

SPECIAL FUNCTION IDENTITIES FROM SUPERELLIPTIC KUMMER VARIETIES*

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Abstract. We prove that a well-known reduction formula by Barnes and Bailey that implies the factorization of Appell’s generalized hypergeometric series into a product of two Gauss’ hypergeometric functions follows entirely from geometry: we first construct a surface of general type as minimal nonsingular model for a product-quotient surface with only rational double points from a pair of superelliptic curves of genus $2r - 1$ with $r \in \mathbb{N}$. We then show that this generalized Kummer variety is equipped with two fibrations with general fiber of genus $2r - 1$. When periods of a holomorphic two-form over carefully crafted transcendental two-cycles on the generalized Kummer variety are evaluated using either of the two fibrations, the answer must be independent of the fibration and the aforementioned family of special function identities is obtained. This family of identities is a multivariate generalization of Clausen’s Formula. Interestingly, this paper’s finding bridges Ernst Kummer’s two independent lines of research, algebraic transformations for the Gauss’ hypergeometric function and nodal surfaces of degree four in \mathbb{P}^3 .

Key words. Kummer surfaces, product-quotient surfaces, special function identities, hypergeometric functions.

Mathematics Subject Classification. 14D0x, 14J28, 33C65.

1. Introduction. The Appell series F_2 is a hypergeometric series in two variables that was introduced by Paul Appell in 1880 as generalization of Gauss’ hypergeometric series ${}_2F_1$ of one variable. Appell established the set of linear partial differential equations of rank four which this function is a solution of, and found various reduction formulas for this and three closely related series F_1, F_3, F_4 in terms of hypergeometric series of one variable. In 1933 Bailey derived a reduction formula for the Appell series F_4 as product of two Gauss’ hypergeometric functions. Later Bailey himself – when studying unpublished manuscripts left by Barnes after his death – realized that Barnes had already obtained this reduction of F_4 in 1907 [4]. When this reduction formula is combined with another result of Bailey’s relating the Appell series F_4 to F_2 [3, Eq. (3.1)], a corresponding reduction formula for the Appell series F_2 is obtained. However, only such Appell series F_2 are obtained in this way that satisfy the quadric property that four of the linearly independent solutions of the associated system of linear partial differential equations are quadratically related [30]. The explicit form of the reduction formula for the Appell series F_2 appeared recently in [31] where the author investigated specializations of Appell’s functions to univariate hypergeometric functions. The reduction formula for all Appell series F_2 satisfying the quadric property can be regarded as a multivariate generalization of the classical Clausen Identity [12] that relates generalized univariate hypergeometric functions of type ${}_3F_2$ to products of two Gauss’ hypergeometric functions¹. Carlson later classi-

*Received April 11, 2016; accepted for publication April 22, 2016.

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¹There are other ways of generalizing of this identity: in [2] a generalization of Clausen’s Identity to *univariate* generalized hypergeometric functions of rank 4 and 5 was given.

fied all quadratic transformations of so-called double integral averages with two rows and two columns into itself in [9]. It turns out that the multivariate reduction formula of Bailey and Barnes is in fact the only quadratic transformation that relates the Appell series F_2 with two independent variables and two free parameters to a product of two ${}_2F_1$ -series. Clausen's original identity has played a central role in the construction of periods and mirror maps in the context of string theory in [25] and in the proof of the Bieberbach conjecture in [13]. At the same time, this multivariate identity incorporates many other classical identities for Gauss' hypergeometric functions. For example, taken in conjunction with an obvious symmetry of the F_2 -system, the decomposition realizes the most important quadratic identity² for the Gauss' hypergeometric function ${}_2F_1$ which was first derived by Kummer [19]. A complete list of all quadratic identities was later given by Goursat [16]. The importance of these identities was originally founded on the fact that they enable one to determine the value of a hypergeometric function ${}_2F_1(z)$ at $z = -1$ by relating it to the value of another hypergeometric function at $z = 1$ which, in turn, could be evaluated using Gauss' theorem [20]. There are many generalizations giving the value of a hypergeometric function as an algebraic number at special rational values that are based on higher-degree identities.

The reduction formula – which we will also refer to as *Multivariate Clausen Identity* – stems from the fact that a quadric in \mathbb{P}^3 is ruled and decomposes into $\mathbb{P}^1 \times \mathbb{P}^1$. If we consider this ruled quadric the period domain of a family of polarized varieties, then the decomposition formula – as we will show – has a purely algebro-geometric interpretation as period computation on a generalized Kummer variety. In the simplest case, this generalized Kummer variety is an ordinary Kummer surface of two non-isogenous elliptic curves. On the one hand, we can view this Kummer surface as a resolution of the quotient of an Abelian surface by an involution. On the other hand, the Kummer surface can be equipped with the structure of a Jacobian elliptic $K3$ surface of Picard-rank 18 [27]. However, for the purpose of realizing hypergeometric function identities only those Jacobian elliptic fibrations will prove relevant that relate the $K3$ surface to an extremal Jacobian rational elliptic surface by a quadratic twist or a quadratic base transformation and that have non-trivial two-torsion. Equivalently, we can say that the Kummer surface is equipped with two Jacobian elliptic fibrations, an isotrivial fibration and a non-isotrivial one.

The central idea of this article is that non-trivial relations on special functions can be shown to follow entirely from geometry. In particular, periods of a holomorphic two-form over carefully crafted transcendental two-cycles on a Kummer surface can be evaluated using either of the two aforementioned elliptic fibrations. As the answer must be independent of the fibration used we obtain a special function identity that is the Multivariate Clausen Identity. However, the most general version of the Multivariate Clausen Identity which has additional free rational parameters cannot be realized in the framework of a classical Kummer surface. Therefore, we will construct a generalized Kummer variety as minimal nonsingular model for a product-quotient surface with only rational double points from a pair of two highly symmetric curves of genus $2r - 1$ with $r \in \mathbb{N}$, so-called superelliptic curves. Product-quotient surfaces have become an important tool in the construction of smooth connected minimal complex projective surfaces of general type [5, 10, 11]. In fact, the Kummer variety we construct is a family of such surfaces of general type with irregularity $q = 4(r - 1)$

²The quadratic identity in question consists in fact of two quadratic identities, but for complementary moduli of the Gauss' hypergeometric function in question.

and geometric genus $p_g = 1 + 2(r^2 - 1)$. Furthermore, we will establish on such a generalized Kummer variety the structure – not of elliptic fibrations with section – but of fibrations with fibers of genus $2r - 1$. When periods of a holomorphic two-form over carefully crafted transcendental two-cycles on the generalized Kummer variety are evaluated using either of two fibrations, the answer must be independent of the fibration and the aforementioned family of special function identities is obtained. Therefore, the major result of this article is as follows:

THEOREM 1.1.

- (1) *Every Appell hypergeometric function with rational parameters satisfying the quadric property can be obtained as period integral of a suitable holomorphic two-form on the generalized Kummer variety of two superelliptic curves.*
- (2) *The Multivariate Clausen Identity of Barnes and Bailey follows entirely from geometry, i.e., from the existence of an isotrivial fibration and a second non-isotrivial fibration on the generalized Kummer variety of two superelliptic curves.*

The above theorem summarizes the more technical statements that will be proved in Lemma 5.2, Lemma 5.3, and Theorem 5.4. It is interesting that Ernst Kummer was the first person who investigated both quadratic identities for the Gauss' hypergeometric function as well as the nodal surfaces of degree 4 in \mathbb{P}^3 [21] that we call Kummer surfaces today. This article connects precisely these two subjects using today's understanding of variations of Hodge structure and period mappings.

This article is structured as follows: in Section 2 we will review some basic facts about Gauss' hypergeometric functions and the multivariate generalization F_2 . For Appell's hypergeometric system satisfying the quadric property we prove that the associated Pfaffian system decomposes as outer tensor product of two rank-two hypergeometric systems. Next, we will review the Multivariate Clausen Identity that decomposes the holomorphic solution F_2 into a product of two Gauss' hypergeometric functions ${}_2F_1$. In Section 3 we will prove some basic facts about superelliptic curves, the resolution of their singularities, the construction of a holomorphic one-form and the lifting of certain cyclic group actions to the resolution curves, and compute period integrals of the first kind in terms of Gauss' hypergeometric functions. In Section 4 we will construct a generalized Kummer variety as minimal nonsingular model of a product-quotient surface with only rational double points from a pair of two superelliptic curves, determine its Hodge diamond, and the explicit defining equations for certain fibrations with fibers of genus $2r - 1$ on it. In Section 5, we prove that the Multivariate Clausen Identity realizes the equality of periods of a holomorphic two-form evaluated over a suitable two-cycle using the structure of either of the two constructed fibrations.

Acknowledgments. The first author acknowledges support from the Simons Foundation through grant no. 208258. The second author acknowledges support from the National Sciences and Engineering Research Council, the Pacific Institute for Mathematical Sciences, and the Visiting Campobassi Professorship at the University of Maryland. The third author acknowledges the generous support of the University of Alberta's Faculty of Science Visiting Scholar Program.

2. Appell’s hypergeometric function F_2 . Appell’s hypergeometric function F_2 is defined by the following double hypergeometric series

$$F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, z_2\right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta_1)_m (\beta_2)_n}{(\gamma_1)_m (\gamma_2)_n m! n!} z_1^m z_2^n \tag{2.1}$$

that converges absolutely for $|z_1| + |z_2| < 1$. The series is a bivariate generalization of the Gauss’ hypergeometric series

$${}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n \tag{2.2}$$

that converges absolutely for $|z| < 1$. Outside of the radius of convergence we can define both ${}_2F_1$ and F_2 by their analytic continuation. As a multi-valued function of z or z_1 and z_2 , the function ${}_2F_1$ or F_2 , respectively, are analytic everywhere except for possible branch loci. For ${}_2F_1$, the possible branch points are located at $z = 0$, $z = 1$ and $z = \infty$. For F_2 , the possible branch loci are the union of the following lines

$$z_1 = 0, \quad z_1 = 1, \quad z_1 = \infty, \quad z_2 = 0, \quad z_2 = 1, \quad z_2 = \infty, \quad z_1 + z_2 = 1. \tag{2.3}$$

We call the branch obtained by introducing a cut from 1 to ∞ on the real z -axis or the real z_1 - and z_2 -axes, respectively, the principal branch of ${}_2F_1$ and F_2 , respectively. The principal branches of ${}_2F_1$ and F_2 are entire functions in α, β or α, β_1, β_2 , and meromorphic in γ or γ_1, γ_2 with poles for $\gamma, \gamma_1, \gamma_2 = 0, -1, -2, \dots$. Except where indicated otherwise we always use principal branches.

Appell’s function F_2 satisfies a Fuchsian³ system of partial differential equations analogous to the hypergeometric equation for the function ${}_2F_1$. The differential equation satisfied by ${}_2F_1$ is

$$z(1-z) \frac{d^2 F}{dz^2} + (\gamma - (\alpha + \beta + 1)z) \frac{dF}{dz} - \alpha \beta F = 0. \tag{2.4}$$

It is a Fuchsian equation with three regular singularities at $z = 0$, $z = 1$ and $z = \infty$ with local exponent differences equal to $1 - \gamma$, $\gamma - \alpha - \beta$, and $\alpha - \beta$, respectively. The system of partial differential equations satisfied by F_2 is given by

$$\begin{aligned} z_1(1-z_1) \frac{\partial^2 F}{\partial z_1^2} - z_1 z_2 \frac{\partial^2 F}{\partial z_1 \partial z_2} + (\gamma_1 - (\alpha + \beta_1 + 1)z_1) \frac{\partial F}{\partial z_1} - \beta_1 z_2 \frac{\partial F}{\partial z_2} - \alpha \beta_1 F &= 0, \\ z_2(1-z_2) \frac{\partial^2 F}{\partial z_2^2} - z_1 z_2 \frac{\partial^2 F}{\partial z_1 \partial z_2} + (\gamma_2 - (\alpha + \beta_2 + 1)z_2) \frac{\partial F}{\partial z_2} - \beta_2 z_1 \frac{\partial F}{\partial z_1} - \alpha \beta_2 F &= 0. \end{aligned} \tag{2.5}$$

This is a holonomic system of rank 4 whose singular locus on $\mathbb{P}^1 \times \mathbb{P}^1$ is the union of the lines in (2.3). For $\text{Re}(\gamma) > \text{Re}(\beta) > 0$, the Gauss’ hypergeometric function ${}_2F_1$ has an integral representation

$${}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| z\right) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{dx}{x^{1-\beta} (1-x)^{1+\beta-\gamma} (1-zx)^\alpha}. \tag{2.6}$$

We have the following well-known generalization for the Appell hypergeometric function:

³Fuchsian means linear homogeneous and with regular singularities.

LEMMA 2.1. For $\operatorname{Re}(\gamma_1) > \operatorname{Re}(\beta_1) > 0$ and $\operatorname{Re}(\gamma_2) > \operatorname{Re}(\beta_2) > 0$, we have the following integral representation for Appell's hypergeometric series

$$F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, z_2\right) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1 - \beta_1)\Gamma(\gamma_2 - \beta_2)} \times \int_0^1 du \int_0^1 dx \frac{1}{u^{1-\beta_2}(1-u)^{1+\beta_2-\gamma_2} x^{1-\beta_1}(1-x)^{1+\beta_1-\gamma_1} (1-z_1x-z_2u)^\alpha}. \tag{2.7}$$

Proof. The series expansion is obtained for $|z_1| + |z_2| < 1$ as follows

$$\begin{aligned} & \int_0^1 du \int_0^1 dx \frac{1}{u^{1-\beta_2}(1-u)^{1+\beta_2-\gamma_2} x^{1-\beta_1}(1-x)^{1+\beta_1-\gamma_1} (1-z_1x-z_2u)^\alpha} \\ &= \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}}{m!n!} \int_0^1 \frac{(z_1x)^m dx}{x^{1-\beta_1}(1-x)^{1+\beta_1-\gamma_1}} \int_0^1 \frac{(z_2u)^n du}{u^{1-\beta_2}(1u)^{1+\beta_2-\gamma_2}} \\ &= \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}}{m!n!} z_1^m z_2^n \frac{\Gamma(\beta_1+m)\Gamma(\gamma_1-\beta_1)}{\Gamma(\gamma_1+m)} \frac{\Gamma(\beta_2+n)\Gamma(\gamma_2-\beta_2)}{\Gamma(\gamma_2+n)} \\ &= \frac{\Gamma(\beta_1)\Gamma(\gamma_1-\beta_1)\Gamma(\beta_2)\Gamma(\gamma_2-\beta_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \sum_{m,p \geq 0} \frac{(\alpha)_{m+p}(\beta_1)_m(\beta_2)_n}{m!(\gamma_1)_m n!(\gamma_2)_n} z_1^m z_2^n. \end{aligned} \tag{2.8}$$

This proves the lemma. \square

The connection between Gauss' hypergeometric function ${}_2F_1$ and Appell's hypergeometric function F_2 is given by an integral transform:

COROLLARY 2.2. For $\operatorname{Re}(\gamma_1) > \operatorname{Re}(\beta_1) > 0$ and $\operatorname{Re}(\gamma_2) > \operatorname{Re}(\beta_2) > 0$, we have the following integral relation between Gauss' hypergeometric function ${}_2F_1$ and Appell's hypergeometric function F_2 :

$$\begin{aligned} & \frac{1}{A^\alpha} F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| \frac{1}{A}, 1 - \frac{B}{A}\right) = -\frac{\Gamma(\gamma_2)(A-B)^{1-\gamma_2}}{\Gamma(\beta_2)\Gamma(\gamma_2-\beta_2)} \\ & \times \int_A^B \frac{dU}{(A-U)^{1-\beta_2}(U-B)^{1+\beta_2-\gamma_2} U^\alpha} {}_2F_1\left(\begin{matrix} \alpha, \beta_1 \\ \gamma_1 \end{matrix} \middle| \frac{1}{U}\right). \end{aligned} \tag{2.9}$$

Proof. Using the variable $U = (1 - z_2u)/z_1$ or $u = (1 - z_1U)/z_2$, we obtain

$$\begin{aligned} & \frac{1}{(1-z_2u)^\alpha} \int_0^1 \frac{dx}{x^{1-\beta_1}(1-x)^{1+\beta_1-\gamma_1} (1-\frac{z_1}{1-z_2u}x)^\alpha} \\ &= \frac{\Gamma(\beta_1)\Gamma(\gamma_1-\beta_1)}{\Gamma(\gamma_1)} \frac{1}{z_1^\alpha U^\alpha} {}_2F_1\left(\begin{matrix} \alpha, \beta_1 \\ \gamma_1 \end{matrix} \middle| \frac{1}{U}\right). \end{aligned}$$

Setting $z_1 = 1/A$ and $z_2 = 1 - B/A$ or, equivalently, $A = 1/z_1$ and $B = (1 - z_2)/z_1$, we obtain

$$\begin{aligned} & \int_0^1 du \frac{1}{u^{1-\beta_2}(1-u)^{1+\beta_2-\gamma_2}} \frac{1}{z_1^\alpha U^\alpha} {}_2F_1\left(\begin{matrix} \alpha, \beta_1 \\ \gamma_1 \end{matrix} \middle| \frac{1}{U}\right) \\ &= -\frac{\left(\frac{z_2}{z_1}\right)^{1-\gamma_2}}{z_1^\alpha} \int_A^B \frac{dU}{(A-U)^{1-\beta_2}(U-B)^{1+\beta_2-\gamma_2} U^\alpha} {}_2F_1\left(\begin{matrix} \alpha, \beta_1 \\ \gamma_1 \end{matrix} \middle| \frac{1}{U}\right). \end{aligned} \tag{2.10}$$

Equation (2.9) follows. \square

REMARK 1. Equation (2.7) is the two-parameter generalization of the *Euler integral transform* formula relating ${}_2F_1$ and ${}_3F_2$. To see this in more detail, let us start with the classical Euler integral transform formula, i.e.,

$$\begin{aligned}
 {}_3F_2\left(\begin{matrix} \alpha, \beta_1, 1 + \alpha - \gamma_2 \\ \gamma_1, 1 + \alpha - \gamma_2 + \beta_2 \end{matrix} \middle| z_1\right) &= \frac{\Gamma(\gamma_1)}{\Gamma(\beta_1)\Gamma(\gamma_1 - \beta_1)} \\
 \times \int_0^1 \frac{dx}{x^{1-\beta_1}(1-x)^{1+\beta_1-\gamma_1}} {}_2F_1\left(\begin{matrix} \alpha, 1 + \alpha - \gamma_2 \\ 1 + \alpha - \gamma_2 + \beta_2 \end{matrix} \middle| z_1 x\right). & \tag{2.11}
 \end{aligned}$$

We compare Equation (2.11) with an expression similar to Equation (2.7), but where we have replaced the ramification point $x = 1$ by $x = \infty$ as upper integration limit⁴ and changed the normalization factor, i.e.,

$$\begin{aligned}
 \tilde{F}_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, z_2\right) &= \frac{(-1)^{-\beta_2} \Gamma(\gamma_1) \Gamma(1 + \alpha - \gamma_2)}{\Gamma(\beta_1) \Gamma(\beta_2) \Gamma(\gamma_1 - \beta_1) \Gamma(1 + \alpha - \gamma_2 + \beta_2)} \\
 \times \int_0^\infty du \int_0^1 dx \frac{1}{u^{1-\beta_2} (1-u)^{1+\beta_2-\gamma_2} x^{1-\beta_1} (1-x)^{1+\beta_1-\gamma_1} (1-z_1 x - z_2 u)^\alpha}. & \tag{2.12}
 \end{aligned}$$

Mapping $u \mapsto 1 - 1/u$, we obtain

$$\begin{aligned}
 \tilde{F}_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, z_2\right) &= \frac{\Gamma(\gamma_1)}{\Gamma(\beta_1) \Gamma(\gamma_1 - \beta_2)} \\
 \times \int_0^1 \frac{dx}{x^{1-\beta_1} (1-x)^{1+\beta_1-\gamma_2}} {}_2F_1\left(\begin{matrix} \alpha, 1 + \alpha - \gamma_2 \\ 1 + \alpha - \gamma_2 + \beta_2 \end{matrix} \middle| z_1 x + (z_2 - 1)\right), & \tag{2.13}
 \end{aligned}$$

and, therefore,

$$\tilde{F}_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, 1\right) = {}_3F_2\left(\begin{matrix} \alpha, \beta_1, 1 + \alpha - \gamma_2 \\ \gamma_1, 1 + \alpha - \gamma_2 + \beta_2 \end{matrix} \middle| z_1\right). \tag{2.14}$$

The functions F_2 and \tilde{F}_2 satisfy the same system of linear partial differential equations, but different boundary conditions. This is in agreement with [31, Thm. 2.1] where it was shown that the two restrictions

$${}_3F_2\left(\begin{matrix} \alpha, \beta_1, 1 + \alpha - \gamma_2 \\ \gamma_1, 1 + \alpha - \gamma_2 + \beta_2 \end{matrix} \middle| z_1\right) \quad \text{and} \quad F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, 1\right) \tag{2.15}$$

satisfy the same ordinary differential equation.

