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THE MEAN CURVATURE FLOW FOR INVARIANT HYPERSURFACES IN A HILBERT SPACE WITH AN ALMOST FREE GROUP ACTION*

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Abstract. In this paper, we study the regularized mean curvature flow starting from invariant hypersurfaces in a Hilbert space equipped with an isometric almost free Hilbert Lie group action whose orbits are minimal regularizable submanifolds, where "almost free" means that the stabilizers of the group action are finite. First we obtain the evolution equations for some geometric quantities along the regularized mean curvature flow. Next, by using the evolution equations, we prove a horizontally strongly convexity preservability theorem for the regularized mean curvature flow. From this theorem, we derive the strongly convexity preservability theorem for the mean curvature flow starting from compact Riemannian suborbifolds in the orbit space (which is a Riemannian orbifold) of the Hilbert Lie group action.

Key words. Regularized mean curvature flow, horizontally strongly convexity, Riemannian suborbifold.

Mathematics Subject Classification. 53C44, 53C42.

1. Introduction. R. S. Hamilton ([Ha]) proved the existenceness and the uniqueness (in short time) of solutions satisfying any initial condition of a weakly parabolic equation for sections of a finite dimensional vector bundle. The Ricci flow equation for Riemannian metrics on a fixed compact manifold M is a weakly parabolic equation, where we note that the Riemannian metrics are sections of the $(0,2)$-tensor bundle $T^{(0,2)}M$ of M. Let f_t $(0 \leq t < T)$ be a C^∞-family of immersions of M into the m-dimensional Euclidean space $\mathbb{R}^m$. Define a map $F : M \times [0,T) \to \mathbb{R}^m$ by $F(x,t) := f_t(x)$ $((x,t) \in M \times [0,T))$. The mean curvature flow equation is described as

$$\frac{\partial F}{\partial t} = \triangle_t f_t,$$

where $\triangle_t$ is the Laplacian operator of the metric g_t on M induced from the Euclidean metric of $\mathbb{R}^m$ by f_t. Here we note that $\triangle_t f_t$ is equal to the mean curvature vector of f_t. This evolution equation also is a weakly parabolic equation, where we note that the immersions f_t's are regarded as sections of the trivial bundle $M \times \mathbb{R}^m$ over M under the identification of f_t and its graph immersion $\mathrm{id}_M \times f : M \to M \times \mathbb{R}^m$ (id_M : the identity map of M). Hence we can apply the Hamilton's result to this evolution equation and hence can show the existenceness and the uniqueness (in short time) of solution of this evolution equation satisfying any initial condition. In this paper, we consider the case where the ambient space is a (separable infinite dimensional) Hilbert space V. Let M be a Hilbert manifold and f_t $(0 \leq t < T)$ be a C^∞-family of immersions of M into V. Assume that f_t is regularizable, where "regularizability" means that the codimension of f is finite, for each normal vector v of M, the shape operator A_v is a compact operator, and that the regularized trace $\mathrm{Tr}_r\, A_v$ of A_v and the trace $\mathrm{Tr}\, A_v^2$ of A_v^2 exist. Note that the notions of the regularized trace and the regularized mean curvature vector were introduced in [HLO] (see the next section

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about the definitions of these notions). Denote by H_t the regularized mean curvature vector of f_t. Define a map $F : M \times [0, T) \to V$ as above in terms of f_t's. We call f_t's $(0 \le t < T)$ the *regularized mean curvature flow* if the following evolution equation holds:

$$\frac{\partial F}{\partial t} = \triangle_t^r f_t. \tag{1.1}$$

Here $\triangle_t^r f_t$ is defined as the vector field along f_t satisfying

$$\langle \triangle_t^r f_t, v\rangle := \mathrm{Tr}_r \langle (\nabla^t df_t)(\cdot, \cdot), v\rangle^\sharp \quad (\forall\, v \in V),$$

where ∇^t is the Riemannian connection of the metric g_t on M induced from the metric $\langle\ ,\ \rangle$ of V by f_t, $\langle (\nabla^t df_t)(\cdot, \cdot), v\rangle^\sharp$ is the $(1,1)$-tensor field on M defined by $g_t(\langle (\nabla^t df_t)(\cdot, \cdot), v\rangle^\sharp(X), Y) = \langle (\nabla^t df_t)(X, Y), v\rangle$ $(X, Y \in TM)$ and $\mathrm{Tr}_r(\cdot)$ is the regularized trace of $(\cdot)$. Note that $\triangle_t^r f_t$ is equal to H_t. In general, the existenceness and the uniqueness (in short time) of solutions of this evolution equation satisfying any initial condition has not been shown yet. For we cannot apply the Hamilton's result to this evolution equation because it is regarded as the evolution equation for sections of the *infinite* dimensional vector bundle $M \times V$ over M. However we can show the existenceness and the uniqueness (in short time) of solutions of this evolution equation in special case. In this paper, we consider a isometric almost free action of a Hilbert Lie group G on a Hilbert space V whose orbits are regularized minimal, that is, they are regularizable submanifold and their regularized mean curvature vectors vanish, where "almost free" means that the stabilizers of the action are finite. Let $M(\subset V)$ be a G-invariant submanifold in V. Assume that the image of M by the orbit map of the G-action is compact. Let f be the inclusion map of M into V. We first show that the regularized mean curvature flow starting from M exists uniquely in short time (see Proposition 4.1). In particular, we consider the case where M is a hypersurface. The first purpose of this paper is to obtain the evolution equations for various geometrical quantities along the regularized mean curvature flow starting from G-invariant hypersurfaces (see Section 4). The second purpose is to prove a maximum principal for an evolution equation related to a G'-invariant symmetric $(0,2)$-tensor fields S_t's on a Hilbert manifold M equipped with an isometric almost free Hilbert Lie group action G' such that M/G' is a finite dimensional compact Riemannian orbifold (see Section 5). The third purpose is to prove a horizontally strongly convexity preservability theorem for the regularized mean curvature flow starting from the above invariant hypersurface by using the evolution equations in Section 4 and imitating the discussion in the proof of a maximum principal in Section 5 (see Section 6). From this theorem, we derive the strongly convexity preservability theorem for the mean curvature flow starting from compact Riemannian suborbifolds in the orbit space V/G (which is a Riemannian orbifold) (see Section 7).

2. The regularized mean curvature flow. Let f_t $(0 \le t < T)$ be a one-parameter C^∞-family of immersions of a manifold M into a (finite dimensional) Riemannian manifold N, where T is a positive constant or $T = \infty$. Denote by H_t the mean curvature vector of f_t. Define a map $F : M \times [0, T) \to N$ by $F(x, t) = f_t(x)$ $((x, t) \in M \times [0, T))$. If, for each $t \in [0, T)$, $\dfrac{\partial F}{\partial t} = H_t$ holds, then f_t $(0 \le t < T)$ is called a *mean curvature flow*.

Let f be an immersion of an (infinite dimensional) Hilbert manifold M into a Hilbert space V and A the shape tensor of f. If $\mathrm{codim}\, M < \infty$ and A_v is a compact operator for each normal vector v of f, then M is called a *Fredholm submanifold*. In this paper, we then call f a *Fredholm immersion*. Furthermore, if, for each normal vector v of M, the regularized trace $\mathrm{Tr}_r\, A_v$ and $\mathrm{Tr}\, A_v^2$ exist, then M is called *regularizable submanifold*, where $\mathrm{Tr}_r\, A_v$ is defined by $\mathrm{Tr}_r\, A_v := \sum_{i=1}^{\infty}(\mu_i^+ + \mu_i^-)$ ($\mu_1^- \leq \mu_2^- \leq \cdots \leq 0 \leq \cdots \leq \mu_2^+ \leq \mu_1^+$: the spectrum of A_v). Note that the notion of the regularized trace was defined in [HLO] and that it differs from the trace defined in terms of the zeta function in [KT]. In this paper, we then call f *regularizable immersion*. If f is a regulalizable immersion, then the *regularized mean curvature vector* H of f is defined by $\langle H, v\rangle = \mathrm{Tr}_r\, A_v$ ($\forall\, v \in T^\perp M$), where $\langle\ ,\ \rangle$ is the inner product of V and $T^\perp M$ is the normal bundle of f. If $H = 0$, then f is said to be *minimal*. In particular, if f is of codimension one, then we call the norm $||H||$ of H the *regularized mean curvature function* of f.

Let f_t ($0 \leq t < T$) be a C^∞-family of regularizable immersions of M into V. Denote by H_t the regularized mean curvature vector of f_t. Define a map $F : M \times [0, T) \to V$ by $F(x, t) := f_t(x)$ ($(x, t) \in M \times [0, T)$). If $\frac{\partial F}{\partial t} = H_t$ holds, then we call f_t ($0 \leq t < T$) the *regularized mean curvature flow*. It has not been known whether the regulalized mean curvature flow starting from any regularizable hypersurface exists uniquely in short time. However its existence and uniqueness (in short time) is shown in a special case (see Proposition 4.1).

3. The mean curvature flow in Riemannian orbifolds. In this section, we shall define the notion of the mean curvaure flow starting from a suborbifold in a Riemannian orbifold. First we recall the notions of a Riemannian orbifold and a suborbifold following to [AK, BB, GKP, Sa, Sh, Th]. Let M be a paracompact Hausdorff space and $(U, \phi, \widetilde{U}/\Gamma)$ a triple satisfying the following conditions:

(i) U is an open set of M,
(ii) $\widehat{U}$ is an open set of $\mathbb{R}^n$ and Γ is a finite subgroup of the C^k-diffeomorphism group $\mathrm{Diff}^k(\widehat{U})$ of $\widehat{U}$,
(iii) ϕ is a homeomorphism of U onto $\widehat{U}/\Gamma$.

Such a triple $(U, \phi, \widehat{U}/\Gamma)$ is called an *n-dimensional orbifold chart*. Let $\mathcal{O} := \{(U_\lambda, \phi_\lambda, \widehat{U}/\Gamma_\lambda) \,|\, \lambda \in \Lambda\}$ be a family of n-dimensional orbifold charts of M satisfying the following conditions:

(O1) $\{U_\lambda \,|\, \lambda \in \Lambda\}$ is an open covering of M,
(O2) For $\lambda, \mu \in \Lambda$ with $U_\lambda \cap U_\mu \neq \emptyset$, there exists an n-dimensional orbifold chart $(W, \psi, \widehat{W}/\Gamma')$ such that C^k-embeddings $\rho_\lambda : \widehat{W} \hookrightarrow \widehat{U}_\lambda$ and $\rho_\mu : \widehat{W} \hookrightarrow \widehat{U}_\mu$ satisfying $\phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} \circ \rho_\lambda = \psi^{-1} \circ \pi_{\Gamma'}$ and $\phi_\mu^{-1} \circ \pi_{\Gamma_\mu} \circ \rho_\mu = \psi^{-1} \circ \pi_{\Gamma'}$, where π_{Γ_λ}, π_{Γ_μ} and $\pi_{\Gamma'}$ are the orbit maps of Γ_λ, Γ_μ and Γ', respectively.

Such a family $\mathcal{O}$ is called an *n-dimensional C^k-orbifold atlas* of M and the pair $(M, \mathcal{O})$ is called an *n-dimensional C^k-orbifold*. Let $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ be an n-dimensional orbifold chart around $x \in M$. Then the group $(\Gamma_\lambda)_{\widehat{x}} := \{b \in \Gamma_\lambda \,|\, b(\widehat{x}) = \widehat{x}\}$ is unique for x up to the conjugation, where $\widehat{x}$ is a point of $\widehat{U}_\lambda$ with $(\phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda})(\widehat{x}) = x$. Denote by $(\Gamma_\lambda)_x$ the conjugate class of this group $(\Gamma_\lambda)_{\widehat{x}}$, This conjugate class is called the *local group at x*. If the local group at x is not trivial, then x is called a *singular point* of $(M, \mathcal{O})$. Denote by $\mathrm{Sing}(M, \mathcal{O})$ (or $\mathrm{Sing}(M)$) the set of all singular points of $(M, \mathcal{O})$. This set $\mathrm{Sing}(M, \mathcal{O})$ is called the *singular set* of $(M, \mathcal{O})$. Let $x \in M$ and $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ an orbifold chart around x. Take $\widehat{x}_\lambda \in \widehat{U}_\lambda$ with $\pi_{\Gamma_\lambda}(\widehat{x}_\lambda) = x$.

The group $(\Gamma_\lambda)_{\widehat{x}_\lambda}$ acts on $T_{\widehat{x}_\lambda}\widehat{U}_\lambda$ naturally. Denote by $\mathcal{O}_x$ the subfamily of $\mathcal{O}$ consisting of all orbifold charts around x. Give $\mathcal{T}_x := \bigoplus_{(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)\in\mathcal{O}_x} T_{\widehat{x}_\lambda}\widehat{U}_\lambda/(\Gamma_\lambda)_{\widehat{x}_\lambda}$ an equivalence relation $\sim$ as follows. Let $(U_{\lambda_1},\phi_{\lambda_1},\widehat{U}_{\lambda_1}/\Gamma_{\lambda_1})$ and $(U_{\lambda_2},\phi_{\lambda_2},\widehat{U}_{\lambda_2}/\Gamma_{\lambda_2})$ be members of $\mathcal{O}_x$. Let η be the diffeomorphism of a sufficiently small neighborhood of $\widehat{x}_{\lambda_1}$ in $\widehat{U}_{\lambda_1}$ into $\widehat{U}_{\lambda_2}$ satisfying $\phi_{\lambda_2}^{-1}\circ\pi_{\Gamma_{\lambda_2}}\circ\eta = \phi_{\lambda_1}^{-1}\circ\pi_{\Gamma_{\lambda_1}}$. Define an equivalence relatiton $\sim$ in $\mathcal{T}_x$ as the relation generated by

$$[v_1]\sim[v_2] \underset{\text{def}}{\iff} [v_2]=[\eta_*(v_1)]$$
$$([v_1]\in T_{\widehat{x}_{\lambda_1}}\widehat{U}_{\lambda_1}/(\Gamma_{\lambda_1})_{\widehat{x}_{\lambda_1}},\ [v_2]\in T_{\widehat{x}_{\lambda_2}}\widehat{U}_{\lambda_2}/(\Gamma_{\lambda_2})_{\widehat{x}_{\lambda_2}}),$$

where $[v_i]$ $(i=1,2)$ is the $(\Gamma_{\lambda_i})_{\widehat{x}_{\lambda_i}}$-orbits through $v_i\in T_{\widehat{x}_{\lambda_i}}\widehat{U}_{\lambda_i}$. We call the quotient space $\mathcal{T}_x/\sim$ the *orbitangent space of* M *at* x and denote it by T_xM. If $(M,\mathcal{O})$ is of class C^k $(r\geq 1)$, then $TM := \bigoplus_{x\in M} T_xM$ is a C^{r-1}-orbifold in a natural manner. We call TM the *orbitangent bundle of* M. The group $(\Gamma_\lambda)_{\widehat{x}_\lambda}$ acts on the (r,s)-tensor space $T^{(r,s)}_{\widehat{x}_\lambda}\widehat{U}_\lambda$ of $T_{\widehat{x}_\lambda}\widehat{U}_\lambda$ naturally. Give $\mathcal{T}^{(r,s)}_x := \bigoplus_{(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)\in\mathcal{O}_x} T^{(r,s)}_{\widehat{x}_\lambda}\widehat{U}_\lambda/(\Gamma_\lambda)_{\widehat{x}_\lambda}$ an equivalence relation $\sim$ as above. We call the quotient space $\mathcal{T}^{(r,s)}_x/\sim$ the (r,s)-*orbitensor space of* M *at* x and denote it by $T^{(r,s)}_xM$. If $(M,\mathcal{O})$ is of class C^k $(r\geq 1)$, then $T^{(r,s)}M := \bigoplus_{x\in M} T^{(r,s)}_xM$ is a C^{r-1}-orbifold in a natural manner. We call $T^{(r,s)}M$ the (r,s)-*orbitensor bundle of* M.

Let $(M,\mathcal{O}_M)$ and $(N,\mathcal{O}_N)$ be orbifolds, and f a map from M to N. If, for each $x\in M$ and each pair of an orbifold chart $(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)$ of $(M,\mathcal{O}_M)$ around x and an orbifold chart $(V_\mu,\psi_\mu,\widehat{V}_\mu/\Gamma'_\mu)$ of $(N,\mathcal{O}_N)$ around $f(x)$ $(f(U_\lambda)\subset V_\mu)$, there exists a C^k-map $\widehat{f}_{\lambda,\mu}:\widehat{U}_\lambda\to\widehat{V}_\mu$ with $f\circ\phi_\lambda^{-1}\circ\pi_{\Gamma_\lambda} = \psi_\mu^{-1}\circ\pi_{\Gamma'_\mu}\circ\widehat{f}_{\lambda,\mu}$, then f is called a C^k-*orbimap* (or simply a C^k-*map*). Also $\widehat{f}_{\lambda,\mu}$ is called a *local lift* of f with respect to $(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)$ and $(V_\mu,\psi_\mu,\widehat{V}_\mu/\Gamma'_\mu)$. Furthermore, if each local lift $\widehat{f}_{\lambda,\mu}$ is an immersion, then f is called a C^k-*orbiimmersion* (or simply a C^k-*immersion*) and $(M,\mathcal{O}_M)$ is called a C^k-*(immersed) suborbifold* in $(N,\mathcal{O}_N,g)$. Similarly, if each local lift $\widehat{f}_{\lambda,\mu}$ is a submersion, then f is called a C^k-*orbisubmersion*.

