denote by ε the number of the (-1)-curves in fibers contracted by ρ. More precisely, ρ can be decomposed as $\rho_1 : \widetilde{S} \to S'$ and $\rho_2 : S' \to S$ where ρ_1 contracts all (-1)-curves in fibers occurring in exceptional sets of the singularities of S_0 and ρ_2 contracts all extra (-1)-curves in fibers. Let ε_i be the number of (-1)-curves contracted by ρ_i $(i = 1, 2)$. One has $\varepsilon = \varepsilon_1 + \varepsilon_2$. Obviously, if $\gamma > 0$, then $\varepsilon_2 = 0$, i.e., there is no extra contractions.

From the formulae for relatively numerical invariants, we have

$$K_{P_0}^2 = K_{P_0/C}^2 + 8(\gamma - 1)(b - 1),$$
$$\chi(\mathcal{O}_{P_0}) = \chi_\varphi + (\gamma - 1)(b - 1).$$

and

$$K_f^2 = K_{\tilde{S}/C}^2 + \varepsilon,$$
$$\chi_f = \chi_{\tilde{f}}.$$

Therefore, the formulae (8) and (9) imply that

$$K_f^2 = 3K_{P_0/C}^2 + \frac{4}{3}D_2^2 + 4D_2K_{P_0/C} - \sum_{i=0}^{k-1}(\frac{4n_i(n_i - 3)}{3} + 3) + \varepsilon, \tag{12}$$

$$\chi_f = 3\chi_\varphi + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_{P_0/C} - \sum_{i=0}^{k-1}\frac{n_i(5n_i - 9)}{18}. \tag{13}$$

Proof of Theorem 1.1. Let

$$\lambda = \frac{24(g - 1)}{5g - 6\gamma + 1}.$$

By formulae (12) and (13), we have

$$K_f^2 - \lambda\chi_f = 3(K_{P_0/C}^2 - \lambda\chi_\varphi) + (\frac{4}{3} - \frac{5\lambda}{18})D_2^2 + (4 - \frac{\lambda}{2})D_2K_{P_0/C} + \tau(\lambda), \tag{14}$$

where

$$\tau(\lambda) = \sum_{i=0}^{k-1}((\frac{5}{18}\lambda - \frac{4}{3})n_i^2 + (4 - \frac{\lambda}{2})n_i - 3) + \varepsilon. \tag{15}$$

It is easy to see that theorem 1.1 follows from the following two claims:

Claim 1.

$$K_f^2 - \lambda\chi_f - \tau(\lambda) \geq 0.$$

Moreover, if $\gamma \neq 1$, then the equality holds if and only if φ is locally trivial, and $D_2 \equiv_{\mathrm{num}} aK_{P_0/C} + bF_\varphi$, for some $a, b \in \mathbb{Q}$.

Claim 2.

$$\tau(\lambda) \geq 0.$$

Moreover, if $(g,\gamma) \neq (3,0)$, then $\tau(\lambda) = 0$ if and only if all singularities of $\pi : S_0 \to P_0$ are rational double points.

Proof of Claim 1. By Hurwitz's formula, we have

$$2g - 2 = 3(2\gamma - 2) + 2D_2F_\varphi, \tag{16}$$

where F_φ is a general fiber of $\varphi : P_0 \to C$. Hence

$$D_2K_{P_0} = D_2K_{P_0/C} + 2(g-1)(b-1) - 6(\gamma-1)(b-1). \tag{17}$$

It then follows from (16) and (17) that

$$\begin{aligned}&(\frac{4}{3} - \frac{5\lambda}{18})D_2^2 + (4 - \frac{\lambda}{2})D_2K_{P_0/C}\\ &= \frac{4}{5g-6\gamma+1}(2(D_2F_\varphi)(K_{P_0/C}D_2) - (K_{P_0/C}F_\varphi)(D_2^2)).\end{aligned} \tag{18}$$

We will divide the proof of Claim 1 into three cases:

Case A. Assume that $\gamma = 0$.

In this case, $\lambda = \lambda_0 = \frac{24(g-1)}{5g+1}$, and $\varphi : P_0 \to C$ is a ruled surface. Thus $K_{P_0/C}^2 = \chi_\varphi = 0$.

Since the Picard number of P_0 is 2, the determinant of the intersection matrix of $K_{P_0/C}, F_\varphi, D_2$ is zero, where F_φ is a general fiber of $\varphi : P_0 \to C$. Thus one has

$$D_2^2 = -(D_2F_\varphi)(D_2K_{P_0/C}),$$

that is, $D_2K_{P_0/C} = -\frac{D_2^2}{g+2}$ by formula (16).