REMARK 2. Equation (2.9) is of particular geometric importance since it makes apparent that two linear transformation formulas for Appell’s function F_2 are induced by the maps $(A, B) \mapsto (B, A)$ and $(A, B) \mapsto (1 - A, 1 - B)$ on the variables. In fact, the variables (A, B) will later be identified with geometric moduli. Concretely, the right hand side of Equation (2.9) is invariant under interchanging A and B if we map $(\beta_2, \gamma_2) \mapsto (\gamma_2 - \beta_2, \gamma_2)$ as well. This proves the well-known identity

$$\frac{1}{A^\alpha} F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| \frac{1}{A}, 1 - \frac{B}{A}\right) = \frac{1}{B^\alpha} F_2\left(\begin{matrix} \alpha; \beta_1, \gamma_2 - \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| \frac{1}{B}, 1 - \frac{A}{B}\right). \tag{2.16}$$

⁴As we will explain later, this corresponds to a different choice of A -cycle in the fiber of a variety fibered over \mathbb{P}^2 whose Picard-Fuchs equation is given by Appell’s hypergeometric system.

Similarly, using the classical linear transformation for the hypergeometric function

$${}_2F_1\left(\begin{matrix} \alpha, \beta_1 \\ \gamma_1 \end{matrix} \middle| \frac{1}{U}\right) = \left(\frac{U}{U-1}\right)^\alpha {}_2F_1\left(\begin{matrix} \alpha, \gamma_1 - \beta_1 \\ \gamma_1 \end{matrix} \middle| \frac{1}{1-U}\right), \tag{2.17}$$

we can check that right hand side of Equation (2.9) is also invariant under mapping $U \mapsto 1-U, (A, B) \mapsto (1-A, 1-B)$ if we map $(\beta_1, \gamma_1) \mapsto (\gamma_1 - \beta_1, \gamma_1)$ as well, proving the well-known identity

$$\begin{aligned} & \frac{1}{A^\alpha} F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| \frac{1}{A}, 1 - \frac{B}{A}\right) \\ &= \frac{1}{(A-1)^\alpha} F_2\left(\begin{matrix} \alpha; \gamma_1 - \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| \frac{1}{1-A}, 1 - \frac{1-B}{1-A}\right). \end{aligned} \tag{2.18}$$

2.1. Quadratic relation between solutions. In [30] Sasaki and Yoshida studied several systems of linear differential equations in two variables holonomic of rank 4. Using a differential geometric technique they determined in terms of the coefficients of the differential equations the *quadric property* condition that the four linearly independent solutions are quadratically related. For Appell’s hypergeometric system the quadric condition is as follows:

PROPOSITION 2.3 (Sasaki, Yoshida). *Appell’s hypergeometric system satisfies the quadric property if and only if*

$$\alpha = \beta_1 + \beta_2 - \frac{1}{2}, \gamma_1 = 2\beta_1, \gamma_2 = 2\beta_2. \tag{2.19}$$

REMARK 3. For Appell’s hypergeometric system satisfying the quadric property the transformations $(A, B) \mapsto (B, A)$ and $(A, B) \mapsto (1-A, 1-B)$ are isometries. This is obvious from Equations (2.16) and (2.18).

REMARK 4. The condition in Eq. (2.19) – implying that the four linearly independent solutions of Appell’s hypergeometric system F_2 are quadratically related – was used in [30, Sec. 5.4] to regain Bailey’s formula [3, Eq. (3.1)] relating the Appell series F_2 and F_4 from a geometric point of view as rational map from \mathbb{P}^2 to \mathbb{P}^2 .

In the following, we will be exclusively concerned with Appell’s hypergeometric system satisfying this quadric property. Geometrically, the quadric condition will correspond to one of the Hodge-Riemann relations for a polarized Hodge structure. The system (2.5) of linear differential equations satisfied by the Appell hypergeometric function

$$F_2\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}; \beta_1, \beta_2 \\ 2\beta_1, 2\beta_2 \end{matrix} \middle| z_1, z_2\right)$$

can be written as Pfaffian system for the vector-valued function

$$\vec{F} = \langle F, \theta_{z_1} F, \theta_{z_2} F, \theta_{z_1} \theta_{z_2} F \rangle^t$$

with $\theta_{z_i} = z_i \partial_{z_i}$. The Pfaffian system associated with (2.5) is the rank-four system

$$d\vec{F} = \Omega^{(F_2)} \cdot \vec{F} \tag{2.20}$$

where the connection matrix $\Omega^{(F_2)}$ is given in Equation (2.35). Next, we introduce two copies of the rank-two Pfaffian system associated with the Gauss' hypergeometric function, i.e., for

$$\vec{f}_{\Lambda_i} = \langle f(\Lambda_i), \theta_{\Lambda_i} f(\Lambda_i) \rangle^t$$

where

$$f(\Lambda_i) = {}_2F_1 \left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_i^2 \right)$$

for $i = 1, 2$. The two systems have the form

$$d\vec{f}_{\Lambda_i} = \Omega_{\Lambda_i}^{(2F_1)} \cdot \vec{f}_{\Lambda_i} \tag{2.21}$$

with connection matrices given in Equation (2.36). The outer tensor product of the two rank-two Pfaffian systems is constructed by introducing $\vec{H} = \vec{f}_{\Lambda_1} \boxtimes \vec{f}_{\Lambda_2}$, i.e.,

$$\vec{H} = \langle f(\Lambda_1) f(\Lambda_2), \theta_{\Lambda_1} f(\Lambda_1) f(\Lambda_2), f(\Lambda_1) \theta_{\Lambda_2} f(\Lambda_2), \theta_{\Lambda_1} f(\Lambda_1) \theta_{\Lambda_2} f(\Lambda_2) \rangle^t .$$

The associated Pfaffian system is the rank-four system

$$d\vec{H} = \Omega^{(2F_1 \otimes 2F_1)} \cdot \vec{H} \tag{2.22}$$

with the connection form

$$\Omega^{(2F_1 \otimes 2F_1)} = \Omega_{\Lambda_1}^{(2F_1)} \boxtimes \mathbb{I} + \mathbb{I} \boxtimes \Omega_{\Lambda_2}^{(2F_1)} . \tag{2.23}$$

The connection matrix is given in Equation (2.37). Conversely, the quadric property implies that the Pfaffian system for Appell's hypergeometric system can be decomposed as outer tensor product of two rank-two Fuchsian systems. In particular, we have the following proposition:

PROPOSITION 2.4. *The connection form of Appell's hypergeometric system satisfying the quadric condition decomposes as*

$$\Omega_{\Lambda_1}^{(2F_1)} \boxtimes \mathbb{I} + \mathbb{I} \boxtimes \Omega_{\Lambda_2}^{(2F_1)} = g^{-1} \cdot \Omega^{(F_2)} \Big|_{(z_1, z_2)=T(\Lambda_1, \Lambda_2)} \cdot g + g^{-1} \cdot dg \tag{2.24}$$

where the gauge transformation g is given in Equation (2.38) and the transformation T is given by

$$\begin{aligned} z_1 &= 4 \frac{\Lambda_1 \Lambda_2}{(\Lambda_1 + \Lambda_2)^2} , \\ z_2 &= - \frac{(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)}{(\Lambda_1 + \Lambda_2)^2} . \end{aligned} \tag{2.25}$$

Proof. By direct computation using the given matrices. \square

The theorem proves that two Pfaffian systems are equivalent. However, even more is true: one can explicitly relate certain holomorphic solutions of the two Pfaffian systems directly. To such extent, Vidunas derived an explicit formula in [31, Eq. (35)]. Here, we present his formula using the variables A, B of Corollary 2.2 and correct a

typographic error. As mentioned in the introduction, Equation (2.26) is Bailey’s famous reduction formula for the Appell series F_4 [4, Eq. (1.1)] when combined with another result of Bailey’s relating the Appell series F_2 and F_4 [3, Eq. (3.1)].

THEOREM 2.5 (Multivariate Clausen Identity). *For $\text{Re}(\beta_1), \text{Re}(\beta_2) > 0$, $|z_1| + |z_2| < 1$, $|\Lambda_1^2| < 1$, and $|1 - \Lambda_2^2| < 1$, Appell’s hypergeometric series factors into two hypergeometric functions according to*

$$\begin{aligned}
 &F_2\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}; \beta_1, \beta_2 \\ 2\beta_1, 2\beta_2 \end{matrix} \middle| z_1, z_2\right) \\
 &= (\Lambda_1 + \Lambda_2)^{2\beta_1+2\beta_2-1} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2\right) \\
 &\quad \times {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \Lambda_2^2\right)
 \end{aligned} \tag{2.26}$$

with

$$(z_1, z_2) = \left(\frac{4 \Lambda_1 \Lambda_2}{(\Lambda_1 + \Lambda_2)^2}, -\frac{(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)}{(\Lambda_1 + \Lambda_2)^2} \right).$$

Proof. The identity (2.26) is equivalent to

$$\begin{aligned}
 &\frac{1}{A^{\beta_1+\beta_2-\frac{1}{2}}} F_2\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}; \beta_1, \beta_2 \\ 2\beta_1, 2\beta_2 \end{matrix} \middle| \frac{1}{A}, 1 - \frac{B}{A}\right) \\
 &= 2^{2\beta_1+2\beta_2-1} (\Lambda_1 \Lambda_2)^{\beta_1+\beta_2-\frac{1}{2}} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2\right) \\
 &\quad \times {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \Lambda_2^2\right),
 \end{aligned} \tag{2.27}$$

where

$$(z_1, z_2) = \left(\frac{1}{A}, 1 - \frac{B}{A} \right), \quad (A, B) = \left(\frac{1}{z_1}, \frac{1 - z_2}{z_1} \right)$$

and

$$(z_1, z_2) = \left(\frac{4 \Lambda_1 \Lambda_2}{(\Lambda_1 + \Lambda_2)^2}, -\frac{(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)}{(\Lambda_1 + \Lambda_2)^2} \right), \quad (A, B) = \left(\frac{(\Lambda_1 + \Lambda_2)^2}{4\Lambda_1 \Lambda_2}, \frac{(\Lambda_1 \Lambda_2 + 1)^2}{4\Lambda_1 \Lambda_2} \right).$$

We already know that both sides of Equation (2.27) satisfy the same system of linear differential equations of rank four. In fact, the functions

$${}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2\right) \quad \text{and} \quad {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \Lambda_2^2\right)$$

satisfy the same ordinary differential equation but have a different behavior at the ramification points. Expanding both sides of Equation (2.27) in terms of $\epsilon_1 = \Lambda_1$ and $\epsilon_2 = 1 - \Lambda_2$ and showing agreement of the first terms proves equality. \square

REMARK 5. Note that the right hand side of Equation (2.26) is up to a renormalization factor a special solution to the (outer) tensor product of two copies of solutions to the same ordinary differential equation as the hypergeometric function

$$f(\Lambda_i) = {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_i^2\right).$$

The aforementioned factor is precisely the entry g_{11} of the gauge transformation g from Proposition 2.4. In fact, the entry g_{11} together with the coordinate transformation (2.25) completely determines the entire matrix g . This shows that Equation (2.26) implies Equation (2.24).

REMARK 6. Different branches for the renormalization factor on the right hand side of Equation (2.26) are connected by *duality transformations* on the moduli Λ_1 and Λ_2 to the particular branch chosen in Theorem 2.5. If we write

$$\begin{aligned} & F_2\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}; \beta_1, \beta_2 \\ 2\beta_1, 2\beta_2 \end{matrix} \middle| z_1, z_2\right) \\ &= h^{2\beta_1+2\beta_2-1} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2\right) {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \Lambda_2^2\right), \end{aligned} \tag{2.28}$$

instead of Equation (2.26), then the dependence of (A, B) , (z_1, z_2) on Λ_1 and Λ_2 and the matching branches for $g_{11} = h^{2\beta_1+2\beta_2-1}$ are given in Table 1.

duality transformation	A	B	z_1	z_2	h
$(\Lambda_1, \Lambda_2) \mapsto (\pm\Lambda_1^{\pm 1}, \pm\Lambda_2^{\pm 1})$	$\frac{(\Lambda_1+\Lambda_2)^2}{4\Lambda_1\Lambda_2}$	$\frac{(\Lambda_1\Lambda_2+1)^2}{4\Lambda_1\Lambda_2}$	$\frac{4\Lambda_1\Lambda_2}{(\Lambda_1+\Lambda_2)^2}$	$-\frac{(\Lambda_1^2-1)(\Lambda_2^2-1)}{(\Lambda_1+\Lambda_2)^2}$	$\Lambda_1 + \Lambda_2$
$(\Lambda_1, \Lambda_2) \mapsto (\mp\Lambda_1, \pm\Lambda_2)$ $(A, B) \mapsto (1 - A, 1 - B)$	$-\frac{(\Lambda_1-\Lambda_2)^2}{4\Lambda_1\Lambda_2}$	$-\frac{(\Lambda_1\Lambda_2-1)^2}{4\Lambda_1\Lambda_2}$	$-\frac{4\Lambda_1\Lambda_2}{(\Lambda_1-\Lambda_2)^2}$	$-\frac{(\Lambda_1^2-1)(\Lambda_2^2-1)}{(\Lambda_1-\Lambda_2)^2}$	$-\Lambda_1 + \Lambda_2$
$(\Lambda_1, \Lambda_2) \mapsto (\Lambda_1^{\mp 1}, \Lambda_2^{\pm 1})$ $(A, B) \mapsto (B, A)$	$\frac{(\Lambda_1\Lambda_2+1)^2}{4\Lambda_1\Lambda_2}$	$\frac{(\Lambda_1+\Lambda_2)^2}{4\Lambda_1\Lambda_2}$	$\frac{4\Lambda_1\Lambda_2}{(\Lambda_1\Lambda_2+1)^2}$	$\frac{(\Lambda_1^2-1)(\Lambda_2^2-1)}{(\Lambda_1\Lambda_2+1)^2}$	$\Lambda_1\Lambda_2 + 1$
$(\Lambda_1, \Lambda_2) \mapsto (\mp\Lambda_1^{\mp 1}, \pm\Lambda_2^{\pm 1})$ $(A, B) \mapsto (1 - B, 1 - A)$	$-\frac{(\Lambda_1\Lambda_2-1)^2}{4\Lambda_1\Lambda_2}$	$-\frac{(\Lambda_1-\Lambda_2)^2}{4\Lambda_1\Lambda_2}$	$-\frac{4\Lambda_1\Lambda_2}{(\Lambda_1\Lambda_2-1)^2}$	$\frac{(\Lambda_1^2-1)(\Lambda_2^2-1)}{(\Lambda_1\Lambda_2-1)^2}$	$-\Lambda_1\Lambda_2 + 1$

Table 1: Relation between variables and moduli

REMARK 7. The left hand side of Equation (2.26) has an obvious symmetry given by interchanging $(z_1, z_2) \mapsto (z_2, z_1)$ and simultaneously swapping $(\beta_1, \beta_2) \mapsto (\beta_2, \beta_1)$. From the integral representation in Equation (2.7) it is clear that this is equivalent to interchanging the roles of the variables x and u . In terms of the coordinates (A, B) used in Equation 2.27 and (Λ_1, Λ_2) , respectively, interchanging variables amounts to symmetry transformations

$$(A, B) \mapsto \left(\frac{A}{A-B}, \frac{A-1}{A-B}\right), \quad (\Lambda_1, \Lambda_2) \mapsto \left(-\frac{1-\Lambda_2}{1+\Lambda_2}, \frac{1+\Lambda_1}{1-\Lambda_1}\right). \tag{2.29}$$

Invariance of the right hand side of Equation (2.26) under transformation (2.29) using

$$\begin{aligned} & \left(-\frac{1-\Lambda_2}{1+\Lambda_2} + \frac{1+\Lambda_1}{1-\Lambda_1}\right)^{2\beta_1+2\beta_2-1} \left(\frac{1+\Lambda_2}{2}\right)^{2\beta_1+2\beta_2-1} (1-\Lambda_1)^{2\beta_1+2\beta_2-1} \\ &= \left(\Lambda_1 + \Lambda_2\right)^{2\beta_1+2\beta_2-1} \end{aligned}$$

recovers the following two quadratic identities for the Gauss' hypergeometric function that are due to Kummer [19, Eq. (41) and (51)]:

$$\begin{aligned}
 & {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \left(\frac{1 - \Lambda_2}{1 + \Lambda_2}\right)^2\right) \\
 &= \left(\frac{1 + \Lambda_2}{2}\right)^{2\beta_1 + 2\beta_2 - 1} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_1 \\ 2\beta_1 \end{matrix} \middle| 1 - \Lambda_2^2\right), \\
 & {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \left(\frac{1 + \Lambda_1}{1 - \Lambda_1}\right)^2\right) \\
 &= (1 - \Lambda_1)^{2\beta_1 + 2\beta_2 - 1} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_1 \\ \beta_2 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2\right).
 \end{aligned} \tag{2.30}$$

REMARK 8. The generalized univariate hypergeometric function ${}_3F_2$ satisfies the so-called *Clausen Identity*

$$\begin{aligned}
 & {}_3F_2\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_1, \beta_1 - \beta_2 + \frac{1}{2} \\ 2\beta_1, \beta_1 + \frac{1}{2} \end{matrix} \middle| z_1\right) \\
 &= {}_2F_1\left(\begin{matrix} \frac{\beta_1}{2} + \frac{\beta_2}{2} - \frac{1}{4}, \frac{\beta_1}{2} - \frac{\beta_2}{2} + \frac{1}{4} \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| z_1\right)^2
 \end{aligned} \tag{2.31}$$

if and only if its parameters satisfy the quadric property in Equation (2.19). Using the quadratic transformation formula [1, Eq. (15.3.23)], the Clausen identity (2.31) simplifies to

$$\begin{aligned}
 & {}_3F_2\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_1, \beta_1 - \beta_2 + \frac{1}{2} \\ 2\beta_1, \beta_1 + \frac{1}{2} \end{matrix} \middle| z_1\right) \\
 &= (1 - \Lambda_1^2)^{2\beta_1 + 2\beta_2 - 1} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2\right)^2
 \end{aligned} \tag{2.32}$$

with $z_1 = -4\Lambda_1^2/(1 - \Lambda_1^2)^2$. Note that the same relation between z_1 and Λ_1 as well as the renormalization factor can be obtained by setting $\Lambda_2 = \Lambda_1$ and $z_2 = 1$ in Table 1 (fourth line), i.e.,

$$\begin{aligned}
 & F_2\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}; \beta_1, \beta_2 \\ 2\beta_1, 2\beta_2 \end{matrix} \middle| z_1, 1\right) \\
 &= (1 - \Lambda_1^2)^{2\beta_1 + 2\beta_2 - 1} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2\right) \\
 & \quad \times {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \Lambda_1^2\right).
 \end{aligned} \tag{2.33}$$

Therefore, using Equation (2.14) we see that the two restrictions

$$\tilde{F}_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, 1\right) \quad \text{and} \quad F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{matrix} \middle| z_1, 1\right) \tag{2.34}$$

satisfy the analogous factorization formulas but different boundary conditions. By analogy, we call the (unrestricted) Equation (2.26) the *Multivariate Clausen Identity*.

