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TRANSLATING SOLITONS IN ARBITRARY CODIMENSION*

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Abstract. We study the translating solitons of the mean curvature flow. Although many authors study translating solitons in codimension one, there are few references and examples for higher codimensional cases except for Lagrangian translating solitons. First we observe non-trivial examples of translating solitons in arbitrary codimension. We will see that they have the property called parallel principal normal (PPN). Inspired by this fact and the work of Smoczyk for self-shrinkers in 2005, we then characterize the complete translating solitons with PPN.

Key words. Mean curvature flow, translating soliton, parallel principal normal.

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1. Introduction and the main theorem.

1.1. Translating solitons. Let $F : M^m \to \mathbb{R}^n$ be a smooth immersion of an m-dimensional Riemannian submanifold M^m in $\mathbb{R}^n$, $m \geq 2$, $n \geq 3$, with the second fundamental form B, where $p := n - m$ is the codimension of M^m. A translating soliton is a submanifold in $\mathbb{R}^n$ which satisfies the following equation:

$$H = T^{\perp}, \tag{1.1}$$

where $H := \mathrm{Trace}B \in T^{\perp}M$ is the mean curvature vector of M^m, and $T^{\perp}$ is the normal component of a constant vector T in $\mathbb{R}^n$.

Translating solitons arise from the theory of mean curvature flow. In general this flow has finite time singularities which are classified into two categories: Type I and Type II. By rescaling at a singularity we obtain a limiting submanifold. It is well known that a limiting submanifold of Type I must be a self-shrinker that satisfies the equation $H = -1/2F^{\perp}$. On the other hand a limiting submanifold of Type II must be an eternal solution (which exists for all the time $-\infty < t < \infty$). Translating solitons are important examples of eternal solutions. They do not change their shape under the mean curvature flow up to translation in Euclidean space.

REMARK 1.1. In the case of hypersurfaces ($p = 1$), Huisken-Sinestrari [6] showed that if the initial surface is mean convex and the singularity is of Type II, then any limit hypersurface is a convex eternal solution and must be a translating soliton.

Although many authors study translating solitons in the case $p = 1$, few results are known about higher codimensional cases except for Lagrangian translating solitons. Our aim in this paper is to introduce simple (but non-trivial) examples of translating solitons in arbitrary dimension and codimension ($m \geq 2, p \geq 1$) and observe their properties.

In what follows we assume that the mean curvature vector does not vanish anywhere on M^m. Now we can define the special direction globally:

$$\nu := \frac{H}{|H|}, \tag{1.2}$$

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which is called the principal normal (or normalized mean curvature vector).

Definition 1.2. We say that a submanifold has a parallel principal normal (we write PPN, for short) if the principal normal ν is parallel with respect to the normal connection of $T^{\perp}M$, that is,

$$\nabla^{\perp}\nu \equiv 0. \tag{1.3}$$

We also need another concept of a normal bundle.

Definition 1.3. We say that a submanifold has a flat normal bundle if the normal curvature of $T^{\perp}M$ with respect to $\nabla^{\perp}$ vanishes identically, that is $R^{\perp} \equiv 0$.

Remark 1.4. (See also [13], Remarks, p.5) All hypersurfaces trivially have a PPN and a flat normal bundle. We also know that for codimension $p = 2$, PPN implies the flatness of the normal bundle.

The examples we introduce in the later sections are submanifolds with PPN or flat normal bundle. Conversely, by observing these examples, we characterize translating solitons with PPN. To do so, we use the technique by Huisken [5] for self-shrinkers in codimension one. Smoczyk [13] developed Huisken's technique to higher codimensional self-shrinkers with PPN. Moreover Martin, Savas-Halilaj and Smoczyk [9] used similar technique for translating solitons in codimension one. Fortunately their technique is applicable to translating solitons in higher codimension with PPN. Our main results are the following.

Theorem A. *Let N^{m-1} be any complete minimal submanifold of the unit sphere $S^{n-2}(1) \subset \mathbb{R}^{n-1}$, and $r(t) : \mathbb{R}_+ \to \mathbb{R}$ be a function satisfying the ODE*

$$\ddot{r}(t) = \big(1 + \dot{r}(t)^2\big)\left(1 - \frac{(m-1)\dot{r}(t)}{t}\right).$$

Set $M^m := \mathbb{R}_+ \times N^{m-1}$ and define an immersion $F : M^m \to \mathbb{R}^n$ by

$$F(t, q) := \big(tq, r(t)\big) \in \mathbb{R}^n,$$

where $q \in N^{m-1} \subset S^{n-2}(1) \subset \mathbb{R}^{n-1}$ and $t \in \mathbb{R}_+$. Then a submanifold $F(M^m) \subset \mathbb{R}^n$ is a translating soliton with the direction $T = (0, \cdots, 0, 1) \in \mathbb{R}^n$.