In the sequel, we assume that $r=\infty$. Denote by pr_{TM} and $\mathrm{pr}_{T^{(r,s)}M}$ the natural projections of TM and $T^{(r,s)}M$ onto M, respectively. These are C^∞-orbimaps. We call a C^k-orbimap $X:M\to TM$ with $\mathrm{pr}_{TM}\circ X=\mathrm{id}$ a C^k-*orbitangent vector field* on $(M,\mathcal{O}_M)$ and a C^k-orbimap $S:M\to T^{(r,s)}M$ with $\mathrm{pr}_{T^{(r,s)}M}\circ S=\mathrm{id}$ a (r,s)-*orbitensor field* of class C^k on $(M,\mathcal{O}_M)$. If a (r,s)-orbitensor field g of class C^k on $(M,\mathcal{O}_M)$ is positive definite and symmetric, then we call g a C^k-*Riemannian orbimetric* and $(M,\mathcal{O}_{\mathcal{M}},g)$ a C^k-*Riemannian orbifold*.

Let f be a C^∞-*orbiimmersion* of an C^∞-orbifold $(M,\mathcal{O}_M)$ into C^∞-Riemannian orbifold $(N,\mathcal{O}_N,g)$. Take an orbifold chart $(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)$ of M around x and an orbifold chart $(V_\mu,\psi_\mu,\widehat{V}_\mu/\Gamma'_\mu)$ of N around $f(x)$ with $f(U_\lambda)\subset V_\mu$. Let $\widehat{f}_{\lambda,\mu}$ be the local lift of f with respect to these orbifold charts and $\widehat{g}_\mu$ that of g to $\widehat{V}_\mu$. Denote by $(T^\perp_{\widehat{x}_\lambda}\widehat{U}_\lambda)_\mu$ the orthogonal complement of $(\widehat{f}_{\lambda,\mu})_*(T_{\widehat{x}_\lambda}\widehat{U}_\lambda)$ in $(T_{\widehat{f(x)}_\mu}\widehat{V}_\mu,(\widehat{g}_\mu)_{\widehat{f(x)}_\mu})$. The group $(\Gamma'_\mu)_{\widehat{f(x)}_\mu}$ acts on $(T^\perp_{\widehat{x}_\lambda}\widehat{U}_\lambda)_\mu$ naturally. Give

$$\mathcal{T}^\perp_x := \bigoplus_{(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)\in\mathcal{O}_{M,x}}\ \bigoplus_{(V_\mu,\psi_\mu,\widehat{V}_\mu/\Gamma_\mu)\in\mathcal{O}_{N,f(x)}} (T^\perp_{\widehat{x}_\lambda}\widehat{U}_\lambda)_\mu/(\Gamma'_\mu)_{\widehat{f(x)}_\mu}$$

an equivalence relation $\sim$ as follows. Let $(U_{\lambda_i}, \phi_{\lambda_i}, \widehat{U}_{\lambda_i}/\Gamma_{\lambda_i})$ $(i=1,2)$ be members of $\mathcal{O}_{M,x}$ and $(V_{\mu_i}, \psi_{\mu_i}, \widehat{V}_{\mu_i}/\Gamma'_{\mu_i})$ $(i=1,2)$ members of $\mathcal{O}_{N,f(x)}$ with $f(U_{\lambda_i}) \subset V_{\mu_i}$. Let η_{μ_1,μ_2} be the diffeomorphism of a sufficiently small neighborhood of $\widehat{f(x)}_{\mu_1}$ in $\widehat{V}_{\mu_1}$ into $\widehat{V}_{\mu_2}$ satisfying $\psi_{\mu_2}^{-1} \circ \pi_{\Gamma'_{\mu_2}} \circ \eta_{\mu_1,\mu_2} = \psi_{\mu_1}^{-1} \circ \pi_{\Gamma'_{\mu_1}}$. Define an equivalence relatiton $\sim$ in $\mathcal{T}_x^\perp$ as the relation generated by

$$[\xi_1] \sim [\xi_2] \underset{\text{def}}{\Longleftrightarrow} [\xi_2] = [(\eta_{\mu_1,\mu_2})_*(\xi_1)]$$
$$([\xi_1] \in (T^\perp_{\widehat{x}_{\lambda_1}} \widehat{U}_{\lambda_1})_{\mu_1}/(\Gamma'_{\mu_1})_{\widehat{f(x)}_{\mu_1}},\ [\xi_2] \in (T_{\widehat{x}_{\lambda_2}} \widehat{U}_{\lambda_2})_{\mu_2}/(\Gamma'_{\mu_2})_{\widehat{f(x)}_{\mu_2}}),$$

where $[\xi_i]$ $(i=1,2)$ is the $(\Gamma'_{\mu_i})_{\widehat{f(x)}_{\mu_i}}$-orbits through $\xi_i \in (T^\perp_{\widehat{x}_{\lambda_i}} \widehat{U}_{\lambda_i})_{\mu_i}$. We call the quotient space $\mathcal{T}_x^\perp/\sim$ the *orbinormal space of* M *at* x and denote it by $T_x^\perp M$. If f is of class C^∞, then $T^\perp M := \underset{x\in M}{\oplus} T_x^\perp M$ is a C^∞-orbifold in a natural manner. We call $T^\perp M$ the *orbinormal bundle* of M. Denote by $\mathrm{pr}_{T^\perp M}$ the natural projection of $T^\perp M$ onto M. This is C^∞-orbisubmersion. We call a C^k-orbimap $\xi : M \to T^\perp M$ with $\mathrm{pr}_{T^\perp M} \circ \xi = \mathrm{id}$ a C^k-*orbinormal vector field* of $(M, \mathcal{O}_M)$ in $(N, \mathcal{O}_N, g)$.

Take an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of M around x and an orbifold chart $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of N around $f(x)$ with $f(U_\lambda) \subset V_\mu$. Denote by $(T^\perp_{\widehat{x}_\lambda} \widehat{U}_\lambda)_\mu^{(r,s)}$ the (r,s)-tensor space of $(T^\perp_{\widehat{x}_\lambda} \widehat{U}_\lambda)_\mu$. The group $(\Gamma'_\mu)_{\widehat{f(x)}_\mu}$ acts on $(T^\perp_{\widehat{x}_\lambda} \widehat{U}_\lambda)_\mu^{(r,s)}$ naturally. Give

$$(\mathcal{T}_x^\perp)^{(r,s)} := \underset{(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)\in\mathcal{O}_{M,x}}{\oplus} \ \underset{(V_\mu,\psi_\mu,\widehat{V}_\mu/\Gamma_\mu)\in\mathcal{O}_{N,f(x)}}{\oplus} (T^\perp_{\widehat{x}_\lambda} \widehat{U}_\lambda)_\mu^{(r,s)}/(\Gamma'_\mu)_{\widehat{f(x)}_\mu}$$

an equivalence relation $\sim$ as follows. Let $(U_{\lambda_i}, \phi_{\lambda_i}, \widehat{U}_{\lambda_i}/\Gamma_{\lambda_i})$, $(V_{\mu_i}, \psi_{\mu_i}, \widehat{V}_{\mu_i}/\Gamma'_{\mu_i})$ $(i=1,2)$ and η_{μ_1,μ_2} be as above. Define an equivalence relatiton $\sim$ in $(\mathcal{T}_x^\perp)^{(r,s)}$ as the relation generated by

$$[S_1] \sim [S_2] \underset{\text{def}}{\Longleftrightarrow} [S_1] = [(\eta_{\mu_1,\mu_2})^*(S_1)]$$
$$([S_1] \in (T^\perp_{\widehat{x}_{\lambda_1}} \widehat{U}_{\lambda_1})_{\mu_1}^{(r,s)}/(\Gamma'_{\mu_1})_{\widehat{f(x)}_{\mu_1}},\ [S_2] \in (T_{\widehat{x}_{\lambda_2}} \widehat{U}_{\lambda_2})_{\mu_2}^{(r,s)}/(\Gamma'_{\mu_2})_{\widehat{f(x)}_{\mu_2}}),$$

where $[S_i]$ $(i=1,2)$ is the $(\Gamma'_{\mu_i})_{\widehat{f(x)}_{\mu_i}}$-orbits through $S_i \in (T^\perp_{\widehat{x}_{\lambda_i}} \widehat{U}_{\lambda_i})_{\mu_i}^{(r,s)}$. We denote the quotient space $(\mathcal{T}_x^\perp)^{(r,s)}/\sim$ by $(T_x^\perp M)^{(r,s)}$. If f is of class C^∞, then $(T^\perp M)^{(r,s)} := \underset{x\in M}{\oplus} (T_x^\perp M)^{(r,s)}$ is a C^∞-orbifold in a natural manner. We call $(T^\perp M)^{(r,s)}$ the (r,s)-*orbitensor bundle* of $T^\perp M$. Denote by $\mathrm{pr}_{(T^\perp M)^{(r,s)}}$ the natural projection of $(T^\perp M)^{(r,s)}$ onto M. This is C^∞-orbisubmersion.

Next we shall define the tensor product $T^{(r,s)}M \otimes (T^\perp M)^{(s',t')}$ of $T^{(r,s)}M$ and $(T^\perp M)^{(s',t')}$. Take an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of M around x and an orbifold chart $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of N around $f(x)$ with $f(U_\lambda) \subset V_\mu$. The group $(\Gamma_\lambda)_{\widehat{x}_\lambda} \times (\Gamma'_\mu)_{\widehat{f(x)}_\mu}$ acts on $(T^{(r,s)}_{\widehat{x}_\lambda} \widehat{U}_\lambda) \otimes (T^\perp_{\widehat{x}_\lambda} \widehat{U}_\lambda)_\mu^{(s',t')}$ naturally. Give

$$\mathcal{T}_x^{(r,s)} \otimes (\mathcal{T}_x^\perp)^{(s',t')} := \underset{(U_\lambda,\phi_\lambda,\widehat{U}_\lambda/\Gamma_\lambda)\in\mathcal{O}_{M,x}}{\oplus} \ \underset{(V_\mu,\psi_\mu,\widehat{V}_\mu/\Gamma_\mu)\in\mathcal{O}_{N,f(x)}}{\oplus} \left((T^{(r,s)}_{\widehat{x}_\lambda} \widehat{U}_\lambda) \otimes (T^\perp_{\widehat{x}_\lambda} \widehat{U}_\lambda)_\mu^{(s',t')}\right)/((\Gamma_\lambda)_{\widehat{x}_\lambda} \times (\Gamma'_\mu)_{\widehat{f(x)}_\mu})$$

an equivalence relation $\sim$ as follows. Let $(U_{\lambda_i}, \phi_{\lambda_i}, \widehat{U}_{\lambda_i}/\Gamma_{\lambda_i})$, $(V_{\mu_i}, \psi_{\mu_i}, \widehat{V}_{\mu_i}/\Gamma'_{\mu_i})$ $(i = 1, 2)$ and η_{μ_1,μ_2} be as above. Also let $\eta_{\lambda_1,\lambda_2}$ be a diffeomorphism defined in similar to η_{μ_1,μ_2}. Define an equivalence relation $\sim$ in $\mathcal{T}_x^{(r,s)} \otimes (\mathcal{T}_x^{\perp})^{(s',t')}$ as the relation generated by

$$[S_1] \sim [S_2] \underset{\text{def}}{\Longleftrightarrow} [S_1] = [(\eta^*_{\lambda_1,\lambda_2} \otimes \eta^*_{\mu_1,\mu_2}) S_2]$$
$$([S_1] \in (T^{(r,s)}_{\widehat{x}_{\lambda_1}} \widehat{U}_{\lambda_1}) \otimes (T^{\perp}_{\widehat{x}_{\lambda_1}} \widehat{U}_{\lambda_1})^{(s',t')}_{\mu_1}), \quad [S_2] \in (T^{(r,s)}_{\widehat{x}_{\lambda_2}} \widehat{U}_{\lambda_2}) \otimes (T^{\perp}_{\widehat{x}_{\lambda_2}} \widehat{U}_{\lambda_2})^{(s',t')}_{\mu_2})),$$

where $[S_i]$ $(i = 1, 2)$ is the $((\Gamma_{\lambda_i})_{\widehat{x}_{\lambda_i}} \times (\Gamma'_{\mu_i})_{\widehat{f(x)}_{\mu_i}})$-orbits through $S_i \in ((T^{(r,s)}_{\widehat{x}_{\lambda_i}} \widehat{U}_{\lambda_i}) \otimes (T^{\perp}_{\widehat{x}_{\lambda_i}} \widehat{U}_{\lambda_i})^{(s',t')}_{\mu_i}))$. We denote the quotient space $(\mathcal{T}_x^{(r,s)} \otimes (\mathcal{T}_x^{\perp})^{(s',t')})/\sim$ by $T_x^{(r,s)} M \otimes (T_x^{\perp} M)^{(s',t')}$. Set $T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')} := \bigoplus_{x \in M} \left(T_x^{(r,s)} M \otimes (T_x^{\perp} M)^{(s',t')} \right)$. If f is of class C^∞, then $T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')}$ is a C^∞-orbifold in a natural manner. We call $T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')}$ the *orbitensor product bundle* of $T^{(r,s)} M$ and $(T^{\perp} M)^{(s',t')}$. Denote by $\mathrm{pr}_{T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')}}$ the natural projection of $T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')}$ onto M. This is a C^∞-orbisubmersion. We call a C^k-orbimap $S : M \to T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')}$ with $\mathrm{pr}_{T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')}} \circ S = \mathrm{id}$ a *C^k-section of* $T^{(r,s)} M \otimes (T^{\perp} M)^{(s',t')}$. Let g, h, A, H and ξ be the induced metric, the second fundamental form, the shape tensor, the mean curvature and a unit normal vector field of the immersion $f|_{M \setminus \mathrm{Sing}(M)} : M \setminus \mathrm{Sing}(M) \hookrightarrow N \setminus \mathrm{Sing}(N)$, respectively. It is easy to show that g, h, A and H extend a $(0, 2)$-orbitensor field of class C^∞ on $(M, \mathcal{O}_M)$, a C^k-section of $T^{(0,2)} M \otimes T^{\perp} M$, a C^k-section of $T^{(1,1)} M \otimes (T^{\perp} M)^{(0,1)}$ and a C^∞-orbinormal vector field on $(M, \mathcal{O}_M)$. We denote these extensions by the same symbols. We call these extensions g, h, A and H the *induced orbimetric*, the *second fundamental orbiform*, the *shape orbitensor* and the *mean curvature orbifunction* of f. Here we note that ξ does not necessarily extend a C^∞-orbinormal vector field on $(M, \mathcal{O})$ (see Fig. 2).

Now we shall define the notion of the mean curvature flow starting from a C^∞-suborbifold in a C^∞-Riemannian orbifold. Let f_t $(0 \le t < T)$ be a C^∞-family of C^∞-orbiimmersions of a C^∞-orbifold $(M, \mathcal{O}_M)$ into a C^∞-Riemannian orbifold $(N, \mathcal{O}_N, g)$. Assume that, for each $(x_0, t_0) \in M \times [0, T)$ and each pair of an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathcal{O}_M)$ around x_0 and an orbifold chart $(V_\mu, \phi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathcal{O}_N)$ around $f_{t_0}(x_0)$ such that $f_t(U_\lambda) \subset V_\mu$ for any $t \in [t_0, t_0 + \varepsilon)$ (ε : a sufficiently small positive number), there exists local lifts $(\widehat{f}_t)_{\lambda,\mu} : \widehat{U}_\lambda \to \widehat{V}_\mu$ of f_t $(t \in [t_0, t_0 + \varepsilon))$ such that they give the mean curvature flow in $(\widehat{V}_\mu, \widehat{g}_\mu)$, where $\widehat{g}_\mu$ is the local lift of g to $\widehat{V}_\mu$. Then we call f_t $(0 \le t < T)$ the *mean curvature flow* in $(N, \mathcal{O}_N, g)$.