2.2. Connection matrices. The connection matrix for the Pfaffian system in Equation (2.20) is given by

$$\Omega^{(F_2)} = \begin{pmatrix} 0 & \frac{dz_1}{z_1} & \frac{dz_2}{z_2} & 0 \\ -\frac{\alpha \beta_1 dz_1}{z_1 - 1} & \frac{z_1}{z_1(z_1 - 1)} - \frac{(\alpha + \beta_1) z_1 - 2\beta_1 + 1}{z_1(z_1 - 1)} dz_1 & -\frac{\beta_1 dz_1}{\beta_1 dz_1} & -\frac{dz_1}{z_1 - 1} + \frac{dz_2}{z_1} \\ -\frac{\alpha \beta_2 dz_2}{z_2 - 1} & -\frac{\beta_2 dz_2}{\beta_2 dz_2} & -\frac{z_2 - 1}{((\alpha + \beta_2) z_2 - 2\beta_2 + 1) dz_2} & -\frac{dz_2}{z_2 - 1} + \frac{dz_1}{z_1} \\ \frac{\alpha \beta_1 \beta_2 z_2 dz_1}{(z_1 - 1)(z_1 + z_2 - 1)} + \frac{\beta_2(\alpha - \beta_1 + 1) z_2 dz_1}{(z_1 - 1)(z_1 + z_2 - 1)} & \frac{\beta_2((\alpha - 2\beta_1 + 1) z_2 - \beta_1 z_1 - \alpha + 2\beta_2 - 1) dz_2}{(z_2 - 1)(z_1 + z_2 - 1)} & -\frac{\beta_1(\alpha - \beta_2 + 1) z_1 dz_2}{(z_2 - 1)(z_1 + z_2 - 1)} - \frac{\beta_2 z_2(z_2 - 1)}{(z_1 - 1)(z_1 + z_2 - 1)} & \Omega_{4,4}^{(F_2)} \end{pmatrix} \tag{2.35}$$

where $\alpha = \beta_1 + \beta_2 - \frac{1}{2}$ and

$$\Omega_{4,4}^{(F_2)} = -\frac{((\alpha + \beta_1 - 2\beta_2 + 1) z_1^2 + ((2\beta_1 - \beta_2 - 1) z_2 - \alpha - 3\beta_1 + 2\beta_2) z_1 + (-2\beta_1 + 1) z_2 + 2\beta_1 - 1) dz_1}{z_1(z_1 - 1)(z_1 + z_2 - 1)} - \frac{((\alpha + \beta_2 - 2\beta_1 + 1) z_2^2 + ((2\beta_2 - \beta_1 - 1) z_1 - \alpha - 3\beta_2 + 2\beta_1) z_2 + (-2\beta_2 + 1) z_1 + 2\beta_2 - 1) dz_2}{z_2(z_2 - 1)(z_1 + z_2 - 1)}.$$

The connection matrix for the Pfaffian system in Equation (2.21) is given by

$$\Omega_{\Lambda_i}^{(2F_1)} = \begin{pmatrix} 0 & \frac{d\Lambda_i}{\Lambda_i} \\ -\frac{2(2\beta_1 + 2\beta_2 - 1)\beta_2 \Lambda_i d\Lambda_i}{\Lambda_i^2 - 1} & -\frac{((2\beta_1 + 4\beta_2 - 1)\Lambda_i^2 - 2\beta_1 + 1) d\Lambda_i}{\Lambda_i(\Lambda_i^2 - 1)} \end{pmatrix}. \tag{2.36}$$

The connection matrix for the outer tensor product of two copies of the above Pfaffian system is given by

$$\Omega^{(2F_1 \otimes 2F_1)} = \begin{pmatrix} 0 & \frac{d\Lambda_1}{\Lambda_1} & \frac{d\Lambda_2}{\Lambda_2} & 0 \\ -\frac{2(2\beta_1 + 2\beta_2 - 1)\beta_2 \Lambda_1 d\Lambda_1}{\Lambda_1^2 - 1} & -\frac{((2\beta_1 + 4\beta_2 - 1)\Lambda_1^2 - 2\beta_1 + 1) d\Lambda_1}{\Lambda_1(\Lambda_1^2 - 1)} & 0 & \frac{d\Lambda_2}{\Lambda_2} \\ -\frac{2(2\beta_1 + 2\beta_2 - 1)\beta_2 \Lambda_2 d\Lambda_2}{\Lambda_2^2 - 1} & 0 & -\frac{((2\beta_1 + 4\beta_2 - 1)\Lambda_2^2 - 2\beta_1 + 1) d\Lambda_2}{\Lambda_2(\Lambda_2^2 - 1)} & \frac{d\Lambda_1}{\Lambda_1} \\ 0 & -\frac{2(2\beta_1 + 2\beta_2 - 1)\beta_2 \Lambda_2 d\Lambda_2}{\Lambda_2^2 - 1} & \frac{2(2\beta_1 + 2\beta_2 - 1)\beta_2 \Lambda_1 d\Lambda_1}{\Lambda_1^2 - 1} & -\frac{((2\beta_1 + 4\beta_2 - 1)\Lambda_1^2 - 2\beta_1 + 1) d\Lambda_1}{\Lambda_1(\Lambda_1^2 - 1)} - \frac{((2\beta_1 + 4\beta_2 - 1)\Lambda_2^2 - 2\beta_1 + 1) d\Lambda_2}{\Lambda_2(\Lambda_2^2 - 1)} \end{pmatrix} \tag{2.37}$$

The gauge transformation relating connection matrices (2.35) and (2.37) is given by

$$g = \begin{pmatrix} \frac{(\Lambda_1 + \Lambda_2)^{2\alpha}}{-2\alpha(\Lambda_1 + \Lambda_2)^{2\alpha}\Lambda_1\Lambda_2} & 0 & 0 & 0 \\ \frac{\Lambda_1\Lambda_2 - 1}{\alpha(\Lambda_1 + \Lambda_2)^{2\alpha}(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)} & -\frac{\Lambda_2(\Lambda_1^2 - 1)(\Lambda_1 + \Lambda_2)^{2\alpha}}{(\Lambda_1\Lambda_2 - 1)(\Lambda_1 - \Lambda_2)} & \frac{\Lambda_1(\Lambda_2^2 - 1)(\Lambda_1 + \Lambda_2)^{2\alpha}}{(\Lambda_1\Lambda_2 - 1)(\Lambda_1 - \Lambda_2)} & 0 \\ -\frac{\Lambda_1^2\Lambda_2 - 1}{\alpha\Lambda_1\Lambda_2(\Lambda_1^2 - 1)(\Lambda_1 + \Lambda_2)^{2\alpha}((2\beta_1 - 1)\Lambda_1\Lambda_2 - 2\beta_1 - 1)} & \frac{\Lambda_1^2\Lambda_2^2 - 1}{2(\Lambda_1^2\Lambda_2^2 - 1)} & \frac{(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)(\Lambda_1 + \Lambda_2)^{2\alpha}}{2(\Lambda_1^2\Lambda_2^2 - 1)} & -\frac{(\Lambda_1 + \Lambda_2)^{2\alpha}(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)}{2(\Lambda_1\Lambda_2 - 1)^2} \end{pmatrix} \quad (2.38)$$

with $\alpha = \beta_1 + \beta_2 - \frac{1}{2}$ and

$$g_{42} = \frac{\Lambda_2(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)(\Lambda_1 + \Lambda_2)^{2\alpha}(2\alpha\Lambda_1^2\Lambda_2^2 - ((2\beta_1 - 1)\Lambda_1^2 + 2\beta_1 + 1)\Lambda_1\Lambda_2 + (2\beta_1 + 1)\Lambda_1^2 - 2\beta_2)}{2(\Lambda_1\Lambda_2 - 1)^3(\Lambda_1\Lambda_2 + 1)(\Lambda_1 - \Lambda_2)},$$

$$g_{43} = -\frac{\Lambda_1(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)(\Lambda_1 + \Lambda_2)^{2\alpha}(2\alpha\Lambda_1^2\Lambda_2^2 - ((2\beta_1 - 1)\Lambda_2^2 + 2\beta_1 + 1)\Lambda_1\Lambda_2 + (2\beta_1 + 1)\Lambda_2^2 - 2\beta_2)}{2(\Lambda_1\Lambda_2 - 1)^3(\Lambda_1\Lambda_2 + 1)(\Lambda_1 - \Lambda_2)}.$$

3. Superelliptic curves and their periods. In this section we discuss some basic facts about superelliptic curves and their periods. Superelliptic curves and Jacobians of curves with superelliptic components have been studied extensively from an arithmetic point of view (cf. [15, 6, 7]). However, our main focus will be on the computation of periods of the first kind for such superelliptic curves.

3.1. A generalization of the Legendre normal form. Every elliptic curve, i.e., smooth projective algebraic curve of genus one, can be written in Legendre normal form as

$$y^2 = x(x-1)(x-\lambda) \tag{3.1}$$

with $\lambda \notin \{0, 1, \infty\}$.

DEFINITION 3.1. For four positive integers $r, s, p, q \in \mathbb{N}$ with $2s-p > 0$, $p+q-s > 0$, $q-r+s > 0$ let $SE(\lambda)_{s,p,q}^{2r}$ be the plane algebraic curve given in projective coordinates $[X : Y : Z] \in \mathbb{P}^2$ by

$$SE(\lambda)_{s,p,q}^{2r} : Y^{2r} Z^{3s-2r-p+q} = X^{p+q-s} (X-Z)^{2s-p} (X-\lambda Z)^{2s-p}. \tag{3.2}$$

REMARK 9. Unless necessary we will not distinguish between the plane algebraic curve in Equation (3.2) and the affine curve given by

$$SE(\lambda)_{s,p,q}^{2r} : y^{2r} = x^{p+q-s} (x-1)^{2s-p} (x-\lambda)^{2s-p}. \tag{3.3}$$

To generalize the Legendre normal form in Equation (3.1) we choose three positive integers $r, p, q \in \mathbb{N}$ in the range

$$0 < p, q < 2r, \quad -r < p - q < r, \quad r < p + q < 3r, \tag{3.4}$$

such that the following divisibility constraints hold

$$(p, 2r) = 1, \quad (p + q - r, 2r) = 1, \quad (p - q + r, 2r) = 1. \tag{3.5}$$

We will now construct a smooth irreducible curve C over $\mathbb{C}(\lambda)$ associated with the curve $SE(\lambda)_{r,p,q}^{2r}$.

REMARK 10. For $r = 1$ it follows $p = q = 1$ and Equation (3.3) is the classical Legendre normal form of an elliptic curve.

REMARK 11. The inequalities and divisibility constraints simplify considerably in two special cases: in the case $q = r$, all inequalities and divisibility constraints simplify to $0 < p < 2r$ with $(p, 2r) = 1$. Similarly, in the case $q = 3r - 2p$, all inequalities and divisibility constraints simplify to $2r < 3p < 4r$ with $(3p, 2r) = 1$.

LEMMA 3.2. The minimal resolution of the curve $SE(\lambda)_{r,p,q}^{2r}$ is a smooth irreducible curve of genus $2r - 1$ if the integers (r, p, q) satisfy the inequalities (3.4), the divisibility constraints (3.5), and $\lambda \in \mathbb{C} \setminus \{0, 1, \infty\}$.

Proof. The reduced discriminant of Equation (3.3) vanishes for $\lambda \in \{0, 1, \infty\}$. We consider the projective curve in \mathbb{P}^2 given by

$$SE(\lambda)_{r,p,q}^{2r} : Y^{2r} Z^{r-p+q} = X^{p+q-r} (X-Z)^{2r-p} (X-\lambda Z)^{2r-p}. \tag{3.6}$$

Given $\lambda \in \mathbb{C} \setminus \{0, 1, \infty\}$, a straightforward calculation shows that (3.6) is smooth if $r = 1$ which in turn implies $p = q = 1$, i.e., the curve is the classical Legendre form of a smooth elliptic curve. In this case, the curve (3.6) is the desired smooth projective model.

Let us now assume that $r > 1$. The curve (3.6) is singular and has at most the four isolated singular points

$$[X : Y : Z] \in \left\{ [0 : 0 : 1], [1 : 0 : 1], [\lambda : 0 : 1], [0 : 1 : 0] \right\}, \tag{3.7}$$

and is smooth everywhere else. The homogeneous polynomial

$$F(X, Y, Z) = -Y^{2r} Z^{r-p+q} + X^{p+q-r} (X - Z)^{2r-p} (X - \lambda Z)^{2r-p} \tag{3.8}$$

is irreducible. We first observe that the only point at infinity is $[X : Y : Z] = [0 : 1 : 0]$, and it is a singular point if $r - p + q > 1$. Setting $Y = 1$ in Equation (3.8) we can read off the multiplicity $r - p + q$ of $SE(\lambda)_{r,p,q}^{2r}$ at $[0 : 1 : 0]$. A generic line through the point $[0 : 1 : 0]$ is of the form $X/\xi = Z/\kappa$ and $Y = 1$. Thus, any point $[\rho\xi : 1 : \rho\kappa]$ on this line also lies on the curve if ρ is a root of the equation

$$0 = -\rho^{r-p+q} \kappa^{r-p+q} + \rho^{3r-p+q} \xi^{p+q-r} (\xi - \kappa)^{2r-p} (\xi - \lambda \kappa)^{2r-p}.$$

For the equation to have additional roots, κ must vanish and the line $[\rho\xi : 1 : 0]$ then has $3r - p + q$ of its intersections coincident with the curve in $[0 : 1 : 0]$, i.e., it is a nodal tangent. Moreover, because of the coefficient being κ^{r-p+q} , all of the nodal tangents coincide. Any irreducible plane curve can be transformed, by a finite succession of standard quadratic transformations, into a curve whose multiple points are all ordinary, i.e., all nodal tangents are distinct at these points. Blowing-up then resolves the singularity into a set of simple points on the transform corresponding to directions of the nodal tangents. Because of the relation

$$(3r - p + q, r - p + q) = (2r, r - p + q) = (2r, -r - p + q) = (2r, p - q + r) = 1$$

there is only one point above infinity on the blow-up. In other words, the point over infinity is totally ramified. We then proceed in the same way for the remaining singular points. The points in Equation (3.7) are found to have multiplicities

$$p + q - r, 2r - p, 2r - p, r - p + q. \tag{3.9}$$

Requiring total ramification over each singular point leads to the divisibility constraints (3.5). We thus obtain a smooth irreducible model. In particular, the degree $2r$ map $\phi : SE(\lambda)_{r,p,q}^{2r} \rightarrow \mathbb{P}^1$ given by $\phi : [X : Y : Z] \mapsto [X : Z]$ is totally ramified at the singular points. The Riemann-Hurwitz formula implies

$$2(1 - g) = 2(2r)(1 - 0) - 4(2r - 1). \tag{3.10}$$

Hence, we find $g = 2r - 1$. \square

To understand the resolution process from Lemma 3.2 from the point of local normalization, we compute the Puiseux expansions for the defining polynomial f in Equation (3.8) in terms of a local parameter $z_0, z_1, z_\lambda, z_\infty \in \mathbb{C}$ around each singular point given in Equation (3.7). The leading terms that determine the analytic equivalence class of each singularity are listed in Table 2 where $y_0(z^{2r}), y_1(z^{2r}), y_\lambda(z^{2r}), x_\infty(z^{2r})$ are – by construction – convergent power series in z^{2r} .

$[X : Y : Z]$	$[0 : 0 : 1]$	$[1 : 0 : 1]$	$[\lambda : 0 : 1]$	$[0 : 1 : 0]$
multiplicity	$p + q - r$	$2r - p$	$2r - p$	$r - p + q$
Puiseux coefficients	$(2r, p + q - r)$	$(2r, 2r - p)$	$(2r, 2r - p)$	$(3r - p + q, r - p + q)$
Puiseux expansion	$X = z_0^{2r}$ $Y = \lambda^{\frac{p+q}{2r}} z_0^{p+q-r} (1 + y_0(z_0^{2r}))$ $Z = 1$	$X = 1 + z_1^{2r}$ $Y = (1 - \lambda)^{\frac{2r-p}{2r}} z_1^{2r-p} (1 + y_1(z_1^{2r}))$ $Z = 1$	$X = \lambda + z_\lambda^{2r}$ $Y = (\lambda(\lambda - 1))^{\frac{2r-p}{2r}} z_\lambda^{2r-p} (1 + y_\lambda(z_\lambda^{2r}))$ $Z = 1$	$X = z_\infty^{r-p+q} (1 + x_\infty(z_\infty^{2r}))$ $Y = 1$ $Z = z_\infty^{r-p+q}$

Table 2: *Puiseux expansions about singular points*

We observe that each singularity is locally analytically equivalent to a neighborhood of the origin for the curve $x^a - y^b = 0$ in \mathbb{C}^2 where a, b are relatively prime integers. By [8, Prop. 1, p. 224] such a singularity is topologically equivalent to a cone over a torus knot of type (a, b) . The resolution graph is obtained using a Theorem of Enriques and Chisini [8, Thm. 12, p. 516] by running an Euclidean division algorithm on the two integers a, b . The corresponding blow-ups are described in [8, Example 2, p. 471] in detail. It is then immediate from Kummer theory that the degree $2r$ map $\phi : SE(\lambda)_{r,p,q}^{2r} \rightarrow \mathbb{P}^1$ given by $\phi : [X : Y : Z] \mapsto [X : Z]$ is totally ramified at the singular points: over the points $[0 : 1]$, $[1 : 0]$, $[1 : 1]$, and $[\lambda : 1]$, the extension of (fraction fields of) local rings is simply adjoining a $2r$ th root of an element whose order of vanishing is relatively prime to $2r$ if the divisibility constraints (3.5) hold.

Next we prove that the involution automorphism on the curve $SE(\lambda)_{r,p,q}^{2r}$ given by

$$\iota : (x, y) \mapsto (x, -y) , \tag{3.11}$$

lifts to its minimal resolution constructed in Lemma 3.2.

LEMMA 3.3. *The involution automorphism (3.11) lifts to the minimal resolution of the curve $SE(\lambda)_{r,p,q}^{2r}$ constructed in Lemma 3.2.*

Proof. We have to prove that we can lift the involution in Equation (3.11) to local coordinates on the normalization of neighborhoods of the singular points. We start by constructing the local normalization of a neighborhood of the singular point $[X : Y : Z] = [0 : 0 : 1]$ or $(x, y) = (0, 0)$. For $\epsilon > 0$ and $\delta > 0$ we consider the neighborhoods

$$U_{\epsilon,\delta} := \left\{ (x, y) \in \mathbb{C}^2 \mid |x| < \delta, |y| < \epsilon, f(x, y, 1) = 0 \right\} ,$$

where the defining polynomial f was given in Equation (3.8) and

$$B_\delta := \left\{ z_0 \in \mathbb{C} \mid |z_0| < \delta^{\frac{1}{2r}} \right\} .$$

By the local normalization theorem [8, Thm. 1], there is and $\epsilon_0 > 0$ such that for each $0 < \epsilon < \epsilon_0$ there is a $\delta > 0$ such that the mapping $\pi_0 : B_\delta \rightarrow \mathbb{C}^2$ with

$$\pi_0(z_0) = (z_0^{2r}, \lambda^{2r-p} z_0^{p+q-r} (1 + y_0(z_0^{2r}))$$

is holomorphic and onto $U_{\epsilon,\delta}$. Moreover, the restriction

$$\pi_0 : B_\delta \setminus \{0\} \rightarrow U_{\epsilon,\delta} \setminus \{0\}$$

is biholomorphic with $\pi_0^{-1}(0) = 0$. Note that the action $z_0 \mapsto -z_0$ on B_δ after projection matches the involution in Equation (3.11) since $p + q - r$ is odd which

follows from the condition $(p + q - r, 2r) = 1$. Similarly, we proceed for the singular points at $(x, y) = (1, 0)$ and $(x, y) = (\lambda, 0)$.

For the singular point $[X : Y : Z] = [0 : 1 : 0]$, we note that the involution on the curve $SE(\lambda)_{r,p,q}^{2r}$ in projective coordinates is given by

$$\iota : [X : Y : Z] \mapsto [-X : Y : -Z]. \tag{3.12}$$

The local normalization theorem then provides a local coordinate z_∞ on the normalization and the biholomorphic mapping

$$\pi_\infty(z_\infty) = (z_\infty^{r-p+q} (1 + x_\infty(z_\infty^{2r})), z_\infty^{3r-p+q}) = \left(\frac{X}{Y}, \frac{Z}{Y} \right).$$

The action $z_\infty \mapsto -z_\infty$ after projection matches the involution automorphism in Equation (3.12) since $r - p + q$ and $3r - p + q$ are odd. \square

Note that Lemma 3.3 does not guarantee that the action of the bigger group $\mathbb{Z}_{2r} : (x, y) \mapsto (x, \rho_{2r}y)$ lifts to the minimal resolution for general integers p, q, r with $\rho_{2r} = \exp\left(\frac{2\pi i}{2r}\right)$ as well. However, in the special case $q = r$ we can give lift the action and give a precise description on the resolution curve.