We describe the details of this theorem in Section 3. By using Theorem A we can construct many complete translating solitons with flat normal bundle in codimension two (see Example 3.4 and Corollary 3.5).

To characterize a submanifold with PPN, we use the quantity $P := \langle B, H\rangle$, that is, the second fundamental form with respect to H.

Theorem B. *A complete translating soliton M^m with PPN in arbitrary codimension such that $|P^2|/|H|^4$ attains its maximum in M^m can only be the product immersion consisting of the grim reaper $\gamma \subset \mathbb{R}^2$ and a complete minimal submanifold $L^{m-1} \subset \mathbb{R}^{n-2}$.*

Remark 1.5. Recently we showed a Bernstein type theorem (a non existence result) for translating solitons with flat normal bundle in arbitrary codimension [7]. Xin obtained a more general result without flat normal condition [15]. Furthermore we also showed a time-dependent Bernstein theorem for eternal solutions in codimension one [8].

2. Preliminaries. We follow the setting in Smoczyk [13]. Since we consider isometric immersions we locally identify M^m with $F(M^m)$. Let ∇ and $\overline{\nabla}$ be Levi-Civita connections on M^m and $\mathbb{R}^n$, respectively. The second fundamental form B is defined by $B(X,Y) = (\overline{\nabla}_X Y)^\perp$ for any tangent vectors X, Y of M^m. For notational simplicity, we use ∇ for natural connections on various bundles except for the normal connection $\nabla^\perp$. For $\xi \in T^\perp M$ the shape operator $A^\xi : TM \to TM$ is defined by $A^\xi(X) = -(\overline{\nabla}_X \xi)^\top$, where $(\cdots)^\top$ denotes the tangent component. The identity $\langle B(X,Y), \xi\rangle = \langle A^\xi(X), Y\rangle$ is satisfied. Here $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. Our convention for the curvature tensor R of M^m is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad \forall X, Y, Z \in TM,$$

and the curvature tensor $R^\perp$ of $T^\perp M$ and $\overline{R}$ of $\mathbb{R}^n$ are defined by the same rule. We use the basis with respect to the local coordinate on M^m, $\{\frac{\partial}{\partial x^i}\}$ and $g_{ij} := \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\rangle$ is the induced metric on M^m. We write (g^{ij}) as the inverse matrix of (g_{ij}); then the mean curvature vector is $H = g^{ij}B_{ij}$. Here we use Einstein's notation. It will be convenient to raise and lower indices using the metric tensors g_{ij}, g^{ij}. For instance

$$B^i_j = g^{ki}B_{kj}.$$

We also use a local orthonormal normal frame $\nu_\alpha \in T^\perp M$. We use the following range of indices:

$$1 \le i,j,k \le m, \quad 1 \le \alpha,\beta,\gamma \le n-m.$$

Set $R_{ijkl} := \langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\rangle$, $B_{ij} := B(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, and $R^\perp_{ij} := R^\perp(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. Now the Gauss, Codazzi and Ricci equations are given by:

$$R_{ijkl} = \langle B_{ik}, B_{jl}\rangle - \langle B_{il}, B_{jk}\rangle, \tag{2.1}$$

$$\nabla^\perp_i B_{jk} = \nabla^\perp_j B_{ik}, \tag{2.2}$$

$$R^\perp_{ij} = B_{ik} \wedge B^k_j. \tag{2.3}$$

We define

$$P_{ij} := \langle B_{ij}, H\rangle, \quad Q_{ij} := \langle B^k_i, B_{kj}\rangle, \quad S_{ijkl} := \langle B_{ij}, B_{kl}\rangle.$$

By the Gauss equation, it is easy to check that the Ricci curvature tensor is given by

$$R_{ij} := g^{kl}R_{ikjl} = P_{ij} - Q_{ij}.$$

We know the following Simon's identities. Let $\Delta := g^{ij}\nabla_i\nabla_j$ be the Laplace-Beltrami operator on M^m.

PROPOSITION 2.1 (See [13], Section 2). *On a submanifold $M^m \subset \mathbb{R}^n$, we have*

$$\nabla^\perp_i\nabla^\perp_j H = \Delta^\perp B_{ij} + R_{ikjl}B^{kl} - R^k_i B_{kj} + Q^k_j B_{ki} - S_{ikjl}B^{kl}. \tag{2.4}$$

$$2\langle B^{ij}, \nabla^\perp_i\nabla^\perp_j H\rangle = \Delta|B|^2 - 2|\nabla^\perp B|^2 + 2|S|^2 - 2\langle P, Q\rangle + 2|R^\perp|^2. \tag{2.5}$$

3. Translating solitons in arbitrary codimension. In this section, we investigate examples of translating solitons with PPN or flat normal bundle in arbitrary codimension. Except for hypersurfaces or Lagrangian translating solitons, few examples are known so far. By using well-known wing-like curves of Clutterbuck-Schnürer-Schulze [3], we construct many complete translating solitons with flat normal bundle in arbitrary codimension in the following. This construction is an analogue of the result by Choe-Hoppe [2] for minimal submanifolds.