THEOREM 3.1. *For any C^∞-orbiimmersion f of a compact C^∞-orbifold into a C^∞-Riemannian orbifold, the mean curvature flow starting from f exists uniquely in short time.*

Proof. Let f be a C^∞-orbiimmersion of an n-dimensional compact C^∞-orbifold $(M, \mathcal{O}_M)$ into an $(n + r)$-dimensional C^∞-Riemmannian orbifold $(N, \mathcal{O}_N, g)$. Fix $x_0 \in M$. Take an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathcal{O}_M)$ around x_0 and an orbifold chart $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathcal{O}_N)$ around $f(x_0)$ such that $f(U_\lambda) \subset V_\mu$ and that $\widehat{U}_\lambda$ is relative compact. Also, let $\widehat{f}_{\lambda,\mu} : \widehat{U}_\lambda \hookrightarrow \widehat{V}_\mu$ be a local lift of f and $\widehat{g}_\mu$ a local lift of g (to $\widehat{V}$). Since $\widehat{U}_\lambda$ is relative compact, there exists the mean curvature flow $(\widehat{f}_{\lambda,\mu})_t : \widehat{U}_\lambda \hookrightarrow (\widehat{V}_\mu, \widehat{g}_\mu)$ $(0 \le t < T)$ starting from $\widehat{f}_{\lambda,\mu} : \widehat{U}_\lambda \hookrightarrow (\widehat{V}_\mu, \widehat{g}_\mu)$. Since $\widehat{f}_{\lambda,\mu}$ is projetable to $f|_{U_\lambda}$ and $\widehat{g}_\mu$ is Γ'_μ-invariant, $(\widehat{f}_{\lambda,\mu})_t$ $(0 \le t < T)$ also are projectable to maps of U_λ into V_μ. Denote by $(f_{\lambda,\mu})_t$'s these maps of U_λ into V_μ. It is clear that $(f_{\lambda,\mu})_t$ $(0 \le t < T)$ is the mean curvature flow starting from $f|_{U_\lambda}$. Hence, it follows from the arbitrariness of x_0 and the compactness of M that the mean curvature flow starting from f exists uniquely in short time. □

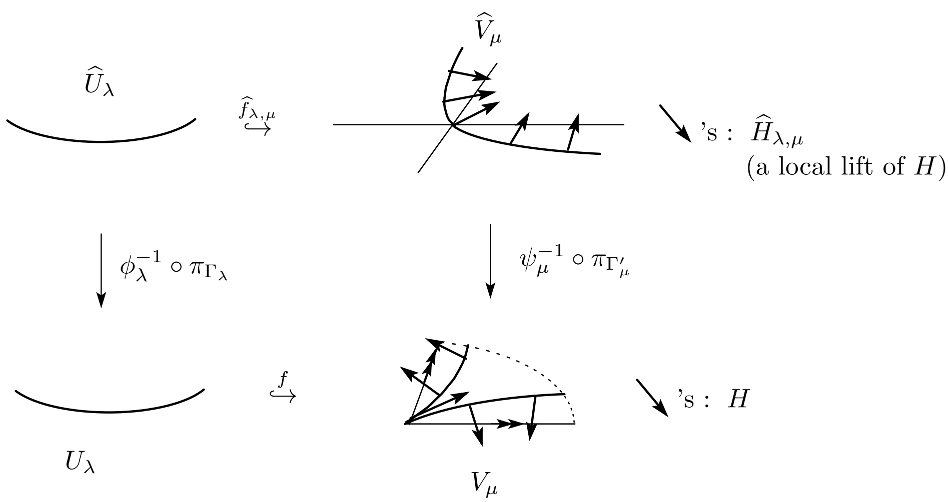

FIG. 1.

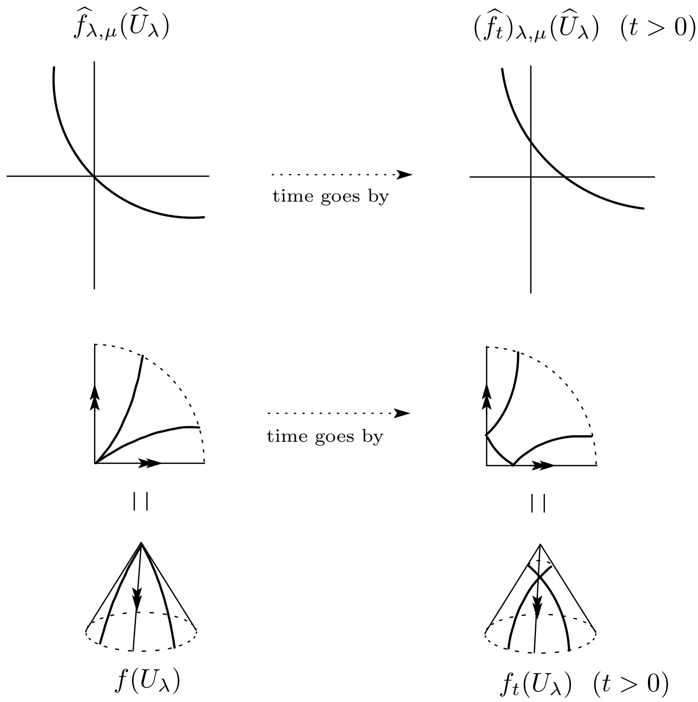

FIG. 1 (CONTINUED).

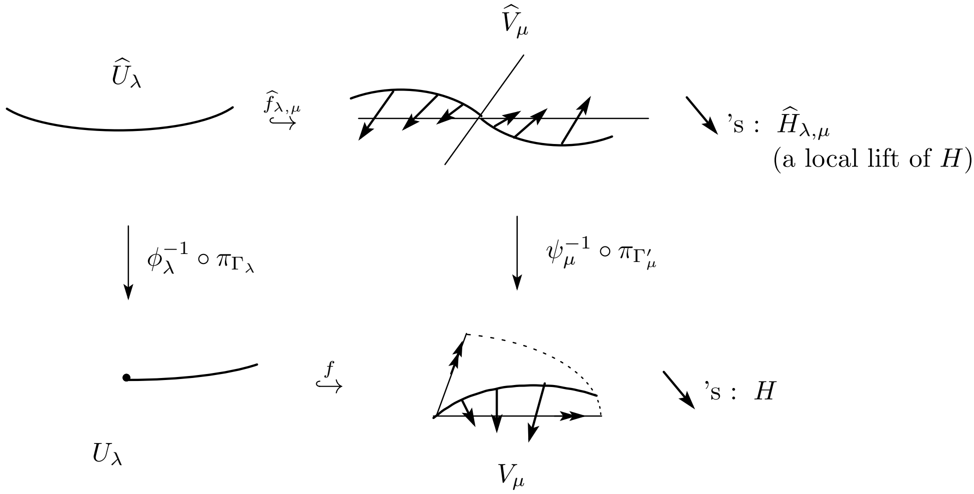

FIG. 2.

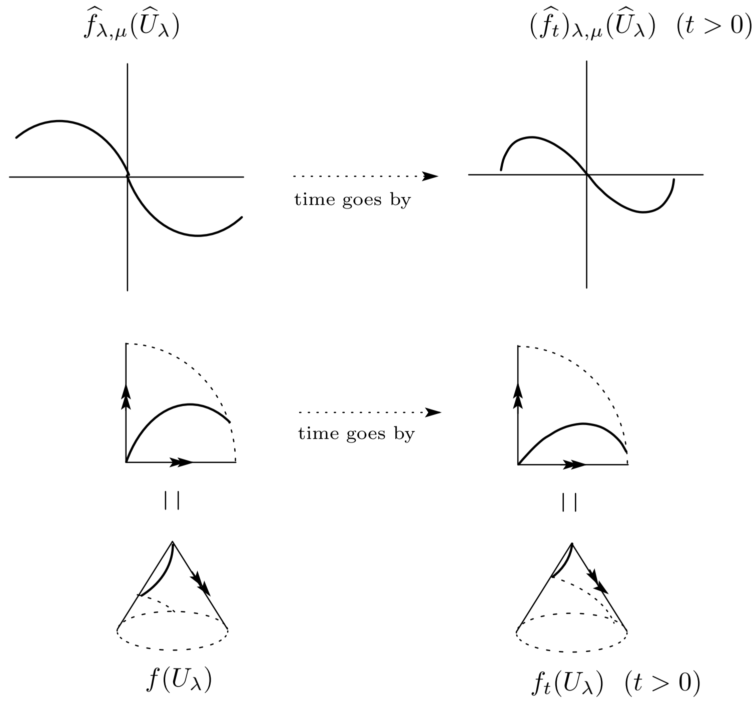

FIG. 2 (CONTINUED).

4. Evolution equations. Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$. The orbit space V/G is a (finite dimensional) C^∞-orbifold. Let $\phi : V \to V/G$ be the orbit map and set $N := V/G$. Here we give an example of such an isometric almost free action of a Hilbert Lie group.

Example. Let G be a compact semi-simple Lie group, K a closed subgroup of G and Γ a finite subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Assume that a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ exists. Let B be the Killing form of $\mathfrak{g}$. Give G the bi-invariant metric induced from B. Let $H^0([0,1],\mathfrak{g})$ be the Hilbert space of all paths in the Lie algebra $\mathfrak{g}$ of G which are L^2-integrable with respect to B. Also, let $H^1([0,1],G)$ the Hilbert Lie group of all paths in G which are of class H^1 with respect to g. This group $H^1([0,1],G)$ acts on $H^0([0,1],\mathfrak{g})$ isometrically and transitively as a gauge action:

$$(a * u)(t) = \mathrm{Ad}_G(a(t))(u(t)) - (R_{a(t)})_*^{-1}(a'(t)) \ (a \in H^1([0,1],G),\ u \in H^0([0,1],\mathfrak{g})),$$

where Ad_G is the adjoint representation of G and $R_{a(t)}$ is the right translation by $a(t)$ and a' is the weak derivative of a. Set $P(G,\Gamma\times K) := \{a \in H^1([0,1],G) \,|\, (a(0),a(1)) \in \Gamma\times K\}$. The group $P(G,\Gamma\times K)$ acts on $H^0([0,1],\mathfrak{g})$ almost freely and isometrically, and the orbit space of this action is diffeomorphic to the orbifold $\Gamma \setminus G / K$. Furthermore, each orbit of this action is regularizable and minimal.

Give N the Riemannian orbimetric such that ϕ is a Riemannian orbisubmersion. Let $f : M \hookrightarrow V$ be a G-invariant submanifold immersion such that $(\phi\circ f)(M)$ is compact. For this immersion f, we can take an orbiimmesion $\overline{f}$ of a compact orbifold $\overline{M}$ into N and an orbifold submersion $\phi_M : M \to \overline{M}$ with $\phi\circ f = \overline{f}\circ\phi_M$. Let $\overline{f}_t$ $(0 \le t < T)$ be the mean curvature flow starting from $\overline{f}$. The existenceness and the uniqueness of this flow in short time is assured by Proposition 3.1. Define a map $\overline{F} : \overline{M}\times[0,T) \to N$ by $\overline{F}(x,t) := \overline{f}_t(x)$ $((x,t) \in \overline{M}\times[0,T))$. Denote by H the regularized mean curvature vector of f and $\overline{H}$ that of $\overline{f}$. Since ϕ has minimal regularizable fibres, H is the horizontal lift of $\overline{H}$. Take $x \in \overline{M}$ and $u \in \phi_M^{-1}(x)$. Define a curve $c_x : [0,T) \to N$ by $c_x(t) := \overline{f}_t(x)$ and let $(c_x)^L_u : [0,T) \to V$ be the horizontal lift of c_x to $f(u)$ satisfying $((c_x)^L_u)'(0) = H_u$. Define an immersion $f_t : M \hookrightarrow V$ by $f_t(u) = (c_x)^L_u(t)$ $(u \in \widetilde{M})$ and a map $F : M\times[0,T) \to V$ by $F(u,t) = f_t(u)$ $((u,t) \in M\times[0,T))$.

Proposition 4.1. *The flow f_t $(0 \le t < T)$ is the regularized mean curvature flow starting from f.*

Proof. Denote by $\overline{H}_t$ the mean curvature vector of $\overline{f}_t$ and H_t the regularized mean curvature vector of f_t. Fix $(u,t) \in M\times[0,T)$. It is clear that $\phi\circ f_t = \overline{f}_t\circ\phi_M$. Hence, since each fibre of ϕ is regularizable and minimal, $(H_t)_u$ coincides with one of the horizontal lifts of $(\overline{H}_t)_{\phi(u)}$ to $(c_{\phi(u)})^L_u(t)$. On the other hand, from the definition of F, we have $\dfrac{\partial F}{\partial t}(u,t) = ((c_{\phi(u)})^L_u)'(t)$, which is one of the horizontal lifts of $(\overline{H}_t)_{\phi(u)}$ to $(c_{\phi(u)})^L_u(t)$. These facts together with $\dfrac{\partial F}{\partial t}(u,0) = H_u$ implies that $\dfrac{\partial F}{\partial t}(u,t) = (H_t)_u$. Thus f_t $(0 \le t < T)$ is the regularized mean curvature flow starting from f. This completes the proof. $\square$

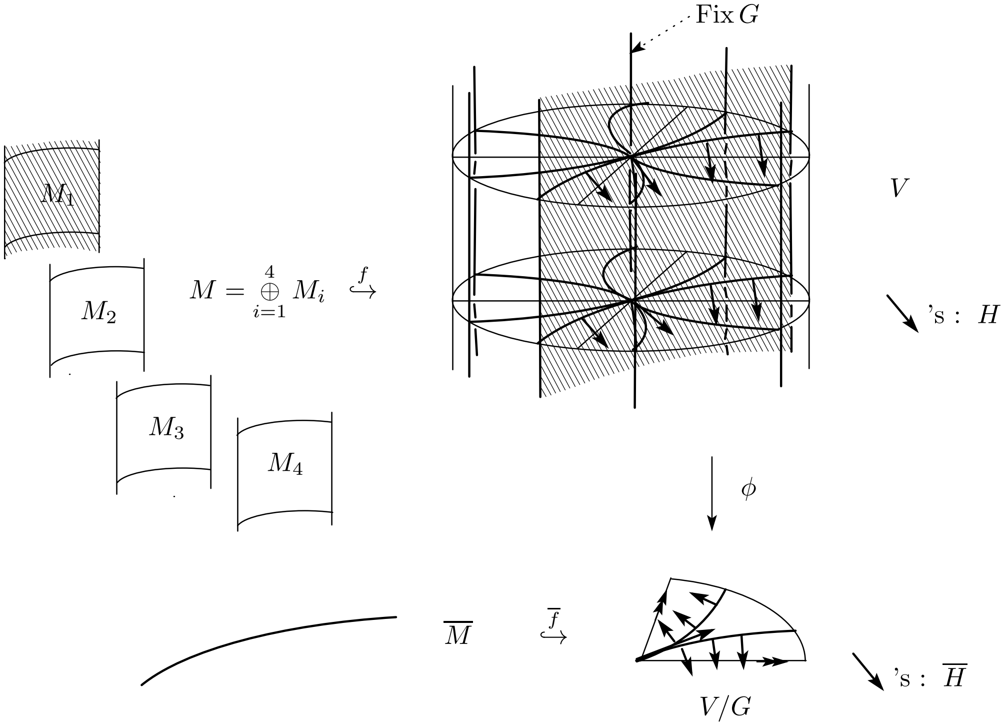

FIG. 3.