LEMMA 3.4. *The action $\mathbb{Z}_{2r} : [X : Y : Z] \mapsto [X : \rho_{2r}^p Y : Z]$ lifts to the minimal resolution of the curve $SE(\lambda)_{r,p,r}^{2r}$ constructed in Lemma 3.2.*

Proof. Using $q = r$ and $(p, 2r) = 1$, the local coordinates z_0, z_1, z_λ , and z_∞ from the proof of Lemma 3.3, and the Puiseux expansions in Table 2, we obtain the following actions on the local normalizations:

$$\begin{array}{llll} z_0 \mapsto \rho_{2r} z_0 & z_1, z_\lambda \mapsto \rho_{2r}^{2r-1} z_1, z_\lambda & z_\infty \mapsto \rho_{2r} z_\infty \\ X \mapsto X & X \mapsto X & X \mapsto (\rho_{2r})^{2r-p} X = \rho_{2r}^{-p} X \\ Y \mapsto \rho_{2r}^p Y & Y \mapsto (\rho_{2r}^{2r-1})^{2r-p} Y = \rho_{2r}^p Y & Y \mapsto Y \\ Z \mapsto Z & Z \mapsto Z & Z \mapsto (\rho_{2r})^{4r-p} Z = \rho_{2r}^{-p} Z \end{array}$$

This proves that the actions on the local normalizations patch together to give a lift of the action

$$[X : Y : Z] \mapsto [X : \rho_{2r}^p Y : Z] = [\rho_{2r}^{-p} X : Y : \rho_{2r}^p Z].$$

\square

REMARK 12. The result of Lemma 3.3 can also be understood from the point of Galois theory: the involution is an element of the Galois group of the extension of function fields, and the Galois group automatically has a well defined action on the integral elements of the field, namely the coordinate ring of the smooth model of $SE(\lambda)_{r,p,r}^{2r}$. The same applies to the result of Lemma 3.4.

DEFINITION 3.5. *We call the smooth irreducible curve of genus $2r - 1$ that is the minimal resolution of the algebraic curve $SE(\lambda)_{r,p,q}^{2r}$ a superelliptic curve assuming that the inequalities (3.4), the divisibility constraints (3.5), and $\lambda \in \mathbb{C} \setminus \{0, 1, \infty\}$ are satisfied.*

REMARK 13. All statements derived so far will also be valid for the smooth irreducible over $\mathbb{C}(\lambda)$ associated with the affine curve of degree $3r + p - q$ given by

$$SE(\lambda)_{r,2r-p,2r-q}^{2r} : y^{2r} = x^{3r-p-q} (x - 1)^p (x - \lambda)^p. \tag{3.13}$$

The affine curve (3.13) is obtained from Equation (3.3) by sending $(p, q) \mapsto (2r-p, 2r-q)$ which leaves the inequalities (3.4) and divisibility constraints (3.5) unchanged.

LEMMA 3.6. *The affine curve $SE(\Lambda^2)_{r,2r-p,2r-q}^{2r}$ in Equation (3.13) is equivalent to the affine curve*

$$y^{2r} = x^{3r-p-q} (x^2 + 2(1-2u)x + 1)^p \tag{3.14}$$

with $u = \frac{(1+\Lambda)^2}{4\Lambda}$. Moreover, the affine curves $SE(\lambda_1)_{r,q,r}^{2r}$ and $SE(\Lambda_2^2)_{r,r,2r-q}^{2r}$ are related by a rational transformation with

$$\lambda_1 = \left(\frac{1 + \Lambda_2}{1 - \Lambda_2} \right)^2. \tag{3.15}$$

Proof. The affine curve

$$SE(\Lambda_2^2)_{r,2r-p,2r-q}^{2r} : \eta_2^{2r} = \zeta_2^{3r-p-q} (\zeta_2 - 1)^p (\zeta_2 - \Lambda_2^2)^p \tag{3.16}$$

is related to the affine curve in Equation (3.14) by the birational transformation

$$x = \frac{\zeta_2}{\Lambda_2}, \quad y = \frac{\eta_2}{\Lambda_2^{\frac{3}{2} + \frac{p}{2r} - \frac{q}{2r}}}, \quad u = \frac{(1 + \Lambda_2)^2}{4\Lambda_2}. \tag{3.17}$$

In turn, the affine curve (3.14) is related to the affine curve

$$\eta_1^{2r} = \zeta_1^{r-p+q} (\zeta_1 - 1)^{3r-p-q} (\zeta_1 - \lambda_1)^{3r-p-q} (\zeta_1^2 - \lambda_1)^{2(p-r)} \tag{3.18}$$

by the rational transformation

$$x = \frac{(\zeta_1 - 1)(\zeta_1 - \lambda_1)}{\zeta_1(1 - \lambda_1)}, \quad y = \frac{(\zeta_1^2 - \lambda_1)\eta_1}{\zeta_1^2(1 - \lambda_1)^{\frac{3}{2} + \frac{p}{2r} - \frac{q}{2r}}}, \quad u = \frac{\lambda_1}{\lambda_1 - 1}. \tag{3.19}$$

For $p = r$, the affine curve (3.18) coincides with $SE(\lambda_1)_{r,q,r}^{2r}$. Therefore, a rational transformation between the affine curves $SE(\lambda_1)_{r,q,r}^{2r}$ and $SE(\Lambda_2^2)_{r,r,2r-q}^{2r}$ is given by

$$\begin{aligned} \zeta_2 &= -\frac{(\zeta_1 - 1)(\zeta_1 - \lambda_1)(1 - \Lambda_2)^2}{4\zeta_1}, \\ \eta_2 &= \frac{(-1)^{\frac{3}{2} + \frac{p}{2r} - \frac{q}{2r}} (\zeta_1^2 - \lambda_1)(1 - \Lambda_2)^{3 + \frac{p}{r} - \frac{q}{r}} \eta_1}{2^{3 + \frac{p}{r} - \frac{q}{r}} \zeta_1^2} \end{aligned} \tag{3.20}$$

with $\lambda_1 = \left(\frac{1+\Lambda_2}{1-\Lambda_2}\right)^2$. \square

FACT 3.7. *For $r = p = q = 1$, the pencil of curves in Equation (3.14) is the modular elliptic surface for $\Gamma_0(4)$, and the pencil of curves in Equation (3.13) is the modular elliptic surface for $\Gamma(2)$. Lemma 3.6 then describes the (invertible) action of the translation by an order-two point, i.e., two-isogeny.*

REMARK 14. Given the transformation (3.15) between the parameters, the relation between the affine curves $SE(\lambda_1)_{r,q,r}^{2r}$ and $SE(\Lambda_2^2)_{r,r,2r-q}^{2r}$ is governed by the following $(8r - q, 4r - q)$ -correspondence in (ζ_1, ζ_2) :

$$\begin{aligned} &(-1)^q 4^{-4r+q} \zeta_1^q (\zeta_1 - 1)^{2r-q} (\zeta_1 - \lambda_1)^{2r-q} (\zeta_1^2 - \lambda_1)^{2r} \\ &= \zeta_1^{4r} \zeta_2^{2r-q} (\zeta_2 - 1)^r (\zeta_2 - \Lambda_2^2)^r (\Lambda_2 - 1)^{-8r+2q}. \end{aligned} \tag{3.21}$$

3.2. Periods for superelliptic curves. In this section we will compute all integrals of the first kind for the curve C_1 constructed in Lemma 3.2 explicitly. Recall that in Lemma 3.2 we constructed the resolution $\psi_1 : C_1 \rightarrow SE(\lambda_1)_{r,p,q}^{2r}$ of the plane algebraic curve $SE(\lambda_1)_{r,p,q}^{2r}$ with affine equation $F(x, y) = 0$ where

$$F(x, y) = -y^{2r} + x^{p+q-r} (x - 1)^{2r-p} (x - \lambda)^{2r-p}. \tag{3.22}$$

The affine coordinates x, y are rational functions on $SE(\lambda_1)_{r,p,q}^{2r}$, and hence give rise to rational functions on C_1 when composed with ψ_1 . Likewise, their differentials dx and dy are meromorphic differentials on C_1 . We have the following lemma:

LEMMA 3.8. *The differential form dx/y is a holomorphic differential one-form on C_1 .*

Proof. We will first discuss the situation for all points in Equation (3.7) with $Z \neq 0$. The form dx/y has poles, if at all, at the points of $SE(\lambda_1)_{r,p,q}^{2r}$ with $y = 0$. These are precisely the singular points. Let t be the local coordinate on C_1 in a neighborhood of the point which is mapped onto any of the singular points by the resolution $\psi_1 : C_1 \rightarrow SE(\lambda_1)_{r,p,q}^{2r}$. In terms of the local coordinate t the resolution is described by the two functions $x = x(t)$ and $y = y(t)$ given in Table 2, and the differential form dx/y equals

$$\frac{dx}{y} = \frac{\dot{x}(t) dt}{y(t)}. \tag{3.23}$$

Hence, the one-form has no pole iff the orders of these functions satisfy

$$(O(x \circ \psi_1) - 1) - O(y \circ \psi_1) \geq 0. \tag{3.24}$$

For the point $(x, y) = (0, 0)$, Equation (3.24) is equivalent to $3r > p + q$. For the point $(x, y) = (1, 0)$ and $(x, y) = (\lambda, 0)$, Equation (3.24) is equivalent to $p > 0$. The inequalities are satisfied by assumption (3.4). To discuss the point at infinity, we substitute $x = X/Z$ and $y = Y/Z$. In terms of the local coordinate t the resolution is described $X = X(t)$, $Y = 1$, and $Z = Z(t)$, and we obtain the differential form

$$\frac{dX}{Y} - \frac{X dZ}{Y Z} = \left(\dot{X}(t) - \frac{X(t) \dot{Z}(t)}{Z(t)} \right) dt.$$

Hence, the one-form has no pole iff

$$(O(X \circ \psi_1) - 1) \geq 0 \tag{3.25}$$

or $r > p - q$ which is satisfied by assumption (3.4). \square

REMARK 15. Notice that at each smooth point of the algebraic curve $SE(\lambda_1)_{r,p,q}^{2r}$ with affine equation $F(x, y) = 0$ of degree $d = 3r - p + q$ we can compute its total differential. We obtain

$$F_x dx - 2r y^{2r-1} dy = 0 \Rightarrow \frac{dx}{y} = 2r y^{2r-2} \frac{dy}{F_x}. \tag{3.26}$$

Because of the inequalities (3.4) we have $0 \leq 2r - 2 \leq d - 3$. Equation (3.26) then gives an equivalent form for the holomorphic one-form dx/y that already appeared in the work of Riemann [29].

We have proved that for the purpose of a period computation there will be no danger from using the curve $SE(\lambda)_{r,p,q}^{2r}$ and treating the points in Equation (3.7) as formal symbols. However, for the multi-valued function y in Lemma 3.2 to be a well-defined on $SE(\lambda_1)_{r,p,q}^{2r}$, we have to introduce branch cuts: we choose as branch cuts the line segments $[e_1, e_2]$ and $[e_3, e_4]$ where e_1, \dots, e_4 are the four branch points in Equation (3.7). In the previous section, we also explained that the minimal resolution C_1 of $SE(\lambda)_{r,p,q}^{2r}$ which does not change the rank of the first homology. Therefore, a basis of one-cycles for the curve C_1 of genus $2r - 1$ in Lemma 3.2 can be constructed by using the curve $SE(\lambda_1)_{r,p,q}^{2r}$ directly.

First we consider the case $r = 1$, i.e., the case of an elliptic curve. In this case a basis of one-cycles is given by the so-called A - and B -cycle. The A -cycle \mathbf{a}_1 is a closed clockwise cycle around the line segment $[e_1, e_2]$ (on one of the two y -sheets) not cutting through the second branch cut. The B -cycle \mathbf{b}_1 is the closed cycle that runs from the first to the second branch cut on the chosen y -sheet and returns on the adjacent sheet – in this case the $(-y)$ -sheet – in an orientation that gives the positive intersection number $\mathbf{a}_1 \circ \mathbf{b}_1 = 1$. The situation is depicted in Figure 1.

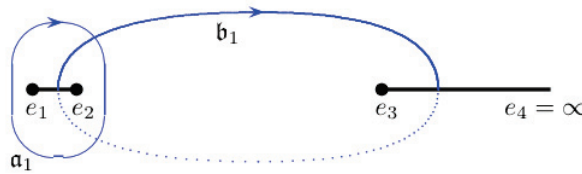


FIG. 1. A - and B -cycle on double-branched cover

For the superelliptic curve $SE(\lambda_1)_{r,p,q}^{2r}$ there are $2r$ choices for the y -sheet in the above construction. Therefore, the construction leads to $2r - 1$ A -cycles \mathbf{a}_i and B -cycles \mathbf{b}_j for $1 \leq i, j \leq 2r - 1$ with intersection numbers

$$\mathbf{a}_i \circ \mathbf{a}_j = 0, \quad \mathbf{b}_i \circ \mathbf{b}_j = 0, \quad \mathbf{a}_i \circ \mathbf{b}_j = \delta_{i,j} - \delta_{i,j+1}.$$

Similar to the elliptic-curve case, there are $2r - 1$ and not $2r$ one-cycles, as the last choice gives a cycle that is already a linear combination of the cycles already constructed. We then have the following lemma computing the periods of the holomorphic one-form dx/y for the resolution C_1 and C_2 of $SE(\lambda)_{r,p,q}^{2r}$ and $SE(\lambda)_{r,2r-p,2r-q}^{2r}$, respectively:

LEMMA 3.9. *Setting $\beta_1 = q/(2r)$ and $\beta_2 = p/(2r)$, we obtain for the periods of dx/y on C_1*

$$\begin{aligned} f_{r,p,q}^{(k)}(\lambda) &= \oint_{\mathbf{a}_k} \frac{dx}{y} \\ &= C_{2r}^{(k)} \frac{\Gamma(\frac{3}{2} - \beta_1 - \beta_2) \Gamma(\beta_2) \lambda^{\frac{1}{2} - \beta_1}}{\Gamma(\frac{3}{2} - \beta_1)} {}_2F_1\left(\begin{matrix} \frac{3}{2} - \beta_1 - \beta_2, 1 - \beta_2 \\ \frac{3}{2} - \beta_1 \end{matrix} \middle| \lambda\right), \end{aligned} \tag{3.27}$$

$$\begin{aligned} f_{r,p,q}^{(k)'}(\lambda) &= \oint_{\mathbf{b}_k} \frac{dx}{y} \\ &= (-1)^{\beta_2} C_{2r}^{(k)} \frac{\Gamma(\beta_2)^2}{\Gamma(2\beta_2) (1 - \lambda)^{1 - 2\beta_2}} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \lambda\right), \end{aligned}$$

and on C_2

$$\begin{aligned}
 f_{r,2r-p,2r-q}^{(k)}(\lambda) &= C_{2r}^{(k)} \frac{\Gamma(\beta_1 + \beta_2 - \frac{1}{2}) \Gamma(1 - \beta_2)}{\Gamma(\beta_1 + \frac{1}{2}) \lambda^{\frac{1}{2} - \beta_1}} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \lambda\right), \\
 f_{r,2r-p,2r-q}^{(k)'}(\lambda) &= (-1)^{1-\beta_2} C_{2r}^{(k)} \frac{\Gamma(1 - \beta_2)^2 (1 - \lambda)^{1-2\beta_2}}{\Gamma(2 - 2\beta_2)} \\
 &\quad \times {}_2F_1\left(\begin{matrix} \frac{3}{2} - \beta_1 - \beta_2, 1 - \beta_2 \\ 2 - 2\beta_2 \end{matrix} \middle| 1 - \lambda\right)
 \end{aligned} \tag{3.28}$$

with $C_{2r}^{(k)} = (\rho_{2r} - 1)/\rho_{2r}^k$, $(-1)^{\beta_2} = \rho_{4r}^p$, $\rho_{2r} = \exp(\frac{2\pi i}{2r})$ for $k = 1, \dots, 2r - 1$.

Proof. First, we note that Equations (3.28) can be obtained from Equations (3.27) by sending $(\beta_1, \beta_2) \mapsto (1 - \beta_1, 1 - \beta_2)$. Therefore, we only have to prove Equations (3.27). We choose $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$, and $e_1 = 0$, $e_2 = \lambda$, $e_3 = 1$, and $e_4 = \infty$. For $x = \delta_1 + i \delta_2$ with $0 < \delta_1 < 1$, we have for y in $SE(\lambda)_{r,p,q}^{2r}$

$$\lim_{\delta_2 \rightarrow 0 \pm} \text{sig}\left(\text{Re}(y^{2r})\right) = 1, \quad \lim_{\delta_2 \rightarrow 0 \pm} \text{Im}(y^{2r}) = 0, \quad \lim_{\delta_2 \rightarrow 0 \pm} \text{sig}\left(\text{Im}(y^{2r})\right) = \pm 1.$$

We then define a multi-valued function y on a $2r$ -fold branched domain by writing

$$y = x^{\beta_1 + \beta_2 - \frac{1}{2}} (1 - x)^{1 - \beta_2} (\lambda - x)^{1 - \beta_2}.$$

We reduce the integration over the A -cycle to an integration along the branch cut and back after inserting the correct phase. We obtain

$$\begin{aligned}
 \oint_{\mathbf{a}_1} \frac{dx}{y} &= \left(1 - e^{-\frac{2\pi i}{2r}}\right) \int_0^\lambda \frac{dx}{x^{\beta_1 + \beta_2 - \frac{1}{2}} (1 - x)^{1 - \beta_2} (\lambda - x)^{1 - \beta_2}} \\
 &= \frac{\left(1 - e^{-\frac{2\pi i}{2r}}\right)}{\lambda^{\beta_1 - \frac{1}{2}}} \int_0^1 \frac{d\tilde{x}}{\tilde{x}^{\beta_1 + \beta_2 - \frac{1}{2}} (1 - \lambda \tilde{x})^{1 - \beta_2} (1 - \tilde{x})^{1 - \beta_2}} \\
 &= \frac{\left(1 - e^{-\frac{2\pi i}{2r}}\right)}{\lambda^{\beta_1 - \frac{1}{2}}} \frac{\Gamma(\frac{3}{2} - \beta_1 - \beta_2) \Gamma(\beta_2)}{\Gamma(\frac{3}{2} - \beta_1)} {}_2F_1\left(\begin{matrix} \frac{3}{2} - \beta_1 - \beta_2, 1 - \beta_2 \\ \frac{3}{2} - \beta_1 \end{matrix} \middle| \lambda\right).
 \end{aligned} \tag{3.29}$$

If \mathbf{a}_1 is replaced by \mathbf{a}_k , y must be replaced by $\rho_{2r}^{k-1}y$ with $\rho_{2r} = \exp \frac{2\pi i}{2r}$ to account for the change in y -sheet.

For $x = \delta_1 + i \delta_2$ with $1 < \delta_1 < \lambda$ and $\delta_2 \approx 0$, we have for y in $SE(\lambda)_{r,p,q}^{2r}$ that $(-1)^{2r-p} \text{Re}(y^{2r}) > 0$. We recall that the B -cycle \mathbf{b}_1 was chosen as the closed cycle that runs from the first to the second branch cut on the chosen y -sheet and returns on the adjacent y -sheet as shown in Figure 1. This implies that

$$\oint_{\mathbf{b}_1} \frac{dx}{y} = \left(1 - \frac{1}{\rho_{2r}}\right) \int_\lambda^1 \frac{dx}{x^{\beta_1 + \beta_2 - \frac{1}{2}} (1 - x)^{1 - \beta_2} (\lambda - x)^{1 - \beta_2}}. \tag{3.30}$$

Setting $x = 1 - (1 - \lambda)\tilde{x}$, we obtain

$$\begin{aligned}
 \oint_{\mathbf{b}_1} \frac{dx}{y} &= e^{-\pi i (1 - \beta_2)} \frac{\left(1 - e^{-\frac{2\pi i}{2r}}\right)}{(1 - \lambda)^{1 - 2\beta_2}} \int_1^0 \frac{d\tilde{x}}{(1 - (1 - \lambda)\tilde{x})^{\beta_1 + \beta_2 - \frac{1}{2}} \tilde{x}^{1 - \beta_2} (1 - \tilde{x})^{1 - \beta_2}} \\
 &= e^{\pi i \beta_2} \left(1 - e^{-\frac{2\pi i}{2r}}\right) \frac{\Gamma(\beta_2)^2}{\Gamma(2\beta_2) (1 - \lambda)^{1 - 2\beta_2}} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \lambda\right).
 \end{aligned} \tag{3.31}$$

If \mathfrak{b}_1 is replaced by \mathfrak{b}_k , y must be replaced by $\rho_{2r}^{k-1}y$ with $\rho_{2r} = \exp \frac{2\pi i}{2r}$ to account for the change in y -sheet.