3.1. Examples of translating solitons in arbitrary codimension. Recall our first main theorem.

THEOREM 3.1 (Theorem A). *Let N^{m-1} be any complete minimal submanifold of the unit sphere $S^{n-2}(1) \subset \mathbb{R}^{n-1}$, and $r(t) : \mathbb{R}_+ \to \mathbb{R}$ be a function satisfying the ODE*

$$\ddot{r}(t) = \big(1 + \dot{r}(t)^2\big)\left(1 - \frac{(m-1)\dot{r}(t)}{t}\right). \tag{3.1}$$

Set $M^m := \mathbb{R}_+ \times N^{m-1}$ and define an immersion $F : M^m \to \mathbb{R}^n$ by

$$F(t, q) := \big(tq, r(t)\big) \in \mathbb{R}^n, \tag{3.2}$$

where $q \in N^{m-1} \subset S^{n-2}(1) \subset \mathbb{R}^{n-1}$ and $t \in \mathbb{R}_+$. Then a submanifold $F(M^m) \subset \mathbb{R}^n$ is a translating soliton with the direction $T = (0, \cdots, 0, 1) \in \mathbb{R}^n$.

Proof. Let $(x^2, \cdots, x^m)$ be a local coordinate of $q \in N^{m-1}$. We easily compute

$$F_1 := \frac{\partial F}{\partial t} = (q, \dot{r}), \quad F_i := \frac{\partial F}{\partial x^i} = (tq_i, 0), \quad 2 \leq i \leq m,$$

where $q_i := \frac{\partial q}{\partial x_i} \in T_q S^{n-2}$. Since $q \in S^{n-2}(1)$, $\langle q, q_i \rangle = 0$. Furthermore

$$F_{11} = (0, \ddot{r}), \quad F_{1j} = (q_j, 0), \quad F_{ij} = (tq_{ij}, 0), \quad 2 \leq i, j \leq m.$$

As an orthonormal frame of $T^\perp M$, we can take

$$\nu_1 = \frac{1}{\sqrt{1 + \dot{r}^2}}(-\dot{r}q, 1), \quad \nu_\alpha = (\widetilde{\nu}_\alpha, 0), \quad 2 \leq \alpha \leq n - m,$$

where $(\widetilde{\nu}_2, \cdots, \widetilde{\nu}_{n-m})$ is normal to N and tangent to $S^{n-2} \subset \mathbb{R}^{n-1}$. Now the metric tensor g of $F(M^m)$ can be computed as follows:

$$(g_{ij}) = \begin{pmatrix} 1 + \dot{r}^2 & O \\ O & t^2(\widetilde{g}_{ij}) \end{pmatrix},$$

where $\widetilde{g}_{ij}$, $(2 \leq i, j \leq m)$ are the components of the metric tensor of N^{m-1}. Let C, D and E be second fundamental forms of $N \subset F(M^m), N \subset S^{n-2}(1)$ and $S^{n-2}(1) \subset \mathbb{R}^n$, respectively. Then the second fundamental form B of $F(M^m) \subset \mathbb{R}^n$ is

$$(h^\alpha_{ij}) = (\langle B_{ij}, \nu_\alpha \rangle) = \begin{pmatrix} h^\alpha_{11} & O \\ O & t(\widetilde{h}^\alpha_{ij}) \end{pmatrix}, \quad 1 \leq \alpha \leq m - n,$$

where $h^1_{11} = \ddot{r}/(1 + \dot{r}^2)^{\frac{1}{2}}$, $h^2_{11} = \cdots = h^{n-m}_{11} = 0$, and

$$\widetilde{h}^\alpha_{ij} = \langle D_{ij} + E_{ij} - C_{ij}, \nu_\alpha \rangle, \quad 2 \leq i, j \leq m.$$