Assume that the codimension of M is equal to one. Denote by $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{V}}$) the horizontal (resp. vertical) distribution of ϕ. Denote by $\mathrm{pr}_{\widetilde{\mathcal{H}}}$ (resp. $\mathrm{pr}_{\widetilde{\mathcal{V}}}$) the orthogonal projection of TV onto $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{V}}$). For simplicity, for $X \in TV$, we denote $\mathrm{pr}_{\widetilde{\mathcal{H}}}(X)$ (resp. $\mathrm{pr}_{\widetilde{\mathcal{V}}}(X)$) by $X_{\widetilde{\mathcal{H}}}$ (resp. $X_{\widetilde{\mathcal{V}}}$). Define a distribution $\mathcal{H}_t$ on M by $f_{t*}((\mathcal{H}_t)_u) = f_{t*}(T_uM) \cap \widetilde{\mathcal{H}}_{f_t(u)}$ $(u \in M)$ and a distribution $\mathcal{V}_t$ on M by $f_{t*}((\mathcal{V}_t)_u) = \widetilde{\mathcal{V}}_{f_t(u)}$ $(u \in M)$. Note that $\mathcal{V}_t$ is independent of the choice of $t \in [0, T)$. Denote by g_t, h_t, A_t, H_t and ξ_t the induced metric, the second fundamental form, the shape tensor and the regularized mean curvature vector and the unit normal vector field of f_t, respectively. The group G acts on M through f_t. Since $\phi : V \to V/G$ is a G-orbibundle and $\widetilde{\mathcal{H}}$ is a connection of the orbibundle, it follows from Proposition 4.1 that this action $G \curvearrowright M$ is independent of the choice of $t \in [0, T)$. It is clear that quantities g_t, h_t, A_t and H_t are G-invariant. Also, let ∇^t be the Riemannian connection of g_t. Let π_M be the projection of $M \times [0, T)$ onto M. For a vector bundle E over M, denote by $\pi_M^* E$ the induced bundle of E by π_M. Also denote by $\Gamma(E)$ the space of all sections of E. Define a section g of $\pi_M^*(T^{(0,2)}M)$ by $g(u, t) = (g_t)_u$ $((u, t) \in M \times [0, T))$, where $T^{(0,2)}M$ is the $(0, 2)$-tensor bundle of M. Similarly, we define a section h of $\pi_M^*(T^{(0,2)}M)$, a section A of $\pi_M^*(T^{(1,1)}M)$, a map $H : M \times [0, T) \to TV$ and a map $\xi : M \times [0, T) \to TV$. We regard H and ξ as V-valued functions over $M \times [0, T)$ under the identification of T_uV's $(u \in V)$ and V. Define a subbundle $\mathcal{H}$ (resp. $\mathcal{V}$) of $\pi_M^* TM$ by $\mathcal{H}_{(u,t)} := (\mathcal{H}_t)_u$ (resp. $\mathcal{V}_{(u,t)} := (\mathcal{V}_t)_u$). Denote by $\mathrm{pr}_{\mathcal{H}}$ (resp. $\mathrm{pr}_{\mathcal{V}}$) the orthogonal projection of $\pi_M^*(TM)$ onto $\mathcal{H}$ (resp. $\mathcal{V}$). For simplicity, for $X \in \pi_M^*(TM)$, we denote $\mathrm{pr}_{\mathcal{H}}(X)$ (resp. $\mathrm{pr}_{\mathcal{V}}(X)$) by $X_{\mathcal{H}}$ (resp. $X_{\mathcal{V}}$). The bundle $\pi_M^*(TM)$ is regarded as a subbundle of $T(M \times [0, T))$. For a section B of $\pi_M^*(T^{(r,s)}M)$, we define $\dfrac{\partial B}{\partial t}$ by

$\left(\frac{\partial B}{\partial t}\right)_{(u,t)} := \frac{dB_{(u,t)}}{dt}$, where the right-hand side of this relation is the derivative of the vector-valued function $t \mapsto B_{(u,t)} (\in T_u^{(r,s)}M)$. Also, we define a section $B_{\mathcal{H}}$ of $\pi_M^*(T^{(r,s)}M)$ by

$$B_{\mathcal{H}} = (\underbrace{\mathrm{pr}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{pr}_{\mathcal{H}}}_{(r-\text{times})}) \circ B \circ (\underbrace{\mathrm{pr}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{pr}_{\mathcal{H}}}_{(s-\text{times})}).$$

The restriction of $B_{\mathcal{H}}$ to $\mathcal{H} \times \cdots \times \mathcal{H}$ (s-times) is regarded as a section of the (r,s)-tensor bundle $\mathcal{H}^{(r,s)}$ of $\mathcal{H}$. This restriction also is denoted by the same symbol $B_{\mathcal{H}}$. For a tangent vector field X on M (or an open set U of M), we define a section $\bar{X}$ of π_M^*TM (or $\pi_M^*TM|_U$) by $\bar{X}_{(u,t)} := X_u$ ($(u,t) \in M \times [0,T)$). Denote by $\widetilde{\nabla}$ the Riemannian connection of V. Define a connection ∇ of π_M^*TM by

$$(\nabla_X Y)_{(\cdot,t)} := \nabla^t_X Y_{(\cdot,t)} \text{ and } \nabla_{\frac{\partial}{\partial t}} Y := \frac{dY_{(u,\cdot)}}{dt}$$

for $X \in T_{(u,t)}(M \times \{t\})$ and $Y \in \Gamma(\pi_M^*TM)$, where $\frac{dY_{(u,t)}}{dt}$ is the derivative of the vector-valued function $t \mapsto Y_{(u,t)} (\in T_uM)$. Define a connection $\nabla^{\mathcal{H}}$ of $\mathcal{H}$ by $\nabla^{\mathcal{H}}_X Y := (\nabla_X Y)_{\mathcal{H}}$ for $X \in T(M \times [0,T))$ and $Y \in \Gamma(\mathcal{H})$. Similarly, define a connection $\nabla^{\mathcal{V}}$ of $\mathcal{V}$ by $\nabla^{\mathcal{V}}_X Y := (\nabla_X Y)_{\mathcal{V}}$ for $X \in T(M \times [0,T))$ and $Y \in \Gamma(\mathcal{V})$. Now we shall derive the evolution equations for some geometric quantities. First we derive the following evolution equation for $g_{\mathcal{H}}$.

LEMMA 4.2. *The sections $(g_{\mathcal{H}})_t$'s of $\pi_M^*(T^{(0,2)}M)$ satisfy the following evolution equation:*

$$\frac{\partial g_{\mathcal{H}}}{\partial t} = -2||H||h_{\mathcal{H}},$$

where $||H|| := \sqrt{g(H,H)}$.

Proof. Take $X, Y \in \Gamma(TM)$. We have

$$\begin{aligned}
&\frac{\partial g_{\mathcal{H}}}{\partial t}(\bar{X},\bar{Y}) = \frac{\partial}{\partial t} g_{\mathcal{H}}(\bar{X},\bar{Y}) = \frac{\partial}{\partial t} g(\bar{X}_{\mathcal{H}},\bar{Y}_{\mathcal{H}}) = \frac{\partial}{\partial t}\langle F_*\bar{X}_{\mathcal{H}}, F_*\bar{Y}_{\mathcal{H}}\rangle \\
&= \langle \frac{\partial}{\partial t}(\bar{X}_{\mathcal{H}}F), \bar{Y}_{\mathcal{H}}F\rangle + \langle \bar{X}_{\mathcal{H}}F, \frac{\partial}{\partial t}(\bar{Y}_{\mathcal{H}}F)\rangle \\
&= \langle \bar{X}_{\mathcal{H}}\left(\frac{\partial F}{\partial t}\right) + \left[\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}\right]F, \bar{Y}_{\mathcal{H}}F\rangle + \langle \bar{X}_{\mathcal{H}}F, \bar{Y}_{\mathcal{H}}\left(\frac{\partial F}{\partial t}\right) + \left[\frac{\partial}{\partial t}, \bar{Y}_{\mathcal{H}}\right]F\rangle \\
&= \langle \bar{X}_{\mathcal{H}}(||H||\xi), \bar{Y}_{\mathcal{H}}F\rangle + \langle \bar{X}_{\mathcal{H}}F, \bar{Y}_{\mathcal{H}}(||H||\xi)\rangle \\
&= -||H||g(A\bar{X}_{\mathcal{H}},\bar{Y}_{\mathcal{H}}) - ||H||g(\bar{X}_{\mathcal{H}},A\bar{Y}_{\mathcal{H}}) = -2||H||h_{\mathcal{H}}(\bar{X},\bar{Y}),
\end{aligned}$$

where we use $\left[\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}\right] \in \mathcal{V}$ and $\left[\frac{\partial}{\partial t}, \bar{Y}_{\mathcal{H}}\right] \in \mathcal{V}$. Thus we obtain the desired evolution equation. □

Next we derive the following evolution equation for ξ.

LEMMA 4.3. *The unit normal vector fields ξ_t's satisfy the following evolution equation:*

$$\frac{\partial \xi}{\partial t} = -F_*(\mathrm{grad}_g ||H||),$$

where $\mathrm{grad}_g(||H||)$ *is the element of* $\pi_M^*(TM)$ *such that* $d||H||(X) = g(\mathrm{grad}_g||H||, X)$ *for any* $X \in \pi_M^*(TM)$.

Proof. Since $\langle \xi, \xi \rangle = 1$, we have $\langle \frac{\partial \xi}{\partial t}, \xi \rangle = 0$. Hence $\frac{\partial \xi}{\partial t}$ is tangent to $f_t(M)$. Take any $(u_0, t_0) \in M \times [0, T)$. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal base of $T_{u_0}M$ with respect to $g_{(u_0,t_0)}$. By the Fourier expanding $\frac{\partial \xi}{\partial t}|_{t=t_0}$, we have

$$\begin{aligned}
\left.\frac{\partial \xi}{\partial t}\right|_{t=t_0} &= \sum \langle \left.\frac{\partial \xi}{\partial t}\right|_{t=t_0}, f_{t_0*}(\bar{e}_i|_{t=t_0}) \rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum \langle \xi_{t_0}, \left.\frac{\partial f_{t*}(\bar{e}_i)}{\partial t}\right|_{t=t_0} \rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum \langle \xi_{t_0}, \left.\frac{\partial}{\partial t}(\bar{e}_i F)\right|_{t=t_0} \rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum \langle \xi_{t_0}, \bar{e}_i \left(\left.\frac{\partial F}{\partial t}\right|_{t=t_0} \right) \rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum \langle \xi_{t_0}, (\bar{e}_i H)|_{t=t_0} \rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum (\bar{e}_i ||H||)|_{t=t_0} f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum g_{t_0}(\mathrm{grad}_{g_{t_0}}||H_{t_0}||, \bar{e}_i|_{t=t_0}) f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -f_{t_0*}(\mathrm{grad}_{g_{t_0}}||H_{t_0}||) = -F_*(\mathrm{grad}_g||H||)|_{t=t_0}
\end{aligned}$$

on U, where we use $\left[\frac{\partial}{\partial t}, \bar{e}_i\right] = 0$. Here we note that $\sum (\cdot)_i$ means $\lim_{k\to\infty} \sum_{i \in S_k} (\cdot)_i$ as $S_k := \{i \mid |(\cdot)_i| > \frac{1}{k}\}$ $(k \in \mathbb{N})$. In particular, we obtain

$$\left(\frac{\partial \xi}{\partial t}\right)_{(u_0,t_0)} = -(F_*((\mathrm{grad}_g||H||))_{(u_0,t_0)}.$$

This completes the proof. □

Let S_t $(0 \le t < T)$ be a C^∞-family of a (r,s)-tensor fields on M and S a section of $\pi_M^*(T^{(r,s)}M)$ defined by $S_{(u,t)} := (S_t)_u$. We define a section $\triangle_{\mathcal{H}} S$ of $\pi_M^*(T^{(r,s)}M)$ by

$$(\triangle_{\mathcal{H}} S)_{(u,t)} := \sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} S,$$

where ∇ is the connection of $\pi_M^*(T^{(r,s)}M)$ (or $\pi_M^*(T^{(r,s+1)}M)$) induced from ∇ and $\{e_1, \cdots, e_n\}$ is an orthonormal base of $\mathcal{H}_{(u,t)}$ with respect to $(g_{\mathcal{H}})_{(u,t)}$. Also, we define a section $\triangle_{\mathcal{H}}^{\mathcal{H}} S_{\mathcal{H}}$ of $\mathcal{H}^{(r,s)}$ by

$$(\triangle_{\mathcal{H}}^{\mathcal{H}} S_{\mathcal{H}})_{(u,t)} := \sum_{i=1}^{n} \nabla_{e_i}^{\mathcal{H}} \nabla_{e_i}^{\mathcal{H}} S_{\mathcal{H}},$$

where $\nabla^{\mathcal{H}}$ is the connection of $\mathcal{H}^{(r,s)}$ (or $\mathcal{H}^{(r,s+1)}$) induced from $\nabla^{\mathcal{H}}$ and $\{e_1, \cdots, e_n\}$ is as above. Let $\mathcal{A}^\phi$ be the section of $T^*V \otimes T^*V \otimes TV$ defined by

$$\mathcal{A}_X^\phi Y := (\widetilde{\nabla}_{X_{\widetilde{\mathcal{H}}}} Y_{\widetilde{\mathcal{H}}})_{\widetilde{\mathcal{V}}} + (\widetilde{\nabla}_{X_{\widetilde{\mathcal{H}}}} Y_{\widetilde{\mathcal{V}}})_{\widetilde{\mathcal{H}}} \quad (X, Y \in TV).$$

Also, let $\mathcal{T}^\phi$ be the section of $T^*V\otimes T^*V\otimes TV$ defined by

$$\mathcal{T}^\phi_X Y := (\widetilde{\nabla}_{X_{\widetilde{\mathcal{V}}}} Y_{\widetilde{\mathcal{H}}})_{\widetilde{\mathcal{V}}} + (\widetilde{\nabla}_{X_{\widetilde{\mathcal{V}}}} Y_{\widetilde{\mathcal{V}}})_{\widetilde{\mathcal{H}}} \quad (X, Y \in TV).$$

Also, let $\mathcal{A}_t$ be the section of $T^*M\otimes T^*M\otimes TM$ defined by

$$(\mathcal{A}_t)_X Y := (\nabla^t_{X_{\mathcal{H}_t}} Y_{\mathcal{H}_t})_{\mathcal{V}_t} + (\nabla^t_{X_{\mathcal{H}_t}} Y_{\mathcal{V}_t})_{\mathcal{H}_t} \quad (X, Y \in TM).$$

Also let $\mathcal{A}$ be the section of $\pi_M^*(T^*M\otimes T^*M\otimes TM)$ defined in terms of $\mathcal{A}_t$'s ($t\in[0,T)$). Also, let $\mathcal{T}_t$ be the section of $T^*M\otimes T^*M\otimes TM$ defined by

$$(\mathcal{T}_t)_X Y := (\nabla^t_{X_{\mathcal{V}_t}} Y_{\mathcal{V}_t})_{\mathcal{H}_t} + (\nabla^t_{X_{\mathcal{V}_t}} Y_{\mathcal{H}_t})_{\mathcal{V}_t} \quad (X, Y \in TM).$$

Also let $\mathcal{T}$ be the section of $\pi_M^*(T^*M\otimes T^*M\otimes TM)$ defined in terms of $\mathcal{T}_t$'s ($t\in[0,T)$). Clearly we have

$$F_*(\mathcal{A}_X Y) = \mathcal{A}^\phi_{F_*X} F_*Y$$

for $X, Y\in\mathcal{H}$ and

$$F_*(\mathcal{T}_W X) = \mathcal{T}^\phi_{F_*W} F_*X$$

for $X\in\mathcal{H}$ and $W\in\mathcal{V}$. Let E be a vector bundle over M. For a section S of $\pi_M^*(T^{(0,r)}M\otimes E)$, we define $\mathrm{Tr}^\bullet_{g_\mathcal{H}} S(\cdots,\overset{j}{\bullet},\cdots,\overset{k}{\bullet},\cdots)$ by

$$(\mathrm{Tr}^\bullet_{g_\mathcal{H}} S(\cdots,\overset{j}{\bullet},\cdots,\overset{k}{\bullet},\cdots))_{(u,t)} = \sum_{i=1}^n S_{(u,t)}(\cdots,\overset{j}{e_i},\cdots,\overset{k}{e_i},\cdots)$$

($(u,t)\in M\times[0,T)$), where $\{e_1,\cdots,e_n\}$ is an orthonormal base of $\mathcal{H}_{(u,t)}$ with respect to $(g_\mathcal{H})_{(u,t)}$, $S(\cdots,\overset{j}{\bullet},\cdots,\overset{k}{\bullet},\cdots)$ means that $\bullet$ is entried into the j-th component and the k-th component of S and $S_{(u,t)}(\cdots,\overset{j}{e_i},\cdots,\overset{k}{e_i},\cdots)$ means that e_i is entried into the j-th component and the k-th component of $S_{(u,t)}$.

Then we have the following relation.

Lemma 4.4. *Let S be a section of $\pi_M^*(T^{(0,2)}M)$ which is symmetric with respect to g. Then we have*

$$\begin{aligned}(\triangle_\mathcal{H} S)_\mathcal{H}(X,Y) = {} & (\triangle^\mathcal{H}_\mathcal{H} S_\mathcal{H})(X,Y)\\ & -2\mathrm{Tr}^\bullet_{g_\mathcal{H}}((\nabla_\bullet S)(\mathcal{A}_\bullet X, Y)) - 2\mathrm{Tr}^\bullet_{g_\mathcal{H}}((\nabla_\bullet S)(\mathcal{A}_\bullet Y, X))\\ & -\mathrm{Tr}^\bullet_{g_\mathcal{H}} S(\mathcal{A}_\bullet(\mathcal{A}_\bullet X), Y) - \mathrm{Tr}^\bullet_{g_\mathcal{H}} S(\mathcal{A}_\bullet(\mathcal{A}_\bullet Y), X)\\ & -\mathrm{Tr}^\bullet_{g_\mathcal{H}} S((\nabla_\bullet\mathcal{A})_\bullet X, Y) - \mathrm{Tr}^\bullet_{g_\mathcal{H}} S((\nabla_\bullet\mathcal{A})_\bullet Y, X)\\ & -2\mathrm{Tr}^\bullet_{g_\mathcal{H}} S(\mathcal{A}_\bullet X, \mathcal{A}_\bullet Y)\end{aligned}$$

for $X, Y\in\mathcal{H}$, where ∇ is the connection of $\pi_M^*(T^{(1,2)}M)$ induced from ∇.