Hence

$$\begin{aligned}K_f^2 - \lambda_0\chi_f - \tau(\lambda_0) &= (\frac{4}{3} - \frac{5\lambda_0}{18})D_2^2 + (4 - \frac{\lambda_0}{2})D_2K_{P_0/C}\\ &= \frac{24(g-1) - (5g+1)\lambda_0}{18(g+2)}\ D_2^2\\ &= 0.\end{aligned}$$

In this case, D_2 is always numerically equivalent to a $\mathbb{Q}$-linear combination of $K_{P_0/C}$ and F_φ.

Case B. Assume $\gamma = 1$.

In this case, $\lambda = \lambda_1 = \frac{24}{5}$ and

$$K_{P_0/C} \equiv (\chi_\varphi + \sum_j \frac{r_j - 1}{r_j})F_\varphi$$

where r_j's are the multiplicities of singular fibers of $\varphi : P_0 \to C$ (see [BHPV04, Corollary V.12.3]). Thus $K^2_{P_0/C} = 0$. One has that

$$\begin{aligned} K_f^2 - \frac{24}{5}\chi_f - \tau(\frac{24}{5}) &= -\frac{72}{5}\chi_\varphi + \frac{8}{5}D_2K_{P_0/C} \\ &= -\frac{72}{5}\chi_\varphi + \frac{8(g-1)}{5}(\chi_\varphi + \sum_j \frac{(r_j-1)}{r_j}) \\ &\geq \frac{8g-80}{5}\chi_\varphi \\ &\geq 0. \end{aligned}$$

The equalities hold if and only if $\chi_\varphi = 0$ and $r_i = 1$ for all i, if and only if φ is locally trivial (see [BHPV04, Theorem III.18.2]).

Case C. Assume $\gamma \geq 2$.

Since $K^2_{P_0/C} \geq 0$, then the Hodge Index Theorem implies that the determinant of the intersection matrix of $K_{P_0/C}, F_\varphi, D_2$ is non-negative. Therefore,

$$2(D_2F_\varphi)(K_{P_0/C}F_\varphi)(K_{P_0/C}D_2) - (K_{P_0/C}F_\varphi)^2D_2^2 \geq (D_2F_\varphi)^2K^2_{P_0/C}.$$

Combining this inequality with (18), one has that

$$(\frac{4}{3} - \frac{5\lambda}{18})D_2^2 + (4 - \frac{\lambda}{2})D_2K_{P_0/C} \geq \frac{2(g-3\gamma+2)^2}{(5g-6\gamma+1)(\gamma-1)}K^2_{P_0/C}.$$

On the other hand, from Xiao's inequality (see [Xia87]), we have

$$K^2_{P_0/C} \geq (4 - \frac{4}{\gamma})\chi_\varphi.$$

Therefore, the following inequality holds

$$\begin{aligned} K_f^2 - \lambda\chi_f - \tau(\lambda) &\geq 3(1 - \frac{\lambda\gamma}{4(\gamma-1)})K^2_{P_0/C} + \frac{2(g-3\gamma+2)^2}{(5g-6\gamma+1)(\gamma-1)}K^2_{P_0/C} \\ &= \frac{(2g-15\gamma-5)(g-1)}{(5g-6\gamma+1)(\gamma-1)}K^2_{P_0/C} \\ &\geq 0. \end{aligned}$$

Looking upon the proof of Case C, one can see clear that the equality holds if and only if φ is locally trivial, and $D_2 \equiv aK_{P_0/C} + bF_\varphi$, for some $a, b \in \mathbb{Q}$.

This completes the proof of Claim 1. ∎

Proof of Claim 2. Define a function

$$f(n) = (\frac{5}{18}\lambda - \frac{4}{3})n^2 + (4 - \frac{\lambda}{2})n - 3.$$

Let l_s be the number of singular points p_i of the branch locus occurring in the canonical resolution such that the corresponding invariants $n_i = s$. Thus

$$\tau(\lambda) = \sum_{s\geq 1} f(s)l_s + \varepsilon.$$

From Lemma 2.4, one has $l_2 \geq 2l_1$ and $\varepsilon \geq \varepsilon_1 \geq l_1$. We divide our proof into two cases.

Case A. Assume that $\gamma > 0$.

In this case $\varepsilon_2 = 0$. Since $f(s) > 0$ for $s \geq 2$ and $f(1) + 2f(2) = -1$, we have

$$\tau(\lambda) \geq (f(1) + 2f(2) + 1)l_1 = 0.$$

$\tau(\lambda) = 0$ if and only if $l_s = 0$ $(s \geq 3)$, $l_2 = 2l_1$ and $\varepsilon = \varepsilon_1 = l_1$. It is equivalent to that each p_i is either of type 1-4 in Section 2 or an infinitely close point of type 1-4. On the other hand, by Lemma 2.2, the point p_i is corresponding to a rational double point if and only if p_i is either of type 1-4 or a infinitely close point of type 1-4. Thus $\tau(\lambda) = 0$ if and only if the singularities of S_0 are at most rational double points.