Since N^{m-1} is a minimal submanifold in $S^{n-2}(1)$, the mean curvature vector of M^m is given by

$$\begin{aligned} H = g^{ij}B_{ij} &= g^{11}B_{11} + \sum_{k,l=2}^{m} g^{kl}B_{kl} = g^{11}h^1_{11}\nu_1 + \frac{1}{t}\sum_{k,l=2}^{m}\widetilde{g}^{kl}\widetilde{h}^{\alpha}_{kl}\nu_\alpha \\ &= g^{11}h^1_{11}\nu_1 + \frac{1}{t}\sum_{k,l=2}^{m}\widetilde{g}^{kl}\langle D_{kl}+E_{kl}-C_{kl},\nu_\alpha\rangle\nu_\alpha \\ &= \frac{\ddot{r}}{(1+\dot{r}^2)^{\frac{3}{2}}}\nu_1 + \frac{1}{t}\langle H_{N;S^{n-2}} + (m-1)(-q,0) + H_{N;F(M)},\nu_\alpha\rangle\nu_\alpha \\ &= \left(\frac{\ddot{r}}{(1+\dot{r}^2)^{\frac{3}{2}}} + \frac{(m-1)\dot{r}}{t\sqrt{1+\dot{r}^2}}\right)\nu_1, \end{aligned}$$

where $H_{N;S^{n-2}}$ and $H_{N;F(M)}$ are the mean curvature vectors of $N \subset S^{n-2}(1)$ and $N \subset F(M^m)$, respectively. On the other hand, since $T = (0, \cdots, 0, 1)$,

$$T^{\perp} = \frac{1}{\sqrt{1+\dot{r}^2}}\nu_1. \tag{3.3}$$

Then, $F(M^m)$ is a translating soliton if $r(t)$ satisfies the differential equation (3.1). □

The equation (3.1) is well known as the equation for the graph of codimension one rotationally symmetric translating solitons in $\mathbb{R}^{m-1}$. If we assume

$$\lim_{t\to 0} r(t) = \lim_{t\to 0}\dot{r}(t) = 0,$$

then there is only one convex solution defined on $\mathbb{R}_+$ growing quadratically at infinity. Its profile curve generates the translating paraboloid. This translating paraboloid is a smooth entire graph. Therefore we can interpret our immersion as a generalization of the translating paraboloid of arbitrary codimension. However, for arbitrary codimension, this generalized translating paraboloid may be non-smooth at the origin $o \in \mathbb{R}^n$ because the profile curve touches the axis of rotation and the norm of the second fundamental form $|B|$ tends to infinity.

Therefore, taking another profile curve which does not touch the axis of rotation we construct complete (smooth) translating solitons in arbitrary codimension. To see this we need the next lemma.

LEMMA 3.2 (Clutterbuck-Schnürer-Schulze [3]). *For any $R > 0$ and any $\varphi_0 \in \mathbb{R}$, the initial value problem*

$$\begin{cases} \dot{\varphi}(t) = (1+\varphi^2)\Big(1 - \dfrac{(m-1)\varphi}{t}\Big), & t \geq R, \\ \varphi(R) = \varphi_0, \end{cases} \tag{3.4}$$

has a unique C^∞ solution φ on $[R,\infty)$. Moreover, as $t \to \infty$, we have an asymptotic expansion

$$\varphi(t) = \frac{t}{m-1} - \frac{1}{t} + O\Big(\frac{1}{t^2}\Big). \tag{3.5}$$

Now we regard the profile curve as the graph on the axis of rotation. More precisely instead of (3.2) we define the immersion as follows:

$$F(y,q) := \big(y, s(y)q\big), \tag{3.6}$$

for some function $s(y)$, where $q \in N^{m-1} \subset S^{n-2}(1) \subset \mathbb{R}^{n-1}, y \in \mathbb{R}$. By a similar computation as in Theorem 3.1, we can take $\nu := \frac{1}{\sqrt{1+\dot{s}^2}}(-\dot{s}, q)$, as the principal normal. The mean curvature vector of this immersion becomes

$$H = \left(\frac{\ddot{s}}{(1+\dot{s}^2)^{\frac{3}{2}}} - \frac{(m-1)}{s\sqrt{1+\dot{s}^2}} \right) \nu.$$

If the immersion is a translating soliton with the direction of $T = (0, \cdots, 0, 1)$, $s(y)$ must satisfy the following ordinary differential equation:

$$\ddot{s}(y) = \left(1 + \dot{s}(y)^2\right)\left(\frac{m-1}{s(y)} - \dot{s}(y) \right). \tag{3.7}$$

Equation (3.7) starting with $s(y_0) = R > 0, \dot{s}(y_0) = 0, y_0 \in \mathbb{R}$ is used by Clutterbuck-Schürer-Schulze. They point out that equation (3.7) has a strictly convex solution $s(y)$ in a small interval around y_0.

Returning to the equation (3.1), by Lemma 3.2, we obtain a curve which has two branches from the initial point (R, y_0) and these branches are asymptotic to the translating paraboloid (see Figure 1). This curve is called the wing-like curve and generates the translating catenoid for the hypersurfaces case which is complete. By the argument so far, our submanifold is a natural generalization of the translating catenoid in arbitrary codimension.