Notice that our choice of branch cuts between $e_1 = 0$ and $e_2 = \lambda$ and $e_3 = 1$ and $e_4 = \infty$, respectively, is compatible with the definition in Section 2 that the complex continuation of the Gauss' hypergeometric function has a branch cut from 1 to ∞ on the real λ -axis. Therefore, all formulas remain consistent if we allow λ to be complex. \square

REMARK 16. In the above proof we have used that on $SE(\lambda)_{r,p,q}^{2r}$ we have

$$\begin{aligned} f_{r,p,q}^{(k)'}(\lambda) &= \oint_{\mathfrak{b}_k} \frac{dx}{y} = C_{2r}^{(k)} \int_1^\lambda \frac{dx}{y} \\ &= (-1)^{\beta_2} C_{2r}^{(k)} \frac{\Gamma(\beta_2)^2}{\Gamma(2\beta_2)(1-\lambda)^{1-2\beta_2}} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1-\lambda\right), \end{aligned} \tag{3.32}$$

and on $SE(\lambda)_{r,2r-p,2r-q}^{2r}$ we have

$$\begin{aligned} f_{r,2r-p,2r-q}^{(k)}(\lambda) &= \oint_{\mathfrak{a}_k} \frac{dx}{y} = C_{2r}^{(k)} \int_0^\lambda \frac{dx}{y} \\ &= C_{2r}^{(k)} \frac{\Gamma(\beta_1 + \beta_2 - \frac{1}{2})\Gamma(1-\beta_2)}{\Gamma(\beta_1 + \frac{1}{2})\lambda^{\frac{1}{2}-\beta_1}} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \lambda\right). \end{aligned} \tag{3.33}$$

REMARK 17. Whenever $k = 1$, i.e., we are looking at the principal branch, we will drop the superscript and write $f_{r,p,q}(\lambda), f'_{r,p,q}(\lambda)$, etc.

REMARK 18. For $p = q = r = k = 1$, the above lemma recovers the well-known period relations for an elliptic curve

$$\oint_{\mathfrak{a}_1} \frac{dx}{y} = 2\pi {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \lambda\right), \quad \oint_{\mathfrak{b}_1} \frac{dx}{y} = 2\pi i {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| 1-\lambda\right). \tag{3.34}$$

REMARK 19. We introduce a τ -parameter as ratio of periods over the A - and B -cycle, i.e.,

$$\tau := \frac{f'_{r,p,q}(\lambda)}{f_{r,p,q}(\lambda)} = \dots = \frac{f_{r,p,q}^{(k)'}(\lambda)}{f_{r,p,q}^{(k)}(\lambda)} \tag{3.35}$$

for $k = 1, \dots, 2r - 1$. Note that for $r = 1$ and $\lambda = \Lambda^2$ the above equation is the well-known relation that express τ as ratio of complete elliptic integrals in Λ , i.e.,

$$2\pi i \tau := -\frac{2\pi K'(\Lambda)}{K(\Lambda)} = 4 \ln \Lambda - 8 \ln 2 + \Lambda^2 + \frac{13}{32} \Lambda^4 + O(\Lambda^6), \tag{3.36}$$

and, by inversion, Λ as ratio of Jacobi theta-function in $q = \exp(2\pi i\tau)$, i.e.,

$$\Lambda = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^2 = 4q^{\frac{1}{4}} \left(1 - 4q^{\frac{1}{2}} + 14q + O(q^{\frac{3}{2}})\right). \tag{3.37}$$

In physics, this is also known as construction of the mirror map [23].

4. Generalized Kummer surfaces. In this section, we will construct a generalized Kummer variety as minimal nonsingular model of a product-quotient surface with only rational double points from a pair of two superelliptic curves, determine its Hodge diamond, and the explicit defining equations for various superelliptic fibrations on it.

As we mentioned in the introduction, Ernst Kummer was the first to study irreducible nodal surfaces of degree four in \mathbb{P}^3 with a maximal possible number of sixteen double points [21]. Each such surface is the quotient of a principally polarized Abelian surface T – the Jacobian $\text{Jac}(C)$ of a smooth hyperelliptic curve C of genus two – by an involution automorphism. The involution has sixteen fixed points, namely the sixteen two-torsion points of the Jacobian. These, it turns out, are exactly the sixteen singular points on the Kummer surface. Resolving these sixteen rational double point singularities gives a $K3$ surface with sixteen disjoint rational curves.

Kummer surfaces have a rich symmetry, the so-called 16_6 -configuration [17]. There are two sets of sixteen (-2) -curves⁵ which we will label $\{Z_{i,j}\}$ and $\{\sigma_{i,j}\}$ with $i, j \in \{1, \dots, 4\}$ such that $Z_{i,j}$ and $\sigma_{k,l}$ intersect if and only if $i = k$ and $j = l$ but not both. The 16 curves $\{Z_{i,j}\}$ are the exceptional divisors corresponding to blow-up of the 16 two-torsion points of the Jacobian, while $\{\sigma_{i,j}\}$ arise from embedding C into $\text{Jac}(C)$ as symmetric theta divisors (cf. [26]). Using curves in the 16_6 -configuration, one can define various elliptic fibrations on the Kummer surface, since all irreducible components of a reducible fiber in an elliptic fibration are (-2) -curves [18].

If we specialize to the case where the Abelian surface $T = E_1 \times E_2$ is the product of two non-isogenous elliptic curves, we obtain from the minimal nonsingular model of the quotient of T by the involution automorphism a $K3$ surface of Picard-rank 18. Now, there is a configuration of twenty-four (-2) -curves, called the double Kummer pencil. It consists of the 16 aforementioned exceptional curves $\{E_{i,j}\}$, plus 8 curves obtained as the images of $\{S_i \times E_2\}$ or $\{E_1 \times S'_j\}$. Here, $\{S_i\}$ and $\{S'_j\}$ denote the two-torsion points on E_1 and E_2 , respectively. Oguiso classified all eleven inequivalent elliptic fibrations arising on such $K3$ surfaces [27]. The simplest of such elliptic fibrations are the ones induced by the projection of T onto its first or second factor and are called the first or second Kummer pencil.

4.1. Generalized Kummer surfaces. In Lemma 3.2 we constructed two smooth irreducible curves C_1 and C_2 of genus $2r - 1$ as minimal resolutions of the two curves $SE(\lambda_1)_{r,p,q}^{2r}$ and $SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$ where

$$\begin{aligned} SE(\Lambda_1^2)_{r,p,q}^{2r} &: \eta_1^{2r} = \zeta_1^{p+q-r} (\zeta_1 - 1)^{2r-p} (\zeta_1 - \Lambda_1^2)^{2r-p}, \\ SE(\Lambda_2^2)_{r,2r-p,2r-q}^{2r} &: \eta_2^{2r} = \zeta_2^{3r-p-q} (\zeta_2 - 1)^p (\zeta_2 - \Lambda_2^2)^p. \end{aligned} \tag{4.1}$$

We now consider the action of the group $G_0 = \mathbb{Z}_2$ on the product $SE(\lambda_1)_{r,p,q}^{2r} \times SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$ given by the involution automorphism

$$\iota : (\zeta_1, \eta_1, \zeta_2, \eta_2) \mapsto (\zeta_1, -\eta_1, \zeta_2, -\eta_2). \tag{4.2}$$

Using Lemma 3.3 on both factors, it follows that the action (4.2) lifts to the product surface $T = C_1 \times C_2$. In the special case $q = r$, we will also consider the action of the group $G = \mathbb{Z}_{2r}$ on $SE(\lambda_1)_{r,p,r}^{2r} \times SE(\lambda_2)_{r,2r-p,r}^{2r}$ generated by

$$(\zeta_1, \eta_1, \zeta_2, \eta_2) \mapsto (\zeta_1, \rho_{2r}^p \eta_1, \zeta_2, \rho_{2r}^{-p} \eta_2). \tag{4.3}$$

⁵A (-2) -curve is a smooth rational curve whose self-intersection number is -2 .

Using Lemma 3.4 on both factors, it follows that the action (4.3) lifts to the product surface $T = C_1 \times C_2$ as well. The action of G_0 (and G in the special case $q = r$) on T is diagonal, but not free. The group G_0 (and G in the special case $q = r$) has 16 fixed points that are obtained by combining the pre-images of the singular points on $SE(\lambda_1)_{r,p,q}^{2r}$ and $SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$ in Equation (3.7). We will now consider the product-quotient surface $V_0 = T/G_0$ (and $V = T/G$ in the special case $q = r$). The singularities of V_0 and V are by construction finite cyclic quotient singularities, which are rational singularities. We have the following lemma:

LEMMA 4.1. *The 16 cyclic quotient singularities on V_0 are ordinary double points. Among the 16 cyclic quotient singularities on V are 8 rational double points of type A_{2r-1} and 8 cyclic quotient singularities of type $\frac{1}{2r}(1, 1)$.*

Proof. Consider a singular point on either V_0 or V . Then, a neighborhood of the singular point is equivalent to the quotient of \mathbb{C}^2 by the action of a linear diagonal automorphism with eigenvalues $\{\exp(\frac{2\pi ia}{n}), \exp(\frac{2\pi ib}{n})\}$ with $(a, n) = (b, n) = 1$ and $n = 2r$ or $n = 2$ if p is on V or V_0 , respectively. In the case of V_0 , the proof of Lemma 3.3 shows that the action of G_0 on the local coordinates of the normalization of $SE(\lambda_1)_{r,p,q}^{2r}$ is given by

$$z_0, z_1, z_\lambda, z_\infty \mapsto -z_0, -z_1, -z_\lambda, -z_\infty . \tag{4.4}$$

As pointed out in Remark 13, the second affine model $SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$ is obtained by sending $(p, q) \mapsto (2r - p, 2r - q)$. It is easy to check that action of G_0 on the local coordinates of the normalization of $SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$ is the same as the one in Equation (4.4). Therefore, we have only rational double points of type A_1 on V_0 .

In the case of V , the proof of Lemma 3.4 shows that the action of $G = \mathbb{Z}_{2r}$ on $SE(\lambda_1)_{r,p,r}^{2r}$ given by

$$[X : Y : Z] \mapsto [X : \rho_{2r}^p Y : Z] = [\rho_{2r}^{-p} X : Y : \rho_{2r}^{-p} Z] , \tag{4.5}$$

can be lifted to an action on its normalization. In the local coordinates it is given by

$$z_0, z_1, z_\lambda, z_\infty \mapsto \rho_{2r} z_0, \rho_{2r}^{2r-1} z_1 = \rho_{2r}^{-1} z_1, \rho_{2r}^{2r-1} z_\lambda = \rho_{2r}^{-1} z_\lambda, \rho_{2r} z_\infty . \tag{4.6}$$

The second affine model $SE(\lambda_2)_{r,2r-p,p}^{2r}$ is obtained by sending $p \rightarrow 2r - p$, but at the same time we are also considering the lift of the different action

$$[X : Y : Z] \mapsto [X : \rho_{2r}^{-p} Y : Z] = [\rho_{2r}^p X : Y : \rho_{2r}^p Z] \tag{4.7}$$

because the actions in Equations (4.5) and (4.7) combine to give the action (4.3). One then checks that action of G on the local coordinates of the normalization of $SE(\lambda_2)_{r,2r-p,p}^{2r}$ is the same as the one in Equation (4.6). It follows that a neighborhood of each singular point is equivalent to the quotient of \mathbb{C}^2 by the action of a linear diagonal automorphism with eigenvalues $\{\exp(\frac{2\pi ia}{2r}), \exp(\frac{2\pi ib}{2r})\}$ with $a, b \in \{-1, 1\}$. If $a \neq b$ the singularity is a rational double point of type A_{2r-1} . \square

We will now consider the minimal resolution $S_0 \rightarrow V_0$ of the rational singularities of the product-quotient surface $V_0 = (C_1 \times C_2)/G_0$ where C_1 and C_2 are the smooth irreducible curves of genus $2r - 1$ and the diagonal action with fixed points of the group $G_0 = \mathbb{Z}_2$. In the special case $q = r$, we will also consider the minimal resolution $S \rightarrow V$ of the rational singularities of the product-quotient surface $V = (C_1 \times C_2)/G$ where

the diagonal action with fixed points of the group $G = \mathbb{Z}_{2r}$ is given by Equation (4.3).

REMARK 20. More details on product-quotient surfaces and a summary on how cyclic quotient singularities can be resolved by so-called Hirzebruch-Jung strings can be found in [5, Sec. 2].

Following [5, Remark 2.5.(2)] we denote by K_{V_0} the divisor class of the canonical Weil divisor on the normal surface corresponding to the inclusion of the smooth locus of V_0 . The intersection product for Weil divisors on a normal surface generally takes values in \mathbb{Q} . We obtain for the self-intersection

$$K_{V_0}^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G_0|} = 16(r - 1)^2. \tag{4.8}$$

We have the following lemma:

LEMMA 4.2. *The surface S_0 has the invariants:*

$$\begin{aligned} K_{S_0}^2 &= 16(r - 1)^2, & e(S_0) &= 24 + 8(r - 1)^2, \\ \chi(\mathcal{O}_{S_0}) &= 2 + 2(r - 1)^2, & \tau(S_0) &= -16. \end{aligned} \tag{4.9}$$

Proof. The singular surface V_0 has only rational double point singularities. Therefore, $K_{V_0}^2$ is an integer and agrees with $K_{S_0}^2$ [28]. Using [5, Cor. 2.7] it follows that

$$e(S_0) = \frac{K_{V_0}^2}{2} + \frac{3}{2}k$$

where $k = 16$ is the number of ordinary double points. The holomorphic Euler characteristic $\chi(\mathcal{O}_{S_0})$ is then obtained by Noether’s formula. The signature is computed by the Thom-Hirzebruch signature index

$$\tau(S_0) = \frac{K_{S_0}^2 - 2e(S_0)}{3} = -k = -16.$$

□

REMARK 21. The same computation cannot be used to obtain the self-intersection of the canonical divisor on the surface S as not all of the singularities of the product-quotient surface V are rational double point singularities. In fact, we have that

$$K_V^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G_0|} = \frac{16(r - 1)^2}{r} \tag{4.10}$$

is an integer only for $r = 1$ if r is odd.

We have the following lemma:

LEMMA 4.3. *It follows that*

$$\pi_1(S_0) = \pi_1(V_0), \quad \pi_1(S) = \pi_1(V). \tag{4.11}$$

In particular, the surfaces S_0 and S have irregularities $q(S_0) = 4(r - 1)$ and $q(S) = 0$, respectively.

Proof. Since the minimal resolution $S_0 \rightarrow V_0$ and $S \rightarrow V$ of the singularities of V_0 and V , respectively, replaces each singular point by a tree of smooth rational curves, the claim about the fundamental group then follows by van Kampen’s theorem. Notice that by construction we have $C_1/G = C_2/G = \mathbb{P}^1$ in the special case $q = r$. By [11, Prop. 3.15] it follows that the surface S falls into the D^0 -case, hence $q(S) = 0$.

Let us now compute the irregularity $q(S_0)$. We obtained S_0 from the singular quotient variety $V_0 = T/G_0$ by blowing up 16 ordinary double point singularities where $T = C_1 \times C_2$ and the action of G_0 was given in Equation (4.2) and could be lifted to T . Alternatively, we can blow up the 16 fixed points of G_0 on T first to construct a surface \tilde{T} whose projection π onto \tilde{T}/G_0 is S_0 . The latter point of view makes it easier to construct a rational basis of the first homology of S_0 as the dimension of the first homology does not change during the process of blowing-up. We denote the two bases for the first homology of the singular plane curves $SE(\lambda_1)_{r,p,q}^{2r}$ and $SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$, respectively, constructed in Section 3.2 by

$$\left(\mathbf{a}_i^{(k)}, \mathbf{b}_j^{(k)} \right)_{i,j=1}^{2r-1} \quad \text{for } k = 1 \text{ and } k = 2.$$

These cycles lift to $4g = 4(2r - 1)$ non-trivial one-cycles $(\gamma_m)_{m=1}^{4(2r-1)}$ on \tilde{T} . The involution (4.2) identifies the non-trivial one-cycles

$$\mathbf{a}_i^{(1)} \times \infty \quad \text{and} \quad \infty \times \mathbf{a}_i^{(2)}$$

with the one-cycles

$$\mathbf{a}_{i+r}^{(1)} \times \infty \quad \text{and} \quad \infty \times \mathbf{a}_{i+r}^{(2)},$$

respectively, for $i = 1, \dots, r - 1$. An analogue statement holds for the B -cycles. Moreover, the one-cycles

$$\mathbf{a}_r^{(1)} \times \infty \quad \text{and} \quad \infty \times \mathbf{a}_r^{(2)}$$

become homotopic to zero on S_0 when the y -sheet is identified with the $(-y)$ -sheet. Therefore, the $4(r - 1)$ one-cycles obtained from the projections of the remaining one-cycles

$$\pi_*(\mathbf{a}_i^{(1)} \times \infty), \quad \pi_*(\infty \times \mathbf{a}_i^{(2)}), \quad \pi_*(\mathbf{b}_i^{(1)} \times \infty), \quad \pi_*(\infty \times \mathbf{b}_i^{(2)})$$

for $i = 1, \dots, r - 1$ form a basis of the first homology of S_0 over \mathbb{Q} . \square

We can now compute the Hodge-diamond of the surface S_0 .

COROLLARY 4.4. *The Hodge-diamond of the surface S_0 is given by*

$$\begin{array}{ccccc} & & 1 & & \\ & q(S_0) & & q(S_0) & \\ p_g(S_0) & & h^{1,1}(S_0) & & p_g(S_0) \\ & q(S_0) & & q(S_0) & \\ & & 1 & & \end{array}$$

with $q(S_0) = 4(r - 1)$, $p_g(S_0) = 1 + 2(r^2 - 1)$, and $h^{1,1}(S_0) = 20 + 2(r^2 - 1)$.

Proof. Lemmas 4.2 and 4.3 imply that

$$2 + 2(r - 1)^2 = \chi(\mathcal{O}_{S_0}) = 1 - q(S_0) + p_g(S_0) = 1 - 4(r - 1) + p_g(S_0),$$

hence $p_g(S_0) = 1 + 2(r^2 - 1)$. The Hodge index theorem implies that the intersection form on $H^{1,1}(S_0) \cap H^2(S_0, \mathbb{R})$ has one positive eigenvalue and $h^{1,1}(S_0) - 1$ negative eigenvalues. Therefore, we have

$$-16 = \tau(S_0) = 2p_g(S_0) + 1 - (h^{1,1}(S_0) - 1) = 4 + 2(r^2 - 1) - h^{1,1}(S_0),$$

hence $h^{1,1}(S_0) = 20 + 2(r^2 - 1)$. \square

DEFINITION 4.5. We call the minimal resolution $\rho : S_0 \rightarrow V_0$ of the 16 rational double point singularities on the product-quotient surface $V_0 = (C_1 \times C_2)/G_0$ a generalized Kummer surface where C_1 and C_2 are the smooth irreducible curves of genus $2r - 1$ obtained as minimal resolution of the two singular curves $SE(\lambda_1)_{r,p,q}^{2r}$ and $SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$, and the diagonal action with fixed points of $G_0 = \mathbb{Z}_2$ is given by Equation (4.2). We will always assume that the modular parameters λ_1 and λ_2 are generic, i.e., not equal to $0, 1, \infty$ and that the integers p, q, r satisfy the inequalities (3.4) and divisibility constraints (3.5).

REMARK 22. If it is necessary we will also write $S_0^{(r,p,q)}(\lambda_1, \lambda_2)$ and $V_0^{(r,p,q)}(\lambda_1, \lambda_2)$. Note that $V_0^{(r,p,q)} \cong V_0^{(r,2r-p,2r-q)}$ and $S_0^{(r,p,q)} \cong S_0^{(r,2r-p,2r-q)}$ since mapping $(p, q) \mapsto (2r - p, 2r - q)$ amounts to interchanging the roles of $(\zeta_1, \eta_1, \lambda_1)$ with $(\zeta_2, \eta_2, \lambda_2)$, but leaves the total space invariant.

In the proof of Lemma 4.3 we constructed a rational basis of $4g = 4(2r - 1)$ non-trivial one-cycles $(\gamma_m)_{m=1}^{4(2r-1)}$ on \tilde{T} where \tilde{T} was obtained by blowing up the 16 fixed points on $C_1 \times C_2$ first such that S_0 is obtained as projection $\pi : \tilde{T} \rightarrow S_0 = \tilde{T}/G_0$. We now look at the second homology of S_0 . We find (1) two-cycles from the exceptional divisors generated during the blow-up constructing S_0 , and (2) two-cycles $(\sigma_{m,n})_{m,n=1}^{4(2r-1)}$ with $m \neq n$ from the projection of two-cycles on \tilde{T} , i.e., $\sigma_{m,n} = \pi_*(\gamma_m \times \gamma_n)$.