Case B. Assume that $\gamma = 0$.

In this case, $\lambda = \frac{24(g-1)}{5g+1}$ and

$$f(n) = -\frac{8}{5g+1}n^2 + \frac{8g+16}{5g+1}n - 3.$$

Since we assume that $\pi_0 : S_0 \to P_0$ is normalized, it follows from Lemma 2.1 that $n_i \leq (g+4)/2$ for all i. By a straightforward computation, one has $f(s) \geq 0$ whenever $s \geq 2$. Furthermore, if $g > 3$, then $f(s) > 0$ for $s \geq 2$. Similar to the discussion in Case A, one finds that $\tau(\lambda) \geq 0$.

In what follows, we assume $g > 3$. Therefore, $\tau(\lambda) = 0$ if and only if $l_s = 0$ $(s \geq 3)$, $l_2 = 2l_1$, $\varepsilon_2 = 0$ and $\varepsilon = \varepsilon_1 = l_1$. Equivalently, the singularities of S_0 are at most rational double points and there is no extra contractions.

Conversely, if all singularities of S_0 are rational double points, then $n_i \leq 2$ for all i, $l_2 = 2l_1$, $\varepsilon_1 = l_1$ by the discussions in Sec. 2.3. From Lemma 3.1 in the following, we see that $\varepsilon_2 = 0$. Therefore, $\tau(\lambda) = 0$. □

LEMMA 3.1. *If $g > 3$, $\gamma = 0$ and all singularities in S_0 are rational double points, then $\varepsilon_2 = 0$.*

Proof. Suppose on the contrary that $\varepsilon_2 > 0$.

We use the notations in the paragraph containing the diagram (11). Let $\tilde{f} : \widetilde{S} \to C$ be the fibration induced by $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$ and Γ be an irreducible component of some fiber of $\tilde{f}$ such that $\Gamma' = \rho_1(\Gamma)$ is a (-1)-curve. Let $f' : S' \to C$ be the fibration induced by ρ_1 and F' be the fiber containing Γ'. By the definition of extra contraction, $\sigma\tilde{\pi}(\Gamma)$ is not a point, hence it is a fiber F_0 of $\varphi_0 : P_0 \to C$. So $\widetilde{F}_0 := \tilde{\pi}(\Gamma)$ is the strict transform of F_0. The fiber of $\tilde{f}$ containing Γ is exactly $\tilde{\pi}^*\sigma^*F_0$. Thus, the multiplicity of Γ' in F' equals the multiplicity of Γ in $\tilde{\pi}^*\widetilde{F}_0$. Since S_0 has at most rational double points, F' consists of some ADE curves and all components of $\rho_1(\tilde{\pi}^*\widetilde{F}_0)$.

Recall that $\tilde{\pi}$ is a totally ramified triple cover. Suppose that $\tilde{\pi}^*\widetilde{F}_0$ is irreducible, i.e., $\tilde{\pi}^*\widetilde{F}_0 = d\Gamma$ $(d = 1$ or $3)$. Since F' consists of Γ' and some ADE curves, and the multiplicity of Γ' in F' is d, one has

$$2g - 2 = K_{S'}F' = dK_{S'} \cdot \Gamma' = -d < 0,$$

which is a contradiction. Hence $\tilde{\pi}^*\widetilde{F}_0 = \Gamma + \Gamma_1 + \Gamma_2$, where Γ_1, Γ_2 are irreducible components isomorphic to Γ as well as $\widetilde{F}_0$.

Suppose that there are at least two singular points of D_2 lying in F_0. Then Γ' meets with at least two (-2)-curves, say C_1, C_2, since the singularities of S_0 are at most rational double points. Thus $(2\Gamma' + C_1 + C_2)^2 \geq 0$. Note that $\Gamma' \cup C_1 \cup C_2$ is in the support of F'. Then F' is proportional to $2\Gamma' + C_1 + C_2$ by [Rei97, Theorem A.7]. Consequently, we would have $g = 0$ by adjunction formula, which is a contradiction. Therefore, there is a unique singular point p_1 of D_2 lying in F_0.

By our assumption and the classification of p_1 in Section 2.3, the ADE curve on S' over p_1, denoted by E, is of type A_2, D_4, E_6, E_8. Let Z be the fundamental cycle supported on E. Since $(\Gamma' + Z)^2 \leq 0$, $\Gamma'^2 = -1$, $Z^2 = -2$ and $\Gamma' Z \geq 1$, we see that $Z \cdot \Gamma' = 1$ and $(\Gamma' + Z)^2 = -1$. It follows that Γ' meets transversely with only one component of E, say C_1, and the multiplicity of C_1 in Z is 1. In particular, E is not of type E_8. Suppose that E is of type E_6. One will find that $\Gamma' + E$ is not semi-negative via contracting repeatedly (-1)-curves from $\Gamma' + E$. That is a contradiction. Suppose that E is of type D_4, then $(2\Gamma' + C_1 + Z)^2 = 0$. So $F' = 2\Gamma' + C_1 + Z$ and $g = 0$. Again, it is a contradiction.