Corollary 3.3. *There exist many complete translating solitons in arbitrary codimension generated by any complete minimal submanifold N^{m-1} in the unit sphere $S^{n-2}(1)$ and a wing-like curve $W(t) \subset \mathbb{R}^2$.*

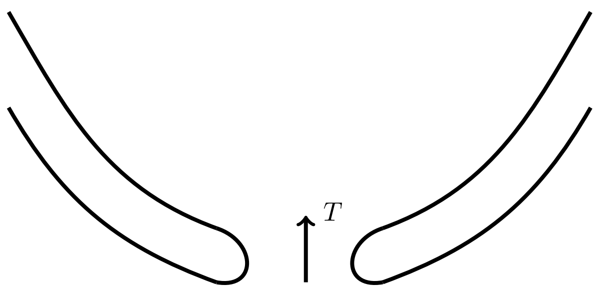

Fig. 1. *Wing-like curves*

If $|H| \neq 0$, we can define the principal normal $\nu := \nu_1$ of an immersion in Theorem 3.1. Note that the principal normal ν is parallel. This is easily checked by a direct calculation as follows:

$$\begin{aligned} \text{PPN} &\iff \nabla^{\perp}_X \nu = 0, \quad \forall X \in TM & (3.8)\\ &\iff \langle \overline{\nabla}_X \nu, \nu \rangle \nu + \langle \overline{\nabla}_X \nu, \nu_2 \rangle \nu_2 + \cdots + \langle \overline{\nabla}_X \nu, \nu_{m-n} \rangle \nu_{m-n} = 0 \\ &\iff \langle \frac{\partial \nu}{\partial t}, \nu_\alpha \rangle = \langle \frac{\partial \nu}{\partial x^i}, \nu_\alpha \rangle = 0, \quad 2 \leq i \leq m, \quad 2 \leq \alpha \leq n-m. & (3.9)\end{aligned}$$

Unfortunately the wing-like curve has the point point $(R, s(y_0))$ where $|H| = 0$, but we can uniquely extend the principal normal because this curve is smooth. Hence we may say this translating soliotn has a PPN.

The following is a simple example.

EXAMPLE 3.4. Let $r(t)$ be a solution of

$$\ddot{r}(t) = \big(1 + \dot{r}(t)^2\big)\left(1 - \frac{2\dot{r}(t)}{t}\right). \tag{3.10}$$

Then the following immersion is a translating soliton in the direction of $T = (0,0,0,0,1) \in \mathbb{R}^5$ with flat normal bundle:

$$F(t,x,y) := \left(\frac{t}{\sqrt{2}}\cos(x), \frac{t}{\sqrt{2}}\sin(x), \frac{t}{\sqrt{2}}\cos(y), \frac{t}{\sqrt{2}}\sin(y), r(t)\right) \in \mathbb{R}^5. \tag{3.11}$$

This immersion is made from the Clifford torus $S^1\big(\frac{1}{\sqrt{2}}\big) \times S^1\big(\frac{1}{\sqrt{2}}\big) \subset S^3(1) \subset \mathbb{R}^4$ and a curve $r(t)$. If we take the wing-like curve $W(t)$ as the solution to equation (3.10), then the immersion is complete. In particular the normal bundle is flat because the second fundamental forms are already diagonalized simultaneously by the local coordinate in (3.11). More generally we know that any codimension two submanifold in $\mathbb{R}^{n-1}$ which is also a hypersurface in $S^{n-2}(1) \subset \mathbb{R}^{n-1}$ has a flat normal bundle. Therefore we conclude the following.

COROLLARY 3.5. *Let $N^{n-3} \subset S^{n-2}(1) \subset \mathbb{R}^{n-1}$ be a complete minimal hypersurface in the unit sphere. Define the immersion by*

$$F(t,q) := \big(tq, r(t)\big) \in \mathbb{R}^n, \tag{3.12}$$

where $t \in \mathbb{R}$ and $q \in N^{n-3} \subset \mathbb{R}^{n-1}$. Assume that $r(t)$ satisfies the equation

$$\ddot{r}(t) = \big(1 + \dot{r}(t)^2\big)\left(1 - \frac{(n-3)\dot{r}(t)}{t}\right). \tag{3.13}$$

Then this immersion is a codimension two complete translating soliton with flat normal bundle if we take a wing-like curve $W(t)$ as a solution to (3.13).

3.2. Examples of translating solitons of product immersion. In this subsection, we study simpler examples than the examples in the previous subsection. First, we review the most famous and simplest translating soliton which is a plane curve called the "grim reaper".

EXAMPLE 3.6.

$$\gamma(x) = \{(x, -\log\cos x) | -\frac{\pi}{2} < x < \frac{\pi}{2}\} \subset \mathbb{R}^2.$$

The grim reaper is the only translating soliton which is a plane curve. Moreover a one dimensional translating soliton in the space $\mathbb{R}^n$ must be planar, that is, the grim reaper (see [1], Lemma 5.2). It is well known that the grim reaper is a geodesic of the Riemannian metric

$$\big\{\mathbb{R}^2, e^{\langle F,T\rangle}\langle\cdot,\cdot\rangle\big\}. \tag{3.14}$$

More generally, a translating submanifold $M^m \subset \mathbb{R}^n$ is a minimal submanifold with respect to the conformal metric $\big\{\mathbb{R}^n, e^{\langle F,T\rangle}\langle\cdot,\cdot\rangle\big\}$.