Proof. Take any $(u_0,t_0)\in M\times[0,T)$. Let $\{e_1,\cdots,e_n\}$ be an orthonormal base of $\mathcal{H}_{(u_0,t_0)}$ with respect to $(g_\mathcal{H})_{(u_0,t_0)}$. Take any $X, Y\in\mathcal{H}_{(u_0,t_0)}$. Let $\widetilde{X}$ be a section of $\mathcal{H}$ on a neighborhood of (u_0,t_0) with $\widetilde{X}_{(u_0,t_0)} = X$ and $(\nabla^\mathcal{H}\widetilde{X})_{(u_0,t_0)} = 0$. Similarly we define $\widetilde{Y}$ and $\widetilde{e}_i$. Let $W = X, Y$ or e_i. Then, it follows from $(\nabla^{\mathcal{H}}_{e_i}\widetilde{W})_{(u_0,t_0)} = 0$, $(\nabla_{e_i}\widetilde{W})_{(u_0,t_0)} = \mathcal{A}_{e_i}W$ and the skew-symmetricness of $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ that

$$\begin{aligned}(\triangle_{\mathcal{H}}S)_{\mathcal{H}}(X,Y) &= \sum_{i=1}^{n}(\nabla_{e_i}\nabla_{e_i}S)(X,Y)\\ &= \sum_{i=1}^{n}(\nabla^{\mathcal{H}}_{e_i}\nabla^{\mathcal{H}}_{e_i}S_{\mathcal{H}})(X,Y)\\ &\quad -2\sum_{i=1}^{n}((\nabla_{e_i}S)(\mathcal{A}_{e_i}X,Y)+(\nabla_{e_i}S)(\mathcal{A}_{e_i}Y,X))\\ &\quad -\sum_{i=1}^{n}(S(\mathcal{A}_{e_i}(\mathcal{A}_{e_i}X),Y)+S(\mathcal{A}_{e_i}(\mathcal{A}_{e_i}Y),X))\\ &\quad -\sum_{i=1}^{n}(S((\nabla_{e_i}\mathcal{A})_{e_i}X,Y)+S((\nabla_{e_i}\mathcal{A})_{e_i}Y,X))\\ &\quad -2\sum_{i=1}^{n}S(\mathcal{A}_{e_i}X,\mathcal{A}_{e_i}Y).\end{aligned}$$

The right-hand side of this relation is equal to the right-hand side of the relation in the statement. This completes the proof. □

Also we have the following Simons-type identity.

LEMMA 4.5. *We have*

$$\triangle_{\mathcal{H}}h = \nabla d||H|| + ||H||(A^2)_\sharp - (\mathrm{Tr}\,(A^2)_{\mathcal{H}})h,$$

where $(A^2)_\sharp$ *is the element of* $\Gamma(\pi_M^*T^{(0,2)}M)$ *defined by* $(A^2)_\sharp(X,Y) := g(A^2X,Y)$ $(X,Y \in \pi_M^*TM)$.

Proof. Take $X,Y,Z,W \in \pi_M^*(TM)$. Since the ambient space V is flat, it follows from the Ricci's identity, the Gauss equation and the Codazzi equation that

$$\begin{aligned}(\nabla_X\nabla_Yh)(Z,W)-(\nabla_Z\nabla_Wh)(X,Y) &= (\nabla_X\nabla_Zh)(Y,W)-(\nabla_Z\nabla_Xh)(Y,W)\\ &= h(X,Y)h(AZ,W)-h(Z,Y)h(AX,W)\\ &\quad +h(X,W)h(AZ,Y)-h(Z,W)h(AX,Y).\end{aligned}$$

By using this relation, we obtain the desired relation. □

NOTE. In the sequel, we omit the notation F_* for simplicity.

Define a section $\mathcal{R}$ of $\pi_M^*(\mathcal{H}^{(0,2)})$ by

$$\begin{aligned}\mathcal{R}(X,Y) := {}&\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}h(\mathcal{A}_\bullet(\mathcal{A}_\bullet X),Y)+\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}h(\mathcal{A}_\bullet(\mathcal{A}_\bullet Y),X)\\ &+\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}h((\nabla_\bullet\mathcal{A})_\bullet X,Y)+\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}h((\nabla_\bullet\mathcal{A})_\bullet Y,X)\\ &+2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}(\nabla_\bullet h)(\mathcal{A}_\bullet X,Y)+2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}(\nabla_\bullet h)(\mathcal{A}_\bullet Y,X)\\ &+2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}h(\mathcal{A}_\bullet X,\mathcal{A}_\bullet Y) \qquad\qquad (X,Y\in\mathcal{H}).\end{aligned}$$

From Lemmas 4.3, 4.4 and 4.5, we derive the following evolution equation for $(h_{\mathcal{H}})_t$'s.

THEOREM 4.6. *The sections $(h_{\mathcal H})_t$'s of $\pi_M^*(T^{(0,2)}M)$ satisfies the following evolution equation:*

$$\frac{\partial h_{\mathcal H}}{\partial t}(X,Y)=(\triangle_{\mathcal H}^{\mathcal H}h_{\mathcal H})(X,Y)-2||H||((A_{\mathcal H})^2)_\sharp(X,Y)-2||H||((\mathcal A_\xi^\phi)^2)_\sharp(X,Y)$$
$$+\mathrm{Tr}\left((A_{\mathcal H})^2-((\mathcal A_\xi^\phi)^2)_{\mathcal H}\right)h_{\mathcal H}(X,Y)-\mathcal R(X,Y)$$

for $X,Y\in\mathcal H$.

Proof. Take $X,Y\in\mathcal H_{(u,t)}$. Easily we have

$$AX=A_{\mathcal H}X+\mathcal A_\xi^\phi X, \tag{4.1}$$

and

$$(A^2)_{\mathcal H}X=(A_{\mathcal H})^2X-(\mathcal A_\xi^\phi)^2X, \tag{4.2}$$

where we use

$$\left(\widetilde\nabla_W\xi\right)_{\widetilde{\mathcal H}}=\left(\widetilde\nabla_\xi W+[W,\xi]\right)_{\widetilde{\mathcal H}}=\left(\widetilde\nabla_\xi W\right)_{\widetilde{\mathcal H}}=\mathcal A_\xi W$$

for $W\in\Gamma(\widetilde{\mathcal V})$ because of $[W,\xi]\in\Gamma(\widetilde{\mathcal V})$. Also, since $\left[\frac{\partial}{\partial t},\bar X_{\mathcal H}\right]\in\mathcal V$, we have

$$\left[\frac{\partial}{\partial t},\bar X_{\mathcal H}\right]=2||H||\mathcal A_\xi^\phi\bar X_{\mathcal H}. \tag{4.3}$$

From Lemma 4.3, (4.1), (4.2) and (4.3), we have

$$\begin{aligned}
&\frac{\partial h_{\mathcal H}}{\partial t}(X,Y)=\frac{\partial}{\partial t}(h_{\mathcal H}(\bar X,\bar Y))=\frac{\partial}{\partial t}\langle\xi,\bar X_{\mathcal H}(\bar Y_{\mathcal H}F)\rangle\\
&=\langle\frac{\partial\xi}{\partial t},\bar X_{\mathcal H}(\bar Y_{\mathcal H}F)\rangle+\langle\xi,\frac{\partial}{\partial t}\left(\bar X_{\mathcal H}(\bar Y_{\mathcal H}F)\right)\rangle\\
&=-\langle F_*(\mathrm{grad}_g||H||),\widetilde\nabla_XF_*\bar Y_{\mathcal H}\rangle+\langle\xi,X\left(\bar Y_{\mathcal H}\left(\frac{\partial F}{\partial t}\right)\right)\rangle\\
&\quad+\langle\xi,X([\frac{\partial}{\partial t},\bar Y_{\mathcal H}]F)\rangle+\langle\xi,[\frac{\partial}{\partial t},\bar X_{\mathcal H}](\bar Y_{\mathcal H}F)\rangle\\
&=-g(\mathrm{grad}_g||H||,\nabla_X\bar Y_{\mathcal H})+X(\bar Y_{\mathcal H}||H||)-||H||\langle\xi,\widetilde\nabla_XF_*(A(\bar Y_{\mathcal H}))\rangle\\
&\quad+\langle\xi,\widetilde\nabla_XF_*([\frac{\partial}{\partial t},\bar Y_{\mathcal H}])\rangle+\langle\xi,\widetilde\nabla_{[\frac{\partial}{\partial t},\bar X_{\mathcal H}]}F_*\bar Y_{\mathcal H}\rangle\\
&=(\nabla d||H||)(X,Y)-||H||h_{\mathcal H}(X,A_{\mathcal H}Y)+||H||h(X,\mathcal A_\xi^\phi Y)+2||H||h(\mathcal A_\xi^\phi X,Y)\\
&=(\nabla d||H||)(X,Y)-||H||g_{\mathcal H}((A_{\mathcal H})^2X,Y)-3||H||g((\mathcal A_\xi^\phi)^2X,Y).
\end{aligned}$$

From this relation and the Simons-type identity in Lemma 4.5, we have

$$\begin{aligned}\frac{\partial h_{\mathcal H}}{\partial t}&=\triangle_{\mathcal H}h-2||H||((A_{\mathcal H})^2)_\sharp-2||H||((\mathcal A_\xi^\phi)^2)_\sharp\\
&\quad+\mathrm{Tr}\left((A_{\mathcal H})^2-((\mathcal A_\xi^\phi)^2)_{\mathcal H}\right)h_{\mathcal H}.\end{aligned} \tag{4.4}$$

Substituting the relation in Lemma 4.4 into (4.4), we obtain the desired relation. □

From Lemma 4.2, we derive the following relation.

LEMMA 4.7. *Let X and Y be local sections of $\mathcal{H}$ such that $g(X,Y)$ is constant. Then we have $g(\nabla_{\frac{\partial}{\partial t}}X,Y)+g(X,\nabla_{\frac{\partial}{\partial t}}Y)=2||H||h(X,Y)$.*

Proof. From Lemma 4.2, we have

$$\begin{aligned}\frac{\partial}{\partial t}g(X,Y)&=\frac{\partial g}{\partial t}(X,Y)+g(\nabla_{\frac{\partial}{\partial t}}X,Y)+g(X,\nabla_{\frac{\partial}{\partial t}}Y)\\&=-2||H||h(X,Y)+g(\nabla_{\frac{\partial}{\partial t}}X,Y)+g(X,\nabla_{\frac{\partial}{\partial t}}Y).\end{aligned}$$

Hence the desired relation follows from the constancy of $g(X,Y)$. □

Next we prepare the following lemma for $\mathcal{R}$.

LEMMA 4.8. *For $X,Y\in\mathcal{H}$, we have*

$$\begin{aligned}\mathcal{R}(X,Y)=&\,2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\left(\langle(\mathcal{A}^{\phi}_{\bullet}X,\mathcal{A}^{\phi}_{\bullet}(A_{\mathcal{H}}Y)\rangle+\langle(\mathcal{A}^{\phi}_{\bullet}Y,\mathcal{A}^{\phi}_{\bullet}(A_{\mathcal{H}}X)\rangle\right)\\&+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\Big(\langle(\mathcal{A}^{\phi}_{\bullet}X,\mathcal{A}^{\phi}_{Y}(A_{\mathcal{H}}\bullet)\rangle+\langle(\mathcal{A}^{\phi}_{\bullet}Y,\mathcal{A}^{\phi}_{X}(A_{\mathcal{H}}\bullet)\rangle\Big)\\&+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\Big(\langle(\widetilde{\nabla}_{\bullet}\mathcal{A}^{\phi})_{\xi}Y,\mathcal{A}^{\phi}_{\bullet}X\rangle+\langle(\widetilde{\nabla}_{\bullet}\mathcal{A}^{\phi})_{\xi}X,\mathcal{A}^{\phi}_{\bullet}Y\rangle\Big)\\&+\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\Big(\langle(\widetilde{\nabla}_{\bullet}\mathcal{A}^{\phi})_{\bullet}X,\mathcal{A}^{\phi}_{\xi}Y\rangle+\langle(\widetilde{\nabla}_{\bullet}\mathcal{A}^{\phi})_{\bullet}Y,\mathcal{A}^{\phi}_{\xi}X\rangle\Big)\\&+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\langle\mathcal{T}^{\phi}_{\mathcal{A}^{\phi}_{\bullet}X}\xi,\mathcal{A}^{\phi}_{\bullet}Y\rangle,\end{aligned}\tag{4.5}$$

where we omit F_*.

Proof. Take $e,X,Y\in\mathcal{H}$. Easily we have

$$\begin{aligned}(\nabla_e h)(\mathcal{A}_eX,Y)=&\,e(\langle\mathcal{A}^{\phi}_eX,\mathcal{A}^{\phi}_{\xi}Y\rangle)-h(\mathcal{A}_e(\mathcal{A}_eX),Y)\\&-h((\nabla_e\mathcal{A})_eX,Y)-h(\mathcal{A}_eX,\mathcal{A}_eY).\end{aligned}\tag{4.6}$$

On the other hand, by simple calculation, we have $((\widetilde{\nabla}_e\mathcal{A}^{\phi})_X\xi)_{\widetilde{\mathcal{V}}}=-((\widetilde{\nabla}_e\mathcal{A}^{\phi})_{\xi}X)_{\widetilde{\mathcal{V}}}$. By using this relation, we can show

$$e(\langle\mathcal{A}^{\phi}_eX,\mathcal{A}^{\phi}_{\xi}Y\rangle)=\langle(\widetilde{\nabla}_e\mathcal{A}^{\phi})_eX,\mathcal{A}^{\phi}_{\xi}Y\rangle+\langle(\widetilde{\nabla}_e\mathcal{A}^{\phi})_{\xi}Y,\mathcal{A}^{\phi}_eX\rangle+h(\mathcal{A}_Y\mathcal{A}_Xe,e\rangle.\tag{4.7}$$

Also, by simple calculations, we have

$$\begin{aligned}h(\mathcal{A}_e(\mathcal{A}_eX),Y)&=-\langle\mathcal{A}^{\phi}_eX,\mathcal{A}^{\phi}_e(A_{\mathcal{H}}Y)\rangle\\h(\mathcal{A}_Y(\mathcal{A}_Xe),e)&=\langle\mathcal{A}^{\phi}_eX,\mathcal{A}^{\phi}_Y(A_{\mathcal{H}}e)\rangle,\\h((\nabla_e\mathcal{A})_eX,Y)&=\langle(\widetilde{\nabla}_e\mathcal{A}^{\phi})_eX,\mathcal{A}^{\phi}_{\xi}Y\rangle,\\h(\mathcal{A}_eX,\mathcal{A}_eY)&=-\langle\mathcal{T}^{\phi}{}_{\mathcal{A}^{\phi}_eX}\xi,\mathcal{A}^{\phi}_eY\rangle.\end{aligned}\tag{4.8}$$

From (4.6), (4.7) and the relations in (4.8), we have the desired relation. □

Also, we prepare the following lemma.

LEMMA 4.9. *For $X,Y,Z\in\mathcal{H}$, we have*

$$2\langle\mathcal{T}^{\phi}_{\mathcal{A}^{\phi}_XY}\xi,\mathcal{A}^{\phi}_XZ\rangle=-\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_X\mathcal{A}^{\phi})_{\xi}Y\rangle+\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_Y\mathcal{A}^{\phi})_{\xi}X\rangle.$$

Proof. Fix $(u_0,t_0)\in M\times[0,T)$. Let $\widetilde{X}$ be an element of $\Gamma(\mathcal{H})$ satisfying $\widetilde{X}_{(u_0,t_0)}=X$ and $(\widetilde{\nabla}^{\mathcal{H}}\widetilde{X})_{(u_0,t_0)}=0$. Let $\widetilde{Y}$ and $\widetilde{Z}$ be similar elements of $\Gamma(\mathcal{H})$ for Y and Z, respectively. At (u_0,t_0), we have

$$\begin{aligned}\langle\mathcal{A}^{\phi}_XZ,\mathcal{A}^{\phi}_Y(A_{\mathcal{H}}X)\rangle &= -\langle\mathcal{A}^{\phi}_XZ,\widetilde{\nabla}_Y(\widetilde{\nabla}_X\xi)-\widetilde{\nabla}_Y(\mathcal{A}^{\phi}_X\xi)\rangle\\ &= -\langle\mathcal{A}^{\phi}_XZ,\widetilde{\nabla}_X(\widetilde{\nabla}_Y\xi)+\widetilde{\nabla}_{[Y,X]}\xi-\widetilde{\nabla}_Y(\mathcal{A}^{\phi}_X\xi)\rangle\\ &=\langle\mathcal{A}^{\phi}_XZ,\widetilde{\nabla}_X(AY)\rangle-2\langle\mathcal{A}^{\phi}_XZ,\widetilde{\nabla}_{\mathcal{A}^{\phi}_YX}\xi\rangle\\ &\quad+\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_Y\mathcal{A}^{\phi})_X\xi\rangle-\langle\mathcal{A}^{\phi}_XZ,\mathcal{A}^{\phi}_X(AY)\rangle\\ &=\langle\mathcal{A}^{\phi}_XZ,\mathcal{A}^{\phi}_Y(A_{\mathcal{H}}X)\rangle-\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_X\mathcal{A}^{\phi})_Y\xi\rangle\\ &\quad-2\langle\mathcal{A}^{\phi}_XZ,\mathcal{T}^{\phi}_{\mathcal{A}^{\phi}_YX}\xi\rangle+\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_Y\mathcal{A}^{\phi})_X\xi\rangle,\end{aligned}\tag{4.9}$$

where we use $(\widetilde{\nabla}^{\mathcal{H}}\widetilde{X})_{(u_0,t_0)}=(\widetilde{\nabla}^{\mathcal{H}}\widetilde{Y})_{(u_0,t_0)}=(\widetilde{\nabla}^{\mathcal{H}}\widetilde{Z})_{(u_0,t_0)}=0$. Also we have

$$\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_X\mathcal{A}^{\phi})_Y\xi\rangle=-\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_X\mathcal{A}^{\phi})_\xi Y\rangle$$

and

$$\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_Y\mathcal{A}^{\phi})_X\xi\rangle=-\langle\mathcal{A}^{\phi}_XZ,(\widetilde{\nabla}_Y\mathcal{A}^{\phi})_\xi X\rangle.$$

Form (4.9) and these relations, we obtain the desired relation. □

LEMMA 4.10. *For $X\in\mathcal{H}$, we have*

$$\begin{aligned}\mathcal{R}(X,X)&=4\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\langle\mathcal{A}^{\phi}_{\bullet}X,\mathcal{A}^{\phi}_{\bullet}(A_{\mathcal{H}}X)\rangle+4\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\langle\mathcal{A}^{\phi}_{\bullet}X,\mathcal{A}^{\phi}_X(A_{\mathcal{H}}\bullet)\rangle\\ &\quad+3\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\langle(\widetilde{\nabla}_{\bullet}\mathcal{A}^{\phi})_\xi X,\mathcal{A}^{\phi}_{\bullet}X\rangle+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\langle(\widetilde{\nabla}_{\bullet}\mathcal{A}^{\phi})_{\bullet}X,\mathcal{A}^{\phi}_\xi X\rangle\\ &\quad+\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\langle\mathcal{A}^{\phi}_{\bullet}X,(\widetilde{\nabla}_X\mathcal{A}^{\phi})_\xi\bullet\rangle\end{aligned}$$

and hence

$$\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}(\bullet,\bullet)=0.$$

Proof. The first relation follows from the relations in Lemmas 4.8 and 4.9 directly. Also, the second relation follows from the first relation directly. □

By using Theorem 4.6 and Lemmas 4.7 and 4.10, we can show the following evolution equation for $||H_t||$'s.