Using Lemma 3.8 the two-form $d\zeta_1/\eta_1 \boxtimes d\zeta_2/\eta_2$ defines a holomorphic differential two-form on $T = C_1 \times C_2$, hence on \tilde{T} , that induces a unique holomorphic two-form ω on $S_0^{(r,p,q)}$ such that

$$\oint_{\sigma_{m,n}} \omega = \iint_{\gamma_m \times \gamma_n} \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2}. \tag{4.12}$$

We have the following lemma:

LEMMA 4.6. The periods of ω over the two-cycles $\sigma_{m,n}$ on $S_0^{(r,p,q)}(\lambda_1, \lambda_2)$ are given by

$$\begin{aligned} F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_1 &= \oint_{\pi_* (\mathfrak{a}_k^{(1)} \times \mathfrak{a}_l^{(2)})} \omega = f_{r,p,q}^{(k)}(\lambda_1) f_{r,2r-p,2r-q}^{(l)}(\lambda_2), \\ F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_2 &= \oint_{\pi_* (\mathfrak{a}_k^{(1)} \times \mathfrak{b}_l^{(2)})} \omega = f_{r,p,q}^{(k)}(\lambda_1) f_{r,2r-p,2r-q}^{(l)'}(\lambda_2), \\ F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_3 &= \oint_{\pi_* (\mathfrak{b}_k^{(1)} \times \mathfrak{a}_l^{(2)})} \omega = f_{r,p,q}^{(k)'}(\lambda_1) f_{r,2r-p,2r-q}^{(l)}(\lambda_2), \\ F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_4 &= \oint_{\pi_* (\mathfrak{b}_k^{(1)} \times \mathfrak{b}_l^{(2)})} \omega = f_{r,p,q}^{(k)'}(\lambda_1) f_{r,2r-p,2r-q}^{(l)'}(\lambda_2), \end{aligned} \tag{4.13}$$

where $1 \leq k, l \leq r - 1$, and $f_{r,p,q}(\lambda)$ and $f'_{r,p,q}(\lambda)$ were computed in Lemma 3.9. The periods in Equations (4.13) satisfy the quadratic relation

$$F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_1 F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_4 - F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_2 F_{r,p,q}^{(k,l)}(\lambda_1, \lambda_2)_3 = 0 .$$

REMARK 23. Whenever $k = 1$, i.e., we are looking at the principal branch, we will drop the superscript and write $F_{r,p,q}(\lambda_1), F'_{r,p,q}(\lambda_1)$, etc.

Proof. First, we observe that on $S_0^{(r,p,q)}$ the two-cycles $\sigma_{m,n}$ are transcendental only if they are among the $4(r - 1)^2$ two-cycles used in Equations (4.13). The lemma and Equations (4.13) then follow from Lemma 3.9. The quadratic period relation is obvious from Equations (4.13). \square

REMARK 24. As special functions in the moduli λ_1, λ_2 , the periods $F_{r,p,q}(\lambda_1, \lambda_2)_l$ for $l = 1, \dots, 4$ satisfy (up to an overall renormalizing function in λ_1, λ_2) the rank-four system described in Proposition 2.4 that is the (outer) tensor product of two hypergeometric functions.

4.2. Fibration structures. Kuwata and Shioda determined explicit formulas determining the elliptic parameter denoted by U and the elliptic fiber coordinates X, Y for each inequivalent Jacobian elliptic fibration labeled $(\mathfrak{J}_i)_{i=1}^{11}$ in [22] on the Kummer surface S_0 that is the minimal nonsingular model for the quotient of a product of two non-isogenous elliptic curves by the involution automorphism. That is, the Kummer surface is a Jacobian elliptic $K3$ surface of Picard-rank 18 with function field

$$\mathbb{C}(S_0) = \mathbb{C}(U, X, Y)$$

with one relation between U, X, Y given in form of a Weierstrass equation. We checked that for all of their 11 elliptic fibrations $(\mathfrak{J}_i)_{i=1}^{11}$ we have

$$\pi^* \left(dU \wedge \frac{dX}{Y} \right) = \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2} \tag{4.14}$$

where π is a projection map analogous to the one used in the proof of Lemma 4.3, and (ζ_1, η_1) and (ζ_2, η_2) are the variables used to define two elliptic curves in Legendre normal form. In this article, we have replaced the elliptic curves in the classical Kummer construction by the two superelliptic curves in Equations (4.1) of genus $2r - 1$. For general values for r, p, q , we still consider ζ_1 and ζ_2 as independent variables, and η_1 and η_2 as variables dependent on r, p, q and ζ_1 or ζ_2 , respectively, by means of Equations (4.1). Thus, the definition for the functions U and X given in [22] then cannot change as we vary r, p, q if we want to maintain the differential relation (4.14). However, we can read off from [22, Table 1] that only for the cases $(\mathfrak{J}_i)_{i=4}^{11}$ are the variables U, X contained in $\mathbb{C}(\zeta_1, \zeta_2)$. Among those, only the five remaining cases $(\mathfrak{J}_i)_{i=4}^8$ have a two-torsion section contained in their Mordell-Weil group which is necessary to determine a superelliptic fibration in the sense of Definition 3.5. We will now derive the equations generalizing fibrations $(\mathfrak{J}_i)_{i=4}^8$ on V_0 . As we will see, for fibrations \mathfrak{J}_5 and \mathfrak{J}_8 we have to specialize to the case $q = 3r - p$ (cf. Remark 11).

The construction for the generalized Kummer variety is slightly more complicated than in the classical case: we started with two singular curves $SE(\lambda_1)_{r,p,q}^{2r}$ and $SE(\lambda_2)_{r,2r-p,2r-q}^{2r}$ with affine coordinates (ζ_1, η_1) and (ζ_2, η_2) , respectively. We constructed their minimal resolutions C_1 and C_2 with normalization maps ψ_1 and ψ_2 , respectively, by Lemma 3.3 and lifted the involution automorphism (4.2) to $T = C_1 \times C_2$.

The coordinates $\{\zeta_1, \eta_1, \zeta_2, \eta_2\}$ when composed with ψ_1 and ψ_2 then are rational functions on T , hence on the space \tilde{T} obtained by blowing up the fixed points of G_0 . Therefore, expressions in these coordinates invariant under the involution descend to rational functions on S_0 , and the fibrations $(\mathfrak{J}_i)_{i=4}^8$ become algebraic relations between these rational functions.

REMARK 25. Note that we will be using the square roots Λ_1 and Λ_2 of the modular parameters λ_1 and λ_2 for the two superelliptic curves in Equations (4.1). We have already seen in Theorem 2.5 that this allows us to find *rational* relations to the variables A, B that will appear as variables in Appell’s hypergeometric series F_2 .

4.2.1. Generalization of fibration \mathfrak{J}_4 . We set $Y = \eta_1 \eta_2$ and $U = \zeta_1, X = \zeta_2$ or $\tilde{U} = \zeta_2, \tilde{X} = \zeta_1$. We have the following lemma:

LEMMA 4.7. *The functions U, X, Y or \tilde{U}, \tilde{X}, Y descend to rational functions on the generalized Kummer variety $S_0^{(r,p,q)}(\Lambda_1^2, \Lambda_2^2)$ satisfying the relation*

$$Y^{2r} = U^{p+q-r} (U - 1)^{2r-p} (U - \Lambda_1^2)^{2r-p} X^{3r-p-q} (X - 1)^p (X - \Lambda_2^2)^p \tag{4.15}$$

or the relation

$$Y^{2r} = \tilde{U}^{3r-p-q} (\tilde{U} - 1)^p (\tilde{U} - \Lambda_2^2)^p \tilde{X}^{p+q-r} (\tilde{X} - 1)^{2r-p} (\tilde{X} - \Lambda_1^2)^{2r-p}, \tag{4.16}$$

respectively, and

$$\pi^* \left(dU \wedge \frac{dX}{Y} \right) = \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2}, \quad \pi^* \left(d\tilde{U} \wedge \frac{d\tilde{X}}{\tilde{Y}} \right) = \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2}. \tag{4.17}$$

REMARK 26. For $r = p = q = 1$, Equations (4.15) and (4.16) coincide with the Jacobian elliptic fibration \mathfrak{J}_4 on a Kummer surface of two non-isogenous elliptic curves established in [22]. Equations (4.15) and (4.16) then describe the so-called first and second Kummer pencil.

4.2.2. Generalization of fibration \mathfrak{J}_5 . We define the fibration parameter

$$U = \frac{(\zeta_1 - \zeta_2) (\Lambda_2^2 \zeta_1 + (\Lambda_1^2 - 1) \zeta_2 - \Lambda_1^2 \Lambda_2^2)}{(\Lambda_2^2 \zeta_1 - \zeta_2) (\zeta_1 + (\Lambda_1^2 - 1) \zeta_2 - \Lambda_1^2)}, \tag{4.18}$$

and the coordinates

$$\begin{aligned} X &= \frac{R (\zeta_1 - \zeta_2) (\zeta_1 - \Lambda_1^2)}{\zeta_1 (\zeta_1 + (\Lambda_1^2 - 1) \zeta_2 - \Lambda_1^2)}, \\ Y &= \frac{R \Lambda_1^2 (1 - \Lambda_1^2) (U - 1) \zeta_2 (\zeta_1 - \zeta_2) \eta_1 \eta_2}{\zeta_1^2 (\Lambda_2^2 \zeta_1 - \zeta_2) (\zeta_1 + (\Lambda_1^2 - 1) \zeta_2 - \Lambda_1^2)^2}, \\ R &= \Lambda_1^2 (\Lambda_2^2 - 1) (U - 1) (U - 1 + \Lambda_1^2 (1 - \Lambda_2^2)) \\ &\quad \times ((\Lambda_1^2 \Lambda_2^2 - \Lambda_1^2 - \Lambda_2^2) U + \Lambda_2^2). \end{aligned} \tag{4.19}$$

We have the following lemma:

LEMMA 4.8. *The functions U, X, Y defined in Equations (4.22) and (4.23) descend to rational functions on $S_0^{(r,p,q=3r-2p)}(\Lambda_1^2, \Lambda_2^2)$ satisfying the relation*

$$\begin{aligned}
 Y^{2r} &= \Lambda_1^{-4p} (1 - \Lambda_1^2)^{2(r-p)} (1 - \Lambda_2^2)^{-2r} (U - 1)^{4(r-p)} X^{2r-p} \\
 &\quad \times (X + \Lambda_1^2 (\Lambda_2^2 - 1) (U - 1) ((\Lambda_1^2 \Lambda_2^2 - 1) U - \Lambda_1^2 + 1) \\
 &\quad \times ((\Lambda_1^2 \Lambda_2^2 - \Lambda_1^2 - \Lambda_2^2) U + \Lambda_2^2))^p \\
 &\quad \times (X + \Lambda_1^2 (\Lambda_2^2 - 1) U (U - 1) (U - 1 + \Lambda_1^2 (1 - \Lambda_2^2)) \\
 &\quad \times ((1 - \Lambda_1^2) \Lambda_2^2 U + \Lambda_1^2 - \Lambda_2^2))^p
 \end{aligned}
 \tag{4.20}$$

and

$$\pi^* \left(dU \wedge \frac{dX}{Y} \right) = \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2} .
 \tag{4.21}$$

Proof. By direct computation using formulas (4.18) and (4.19). \square

REMARK 27. For $r = p = q = 1$, Equation (4.20) coincides with the Jacobian elliptic fibration \mathfrak{J}_5 on a Kummer surface of two non-isogenous elliptic curves established in [22].

4.2.3. Generalization of fibration \mathfrak{J}_6 . We define the fibration parameter

$$U = \frac{\zeta_1}{\zeta_2}
 \tag{4.22}$$

and the coordinates

$$\begin{aligned}
 X &= \frac{\zeta_1 (\zeta_1 - \Lambda_1^2) (\zeta_1 - \zeta_2) (\Lambda_2^2 \zeta_1 - \zeta_2)}{\zeta_2^3 (\zeta_1 - 1)} , \\
 Y &= \frac{(\Lambda_1^2 - 1) \zeta_1^2 (\zeta_1 - \zeta_2) (\Lambda_2^2 \zeta_1 - \zeta_2) \eta_1 \eta_2}{\zeta_2^5 (\zeta_1 - 1)^2} .
 \end{aligned}
 \tag{4.23}$$

We have the following lemma:

LEMMA 4.9. *The functions U, X, Y defined in Equations (4.22) and (4.23) descend to rational functions on $S_0^{(r,p,q)}(\Lambda_1^2, \Lambda_2^2)$ satisfying the relation*

$$\begin{aligned}
 Y^{2r} &= (1 - \Lambda_1^2)^{2(r-p)} U^{-2p+q+r} X^{2r-p} \\
 &\quad \times (X - U (U - \Lambda_1^2) (\Lambda_2^2 U - 1))^p (X - U (U - 1) (\Lambda_2^2 U - \Lambda_1^2))^p
 \end{aligned}
 \tag{4.24}$$

and

$$\pi^* \left(dU \wedge \frac{dX}{Y} \right) = \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2} .
 \tag{4.25}$$

Proof. By direct computation using formulas (4.22) and (4.23). \square

REMARK 28. For $r = p = q = 1$, Equation (4.24) coincides with the Jacobian elliptic fibration \mathfrak{J}_6 on a Kummer surface of two non-isogenous elliptic curves established in [22].

4.2.4. Generalization of fibration \mathfrak{J}_7 . We define the fibration parameter

$$U = \frac{(\zeta_2 - \Lambda_2^2)(\zeta_1 - \zeta_2)}{(\zeta_2 - 1)(\Lambda_2^2 \zeta_1 - \zeta_2)} \tag{4.26}$$

and the coordinates

$$\begin{aligned} X &= \frac{\Lambda_2^2 (\Lambda_2^2 - 1)^2 \zeta_1 \zeta_2 (\zeta_1 - 1)^2 (\zeta_2 - \Lambda_2^2) (\zeta_1 - \zeta_2)}{(\zeta_2 - 1)^3 (\Lambda_2^2 \zeta_1 - \zeta_2)^3}, \\ Y &= -\frac{\Lambda_2^2 (\Lambda_2^2 - 1)^3 (\zeta_1 - 1)^2 \zeta_2 (\zeta_2 - \Lambda_2^2) (\zeta_1 - \zeta_2)^2 \eta_1 \eta_2}{(\Lambda_2^2 \zeta_1 - \zeta_2)^4 (\zeta_2 - 1)^5}. \end{aligned} \tag{4.27}$$

We have the following lemma:

LEMMA 4.10. *The functions U, X, Y defined in Equations (4.26) and (4.27) descend to rational functions on $S_0^{(r,p,q)}(\Lambda_1^2, \Lambda_2^2)$ satisfying the relation*

$$\begin{aligned} Y^{2r} &= \Lambda_2^{2(r-q)} (\Lambda_2^2 - 1)^{2(p-r)} U^{2p-q-r} (U - 1)^{2(r-q)} X^{p+q-r} \\ &\times \left(X^2 - U(U - 1) \left((\Lambda_1^2 \Lambda_2^2 + 1) U - \Lambda_1^2 - \Lambda_2^2 \right) X \right. \\ &\quad \left. + \Lambda_1^2 \Lambda_2^2 U^2 (U - 1)^4 \right)^{2r-p} \end{aligned} \tag{4.28}$$

and

$$\pi^* \left(dU \wedge \frac{dX}{Y} \right) = \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2}. \tag{4.29}$$

Proof. By direct computation using formulas (4.26) and (4.27). \square

REMARK 29. For $r = p = q = 1$, Equation (4.28) coincides with the Jacobian elliptic fibration \mathfrak{J}_7 on a Kummer surface of two non-isogenous elliptic curves established in [22].

4.2.5. Generalization of fibration \mathfrak{J}_8 . We define the fibration parameter

$$U = -\frac{(\zeta_1 - \zeta_2) (\zeta_2 - \Lambda_2^2)}{\Lambda_2^2 (\Lambda_2^2 - 1) \zeta_1 (\zeta_1 - 1)}, \tag{4.30}$$

and the coordinates

$$\begin{aligned} X &= \frac{U ((\Lambda_1^2 - 1) (\Lambda_2^2 - 1) U - 1) (\zeta_2 - 1) (\Lambda_2^2 \zeta_1 - \zeta_2)}{(\Lambda_2^2 - 1) \zeta_2 (\zeta_1 - 1)}, \\ Y &= \frac{\Lambda_2^2 U^3 ((\Lambda_1^2 - 1) (\Lambda_2^2 - 1) U - 1) (\Lambda_2^2 \zeta_1 - \zeta_2) \eta_1 \eta_2}{\zeta_2^2 (\zeta_1 - 1) (\zeta_2 - \Lambda_2^2)}. \end{aligned} \tag{4.31}$$

We have the following lemma:

LEMMA 4.11. *The functions U, X, Y defined in Equations (4.30) and (4.31) descend to rational functions on $S_0^{(r,p,q=3r-2p)}(\Lambda_1^2, \Lambda_2^2)$ satisfying the relation*

$$\begin{aligned} Y^{2r} &= \Lambda_2^{4(p-r)} (1 - \Lambda_1^2)^{2(p-r)} U^{4(p-r)} X^p \\ &\times \left(X^2 - U \left((2\Lambda_1^2 \Lambda_2^2 - \Lambda_1^2 - \Lambda_2^2 + 2) U - 2 \right) X \right. \\ &\quad \left. - U^2 (U - 1) (\Lambda_1^2 \Lambda_2^2 U - 1) \left((\Lambda_1^2 - 1) (\Lambda_2^2 - 1) U - 1 \right) \right)^{2r-p} \end{aligned} \tag{4.32}$$

and

$$\pi^* \left(dU \wedge \frac{dX}{Y} \right) = \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2}. \tag{4.33}$$

Proof. By direct computation using formulas (4.30) and (4.31). \square

REMARK 30. For $r = p = q = 1$, Equation (4.32) coincides with the Jacobian elliptic fibration \mathfrak{J}_8 on a Kummer surface of two non-isogenous elliptic curves established in [22].

4.3. Fibrations by twists and base transformations. We now investigate the generalizations of fibrations \mathfrak{J}_4 , \mathfrak{J}_6 , and \mathfrak{J}_7 given by Equations (4.15), (4.16), (4.24), and (4.28) in more detail. First, we will establish in what sense Equations (4.15), (4.16) and (4.28) define actual fibrations with fibers of genus $2r - 1$ on the generalized Kummer variety S_0 .

PROPOSITION 4.12. *On $S_0^{(r,p,q)}$ there are two isotrivial fibrations $f'_1 : S_0 \rightarrow C'_1$ and $f'_2 : S_0 \rightarrow C'_2$ where C'_1 and C'_2 are smooth irreducible curves of genus $r - 1$ and the fibers are isomorphic to the curves C_2 and C_1 , respectively, of genus $2r - 1$.*

REMARK 31. As pointed out in [11, Rem. 2.5], in the context of isotrivial fibrations – like the ones in Lemma 4.12 – a converse to Arakelov’s theorem does not need to hold, i.e., we have

$$K_{S_0}^2 \neq 8 \left(g(\text{fiber}) - 1 \right) \left(g(\text{base}) - 1 \right) = 32r(r - 1).$$

In fact a small deformation of a constant moduli fibration does not need to be again a constant moduli fibration.

Proof. We first prove a lemma construction the quotient curve C_1/G_0 . In Lemma 3.2 we constructed the non-singular irreducible curve C_1 of genus $2r - 1$ as minimal resolution of the singular curve $SE(\Lambda_1^2)_{r,p,q}^{2r}$ which we denoted by $\psi_1 : C_1 \rightarrow SE(\Lambda_1^2)_{r,p,q}^{2r}$, and also the degree- $2r$ map $[X : Y : Z] \mapsto [X : Z]$ denoted by $\phi_1 : SE(\Lambda_1^2)_{r,p,q}^{2r} \rightarrow \mathbb{P}^1$. Furthermore, in Lemma 3.3 we proved that the involution automorphism (3.11) of the action of $G_0 = \mathbb{Z}_2$ on $SE(\Lambda_1^2)_{r,p,q}^{2r}$ lifts to an automorphism on the minimal resolution C_1 . We first prove the following lemma:

LEMMA 4.13. *The quotient curve C_1/G_0 is isomorphic to the minimal resolution C'_1 of the singular curve $SE(\Lambda_1^2)_{r,p,q}^r$, i.e.,*

$$C_1/G_0 = C'_1 \xrightarrow{\psi'_1} SE(\Lambda_1^2)_{r,p,q}^r, \tag{4.34}$$

and is a smooth irreducible curve of genus $r - 1$.