Before moving on to the next example, we need the following two lemmas.

LEMMA 3.7. *Let $L \subset \mathbb{R}^{n_1} \subset \mathbb{R}^n$ be a submanifold in $\mathbb{R}^{n_1}$. Then we have the following relation*

$$H_{L;\mathbb{R}^{n_1}} = H_{L;\mathbb{R}^n}, \tag{3.15}$$

where $H_{L;\mathbb{R}^{n_1}}$ and $H_{L;\mathbb{R}^n}$ are mean curvature vectors of the immersions $L \subset \mathbb{R}^{n_1}$ and $L \subset \mathbb{R}^n$ respectively.

LEMMA 3.8. *Let $M^m = L^{m_1} \times N^{m_2} \subset \mathbb{R}^n = \mathbb{R}^{n_1+n_2}$ be a product immersion with $L^{m_1} \subset \mathbb{R}^{n_1}$ and $N^{m_2} \subset \mathbb{R}^{n_2}$. Then $L, N \subset M$ are totally geodesic in M and the following relation holds*

$$H_{M;\mathbb{R}^n} = H_{L;\mathbb{R}^n} + H_{N;\mathbb{R}^n}. \tag{3.16}$$

Now we provide an easy example of a translating soliton by taking a product of two manifolds.

PROPOSITION 3.9. *Let $M^m = L^{m-m_1} \times N^{m_1} \subset \mathbb{R}^n$ be a product immersion with $L \subset \mathbb{R}^{n-m_1-1}$ and $N^{m_1} \subset \mathbb{R}^{m_1+1}$. If we assume $L \subset \mathbb{R}^{n-m_1-1}$ to be a complete minimal submanifold and $N \subset \mathbb{R}^{m_1+1}$ is a codimension one complete translating soliton with the direction of $T \in \mathbb{R}^{m_1+1} \subset \mathbb{R}^n$, then the immersion $M \subset \mathbb{R}^n$ is a complete translating soliton with PPN in the direction of $T \in \mathbb{R}^n$.*

Proof. Now $L \subset \mathbb{R}^{n-m_1-1} \subset \mathbb{R}^n$, $N \subset \mathbb{R}^{m_1+1} \subset \mathbb{R}^n$. Furthermore $L, N \subset M$ are totally geodesic. Therefore we can compute the mean curvature vector of M by Lemma 3.7 and Lemma 3.8,

$$H_{M;\mathbb{R}^n} = H_{L;\mathbb{R}^n} + H_{N;\mathbb{R}^n} = H_{L;\mathbb{R}^{n-m_1-1}} + H_{N;\mathbb{R}^{m_1+1}} = T^{\perp}.$$

The principal normal is $\nu = \nu_1 = (0, \widetilde{\nu})$ with $\widetilde{\nu} = \frac{H_{N;\mathbb{R}^{m_1+1}}}{|H_{N;\mathbb{R}^{m_1+1}}|}$. The other unit normals are $\nu_\alpha = (\widetilde{\nu}_\alpha, 0)$, $2 \le \alpha \le n-m$, where $\widetilde{\nu}_\alpha$ are unit normals to $L \subset \mathbb{R}^{n-m_1-1}$. PPN follows similarly as (3.8). ◻

Taking the grim reaper, we obtain the "grim reaper cylinder".

COROLLARY 3.10. *Let $L^{m-1} \subset \mathbb{R}^{n-2}$ be a complete minimal submanifold and $\gamma(t) = \{(t, r(t)) | r(t) = -\log\cos t, -\pi/2 < t < \pi/2\} \subset \mathbb{R}^2$ be a grim reaper. Then the following product immersion*

$$M^m = L^{m-1} \times \gamma \subset \mathbb{R}^{n-2} \times \mathbb{R}^2 = \mathbb{R}^n$$

is a complete translating soliton with PPN in the direction of $T = (0, \cdots, 0, 1) \in \mathbb{R}^n$.