COROLLARY 4.11. *The norms $||H_t||$'s of H_t satisfy the following evolution equation:*

$$\frac{\partial||H||}{\partial t}=\triangle_{\mathcal{H}}||H||+||H||\mathrm{Tr}(A_{\mathcal{H}})^2-3||H||\mathrm{Tr}((\mathcal{A}^{\phi}_\xi)^2)_{\mathcal{H}}.$$

Proof. Fix $(u_0,t_0)\in M\times[0,T)$. Take a local orthonormal frame field $\{e_1,\cdots,e_n\}$ of $\mathcal{H}$ (with respect to g) over a neighborhood U of (u_0,t_0) consisting of the eigenvectors of $A_{\mathcal{H}}$. Since the fibres of ϕ are minimal regularizable submanifolds, we have $||H|| = \sum_{i=1}^{n} h(e_i, e_i)$ on U. Clearly we have

$$\frac{\partial ||H||}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial h_{\mathcal{H}}}{\partial t}(e_i, e_i) + 2h_{\mathcal{H}}(\nabla_{\frac{\partial}{\partial t}} e_i, e_i) \right). \tag{4.10}$$

On the other hand, it follows from Theorem 4.6 that

$$\sum_{i=1}^{n} \frac{\partial h_{\mathcal{H}}}{\partial t}(e_i, e_i) = \triangle_{\mathcal{H}} ||H|| - ||H|| \mathrm{Tr}(A_{\mathcal{H}})^2 - 3||H|| \mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}, \tag{4.11}$$

where we use $\sum_{i=1}^{n} (\triangle^{\mathcal{H}}_{\mathcal{H}} h_{\mathcal{H}})(e_i, e_i) = \triangle_{\mathcal{H}} ||H||$ and $\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} \mathcal{R}(\bullet, \bullet) = 0$ (by Lemma 4.10). Since each e_i is an eigenvector of $A_{\mathcal{H}}$, we have $h(e_i, e_j) = 0$ $(i \neq j)$. By using Lemma 4.7, we can show

$$\sum_{i=1}^{n} h_{\mathcal{H}}(\nabla_{\frac{\partial}{\partial t}} e_i, e_i) = \sum_{i=1}^{n} g(\nabla_{\frac{\partial}{\partial t}} e_i, e_i) h(e_i, e_i) = ||H|| \mathrm{Tr}(A_{\mathcal{H}})^2. \tag{4.12}$$

From (4.10), (4.11) and (4.12), we obtain the desired relation. □

From we derive the following evolution equation for $\mathrm{Tr}(A_{\mathcal{H}})^2_t$.

COROLLARY 4.12. *The quantities* $\mathrm{Tr}(A_{\mathcal{H}})^2_t$*'s satisfy the following evolution equation:*

$$\begin{aligned} \frac{\partial \mathrm{Tr}(A_{\mathcal{H}})^2}{\partial t} = & \triangle_{\mathcal{H}}(\mathrm{Tr}(A_{\mathcal{H}})^2) - 2\mathrm{Tr}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}(\nabla^{\mathcal{H}}_{\bullet} A_{\mathcal{H}} \circ \nabla^{\mathcal{H}}_{\bullet} A_{\mathcal{H}}) \\ & + 2\mathrm{Tr}((A_{\mathcal{H}})^2) \left(\mathrm{Tr}((A_{\mathcal{H}})^2) - \mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} \right) \\ & - 4||H|| \mathrm{Tr} \left(((\mathcal{A}^{\phi}_{\xi})^2) \circ A_{\mathcal{H}} \right) - 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} \mathcal{R}(A_{\mathcal{H}} \bullet, \bullet). \end{aligned}$$

Proof. Fix $(u_0, t_0) \in M \times [0, T)$. Take a local orthonormal frame field $\{e_1, \cdots, e_n\}$ of $\mathcal{H}$ (with respect to $g_{\mathcal{H}}$) over a neighborhood U of (u_0, t_0) consisting of the eigenvectors of $A_{\mathcal{H}}$. From Lemma 4.2, we have

$$\begin{aligned} \frac{\partial h_{\mathcal{H}}}{\partial t}(X, Y) &= \frac{\partial g_{\mathcal{H}}}{\partial t}(A_{\mathcal{H}} X, Y) + g_{\mathcal{H}}(\frac{\partial A_{\mathcal{H}}}{\partial t}(X), Y) \\ &= -2||H|| h_{\mathcal{H}}(A_{\mathcal{H}} X, Y) + g_{\mathcal{H}}(\frac{\partial A_{\mathcal{H}}}{\partial t}(X), Y) \end{aligned} \tag{4.13}$$

for any $X, Y \in \pi^*_M TM$. Since $\{e_1, \cdots, e_n\}$ consists of the eigenvectors of $A_{\mathcal{H}}$, it follows from Lemma 4.7 that

$$g(\nabla_{\frac{\partial}{\partial t}} e_i, e_i) = ||H|| h(e_i, e_i). \tag{4.14}$$

From these relations, Lemmas 4.2 and 4.7, we have

$$
\begin{aligned}
\frac{\partial \mathrm{Tr}(A_{\mathcal{H}})^2}{\partial t} &= \sum_{i=1}^{n} \frac{\partial}{\partial t}(h_{\mathcal{H}}(A_{\mathcal{H}}e_i, e_i)) \\
&= \sum_{i=1}^{n} \left(\frac{\partial h_{\mathcal{H}}}{\partial t}(A_{\mathcal{H}}e_i, e_i) + h_{\mathcal{H}}(\frac{\partial A_{\mathcal{H}}}{\partial t}(e_i), e_i) + 2h_{\mathcal{H}}(A_{\mathcal{H}}e_i, \nabla_{\frac{\partial}{\partial t}} e_i) \right) \\
&= \sum_{i=1}^{n} \left(\frac{\partial h_{\mathcal{H}}}{\partial t}(A_{\mathcal{H}}e_i, e_i) + g_{\mathcal{H}}((\frac{\partial A_{\mathcal{H}}}{\partial t}(e_i), A_{\mathcal{H}}e_i) \right. \\
&\qquad \left. + 2||H||h(e_i, e_i)h_{\mathcal{H}}(A_{\mathcal{H}}e_i, e_i)) \right) \\
&= \sum_{i=1}^{n} \left(2\frac{\partial h_{\mathcal{H}}}{\partial t}(A_{\mathcal{H}}e_i, e_i) + 2||H||g((A_{\mathcal{H}})^3 e_i, e_i) \right. \\
&\qquad \left. + 2||H||h(e_i, e_i)g((A_{\mathcal{H}})^2 e_i, e_i)) \right) \\
&= \sum_{i=1}^{n} \left(2\frac{\partial h_{\mathcal{H}}}{\partial t}(A_{\mathcal{H}}e_i, e_i) + 4||H||g((A_{\mathcal{H}})^3 e_i, e_i) \right).
\end{aligned} \tag{4.15}
$$

Also we have

$$
\sum_{i=1}^{n} (\triangle^{\mathcal{H}}_{\mathcal{H}} h_{\mathcal{H}})(A_{\mathcal{H}}e_i, e_i) = \frac{1}{2}\triangle_{\mathcal{H}} \mathrm{Tr}((A_{\mathcal{H}})^2) - \mathrm{Tr}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} \left(\nabla^{\mathcal{H}}_{\bullet} A_{\mathcal{H}} \circ \nabla^{\mathcal{H}}_{\bullet} A_{\mathcal{H}} \right). \tag{4.16}
$$

From Theorem 4.6, (4.15) and (4.16), we obtain the desired relation. □

By using Corollaries 4.11 and 4.12, we can show the following evolution equation.

COROLLARY 4.13. *The quantities* $\mathrm{Tr}\,(A_{\mathcal{H}})^2_t - \frac{||H_t||^2}{n}$*'s satisfy the following evolution equation:*

$$
\begin{aligned}
\frac{\partial(\mathrm{Tr}\,(A_{\mathcal{H}})^2 - \frac{||H||^2}{n})}{\partial t} &= \triangle_{\mathcal{H}} \left(\mathrm{Tr}(A_{\mathcal{H}})^2 - \frac{||H||^2}{n} \right) + \frac{2}{n} \, ||\mathrm{grad}||H||||^2 \\
&\quad + 2\mathrm{Tr}(A_{\mathcal{H}})^2 \times \left(\mathrm{Tr}(A_{\mathcal{H}})^2 - \frac{||H||^2}{n} \right) \\
&\quad - 2\mathrm{Tr}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} \left(\nabla^{\mathcal{H}} A_{\mathcal{H}} \circ \nabla^{\mathcal{H}} A_{\mathcal{H}} \right) \\
&\quad - 2\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} \times \left(\mathrm{Tr}(A_{\mathcal{H}})^2 - \frac{||H||^2}{n} \right) \\
&\quad - 4||H|| \left(\mathrm{Tr} \left((\mathcal{A}^{\phi}_{\xi})^2 \circ \left(A_{\mathcal{H}} - \frac{||H||}{n}\mathrm{id} \right) \right) \right) \\
&\quad - 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} \mathcal{R} \left(\left(A_{\mathcal{H}} - \frac{||H||}{n}\mathrm{id} \right) \bullet, \bullet \right),
\end{aligned}
$$

where $\mathrm{grad}||H||$ *is the gradient vector field of* $||H||$ *with respect to* g *and* $||\mathrm{grad}||H||||$ *is the norm of* $\mathrm{grad}||H||$ *with respect to* g*.*

Proof. This relation follows directly from Corollaries 4.11, 4.12 and $\triangle_{\mathcal{H}}||H||^2 = 2||H||\triangle_{\mathcal{H}}||H|| + 2\,||\mathrm{grad}||H||||^2$. □

REMARK 4.1. From the evolution equations obtained in this section, the evolution equations for the corresponding geometric quantities of $\overline{f}_t(: \overline{M} \hookrightarrow V/G)$ are derived, respectively. In the case where the G-action is free and hence V/G is a (complete) Riemannian manifold, these derived evolution equations coincide with the evolution equations for the corresponding geometric quantities along the mean curvature flow in a complete Riemannian manifold which were given by Huisken [Hu2]. That is, the discussion in this section give a new proof of the evolution equations in [Hu2] in the case where the ambient complete Riemannian manifold occurs as V/G. In the proof of [Hu2], one need to take local coordinates of the ambient space to derive the evolution equations. On the other hand, in our proof, one need not take local coordinates of the ambient space because the ambient space is a Hilbert space. This is an advantage of our proof.

5. A maximum principle. Let M be a Hilbert manifold and g_t $(0 \leq t < T)$ a C^∞-family of Riemannian metrics on M and $G \curvearrowright M$ a almost free action which is isometric with respect to g_t's $(t \in [0,T))$. Assume that the orbit space M/G is compact. Let $\mathcal{H}_t$ $(0 \leq t < T)$ be the horizontal distribution of the G-action and define a subbundle $\mathcal{H}$ of π_M^*TM by $\mathcal{H}_{(x,t)} := (\mathcal{H}_t)_x$. For a tangent vector field X on M (or an open set U of M), we define a section $\bar{X}$ of π_M^*TM (or $\pi_M^*TM|_U$) by $\bar{X}_{(x,t)} := X_x$ $((x,t) \in M \times [0,T))$. Let ∇^t $(0 \leq t < T)$ be the Riemannian connection of g_t and ∇ the connection of π_M^*TM defined in terms of ∇^t's $(t \in [0,T))$. Define a connection $\nabla^{\mathcal{H}}$ of $\mathcal{H}$ by $\nabla^{\mathcal{H}}_X Y = \mathrm{pr}_{\mathcal{H}}(\nabla_X Y)$ for any $X \in T(M \times [0,T))$ and any $Y \in \Gamma(\mathcal{H})$. For $B \in \Gamma(\pi_M^* T^{(r_0,s_0)}M)$, we define maps $\psi_{B\otimes}$ and $\psi_{\otimes B}$ from $\Gamma(\pi_M^*T^{(r,s)}M)$ to $\Gamma(\pi_M^*T^{(r+r_0,s+s_0)}M)$ by

$$\psi_{B\otimes}(S) := B \otimes S, \quad \text{and} \quad \psi_{\otimes B}(S) := S \otimes B \quad (S \in \Gamma(\pi_M^*T^{(r,s)}M),$$

respectively. Also, we define a map $\psi_{\otimes^k}$ of $\Gamma(\pi_M^*T^{(r,s)}M)$ to $\Gamma(\pi_M^*T^{(kr,ks)}M)$ by

$$\psi_{\otimes^k}(S) := S \otimes \cdots \otimes S \ (k\text{−times}) \quad (S \in \Gamma(\pi_M^*T^{(r,s)}M).$$

Also, we define a map $\psi_{g_{\mathcal{H}},ij}$ $(i < j)$ from $\Gamma(\pi_M^*T^{(0,s)}M)$ (or $\Gamma(\pi_M^*T^{(1,s)}M)$) to $\Gamma(\pi_M^*T^{(0,s-2)}M)$ (or $\Gamma(\pi_M^*T^{(1,s-2)}M)$) by

$$\begin{aligned}&(\psi_{g_{\mathcal{H}},ij}(S))_{(x,t)}(X_1,\cdots,X_{s-2})\\ :=& \sum_{k=1}^n S_{(x,t)}(X_1,\cdots,X_{i-1},e_k,X_{i+1},\cdots,X_{j-1},e_k,X_{j+1},\cdots,X_{s-2})\end{aligned}$$

and define a map $\psi_{\mathcal{H},i}$ from $\Gamma(\pi_M^*T^{(1,s)}M)$ to $\Gamma(\pi_M^*T^{(0,s-1)}M)$ by

$$(\psi_{\mathcal{H},i}(S))_{(x,t)}(X_1,\cdots,X_{s-1}) := \mathrm{Tr}\, S_{(x,t)}(X_1,\cdots,X_{i-1},\bullet,X_i,\cdots,X_{s-1}),$$

where $X_i \in T_xM$ $(i = 1,\cdots,s-1)$ and $\{e_1,\cdots,e_n\}$ is an orthonormal base of $(\mathcal{H}_t)_x$ with respect to g_t. We call a map P from $\Gamma(\pi_M^*T^{(0,s)}M)$ to oneself given by the composition of the above maps of five type *a map of polynomial type*.

In this section, we prove the following maximum principle for a C^∞-family of G-invariant symmetric $(0,2)$-tensor fields on M.

THEOREM 5.1. *Let $S \in \Gamma(\pi_M^*(T^{(0,2)}M))$ such that, for each $t \in [0,T)$, $S_t(:= S_{(\cdot,t)})$ is a G-invariant symmetric $(0,2)$-tensor field on M. Assume that S_t's $(0 \leq t < T)$ satisfy the following evolution equation:*

$$\frac{\partial S_{\mathcal{H}}}{\partial t} = \triangle^{\mathcal{H}}_{\mathcal{H}} S_{\mathcal{H}} + \nabla^{\mathcal{H}}_{\bar{X}_0} S_{\mathcal{H}} + P(S)_{\mathcal{H}}, \tag{5.1}$$

where $X_0 \in \Gamma(TM)$ and P is a map of polynomial type from $\Gamma(\pi_M^*(T^{(0,2)}M))$ to oneself.