We first observe that on the curve $SE(\lambda)_{r,p,q}^{2r}$ the quotient by $G_0 = \mathbb{Z}_2$ can be constructed by using the rational transformation

$$\tilde{X} = X, \quad \tilde{Y}\tilde{Z} = Y^2, \quad \tilde{Z} = Z \tag{4.35}$$

in the equation

$$SE(\lambda)_{r,p,q}^{2r} : Y^{2r} Z^{r-p+q} = X^{p+q-r} (X - Z)^{2r-p} (X - \lambda Z)^{2r-p}. \tag{4.36}$$

We obtain the equation

$$SE(\lambda)_{r,p,q}^r : \tilde{Y}^r \tilde{Z}^{2r-p+q} = \tilde{X}^{p+q-r} (\tilde{X} - \tilde{Z})^{2r-p} (\tilde{X} - \lambda \tilde{Z})^{2r-p}. \tag{4.37}$$

Using Lemma 3.2 we can construct the minimal resolution C'_1 of genus $r - 1$ for the singular curve $SE(\lambda)_{r,p,q}^r$ denoted by $\psi'_1 : C'_1 \rightarrow SE(\lambda)_{r,p,q}^r$, and the ramified degree r map $[\tilde{X} : \tilde{Y} : \tilde{Z}] \mapsto [\tilde{X} : \tilde{Z}]$ denoted by $\phi'_1 : SE(\lambda)_{r,p,q}^r \rightarrow \mathbb{P}^1$ with four totally ramified points. The Puiseux expansions about all singular points of $SE(\lambda)_{r,p,q}^r$ defining the local normalization maps are given in Table 3.

$[\tilde{X} : \tilde{Y} : \tilde{Z}]$	$[0 : 0 : 1]$	$[1 : 0 : 1]$	$[\lambda : 0 : 1]$	$[0 : 1 : 0]$
multiplicity	$p + q - r$	$2r - p$	$2r - p$	$r - p + q$
Puiseux coefficients	$(r, p + q - r)$	$(r, 2r - p)$	$(r, 2r - p)$	$(3r - p + q, 2r - p + q)$
Puiseux expansion	$\tilde{X} = \tilde{z}_0^r$ $\tilde{Y} = \lambda^{\frac{2r-p}{r}} \tilde{z}_0^{p+q-r} (1 + \tilde{y}_0(\tilde{z}_0^r))$ $\tilde{Z} = 1$	$\tilde{X} = 1 + \tilde{z}_1^r$ $\tilde{Y} = (1 - \lambda)^{\frac{2r-p}{r}} \tilde{z}_1^{2r-p} (1 + \tilde{y}_1(\tilde{z}_1^r))$ $\tilde{Z} = 1$	$\tilde{X} = \lambda + \tilde{z}_\lambda^r$ $\tilde{Y} = (\lambda(\lambda - 1))^{\frac{2r-p}{r}} \tilde{z}_\lambda^{2r-p} (1 + \tilde{y}_\lambda(\tilde{z}_\lambda^r))$ $\tilde{Z} = 1$	$\tilde{X} = \tilde{z}_\infty^{2r-p+q} (1 + x_\infty(\tilde{z}_\infty^r))$ $\tilde{Y} = 1$ $\tilde{Z} = \tilde{z}_\infty^{3r-p+q}$

Table 3: Puiseux expansions around singular points

We observe that the Puiseux expansion in Table 2 for $SE(\lambda)_{r,p,q}^{2r}$ coincide with the Puiseux expansion in Table 3 for $SE(\lambda)_{r,p,q}^r$ when using Equations (4.35) together with the transformation between the local normalization parameters given by

$$\tilde{z}_0 = z_0^2, \quad \tilde{z}_1 = z_1^2, \quad \tilde{z}_\lambda = z_\lambda^2, \quad \tilde{z}_\infty = z_\infty^2. \tag{4.38}$$

Notice that by construction from the defining equations for $SE(\lambda)_{r,p,q}^{2r}$ and $SE(\lambda)_{r,p,q}^r$ we have $(1 + y_0(t))^2 = 1 + \tilde{y}_0(t)$, $(1 + y_1(t))^2 = 1 + \tilde{y}_1(t)$, and $(1 + y_\lambda(t))^2 = 1 + \tilde{y}_\lambda(t)$. In Lemma 3.3 we proved that the involution automorphism (3.11) of the action $G_0 = \mathbb{Z}_2$ defined on $SE(\Lambda_1^2)_{r,p,q}^{2r}$ lifts to the minimal resolution as $z_0, z_1, z_\lambda, z_\infty \mapsto -z_0, -z_1, -z_\lambda, -z_\infty$. As Equation (4.38) is invariant under this action, the minimal resolution of $SE(\lambda)_{r,p,q}^r$ is the quotient curve C_1/G_0 . \square

Proof. The generalized Kummer surface S_0 is the minimal resolution of $V_0 = T/G_0$ with $T = C_1 \times C_2$ and $G_0 = \mathbb{Z}_2$ obtained by blowing up 16 ordinary double point singularities (cf. Lemma 4.1). V_0 is the singular double cover of $C_1/G_0 \times C_2/G_0$ whose branch curve is a union of vertical and horizontal curves. By blowing up the branch locus we obtain S_0 still having a map to C_1/G_0 . This establishes the isotrivial fibration $f'_1 : S_0 \rightarrow C_1/G_0$. The other isotrivial fibration is obtained in an analogous manner. The composition

$$C_1/G_0 = C'_1 \xrightarrow{\psi'_1} SE(\Lambda_1^2)_{r,p,q}^r \xrightarrow{\phi'_1} \mathbb{P}^1$$

maps every point on the base curve C'_1 of f'_1 to a point in \mathbb{P}^1 with affine coordinate \tilde{U} . For each fiber away from the ramification points, i.e., $\tilde{U} \notin \{0, 1, \Lambda_1^2, \infty\}$, the fiber coordinates when composed with the normalization map $\psi_2 : C_2 \rightarrow SE(\Lambda_2^2)_{r,2r-p,2r-q}^{2r}$ then satisfy Equation (4.16). \square

Similarly, we have the following lemma describing the non-isotrivial fibration on a generalized Kummer variety:

PROPOSITION 4.14. *Assume that $r > 2$. On the generalized Kummer variety S_0 there is a fibration $\tilde{f} : S_0 \rightarrow \tilde{C}$ such that the genus of the fibres equals $2r - 1$ and \tilde{C} is a smooth irreducible curve of genus $r - 1$. The base curve \tilde{C} admits a meromorphic*

map to \mathbb{P}^1 such that the general fiber is the superelliptic curve associated with the singular curve

$$y^{2r} = (u - A)^{2r-p} (u - B)^{2r-p} x^{3r-p-q} (x^2 + 2(1 - 2u)x + 1)^p, \tag{4.39}$$

where the affine coordinate on the base curve is $u \in \mathbb{P}^1$, and the moduli are related by

$$A = \frac{(\Lambda_1 + \Lambda_2)^2}{4\Lambda_1\Lambda_2}, \quad B = \frac{(\Lambda_1\Lambda_2 + 1)^2}{4\Lambda_1\Lambda_2}. \tag{4.40}$$

It follows that

$$\pi^* \left(du \wedge \frac{dx}{y} \right) = \frac{2^{2-\frac{2p}{r}} \Lambda_1^{\frac{3}{2}-\frac{p}{2r}-\frac{q}{2r}} \Lambda_2^{\frac{1}{2}-\frac{p}{2r}+\frac{q}{2r}}}{(\Lambda_1^2 - 1)^{1-\frac{p}{r}}} \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2}. \tag{4.41}$$

In particular, $du \wedge dx/y$ is a holomorphic differential two-form on S_0 .

Proof. The automorphism of V_0 given by Equations (4.26) and (4.27) lifts to a diffeomorphism Φ of S_0 . Since the isotrivial fibration in Proposition 4.12 is a genus $r - 1$ pencil with $r - 1 \geq 2$, then the Isotropic Subspace Theorem [10, Thm. 1.10] implies that $\Phi f'_1$ determines a genus $r - 1$ pencil as well. Then it follows from Seiberg-Witten theory by [11, Thm. 2.9] that the genus of the fibres of f'_1 and \tilde{f} are equal. The necessary condition of Φ being orientation preserving follows from Equation (4.29).

The projection map in Equation (4.26) from a point on $T = C_1 \times C_2$ to $U \in \mathbb{P}^1$ – where ζ_1 and ζ_2 are understood to be composed with their respective normalization maps $\psi_1 : C_1 \rightarrow SE(\Lambda_1^2)_{r,p,q}^{2r}$ and $\psi_2 : C_2 \rightarrow SE(\Lambda_2^2)_{r,2r-p,2r-q}^{2r}$ for their minimal resolutions – descends to a surjective map $S_0 \rightarrow \mathbb{P}^1$ such that X, Y and U defined in Equation (4.27) and Equation (4.26), respectively, are rational functions on the fiber and base curve, respectively. The transformation

$$x = \frac{X}{\Lambda_1\Lambda_2 U(U-1)^2}, \quad y = -\frac{(A-B)^{2-\frac{p}{r}} \Lambda_2^{1-\frac{q}{r}} Y}{(\Lambda_1\Lambda_2)^{\frac{3}{2}+\frac{p}{2r}-\frac{q}{2r}} (1-\Lambda_2^2)^{1-\frac{p}{r}} U(U-1)^4}, \tag{4.42}$$

is an isomorphism on each fiber for $u \notin \{0, 1, A, B, \infty\}$. Together with Equations (4.40) and

$$u = B + \frac{B-A}{U-1}, \tag{4.43}$$

they relate Equation (4.39) to Equation (4.28). Therefore, using Lemma 3.6 any fiber over $u \notin \{0, 1, A, B, \infty\}$ is a (rescaled) superelliptic curve $SE(\Lambda^2)_{r,2r-p,2r-q}^{2r}$ with $u = (1 + \Lambda)^2/(4\Lambda)$ whose minimal resolution is a smooth irreducible curve of genus $2r - 1$. Equation (4.41) then follows from Equation (4.29) and the transformations in Equations (4.42) and (4.43). Moreover, Equation (4.41) proves that up to a non-vanishing scalar that only depends on the moduli the two-form $du \wedge dx/y$ equals the holomorphic two-form ω already constructed in Section 4.1. \square

REMARK 32. For $r = p = q = 1$, Equation (4.39) defines the Jacobian elliptic fibration \mathfrak{J}_7 on a Kummer surface of two non-isogenous elliptic curves with singular fibers of Kodaira-type I_0^* over $u = A, B$, and of type I_1 and I_4^* over $u = 0, 1$ and $u = \infty$. It is the quadratic twist of the extremal Jacobian elliptic rational surface that is the modular elliptic surface for $\Gamma_0(4)$ (cf. Remark 3.7).

We can also start with the *twisted Legendre pencil of superelliptic curves* given by

$$y^{2r} = (u - A)^{2r-q} (u - B)^{2r-q} x^{2r-p} (x - 1)^{2r-p} (x - u)^{p+q-r} . \tag{4.44}$$

We want to determine under what circumstances the generalized fibration \mathfrak{J}_6 from Section 4.2.3 is the pull-back of the twisted Legendre pencil of superelliptic curves via a morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$, i.e., a fiber product $S_0 \times_{\mathbb{P}^1} \mathbb{P}^1$. If we apply the transformation

$$\begin{aligned} u &= \frac{1}{1 - \tilde{u}} , \\ x &= \frac{1}{1 - \tilde{x}} , \\ y &= \frac{\tilde{y}}{(1 - \tilde{x})^2 (1 - \tilde{u})^2 ((1 - \tilde{a})(1 - \tilde{b}))^{1 - \frac{q}{2r}}} , \\ (A, B) &= \left(\frac{1}{1 - \tilde{A}} , \frac{1}{1 - \tilde{B}} \right) , \end{aligned} \tag{4.45}$$

we obtain the family

$$\tilde{y}^{2r} = (\tilde{u} - 1)^{-p+q+r} (\tilde{u} - \tilde{A})^{2r-q} (\tilde{u} - \tilde{B})^{2r-q} \tilde{x}^{2r-p} (\tilde{x} - 1)^{p-q+r} (\tilde{x} - \tilde{u})^{p+q-r} . \tag{4.46}$$

We now assume that \tilde{u} is given by a base change via the degree-two map

$$U \in \mathbb{P}^1 \mapsto \tilde{u} = \frac{(U - 1)(U - \alpha^2\beta^2)}{(U - \alpha^2)(U - \beta^2)} \in \mathbb{P}^1 , \tag{4.47}$$

such that the pre-images of $\tilde{u} = 0, 1, \infty$ are given by $(1, \alpha^2\beta^2)$, $(0, \infty)$, and (α^2, β^2) , respectively. The branching points, i.e., points where $\partial_U \tilde{u} = 0$, are $U = \pm\alpha\beta$, and the corresponding ramification points are $\tilde{B} = (\alpha\beta - 1)^2 / (\alpha - \beta)^2$ and $\tilde{A} = (\alpha\beta + 1)^2 / (\alpha + \beta)^2$, respectively. Both have ramification index 2. Hence, by the Riemann-Hurwitz formula the map in Equation (4.47) is indeed a morphism from $\mathbb{P}^1 \rightarrow \mathbb{P}^1$. Therefore, if we set

$$\begin{aligned} \tilde{u} &= \frac{(U - 1)(U - \alpha^2\beta^2)}{(U - \alpha^2)(U - \beta^2)} , \\ \tilde{x} &= \frac{\beta^2 X}{U(U - \alpha^2)(U - \beta^2)} , \\ \tilde{y} &= \frac{\beta^{3+\frac{p}{r}-\frac{q}{r}} ((\alpha^2 - 1)(1 - \beta^2))^{\frac{5}{2}-\frac{p}{2r}-\frac{q}{2r}} (U^2 - \alpha^2\beta^2) Y}{U(U - \alpha^2)^3 (U - \beta^2)^3 (\alpha^2 - \beta^2)^{2-\frac{q}{r}}} , \\ (\tilde{A}, \tilde{B}) &= \left(\frac{(\alpha\beta + 1)^2}{(\alpha + \beta)^2} , \frac{(\alpha\beta - 1)^2}{(\alpha - \beta)^2} \right) , \end{aligned} \tag{4.48}$$

and let $(\alpha, \beta) = (\Lambda_1, 1/\Lambda_2)$, we obtain

$$\begin{aligned} Y^{2r} &= (1 - \Lambda_1^2)^{2(r-p)} U^{-2p+q+r} (U - \Lambda_1^2)^{q-r} (\Lambda_2^2 U - 1)^{q-r} \\ &\quad \times (\Lambda_2^2 U^2 - \Lambda_1^2)^{2(r-q)} X^{2r-p} (X - U(U - \Lambda_1^2)(\Lambda_2^2 U - 1))^{p+q-r} \\ &\quad \times (X - U(U - 1)(\Lambda_2^2 U - \Lambda_1^2))^{p+q-r} . \end{aligned} \tag{4.49}$$

Thus, we have proved the following lemma:

LEMMA 4.15. Equation (4.24) on $S_0^{(r,p,q=r)}(\Lambda_1^2, \Lambda_2^2)$ is the pull-back of the twisted Legendre pencil of superelliptic curves given by

$$y^{2r} = (u - A)^r (u - B)^r x^{2r-p} (x - 1)^{2r-p} (x - u)^p, \tag{4.50}$$

via the degree-two base change $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1, U \mapsto u$ given by

$$\frac{1}{1 - u} = \frac{(U - 1)(U - \alpha^2\beta^2)}{(U - \alpha^2)(U - \beta^2)} \tag{4.51}$$

with $(\alpha, \beta) = (\Lambda_1, 1/\Lambda_2)$. The moduli of the Legendre pencil are related to the moduli of $S_0^{(r,p,q=r)}(\Lambda_1^2, \Lambda_2^2)$ by the rational maps

$$A = \frac{(\Lambda_1 \Lambda_2 + 1)^2}{(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)}, \quad B = \frac{(\Lambda_1 \Lambda_2 - 1)^2}{(\Lambda_1^2 - 1)(\Lambda_2^2 - 1)}. \tag{4.52}$$

□

REMARK 33. The family

$$X_3^{2r} = X_1^{2r-p} (1 - X_1)^{2r-p} X_2^{2r-q} (1 - X_2)^{2r-q} (1 - z_1 X_1 - z_2 X_2)^{p+q-r} \tag{4.53}$$

becomes isomorphic to family (4.44), i.e.,

$$y^{2r} = (u - A)^{2r-q} (u - B)^{2r-q} x^{2r-p} (x - 1)^{2r-p} (x - u)^{p+q-r} \tag{4.54}$$

by setting $X_1 = x$ and

$$\begin{aligned} (z_1, z_2) &= \left(\frac{1}{A}, 1 - \frac{B}{A} \right), \\ X_2 &= \frac{u - A}{B - A}, \\ iy &= X_3 A^{\frac{1}{2} - \frac{p}{2r} - \frac{q}{2r}} (A - B)^{2 - \frac{q}{r}}. \end{aligned} \tag{4.55}$$

Lemma (4.15) then shows that Equation (4.53) coincides with Equation (4.24) describing the generalized fibration \mathfrak{J}_6 if and only if $q = r$. On the other hand, interchanging the roles $X_1 \leftrightarrow X_2$ and $z_1 \leftrightarrow z_2$, we obtain the family

$$y^{2r} = (u - A)^{2r-p} (u - B)^{2r-p} x^{2r-q} (x - 1)^{2r-q} (x - u)^{p+q-r}. \tag{4.56}$$

The transformation

$$\begin{aligned} x &\mapsto \frac{x^2 + 2(1 - 2u)x + 1}{4x} + u, \\ y &\mapsto \frac{(x^2 + 1)^{2 - \frac{q}{r}} (x^2 + 2(1 - 2u)x + 1)^{\frac{q}{2r} - \frac{1}{2}} y}{2^{3 + \frac{p}{r} - \frac{q}{r}} x^{3 - \frac{q}{r}}} \end{aligned} \tag{4.57}$$

maps family (4.56) to Equation (4.39) describing the generalized fibration \mathfrak{J}_7 . The transformation (4.57) is rational if and only if $q = r$. Note that in the process of relating Equations (4.24) and (4.39) we have thus interchanged the roles of affine base and fiber coordinates and simultaneously the roles of the superelliptic curves C_1 and C_2 , and applied fiberwise the rational transformation from Lemma 3.6.

REMARK 34. For $r = p = q = 1$, Equation (4.50) defines a Jacobian elliptic fibration, called \mathfrak{J}_6 in [22], on a Kummer surface of two non-isogenous elliptic curves with 2 singular fibers of Kodaira-type I_2^* and 4 singular fibers of type I_2 . This Jacobian elliptic $K3$ surface is obtained by the base transformation in Equation (4.51) from the modular elliptic surface for $\Gamma(2)$ (cf. Remark 3.7). In fact, if we denote by $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ the degree-two map in Equation (4.51), then the pull-back of the Jacobian rational elliptic surface S that is the modular elliptic surface for $\Gamma(2)$ is the Jacobian elliptic fibration \mathfrak{J}_6 . On X , we have the deck transformation ℓ interchanging the pre-images and the fiberwise elliptic involution $\iota : (u, x, y) \mapsto (u, x, -y)$. The composition $j = \iota \circ \ell$ is a Nikulin involution leaving $du \wedge dx/y$ invariant, and the minimal resolution of the quotient X/j is the Jacobian elliptic $K3$ surface $X' \rightarrow \mathbb{P}^1$ in Equation (4.44) with $p = q = r = 1$. The $K3$ surface X' is the quadratic twist of the modular elliptic surface S for $\Gamma(2)$ and thus has fibers of Kodaira-type I_0^* over the two ramification points of f . The situation is summarized in Figure 2. Comparing elliptic surfaces obtained by either a quadratic twist or a degree-two base transformation was also the basis for the iterative procedure in [14] that produced families of elliptically fibered Calabi-Yau n -folds with section from families of elliptic Calabi-Yau varieties of one dimension lower. It also played a key role in relating Yukawa couplings and Seiberg-Witten prepotential in [24].