Therefore, E is of type A_2. Firstly, we claim that $\{p_1\} = D_2 \cap F_0$. Suppose that there is another point $q \in D_2 \cap F_0$. Since p_1 is a unique singular point of D_2 lying in F_0, D_2 is smooth at q. If D_2 meets transversely with F_0 at q, then $\tilde{\pi}^*\widetilde{F}_0$ is locally defined by $z^3 = x$. Hence, $\tilde{\pi}^*\widetilde{F}_0$ is irreducible, which contradicts that $\tilde{\pi}^*\widetilde{F}_0 = \Gamma + \Gamma_1 + \Gamma_2$. If D_2 is tangent to F_0 at q, then Γ' meets with $\rho_1(\Gamma_1)$ and $\rho_1(\Gamma_2)$. Recall that $\Gamma'^2 = -1$. From Zariski's lemma, we know that $\Gamma' F' = 0$. Consequently, the multiplicity of Γ' in F' is at least 2. So the multiplicity of Γ in the fiber $\tilde{\pi}^*\sigma^* F_0$ of $\tilde{f}$ is at least 2. It implies that the multiplicity of Γ in $\tilde{\pi}^*\widetilde{F}_0$ is at least 2, a contradiction. Therefore, $\{p_1\} = D_2 \cap F_0$. By Hurwitz formula and adjunction formula, the intersection number of D_2 and F_0 at p_1 is $g + 2$.

By choosing a suitable local coordinates, one can write the local equation of π near p_1 as $z^3 = x(t + x^{g+1})$, where $t = 0$ is the local equation of F_0 passing through $p_1 = (0, 0)$. By a straightforward resolution, one can see that Γ' is not a (-1)-curve, which contradicts that $\varepsilon_2 > 0$. □

Theorem 1.1 clearly follows from Claim 1 and 2. □

Here we present an example which shows that the bound given in Theorem 1.1 is indeed sharp. The construction of the example comes similarly from [CS08].

EXAMPLE 3.2. Let F and C be smooth curves with genera $\gamma = g(F)$ and $g(C) = 1$ respectively. Let $p_1 : C \times F \to C$ and $p_2 : C \times F \to F$ be the projections, and H_1, H_2 be their general fibers. For sufficiently large integers n and m, the linear system $|3nH_1 + 3mH_2|$ is base-point-free. Hence, by Bertini's Theorem there exists a smooth divisor $D_2 \in |3nH_1 + 3mH_2|$. Since $D_2 \equiv 3L$ for some effective divisor L, we can construct a triple cyclic cover $\pi : S \to C \times F$ branched over D_2. Consider the fibration $f := p_1 \circ \pi : S \to C$. A general fiber is a triple cover over F, and its genus is $g = 3\gamma + 3m - 2$. Note that

$$K_f \sim \pi^*(K_{C\times F/C} + 2nH_1 + 2mH_2)) \sim \pi^*(2nH_1 + (2\gamma + 2m - 2)H_2),$$

which yields that $K_f^2 = 24n(m + \gamma - 1)$, and also we have

$$\chi_f = n(5m + 3\gamma - 3).$$

Therefore, the slope of f is exactly

$$\lambda_f = \frac{24(g-1)}{5g-6\gamma+1}.$$

The following example shows that a triple fibration of type $(g, \gamma) = (3, 0)$ with extreme slope $\lambda_f = 3$ does not need to satisfy satisfy the condition (3) in Theorem 1.1.

Example 3.3. Let

$$\varphi_0 = pr_1 : P_0 = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad (t, x) \to t.$$

Consider the cyclic triple cover $\pi_0 : S_0 \to P_0$ defined by the equation

$$z^3 = t(t-2)x(x+t)(x^2-1).$$

In this case,

$$A_1 = F_0 + F_2 + \Delta_0 + \Delta_1 + \Delta_{-1} + L, \quad A_2 = \Delta_\infty,$$

where F_α (resp. Δ_β) is a fiber (resp. section) defined by $t = \alpha$ (resp. $x = \beta$) and L is the line defined by $x + t = 0$. By a straightforward computation, one has

$$K_f^2 = 12, \quad \chi_f = 4, \quad e_f = 36,$$

where $f : S \to \mathbb{P}^1$ is of genus 3. So $K_f^2 = 3\chi_f$. However, the branch locus contains a singular point p_1 defined by $tx(x+t) = 0$ with $n_1 = 3$. So $\pi^{-1}(p_1) \in S_0$ is not a rational point by Lemma 2.2.

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