(i) *Assume that P satisfies the following condition:*

$$X \in \mathrm{Ker}((S+\varepsilon g)_{\mathcal{H}})_{(x,t)} \;\Rightarrow\; P(S+\varepsilon g)_{(x,t)}(X,X) \geq 0 \quad (\forall\, \varepsilon > 0,\ \ (x,t) \in M \times [0,T)). \tag{$*^+_{S_{\mathcal{H}}}$}$$

If $(S_{\mathcal{H}})_{(\cdot,0)} \geq 0$ (resp. > 0), then $(S_{\mathcal{H}})_{(\cdot,t)} \geq 0$ (resp. > 0) holds for all $t \in [0,T)$.

(ii) *Assume that P satisfies the following condition:*

$$X \in \mathrm{Ker}((S+\varepsilon g)_{\mathcal{H}})_{(x,t)} \;\Rightarrow\; P(S+\varepsilon g)_{(x,t)}(X,X) \leq 0 \quad (\forall\, \varepsilon > 0,\ \ (x,t) \in M \times [0,T)). \tag{$*^-_{S_{\mathcal{H}}}$}$$

If $(S_{\mathcal{H}})_{(\cdot,0)} \leq 0$ (resp. < 0), then $(S_{\mathcal{H}})_{(\cdot,t)} \leq 0$ (resp. < 0) holds for all $t \in [0,T)$.

Proof. First we shall show the part of "If $(S_{\mathcal{H}})_{(\cdot,0)} \geq 0$, then $(S_{\mathcal{H}})_{(\cdot,t)} \geq 0$ holds for all $t \in (0,T)$" in the statement (i). For positive numbers ε and δ, we define $S_{\varepsilon,\delta}$ by $(S_{\varepsilon,\delta})_{(x,t)} := S_{(x,t)} + \varepsilon(\delta + t)g_{(x,t)}$.

Step I. In this step, we show the following statement:

$(*)$ $\exists\, \delta > 0$ s.t. "$((S_{\varepsilon,\delta})_{\mathcal{H}})_{(x,t)} > 0$ $(\forall\, (x,t) \in M \times [0,\delta), \forall\, \varepsilon > 0)$".

Suppose that such a positive number δ does not exists. Fix a sufficiently small positive number δ. For some $\varepsilon_0 > 0$, there exists $(x_0,t_0) \in M \times [0,\delta)$ such that $((S_{\varepsilon_0,\delta})_{\mathcal{H}})_{(x_0,t_0)} = 0$. Here we take t_0 as smally as possible. We have $\mathrm{Ker}((S_{\varepsilon_0,\delta})_{\mathcal{H}})_{(x_0,t_0)} \neq \{0\}$ and $((S_{\varepsilon_0,\delta})_t)_{\mathcal{H}_t} \geq 0\ (\forall\, t \in [0,t_0])$. Take $v_1 \in \mathrm{Ker}((S_{\varepsilon_0,\delta})_{\mathcal{H}})_{(x_0,t_0)})$ with $g_{(x_0,t_0)}(v_1,v_1) = 1$. From the assumption $(*^+_{S_{\mathcal{H}}})$ for P, we have

$$P((S_{\varepsilon_0,\delta})_{(x_0,t_0)})(v_1,v_1) \geq 0. \tag{5.2}$$

The map P is of polynomial type, M/G is compact and S_t is G-invariant. Hence, for each $t \in [0,T)$, there exists a positive constant $C_{\delta,t}$ (depending only on $||(S_{\mathcal{H}})_{(\cdot,t)}||$ and $||((S_{(\varepsilon_0,\delta)})_{\mathcal{H}})_{(\cdot,t)}||$) such that

$$||((P(S_{\varepsilon_0,\delta}))_{\mathcal{H}})_{(\cdot,t)} - ((P(S))_{\mathcal{H}})_{(\cdot,t)}|| \leq C_{\delta,t}||((S_{\varepsilon_0,\delta})_{\mathcal{H}})_{(\cdot,t)} - (S_{\mathcal{H}})_{(\cdot,t)}|| \tag{5.3}$$

on M, where $||\cdot||$ is the pointwise norm of a tensor field $(\cdot)$. We take $C_{\delta,t}$ as smally as possible. Since P is of polynomial type, $\lim_{\delta\to+0} C_{\delta,t}$ exists and $\lim_{\delta\to+0} C_{\delta,t} > 0$. Denote by C_t this limit. Fix $T_1 \in (t_0,T)$. Set

$$C_\delta := \max\left\{ \max_{0\leq t\leq T_1} C_{\delta,t},\ \ \max_{\substack{(x,t) \in M\times[0,T_1] \\ v \in TM \text{ s.t. } g_t(v,v)=1}} \left| \left(\frac{\partial g_{\mathcal{H}}}{\partial t}\right)_{(x,t)} (v,v) \right| \right\}$$

and

$$C := \max\left\{ \max_{0\leq t\leq T_1} C_t,\ \ \max_{\substack{(x,t) \in M\times[0,T_1] \\ v \in TM \text{ s.t. } g_t(v,v)=1}} \left| \left(\frac{\partial g_{\mathcal{H}}}{\partial t}\right)_{(x,t)} (v,v) \right| \right\}.$$

Since C is independent of the choice of δ, we may assume that $C\delta < \frac{1}{4}$ by replacing δ to a smaller positive number if necessary. Furthermore, since $\delta \mapsto C_\delta$ is upper semi-continuous and $\lim_{\delta\to+0} C_{\delta,t} < \infty$, we may assume that $C_\delta\delta < \frac{1}{4}$ by replacing δ to a smaller positive number if necessary. From (5.2) and (5.3), we have

$$P(S)_{(x_0,t_0)}(v_1,v_1) \geq -2C_\delta\varepsilon_0\delta. \tag{5.4}$$

Let X_1 be a section of $\mathcal{H}$ on a normal neighborhood U of (x_0,t_0) in $M\times[0,T)$ such that $(X_1)_{(x_0,t_0)} = v_1$ and that $\nabla^{\mathcal{H}}X_1 = 0$ at (x_0,t_0). Define a function ρ on U by $\rho(x,t) := (S_{\varepsilon_0,\delta})_{(x,t)}((\bar{X}_1)_{(x,t)},(\bar{X}_1)_{(x,t)})$ $((x,t)\in U)$. Since we take (x_0,t_0) and v_1 as above, we have $(\frac{\partial\rho}{\partial t})_{(x_0,t_0)} \leq 0$ (see Fig. 4). Also, we have

$$\left(\frac{\partial\rho}{\partial t}\right)_{(x_0,t_0)} = \left(\frac{\partial S_{\mathcal{H}}}{\partial t}\right)_{(x_0,t_0)}(v_1,v_1) + \varepsilon_0(\delta+t_0)\left(\frac{\partial g_{\mathcal{H}}}{\partial t}\right)_{(x_0,t_0)}(v_1,v_1) + \varepsilon_0.$$

Hence we have

$$\left(\frac{\partial S_{\mathcal{H}}}{\partial t}\right)_{(x_0,t_0)}(v_1,v_1) \leq -\varepsilon_0(\delta+t_0)\left(\frac{\partial g_{\mathcal{H}}}{\partial t}\right)_{(x_0,t_0)}(v_1,v_1) - \varepsilon_0. \tag{5.5}$$

Take $w \in T_{x_0}(M\times\{t_0\})$. Clearly we have $d\rho_{(x_0,t_0)}(w) = 0$. Also we have $d\rho_{(x_0,t_0)}(w) = (\nabla^{\mathcal{H}}_w(S_{\varepsilon_0,\delta})_{\mathcal{H}})_{(x_0,t_0)}(v_1,v_1)$. Hence we have

$$(\nabla^{\mathcal{H}}_w(S_{\varepsilon_0,\delta})_{\mathcal{H}})_{(x_0,t_0)}(v_1,v_1) = 0. \tag{5.6}$$

Clearly we have $(\triangle_{t_0}\rho_{t_0})_{x_0} \geq 0$, where $\triangle_{t_0}$ is the Laplacian operator with respect to g_{t_0}. Also, we have $(\triangle_{t_0}\rho_{t_0})_{x_0} = (\triangle^{\mathcal{H}}_{\mathcal{H}}(S_{\varepsilon_0,\delta})_{\mathcal{H}})(v_1,v_1)$. Hence we have

$$(\triangle^{\mathcal{H}}_{\mathcal{H}}(S_{\varepsilon_0,\delta})_{\mathcal{H}})(v_1,v_1) \geq 0. \tag{5.7}$$

From (5.1), (5.5), (5.6) and (5.7), we have

$$\begin{aligned} P(S)_{(x_0,t_0)}(v_1,v_1) \leq & -\varepsilon_0 + \varepsilon_0(\delta+t_0)\left|(\frac{\partial g_{\mathcal{H}}}{\partial t})_{(x_0,t_0)}(v_1,v_1)\right| \\ \leq & -\varepsilon_0 + 2\varepsilon_0 C_\delta\delta. \end{aligned} \tag{5.8}$$

From (5.4) and (5.8), we have $C_\delta\delta \geq \frac{1}{4}$. This contradicts $C_\delta\delta < \frac{1}{4}$. Therefore the statement $(*)$ is true.

Step II. Let δ be a positive number as in the statement $(*)$. Then, for any $(x,t) \in M\times[0,\delta)$ and any $\varepsilon > 0$, we have $((S_{\varepsilon,\delta})_{\mathcal{H}})_{(x,t)} > 0$. Hence we have $\lim\limits_{\varepsilon\to+0}((S_{\varepsilon,\delta})_{\mathcal{H}})_{(x,t)} = (S_{\mathcal{H}})_{(x,t)} \geq 0$ for any $(x,t)\in M\times[0,\delta)$. Set

$$T_1 := \sup\{t_1 \,|\, (S_{\mathcal{H}})_{(x,t)} \geq 0 \ (\forall\,(x,t)\in M\times[0,t_1]\}.$$

Suppose that $T_1 < T$. Then, by the similar discussion for $(S_{\mathcal{H}})_{(\cdot,T_1)}$ instead of $(S_{\mathcal{H}})_{(\cdot,0)}$, we can show that $(S_{\mathcal{H}})_{(x,t)} \geq 0$ for any $t\in[T_1,T_1+\delta']$ and any $x\in M$, where δ' is some positive number. This contradicts the definition of T_1. Therefore we have $T_1 = T$. Thus we obtain $(S_{\mathcal{H}})_{(\cdot,t)} \geq 0$ for any $t\in[0,T)$.

Similarly we can show the part of "If $(S_{\mathcal{H}})_{(\cdot,0)} > 0$, then $(S_{\mathcal{H}})_{(\cdot,t)} > 0$ holds for all $t\in(0,T)$" in the statement (i) as follows. The map P is of polynomial type, M/G is compact and S_t is G-invariant. Hence it follows from $(S_{\mathcal{H}})_{(\cdot,0)} > 0$ that

$(S_{\mathcal{H}})_{(\cdot,0)} \geq b(g_{\mathcal{H}})_{(\cdot,0)}$ holds for some positive coonstant b. Set $\overline{S} := S - bg$. Then it is easy to show that $\overline{S}$ satisfies

$$\frac{\partial \overline{S}_{\mathcal{H}}}{\partial t} = \triangle^{\mathcal{H}}_{\mathcal{H}} \overline{S}_{\mathcal{H}} + \nabla^{\mathcal{H}}_{\bar{X}_0} \overline{S}_{\mathcal{H}} + \overline{P}(\overline{S})_{\mathcal{H}}$$

for some map $\overline{P}$ of polynomial type satisfying the condition $(*^{\pm}_{\overline{S}_{\mathcal{H}}})$:

$$X \in \mathrm{Ker}((\overline{S} + \varepsilon g)_{\mathcal{H}})_{(x,t)} \ \Rightarrow \ \overline{P}(\overline{S} + \varepsilon g)_{(x,t)}(X, X) \geq 0$$
$$(\forall \varepsilon > 0, \ \ (x,t) \in M \times [0,T)).$$

Hence, it follows from Theorem 5.1 that $(\overline{S}_{\mathcal{H}})_{(\cdot,t)} \geq 0$ (hence $(S_{\mathcal{H}})_{(\cdot,t)} > 0$) holds for all $t \in [0,T)$. The statement (ii) also are derived by the similar discussion. □

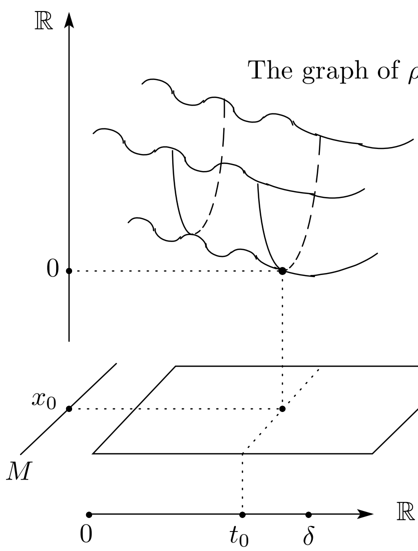

FIG. 4.

REMARK 5.1. (i) According to the proof of the maximum principle by R.S. Hamilton (Theorem 9.1 of [Ha]), we can improve the statement of his maximum principle as follows.

Let g_t $(0 \leq t < T)$ be a C^∞-family of Riemannian metrics on a compact manifold M and S_t $(0 \leq t < T)$ be a C^∞-family of symmetric $(0,2)$-tensor field on M. Assume that S_t's $(0 \leq t < T)$ satisfy the following evolution equation:

$$\frac{\partial S}{\partial t} = \triangle S + \nabla_{\bar{X}_0} S + P(S),$$

where $\triangle S$ is the Laplacian of S with respect to the connection of $\pi^* TM$ defined by the Levi-Civita connections ∇^t's of g_t, $X_0 \in \Gamma(TM)$ and P is a map of polynomial type from $\Gamma(\pi_M^*(T^{(0,2)}M))$ to oneself. Assume that P satisfies the following condition:

$$(*_S) \qquad X \in \mathrm{Ker}(S + \varepsilon g)_{(x,t)} \ \Rightarrow \ P(S + \varepsilon g)_{(x,t)}(X, X) \geq 0$$
$$(\forall \varepsilon > 0, \ \ \forall (x,t) \in M \times [0,T)).$$

If $S_0 \geq 0$ *(resp.* > 0*), then* $S_t \geq 0$ *(resp.* > 0*) holds for all* $t \in [0, T)$*.*

The null-eigenvector condition in [Ha] means the following condition:

$$X \in \mathrm{Ker}(\widehat{S})_{(x,t)} \;\;\Rightarrow\;\; P(\widehat{S})_{(x,t)}(X, X) \geq 0$$
$$(\forall\, \widehat{S} \,:\, \text{symmetric } (0,2)-\text{tensor field on } M, \;\; \forall\,(x,t) \in M \times [0, T)).$$

This condition is stronger than the above condition $(*_S)$. In [Hu1], G. Huisken proved the statement of Theroem 4.3 in [Hu1] by showing that the family $S = (S_{ij} := \frac{h_{ij}}{H} - \varepsilon g_{ij})$ of symmetric $(0,2)$-tensor fields satisfies the above condition $(*_S)$ and applying the maximum principle of R.S. Hamilton. In his proof, it is not shown that the family S satisfies the null-eigenvector condition. The statement of Theorem 4.2 in [Hu2] also was proved by showing that some another family S of symmetric $(0,2)$-tensor fields satisfies the above condition $(*_S)$.

(ii) The constant C_δ in this proof corresponds to the constant C in the proof of Theorem 9.1 in [Ha].

Similarly we obtain the following maximal principle for a C^∞-family of G-invariant functions on M.