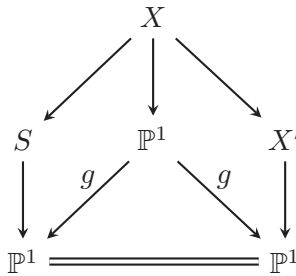


FIG. 2. Relation between twist and elliptic $K3$ surface \mathfrak{J}_6

5. Period computations on a generalized Kummer surface. Equation (4.39) in Lemma 4.14 defines a non-isotrivial fibration on the generalized Kummer surface $S_0^{(r, 2r-p, 2r-q)}(\Lambda_1^2, \Lambda_2^2)$ given by

$$y^{2r} = (u - A)^{2r-p} (u - B)^{2r-p} x^{3r-p-q} (x^2 + 2(1 - 2u)x + 1)^p \tag{5.1}$$

where u is the affine coordinate on a base \mathbb{P}^1 . We now construct a cover by pulling-back Equation (4.39) via the degree-two base change

$$\tilde{g} : \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad z \mapsto u = \frac{(1+z)^2}{4z}.$$

We have the following lemma:

LEMMA 5.1. *The pencil of superelliptic curves over \mathbb{P}^1 given by*

$$\begin{aligned} \tilde{Y}^{2r} &= z^{p+q-r} \left((z - a) \left(z - \frac{1}{a} \right) (z - b) \left(z - \frac{1}{b} \right) \right)^{2r-p} \\ &\times 2^{4p} \tilde{X}^{3r-p-q} (\tilde{X} - 1)^p (\tilde{X} - z^2)^p, \end{aligned} \tag{5.2}$$

is the pull-back of Equation (4.39) on $S_0^{(r,2r-p,2r-q)}(\Lambda_1^2, \Lambda_2^2)$ via the degree-two base change $\tilde{g} : \mathbb{P}^1 \rightarrow \mathbb{P}^1, z \mapsto u$ with the following relations between the affine coordinates on the base and moduli:

$$u = \frac{(1+z)^2}{4z}, \quad A = \frac{(1+a)^2}{4a}, \quad B = \frac{(1+b)^2}{4b}. \tag{5.3}$$

We also have

$$\tilde{g}^* \left(du \wedge \frac{dx}{y} \right) = 4(z^2 - 1) dz \wedge \frac{d\tilde{X}}{\tilde{Y}}. \tag{5.4}$$

Proof. By direct computation using $x = \tilde{X}/z, y = \tilde{Y}/(16z^3)$. \square

REMARK 35. The comparison of Equations (5.3) with Equations (4.40) shows that $a = \Lambda_2/\Lambda_1$ or $a = \Lambda_1/\Lambda_2$ and $b = \Lambda_1 \Lambda_2$ or $b = 1/(\Lambda_1 \Lambda_2)$.

The family of superelliptic curves over \mathbb{P}^1 defined by Equation (5.2) enables us to define a family of closed two-cycles $\Sigma_z^{(k,l)}$ with $1 \leq k, l \leq 2r - 1$ on its total space as follows: first, define a closed one-cycle in each fiber over $z \notin \{0, 1, a^{\pm 1}, b^{\pm 1}, \infty\}$ of the total space as a clockwise one-cycle around the line segment $[0, z^2]$ in the \tilde{X} -plane on a chosen \tilde{Y} -sheet. However, as we move around in the z -plane (which represents moving along the base curve) the overall complex factor in Equation (5.2) changes. Thus, for the multi-valued function \tilde{Y} to become well-defined we have to introduce branch cuts in the z -plane as well. We define a multi-valued function \tilde{Y} by writing

$$\tilde{Y} = 2^{4\beta_2} z^{\beta_1 + \beta_2 - \frac{1}{2}} \prod_{z_0 \in \{a^{\pm 1}, b^{\pm 1}\}} (z - z_0)^{1 - \beta_2} \tilde{X}^{\frac{3}{2} - \beta_1 - \beta_2} (1 - \tilde{X})^{\beta_2} (z^2 - \tilde{X})^{\beta_2} \tag{5.5}$$

with $\beta_1 = q/(2r), \beta_2 = p/(2r)$. We choose one branch cut in the z -plane connecting the points $z = a$ and $z = b$ while not intersecting any other branch points and cuts. We then define the closed two-cycle $\Sigma_z^{(k,l)}$ given two integers $1 \leq k, l \leq 2r - 1$ on the total space as the closed one-cycle that runs suitably close in clockwise orientation around the line segment $[a, b]$ on the k^{th} branch of the z -plane for the multi-valued function

$$z^{\beta_1 + \beta_2 - \frac{1}{2}} \prod_{z_0 \in \{a^{\pm 1}, b^{\pm 1}\}} (z - z_0)^{1 - \beta_2},$$

while taking a clockwise one-cycle around the line segment $[0, z^2]$ on the l^{th} branch of the \tilde{X} -plane for the multi-valued function

$$\tilde{X}^{\frac{3}{2} - \beta_1 - \beta_2} (1 - \tilde{X})^{\beta_2} (z^2 - \tilde{X})^{\beta_2}.$$

We then set $\Sigma_u^{(k,l)} = \tilde{g}_* \Sigma_z^{(k,l)}$ for $1 \leq k, l \leq 2r - 1$. We have the following lemma:

LEMMA 5.2. *The period of the holomorphic two-form $du \wedge dx/y$ over the two-cycle $\Sigma_u^{(k,l)}$ on $S_0^{(r,2r-p,2r-q)}(\Lambda_1^2, \Lambda_2^2)$ equals*

$$\begin{aligned} \oint_{\Sigma_u^{(k,l)}} du \wedge \frac{dx}{y} &= (-1)^{\beta_2} C_{2r}^{(k)} C_{2r}^{(l)} \frac{\Gamma(\alpha) \Gamma(1 - \beta_2) \Gamma(\beta_2)^2}{4^\alpha \Gamma(\beta_1 + \frac{1}{2}) \Gamma(2\beta_2)} \\ &\times \frac{1}{A^\alpha (A - B)^{1 - 2\beta_2}} F_2 \left(\alpha; \beta_1, \beta_2 \mid \frac{1}{A}, 1 - \frac{B}{A} \right) \end{aligned} \tag{5.6}$$

where $C_{2r}^{(k)} = (\rho_{2r} - 1)/\rho_{2r}^k$, $(-1)^{\beta_2} = \rho_{4r}^p$, $\rho_{2r} = \exp(\frac{2\pi i}{2r})$, and $1 \leq k, l \leq 2r - 1$, and the relation between A, B and Λ_1, Λ_2 is given in Equations (4.40).

Proof. We start by transforming the double integral into an iterated integral with each contour reduced to an integration along a branch cut (analogous to the approach in the proof of Lemma 3.9). We obtain

$$\begin{aligned} & \frac{1}{C_{2r}^{(k)} C_{2r}^{(l)}} \oint_{\Sigma_z^{(k,l)}} 4(z^2 - 1) dz \wedge \frac{d\tilde{X}}{\tilde{Y}} \\ &= \frac{1}{C_{2r}^{(k)} C_{2r}^{(l)}} \oint_{\Sigma_u^{(k,l)}} du \wedge \frac{dx}{y} \\ &= \int_A^B \frac{du}{(u - A)^{1-\beta_2} (u - B)^{1-\beta_2}} \int_0^z \frac{dx}{x^{\frac{3}{2}-\beta_1-\beta_2} (x^2 + 2(1 - 2u)x + 1)^{\beta_2}} \tag{5.7} \\ &= -e^{\pi i \beta_2} \int_A^B \frac{du}{(A - u)^{1-\beta_2} (u - B)^{1-\beta_2}} \\ & \quad \times \int_0^z \frac{dx}{x^{\frac{3}{2}-\beta_1-\beta_2} (x^2 + 2(1 - 2u)x + 1)^{\beta_2}}. \end{aligned}$$

Using the relations in Equation (5.3) and the branch of y compatible with Equation (5.5), the inner integral evaluates to

$$\begin{aligned} & \int_0^z \frac{dx}{x^{\frac{3}{2}-\beta_1-\beta_2} (x^2 + 2(1 - 2u)x + 1)^{\beta_2}} \\ &= \int_0^z \frac{dx}{x^{\frac{3}{2}-\beta_1-\beta_2} (z - x)^{\beta_2} (\frac{1}{z} - x)^{\beta_2}} \\ &= z^{\beta_1+\beta_2-\frac{1}{2}} \int_0^1 \frac{d\tilde{x}}{\tilde{x}^{\frac{3}{2}-\beta_1-\beta_2} (1 - \tilde{x})^{\beta_2} (1 - z^2 \tilde{x})^{\beta_2}} \\ &= z^{\beta_1+\beta_2-\frac{1}{2}} \frac{\Gamma(\beta_1 + \beta_2 - \frac{1}{2}) \Gamma(1 - \beta_2)}{\Gamma(\beta_1 + \frac{1}{2})} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| z^2\right). \tag{5.8} \end{aligned}$$

Note that the quadratic identity [1, Eq. (15.3.17)] is equivalent to the following identity for the Gauss' hypergeometric function

$$(1 + z)^{2\beta_1+2\beta_2-1} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| z^2\right) = {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_1 \\ 2\beta_1 \end{matrix} \middle| \frac{1}{u}\right) \tag{5.9}$$

with z and u related by Equation (5.3). Applying this identity to Equation (5.8) we obtain

$$\begin{aligned} & \int_0^z \frac{dx}{x^{\frac{3}{2}-\beta_1-\beta_2} (x^2 + 2(1 - 2u)x + 1)^{\beta_2}} \\ &= \frac{\Gamma(\beta_1 + \beta_2 - \frac{1}{2}) \Gamma(1 - \beta_2)}{\Gamma(\beta_1 + \frac{1}{2})} \left(\frac{1}{4u}\right)^{\beta_1+\beta_2-\frac{1}{2}} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_1 \\ 2\beta_1 \end{matrix} \middle| \frac{1}{u}\right). \tag{5.10} \end{aligned}$$

Combining this result with Equation (2.9) of Corollary 2.2, we obtain

$$\begin{aligned}
 & \oint\!\!\!\oint_{\Sigma_u^{(k,l)}} 4(z^2 - 1) dz \wedge \frac{d\tilde{X}}{\tilde{Y}} \\
 &= \oint\!\!\!\oint_{\Sigma_u^{(k,l)}} du \wedge \frac{dx}{y} \\
 &= -C_{2r}^{(k)} C_{2r}^{(l)} \frac{\Gamma(\alpha) \Gamma(1 - \beta_2)}{4^\alpha \Gamma(\beta_1 + \frac{1}{2})} \\
 &\quad \times e^{\pi i \beta_2} \int_A^B \frac{du}{(A - u)^{1 - \beta_2} (u - B)^{1 - \beta_2} u^\alpha} {}_2F_1\left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_1 \\ 2\beta_1 \end{matrix} \middle| \frac{1}{u}\right) \\
 &= e^{\pi i \beta_2} C_{2r}^{(k)} \frac{\Gamma(\alpha) \Gamma(1 - \beta_2) \Gamma(\beta_2)^2}{4^\alpha \Gamma(\beta_1 + \frac{1}{2}) \Gamma(2\beta_2)} \\
 &\quad \times \frac{1}{A^\alpha (A - B)^{1 - 2\beta_2}} F_2\left(\begin{matrix} \alpha; \beta_1, \beta_2 \\ 2\beta_1, 2\beta_2 \end{matrix} \middle| \frac{1}{A}, 1 - \frac{B}{A}\right).
 \end{aligned} \tag{5.11}$$

□

We now relate the periods over the two-cycles $\Sigma_u^{(k,l)}$ – defined naturally from the point of view of the generalized fibration \mathfrak{J}_7 – to the periods over the two-cycles $\sigma_{i,j}$ – defined naturally from the point of view of the generalized Kummer construction in Section 4.1. We have the following lemma:

LEMMA 5.3. *On the generalized Kummer surface $S_0^{(r,2r-p,2r-q)}(\Lambda_1^2, \Lambda_2^2)$, the periods of the holomorphic two-form ω over the two-cycles $\sigma_{i,j}$ (defined in Section 4.1) are related to the periods of the holomorphic two-form $dU \wedge dX/Y$ over two-cycles $\Sigma_u^{(i+k-1, j+r+1-k)}$ by*

$$\oint\!\!\!\oint_{\sigma_{i,j}} \omega = \oint\!\!\!\oint_{\Sigma_u^{(i+k-1, j+r+1-k)}} dU \wedge \frac{dX}{Y} \tag{5.12}$$

with $1 \leq i, j \leq r - 1$ and for all k with $1 \leq k \leq r - 1$.

Proof. We will use the variables U and X from Equations (4.26) and (4.27), i.e.,

$$\begin{aligned}
 U &= \frac{(\zeta_2 - \Lambda_2^2)(\zeta_1 - \zeta_2)}{(\zeta_2 - 1)(\Lambda_2^2 \zeta_1 - \zeta_2)}, \\
 x &= \frac{X}{\Lambda_1 \Lambda_2 U (U - 1)^2} = \frac{\Lambda_2 \zeta_1}{\Lambda_1 \zeta_2}.
 \end{aligned} \tag{5.13}$$

Note that because of Equation (4.43) the limits $u = A$ and $u = B$ are equivalent to $U = 0$ and $U = \infty$, respectively. Solving Equations (5.13) for (ζ_1, ζ_2) , we obtain

$$\begin{aligned}
 \zeta_1 &= \frac{\Lambda_1 x (\Lambda_1 \Lambda_2 U x - \Lambda_1 \Lambda_2 x - U + \Lambda_2^2)}{\Lambda_1 \Lambda_2^2 U x - \Lambda_1 x - \Lambda_2 U + \Lambda_2}, \\
 \zeta_2 &= \frac{\Lambda_2 (\Lambda_1 \Lambda_2 U x - \Lambda_1 \Lambda_2 x - U + \Lambda_2^2)}{\Lambda_1 \Lambda_2^2 U x - \Lambda_1 x - \Lambda_2 U + \Lambda_2}.
 \end{aligned} \tag{5.14}$$

Thus, the line with $U = 0$ and x unrestricted corresponds to the line with $\zeta_2 = \Lambda_2^2$ and $\zeta_1 = \Lambda_1^2 \Lambda_2^2 x$, i.e., ζ_1 unrestricted. Similarly, the line with $U = \infty$ and x unrestricted

corresponds to the line with $\zeta_2 = 1$ and $\zeta_1 = \Lambda_1^2 x$, i.e., ζ_1 unrestricted. On the other hand, the line with $x = 0$ and U unrestricted corresponds to the line with $\zeta_1 = 0$ and ζ_2 unrestricted. Similarly, the line with $x = z$ and U unrestricted corresponds to the line with $\zeta_1 = \Lambda_1^2$ and ζ_2 unrestricted. Combining these results with Lemma 5.3, Lemma 4.14, and Remark 16, we obtain

$$\oint_{\Sigma_u^{(i+k-1, j+r+1-k)}} dU \wedge \frac{dX}{Y} = -C_{2r}^{(j)} \int_{\Lambda_2^2}^1 \frac{d\zeta_2}{\eta_2} C_{2r}^{(i)} \int_0^{\Lambda_1^2} \frac{d\zeta_1}{\eta_1} = \oint_{b_j} \frac{d\zeta_2}{\eta_2} \cdot \oint_{a_i} \frac{d\zeta_1}{\eta_1}, \tag{5.15}$$

where we have used $C_{2r}^{(j+r+1-k)} C_{2r}^{(i+k-1)} = -C_{2r}^{(i)} C_{2r}^{(j)}$. The lemma follows from

$$\oint_{\sigma_{i,j}} \omega = \iint_{\gamma_i \times \gamma_j} \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2} = \oint_{a_i \times b_j} \frac{d\zeta_1}{\eta_1} \boxtimes \frac{d\zeta_2}{\eta_2}.$$

□

REMARK 36. Equation (5.12) simply states that the periods of a fixed holomorphic two-form on the generalized Kummer surface must be the same when evaluated using either the fibration from Section 4.2.4 or the isotrivial fibration from Section 4.2.1.

We have arrived at the following theorem:

THEOREM 5.4. *The Multivariate Clausen Identity (2.26) is equivalent to Equation (5.12).*

Proof. The generalized Kummer surface $S_0^{(r, 2r-p, 2r-q)}$ was constructed from the two curves

$$\begin{aligned} SE(\Lambda_1^2)_{r, 2r-p, 2r-q}^{2r} : \eta_1^{2r} &= \zeta_1^{3r-p-q} (\zeta_1 - 1)^p (\zeta_1 - \Lambda_1^2)^p, \\ SE(\Lambda_2^2)_{r, p, q}^{2r} : \eta_2^{2r} &= \zeta_2^{p+q-r} (\zeta_2 - 1)^{2r-p} (\zeta_2 - \Lambda_2^2)^{2r-p}. \end{aligned} \tag{5.16}$$

With $\beta_1 = q/(2r)$, $\beta_2 = p/(2r)$ and $\alpha = \beta_1 + \beta_2 - \frac{1}{2}$, we found

$$\frac{\Gamma(\beta_1 + \frac{1}{2})}{\Gamma(\alpha) \Gamma(1 - \beta_2)} \frac{1}{\Lambda_1^{2\beta_1-1}} \int_0^{\Lambda_1^2} \frac{d\zeta_1}{\eta_1} = {}_2F_1 \left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \middle| \Lambda_1^2 \right). \tag{5.17}$$

Similarly, using $\zeta_2 = 1 - (1 - \Lambda_2^2) \zeta_2$ we derived

$$\frac{\Gamma(2\beta_2)}{\Gamma(\beta_2)^2} \frac{(-1)^{1-\beta_2}}{(1 - \Lambda_2^2)^{2\beta_2-1}} \int_{\Lambda_2^2}^1 \frac{d\zeta_2}{\eta_2} = {}_2F_1 \left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \middle| 1 - \Lambda_2^2 \right). \tag{5.18}$$

Moreover, when written out explicitly Equation (5.12) is equivalent to

$$\begin{aligned} & \frac{(-1)^{-\beta_2}}{C_{2r}^{(j+r+1-k)} C_{2r}^{(i+k-1)}} \frac{\Gamma(\beta_1 + \frac{1}{2}) \Gamma(2\beta_2)}{\Gamma(\alpha) \Gamma(1 - \beta_2) \Gamma(\beta_2)^2} \\ & \times (A - B)^{1-2\beta_2} \oint_{\Sigma_u^{(i+k-1, j+r+1-k)}} du \wedge \frac{dx}{y} \\ & = (\Lambda_1 \Lambda_2)^\alpha \left(\frac{\Gamma(\beta_1 + \frac{1}{2})}{\Gamma(\alpha) \Gamma(1 - \beta_2)} \frac{1}{\Lambda_1^{2\beta_1-1}} \int_0^{\Lambda_1^2} \frac{d\zeta_1}{\eta_1} \right) \\ & \times \left(\frac{\Gamma(2\beta_2)}{\Gamma(\beta_2)^2} \frac{(-1)^{1-\beta_2}}{(1 - \Lambda_2^2)^{2\beta_2-1}} \int_{\Lambda_2^2}^1 \frac{d\zeta_2}{\eta_2} \right). \end{aligned} \tag{5.19}$$

where we have used Equation (4.41) and

$$\begin{aligned} & \frac{\beta^{2-\frac{2p}{r}} \Lambda_1^{\frac{3}{2}-\frac{p}{2r}-\frac{q}{2r}} \Lambda_2^{\frac{1}{2}-\frac{p}{2r}+\frac{q}{2r}}}{(\Lambda_1^2 - 1)^{1-\frac{p}{r}}} (A - B)^{1-2\beta_2} \\ &= (\Lambda_1 \Lambda_2)^\alpha \frac{1}{\Lambda_1^{2\beta_1-1}} \frac{1}{(1 - \Lambda_2^2)^{2\beta_2-1}}. \end{aligned} \tag{5.20}$$

We simplify the left hand side using the result of Lemma 5.2 and the right hand side using Equations (5.18) and (5.19). We obtain

$$\begin{aligned} & \frac{1}{(4A)^\alpha} F_2 \left(\alpha; \beta_1, \beta_2 \mid \frac{1}{A}, 1 - \frac{B}{A} \right) \\ &= (\Lambda_1 \Lambda_2)^\alpha {}_2F_1 \left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ \beta_1 + \frac{1}{2} \end{matrix} \mid \Lambda_1^2 \right) {}_2F_1 \left(\begin{matrix} \beta_1 + \beta_2 - \frac{1}{2}, \beta_2 \\ 2\beta_2 \end{matrix} \mid 1 - \Lambda_2^2 \right). \end{aligned} \tag{5.21}$$

This is Equation (2.27) which is equivalent to the Multivariate Clausen Identity (2.26) and thus implies Theorem 2.5. \square

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$$y = 1 + \frac{\alpha}{1} \cdot \frac{\beta}{\gamma} x + \frac{\alpha \cdot \alpha + 1}{1.2} \cdot \frac{\beta \cdot \beta + 1}{\gamma \cdot \gamma + 1} x^2 + \dots$$

ein Quadrat von der Form

$$z = 1 + \frac{\alpha'}{1} \cdot \frac{\beta'}{\gamma'} \cdot \frac{\delta'}{\epsilon'} x + \frac{\alpha' \cdot \alpha' + 1}{1.2} \cdot \frac{\beta' \cdot \beta' + 1}{\gamma' \cdot \gamma' + 1} \cdot \frac{\delta' \cdot \delta' + 1}{\epsilon' \cdot \epsilon' + 1} x^2 + \dots$$

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