We call this translating soliton *generalized grim reaper cylinder.* Let $(x^1, \cdots, x^{m-1})$ and $t = x^m$ be local coordinates of L^{m-1} and γ respectively. Generalized grim reaper cylinders have the property $|P|^2 = |H|^4$. In fact, the second fundamental forms are

$$(h^\alpha_{ij}) = (\langle B_{ij}, \nu_\alpha \rangle) = \begin{pmatrix} (h^\alpha_{ij}) & O \\ O & h^\alpha_{mm} \end{pmatrix}, \qquad 1 \le \alpha \le n-m,$$

where

$$h^{n-m}_{mm} = \frac{\ddot{r}}{\sqrt{1+\dot{r}^2}}, \quad h^{n-m}_{ij} = 0 \ (1 \le i, j \le m-1), \quad h^{\alpha}_{mm} = 0 \ (1 \le \alpha \le n-m-1),$$

and h^{α}_{ij} $(1 \le i, j \le m-1, 1 \le \alpha \le n-m-1)$ is the second fundamental form of the immersion $L^{m-1} \subset \mathbb{R}^{n-2}$. From

$$P = \langle B, H_{M;\mathbb{R}^n} \rangle = \langle B, H_{\gamma;\mathbb{R}^2} \rangle = |H_{\gamma;\mathbb{R}^2}| B^{\nu},$$

we obtain $P_{mm} = |H|^2$, $P_{ij} = 0, (i, j : \text{otherwise})$ and $|P|^2 = |H|^4$.

Remark 3.11. In codimension one case, $|P|^2 = |H|^4$ implies $|B|^2 = |H|^2$. This means the flatness of the scalar curvature of $M^{n-1} \subset \mathbb{R}^n$.

Example 3.12. Let $N_1 \subset \mathbb{R}^{n_1}$ and $N_2 \subset \mathbb{R}^{n_2}$ be codimension one complete translating solitons in the directions of T_1 and T_2 respectively. Then a product immersion of these two translating solitons

$$M := N_1 \times N_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n, \quad (n = n_1 + n_2) \tag{3.17}$$

is a codimension two complete translating soliton in $\mathbb{R}^n$ with flat normal bundle in the direction of $T = T_1 + T_2 \in \mathbb{R}^n$. In fact, from Lemma 3.7 and Lemma 3.8, we compute

$$\begin{aligned} H_{M;\mathbb{R}^n} &= H_{N_1;\mathbb{R}^n} + H_{N_2;\mathbb{R}^n} \\ &= H_{N_1;\mathbb{R}^{n_1}} + H_{N_2;\mathbb{R}^{n_2}} \\ &= T_1^{\perp} + T_2^{\perp} = T^{\perp}, \end{aligned}$$

and $\{T_1^{\perp}/|T_1^{\perp}|, T_2^{\perp}/|T_2^{\perp}|\}$ is a parallel orthonormal frame of $T^{\perp}M$.

4. Translating solitons with PPN. Inspired by Corollary 3.10, we investigate a translating soliton with PPN which has the property that $|P|^2/|H|^4$ attains its maximum. To do so, we calculate the Laplacian of $|P|^2/|H|^4$. Our proof is essentially due to Huisken [5] for self-shrinkers in codimension one and Smoczyk in higher codimension [13]. Recently Martin, Savas-Halilaj and Smoczyk [9] used same technique for translating solitons in codimension one.

In Section 4 and Section 5, we always assume $|H| \neq 0$. If M^m is a submanifold with PPN, we see

$$\nabla_i^{\perp} H = (\nabla_i |H|)\nu, \tag{4.1}$$

where ν is the principal normal.

Definition 4.1. Let $F : M^m \to \mathbb{R}^n$ be an isometric immersion and $T \in \mathbb{R}^n$ be a fixed vector with unit length. We define the height function in the direction of T and its differential,

$$\begin{aligned} u &:= \langle F, T \rangle, \\ \varphi &:= du. \end{aligned}$$

Lemma 4.2. *On a translating soliton $M^m \subset \mathbb{R}^n$, the following relations hold:*

$$\nabla u = T^\top, \tag{4.2}$$
$$|\nabla u|^2 = |T|^2 - |H|^2, \tag{4.3}$$
$$\nabla du = \nabla \varphi = P, \tag{4.4}$$
$$\Delta u = |H|^2 = |T|^2 - |\nabla u|^2. \tag{4.5}$$

Proof. We use $\partial_i := \partial/\partial x^i$. For any $X \in TM$,

$$\langle \nabla u, X\rangle = Xu = \langle X, T\rangle = \langle T^\top, X\rangle.$$

This implies the first relation. The second one follows easily from

$$T = T^\top + T^\perp = \nabla u + H.$$

The third one is

$$\nabla du(\partial_i, \partial_j) = \langle \partial_i\partial_j F - \nabla_{\partial i}\partial_j, T\rangle = \langle B_{ij}, H\rangle = P_{ij}.$$