THEOREM 5.2. *Let* ρ *be a* C^∞*-function over* $M \times [0, T)$ *such that, for each* $t \in [0, T)$*,* $\rho_t (:= \rho(\cdot, t))$ *is a* G*-invariant function on* M*. Assume that* ρ_t*'s* $(0 \leq t < T)$ *satisfy the following evolution equation:*

$$\frac{\partial \rho}{\partial t} = \triangle_{\mathcal{H}} \rho + d\rho(\bar{X}_0) + P(\rho),$$

where $X_0 \in \Gamma(TM)$ *and* P *is a map of polynomial type from* $C^\infty(M \times [0, T))$ *to oneself.*

(i) *Assume that* P *satisfies the following condition:*

$$(\rho + \varepsilon)_{(x,t)} = 0 \;\Rightarrow\; P(\rho + \varepsilon)_{(x,t)} \geq 0$$
$$(\forall\, \varepsilon > 0, \;\; (x,t) \in M \times [0, T)).$$

If $\rho_0 \geq 0$ *(resp.* > 0*), then* $\rho_t \geq 0$ *(resp.* > 0*) holds for all* $t \in [0, T)$*.*

(ii) *Assume that* P *satisfies the following condition:*

$$(\rho + \varepsilon)_{(x,t)} = 0 \;\Rightarrow\; P(\rho + \varepsilon)_{(x,t)} \leq 0$$
$$(\forall\, \varepsilon > 0, \;\; (x,t) \in M \times [0, T)).$$

If $\rho_0 \leq 0$ *(resp.* < 0*), then* $\rho_t \leq 0$ *(resp.* < 0*) holds for all* $t \in [0, T)$*.*

6. Horizontally strongly convexity preservability theorem. Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$ and $\phi : V \to V/G$ the orbit map. Denote by $\widetilde{\nabla}$ the Riemannian connection of V. Set $n := \dim V/G - 1$. Let $M (\subset V)$ be a G-invariant hypersurface in V such that $\phi(M)$ is compact. Let f be an inclusion map of M into V and f_t $(0 \leq t < T)$ the regularized mean curvature flow starting from f. We use the notations in Section 4. In the sequel, we omit the notation f_{t*} for simplicity. For each $u \in V$, we set

$$L := \sup_{u \in V} \max_{(X_1, \cdots, X_5) \in (\widetilde{\mathcal{H}}_1)^5_u} |\langle \mathcal{A}^\phi_{X_1}((\widetilde{\nabla}_{X_2} \mathcal{A}^\phi)_{X_3} X_4),\, X_5 \rangle|,$$

where $\widetilde{\mathcal{H}}_1 := \{X \in \widetilde{\mathcal{H}} \mid \ ||X|| = 1\}$. Assume that $L < \infty$. Note that $L < \infty$ in the case where V/G is compact. In this section, we prove the following horizontally strongly convexity preservability theorem by using results stated in Section 4 and Theorem 5.1.

THEOREM 6.1. *If M satisfies $||H_0||^2(h_{\mathcal{H}})_{(\cdot,0)} > 2n^2L(g_{\mathcal{H}})_{(\cdot,0)}$, then $T < \infty$ holds and $||H_t||^2(h_{\mathcal{H}})_{(\cdot,t)} > 2n^2L(g_{\mathcal{H}})_{(\cdot,t)}$ holds for all $t \in [0,T)$.*

Proof. Since $\mathcal{A}^{\phi}_{\xi}$ is skew-symmetric, we have

$$\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} \leq 0. \tag{6.1}$$

From Corollary 4.11, $\mathrm{Tr}(A_{\mathcal{H}})^2 \geq \frac{||H||^2}{n}$ and (6.1), we have

$$\frac{\partial ||H||}{\partial t} \geq \triangle_{\mathcal{H}}||H|| + \frac{||H||^3}{n}. \tag{6.2}$$

Define a function ρ over $[0,T)$ by $\rho(t) := \min ||H_t||$. Form (6.2), we have $\frac{d\rho}{dt} \geq \frac{1}{n}\rho^3$. Also we have $\rho(0) > 0$ by the assumption. Hence we obtain $T \leq \frac{n}{2\rho(0)^2}$. Set $S := \dfrac{1}{||H||}h - \dfrac{2n^2L}{||H||^3}g$ and $S_\varepsilon := S + \varepsilon g$, where ε is a positive constant. Take $X, Y \in \mathcal{H}$. By using Lemma 4.2, Theorem 4.6, Corollary 4.8 and Lemma 4.10, we can show

$$\begin{aligned}
&\frac{\partial (S_\varepsilon)_{\mathcal{H}}}{\partial t}(X,Y)\\
&= \frac{1}{||H||}(\triangle^{\mathcal{H}}_{\mathcal{H}} h_{\mathcal{H}})(X,Y) - 2((A_{\mathcal{H}})^2)_\sharp(X,Y) - 2((\mathcal{A}^{\phi}_{\xi})^2)_\sharp(X,Y)\\
&\quad - \frac{1}{||H||^2}\left(\triangle^{\mathcal{H}}_{\mathcal{H}}||H|| - 2||H||\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} - 4n^2L\right) h_{\mathcal{H}}(X,Y)\\
&\quad - \frac{1}{||H||}\mathcal{R}(X,Y) - 2\varepsilon||H||h_{\mathcal{H}}(X,Y)\\
&\quad + \frac{3n^2L}{||H||^4}\left(\triangle^{\mathcal{H}}_{\mathcal{H}}||H|| + ||H||\mathrm{Tr}(A_{\mathcal{H}})^2 - 3||H||\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}\right) g_{\mathcal{H}}(X,Y).
\end{aligned} \tag{6.3}$$

Also, we have

$$\begin{aligned}
(\nabla^{\mathcal{H}}_{\mathrm{grad}||H||}(S_\varepsilon)_{\mathcal{H}})(X,Y) &= \frac{1}{||H||}(\nabla^{\mathcal{H}}_{\mathrm{grad}||H||}h_{\mathcal{H}})(X,Y)\\
&\quad - \frac{||\mathrm{grad}||H||\,||^2}{||H||^2}h(X,Y)\\
&\quad + \frac{3n^2L}{||H||^4}||\mathrm{grad}||H||\,||^2 g(X,Y)
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
(\triangle^{\mathcal{H}}_{\mathcal{H}}(S_\varepsilon)_{\mathcal{H}})(X,Y) &= \frac{1}{||H||}(\triangle^{\mathcal{H}}_{\mathcal{H}}h_{\mathcal{H}})(X,Y) - \frac{2}{||H||^2}(\nabla^{\mathcal{H}}_{\mathrm{grad}||H||}h_{\mathcal{H}})(X,Y)\\
&\quad + \frac{1}{||H||^3}\left(2||\mathrm{grad}||H||\,||^2 - ||H||\triangle^{\mathcal{H}}_{\mathcal{H}}||H||\right) h_{\mathcal{H}}(X,Y)\\
&\quad + \frac{3n^2L}{||H||^5}\left(-4||\mathrm{grad}||H||\,||^2 + ||H||\triangle^{\mathcal{H}}_{\mathcal{H}}||H||\right) g_{\mathcal{H}}(X,Y).
\end{aligned} \tag{6.5}$$

From (6.3), (6.4) and (6.5), we have

$$\begin{aligned}\frac{\partial(S_\varepsilon)_{\mathcal H}}{\partial t}(X,Y) = \triangle^{\mathcal H}_{\mathcal H}(S_\varepsilon)_{\mathcal H}(X,Y) + \frac{2}{||H||}(\nabla^{\mathcal H}_{\mathrm{grad}||H||}(S_\varepsilon)_{\mathcal H})(X,Y)\\ + P(S_\varepsilon)(X,Y),\end{aligned} \tag{6.6}$$

where $P(S_\varepsilon)$ is defined by

$$\begin{aligned}P(S_\varepsilon)(Z,W) := &-2((A_{\mathcal H})^2)_\sharp(Z,W) - 2((\mathcal A^\phi_\xi)^2)_\sharp(Z,W) - \frac{1}{||H||}\mathcal R(Z,W)\\ &+ \frac{1}{||H||^2}\left(2||H||\mathrm{Tr}((\mathcal A^\phi_\xi)^2)_{\mathcal H} + 4n^2L\right)h_{\mathcal H}(Z,W)\\ &+ \frac{6n^2L}{||H||^3}\left(\mathrm{Tr}(A_{\mathcal H})^2 - 3\mathrm{Tr}((\mathcal A^\phi_\xi)^2)_{\mathcal H} + \frac{2||\mathrm{grad}\,||H||\,||^2}{||H||^2}\right)g_{\mathcal H}(Z,W)\\ &- 2\varepsilon||H||h_{\mathcal H}(Z,W)\end{aligned}$$

for $Z,W\in\pi_M^*TM$. Fix any positive constant ε_0 and any $(x_0,t_0)\in M\times[0,T)$. Assume that $\mathrm{Ker}((S_{\varepsilon_0})_{\mathcal H})_{(x_0,t_0)}\neq\{0\}$. Take $X_0\in\mathrm{Ker}((S_{\varepsilon_0})_{\mathcal H})_{(x_0,t_0)})$ with $g(X_0,X_0)=1$. Since

$$h(X_0,Y) = \left(\frac{2n^2L}{||H||^2} - \varepsilon_0||H||\right)g(X_0,Y)\quad(\forall\,Y\in\mathcal H),$$

we have

$$A_{\mathcal H}X_0 = \left(\frac{2n^2L}{||H||^2} - \varepsilon_0||H||\right)X_0.$$

For simplicity, we set $\lambda_1 := \frac{2n^2L}{||H||^2} - \varepsilon_0||H||$. By using the first relation in Lemma 4.10, we have

$$\begin{aligned}&P(S_{\varepsilon_0})(X_0,X_0)\\ &= \frac{6n^2L}{||H||^3}\mathrm{Tr}(A_{\mathcal H})^2 + \frac{12n^2L}{||H||^5}||\mathrm{grad}||H||\,||^2\\ &\quad - 2((\mathcal A^\phi_\xi)^2)_\sharp(X_0,X_0) - \left(\frac{14n^2L}{||H||^3} + 2\varepsilon_0\right)\mathrm{Tr}((\mathcal A^\phi_\xi)^2)_{\mathcal H}\\ &\quad + \frac{4}{||H||}\mathrm{Tr}^\bullet_{g_{\mathcal H}}\langle\mathcal A^\phi_{X_0}\bullet, \mathcal A^\phi_{X_0}(A_{\mathcal H}\bullet)\rangle - \frac{4}{||H||}\mathrm{Tr}^\bullet_{g_{\mathcal H}}\langle\mathcal A^\phi_\bullet X_0, \mathcal A^\phi_\bullet(A_{\mathcal H}X_0)\rangle\\ &\quad + \frac{3}{||H||}\mathrm{Tr}^\bullet_{g_{\mathcal H}}\langle(\widetilde\nabla_\bullet\mathcal A^\phi)_{X_0}\xi, \mathcal A^\phi_\bullet X_0\rangle - \frac{1}{||H||}\mathrm{Tr}^\bullet_{g_{\mathcal H}}\langle(\widetilde\nabla_{X_0}\mathcal A^\phi)_\bullet\xi, \mathcal A^\phi_{X_0}\bullet\rangle\\ &\quad + \frac{2}{||H||}\mathrm{Tr}^\bullet_{g_{\mathcal H}}\langle(\widetilde\nabla_\bullet\mathcal A^\phi)_\bullet X_0, \mathcal A^\phi_{X_0}\xi\rangle.\end{aligned} \tag{6.7}$$

Hence, since $\mathrm{Tr}(A_{\mathcal H})^2\geq\frac{||H||^2}{n}$, $((\mathcal A^\phi_\xi)^2)_\sharp(X,X)\leq 0$, $\mathrm{Tr}((\mathcal A^\phi_\xi)^2)_{\mathcal H}\leq 0$ and the definition of L, we have

$$\begin{aligned}P(S_{\varepsilon_0})(X_0,X_0) > \frac{4}{||H||}\mathrm{Tr}^\bullet_{g_{\mathcal H}}\langle\mathcal A^\phi_{X_0}\bullet, \mathcal A^\phi_{X_0}(A_{\mathcal H}\bullet)\rangle\\ - \frac{4}{||H||}\mathrm{Tr}^\bullet_{g_{\mathcal H}}\langle\mathcal A^\phi_\bullet X_0, \mathcal A^\phi_\bullet(A_{\mathcal H}X_0)\rangle.\end{aligned} \tag{6.8}$$

Since $A_{\mathcal H}X_0=\lambda_1X_0$, $X\in\mathrm{Ker}((S_{\varepsilon_0})_{\mathcal H})_{(x_0,t_0)}$ and $((S_{\varepsilon_0})_{\mathcal H})_{(x_0,t_0)}\geq 0$, we may assume that λ_1 is the smallest eigenvalue of $(A_{\mathcal H})_{(x_0,t_0)}$. Let $\{\lambda_i\,|\,i=1,\cdots,n\}$ ($\lambda_1\leq\cdots\leq\lambda_n$) be the set of all eigenvalues of $(A_{\mathcal H})_{(x_0,t_0)}$. Let $\{e_1,\cdots,e_n\}$ be an orthonormal base of $T_{x_0}M$ with respect to $(g_{\mathcal H})_{(x_0,t_0)}$ satisfying $e_1=X_0$ and $A_{\mathcal H}e_i=\lambda_ie_i$ ($i=2,\cdots,n$). Then we have

$$\begin{aligned}&\frac{4}{||H||}\mathrm{Tr}^{\bullet}_{g_{\mathcal H}}\langle\mathcal A^{\phi}_{X_0}\bullet,\mathcal A^{\phi}_{X_0}(A_{\mathcal H}\bullet)\rangle-\frac{4}{||H||}\mathrm{Tr}^{\bullet}_{g_{\mathcal H}}\langle\mathcal A^{\phi}_{\bullet}X_0,\mathcal A^{\phi}_{\bullet}(A_{\mathcal H}X_0)\rangle\\ =\;&\frac{4}{||H||}\sum_{i=1}^{n}(\lambda_i-\lambda_1)\langle\mathcal A^{\phi}_{X_0}e_i,\mathcal A^{\phi}_{X_0}e_i\rangle\geq 0.\end{aligned}$$

From (6.8) and this inequality, we obtain $P(S_{\varepsilon_0})(X_0,X_0)\geq 0$. Hence it follows from the arbitrarinesses of ε_0 and (x_0,t_0) that P satisfies the condition $(*^{+}_{S_{\mathcal H}})$. Therefore it follows from Theorem 5.1 that $(S_{\mathcal H})_{(\cdot,t)}>0$ holds for all $t\in[0,T)$. □

7. Strongly convex preservability theorem in the orbit space. Let V, G and ϕ be as in the previous section. Set $N:=V/G$ and $n:=\dim V/G-1$. Denote by g_N and R_N the Riemannian orbimetric and the curvature orbitensor of N. Also, ∇^N the Riemannian connection of $g_N|_{N\setminus\mathrm{Sing}(N)}$. Denote by $||\nabla^NR_N||$ the norm of ∇^NR_N with respect to g_N. Set $L_N:=\sup||\nabla^NR_N||$. Assume that $L_N<\infty$. Let $\overline{M}$ be a compact suborbifold of codimension one in N immersed by $\overline{f}$ and $\overline{f}_t$ ($t\in[0,T)$) the mean curvature flow starting from $\overline{f}$. Denote by $\overline{g}_t,\overline{h}_t,\overline{A}_t$ and $\overline{H}_t$ be the induced orbimetric, the second fundamental orbiform, the shape orbitensor and the mean curvature orbifunction of $\overline{f}_t$, respectively, and $\overline{\xi}_t$ the unit normal vector field of $\overline{f}_t|_{\overline{M}\setminus\mathrm{Sing}(\overline{M})}$.

From Theorem 6.1, we obtain the following strongly convexity preservability theorem for compact suborbifolds in N.

THEOREM 7.1. *If $\overline{f}$ satisfies $||\overline{H}_0||^2\overline{h}_0>n^2L_N\overline{g}_0$, then $T<\infty$ holds and $||\overline{H}_t||^2\overline{h}_t>n^2L_N\overline{g}_t$ holds for all $t\in[0,T)$.*

Proof. Set $M:=\{(x,u)\in\overline{M}\times V\,|\,\overline{f}(x)=\phi(u)\}$ and define $f:M\to V$ by $f(x,u)=u$ ($(x,u)\in M$). It is clear that f is an immersion. Denote by H_0 the regularized mean curvature vector of f. Define a curve $c_x:[0,T)\to N$ by $c_x(t):=\overline{f}_t(x)$ ($t\in[0,T)$) and let $(c_x)^L_u$ be the horizontal lift of c_x with $(c_x)^L_u(0)=u$ and $((c_x)^L_u)'(0)=(H_0)_{(x,u)}$, where $(x,u)\in M$. Define an immersion $f_t:M\hookrightarrow V$ by $f_t(x,u):=(c_x)^L_u(t)$ ($(x,u)\in M$). Then f_t ($t\in[0,T)$) is the regularized mean curvature flow starting from f (see the proof of Theorem 4.1). Denote by g_t, h_t, A^t and H_t the induced metric, the second fundamental form, the shape tensor and the mean curvature vector of f_t, respectively. By the assumption, $\overline{f}_0$ satisfies $||\overline{H}_0||^2\overline{h}_0>n^2L_N\overline{g}_0$. Also, we can show $L_N=2L$ by long calculation, where L is as in the prevoius section. From these facts, we can show that f_0 satisfies $||H_0||^2(h_{\mathcal H})_0>2n^2L(g_{\mathcal H})_0$. Hence, it follows from Theorem 6.1 that f_t ($t\in[0,T)$) satisfies $||H_t||^2(h_{\mathcal H})_t>2n^2L(g_{\mathcal H})_t$. Furthermore, it follows from this fact that $\overline{f}_t$ ($t\in[0,T)$) satisfies $||\overline{H}_t||^2\overline{h}_t>n^2L_N\overline{g}_t$. □

REMARK 7.1. In the case where the G-action is free and hence N is a (complete) Riemannian manifold, Theorem 7.1 implies the strongly convexity preservability theorem by G. Huisken (see [Hu2, Theorem 4.2]).

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