Finally the last relation follows from

$$\Delta u = \mathrm{Trace}\nabla du = |H|^2 = |T|^2 - |\nabla u|^2.$$

□

Lemma 4.3. *On a translating soliton $M^m \subset \mathbb{R}^n$, the following relations hold:*

$$\nabla_i^\perp H = -\varphi^k B_{ki}, \tag{4.6}$$
$$\nabla_i^\perp \nabla_j^\perp H = -P_i^k B_{kj} - \varphi^k \nabla_k^\perp B_{ij}. \tag{4.7}$$
$$\Delta^\perp H = -P^{ij} B_{ij} - \varphi^k \nabla_k^\perp H. \tag{4.8}$$

Proof. Let $\{\nu_\alpha\} \subset T^\perp M$ be an orthonormal frame. We calculate

$$\begin{aligned}
\nabla_i^\perp H &= \nabla_i^\perp\big(\langle T, \nu_\alpha\rangle \nu_\beta \delta^{\alpha\beta}\big)\\
&= \partial_i \langle T, \nu_\alpha\rangle \nu_\beta \delta^{\alpha\beta} + \langle T, \nu_\alpha\rangle \nabla_i^\perp \nu_\beta \delta^{\alpha\beta}\\
&= \langle T, \overline{\nabla}_{\partial_i}\nu_\alpha\rangle \nu_\beta \delta^{\alpha\beta} + \langle T, \nu_\alpha\rangle \nabla_i^\perp \nu_\beta \delta^{\alpha\beta}\\
&= -\langle \nabla u, A^{\nu_\alpha}(\partial_i)\rangle \nu_\beta \delta^{\alpha\beta} + \langle H, \nabla_i^\perp \nu_\alpha\rangle \nu_\beta \delta^{\alpha\beta} + \langle T, \nu_\alpha\rangle \nabla_i^\perp \nu_\beta \delta^{\alpha\beta}\\
&= -B(\nabla u, \partial_i),
\end{aligned}$$

where A^{ν_α} is the shape operator with respect to ν_α, and we use the following fact in the last line:

$$\begin{aligned}
&\langle H, \nabla_i^\perp \nu_\alpha\rangle \langle \nu_\beta, \nu_\gamma\rangle \delta^{\alpha\beta} + \langle T, \nu_\alpha\rangle \langle \nabla_i^\perp \nu_\beta, \nu_\gamma\rangle \delta^{\alpha\beta}\\
&= \langle H, \nabla_i^\perp \nu_\gamma\rangle - \langle T, \nu_\alpha\rangle \langle \nu_\beta, \nabla_i^\perp \nu_\gamma\rangle \delta^{\alpha\beta}\\
&= \langle H, \nabla_i^\perp \nu_\gamma\rangle - \langle T^\perp, \nabla_i^\perp \nu_\gamma\rangle\\
&= 0.
\end{aligned}$$

Next we show the second relation. Note that $P = \nabla\varphi$, then we have

$$\begin{aligned}\nabla_i^\perp \nabla_j^\perp H &= -\nabla_i^\perp(\varphi^k B_{kj}) \\ &= -\nabla_i \varphi^k B_{kj} - \varphi^k \nabla_i^\perp B_{kj} \\ &= -P_i^k B_{kj} - \varphi^k \nabla_k^\perp B_{ij}.\end{aligned}$$

by using the Codazzi equation (2.2) in the last line. The last relation (4.8) is a direct consequence of the second one (4.7). ☐

LEMMA 4.4. *Let $M^m \subset \mathbb{R}^n$ be a translating soliton with PPN. Then the following formula holds:*

$$\Delta|H|^2 - 2|\nabla|H||^2 + 2|P|^2 + \varphi^k \nabla_k |H|^2 = 0.$$

Proof. From (4.8), we have

$$2\langle \Delta^\perp H, H\rangle + 2P^{ij}\langle B_{ij}, H\rangle + 2\varphi^k \langle \nabla_k^\perp H, H\rangle = 0.$$

The first term in the above relation is

$$2\langle \Delta^\perp H, H\rangle = \Delta|H|^2 - 2|\nabla^\perp H|^2.$$

Since M has a PPN, we know $\nabla^\perp H = \nabla|H|\nu$ with principal normal ν. This completes the proof. ☐

LEMMA 4.5. *On a translating soliton with PPN, the following relations*

$$\begin{aligned}P^{ij} B_{ij} &= \frac{|P|^2}{|H|}\nu, \\ S_{ijkl} P^{ij} P^{kl} &= \frac{|P|^4}{|H|^2}\end{aligned}$$

hold, where $S_{ijkl} = \langle B_{ij}, B_{kl}\rangle$.

Proof. By (4.8) we have

$$P^{ij} B_{ij} = -\Delta^\perp H - \varphi^k \nabla_k^\perp H = -\Big(\Delta|H| + \varphi^k \nabla_k |H|\Big)\nu.$$

Since $P^{ij}B_{ij}$ is a multiple of ν, we can compute as follows:

$$P^{ij} B_{ij} = \langle P^{ij} B_{ij}, \nu\rangle \nu = P^{ij}\langle B_{ij}, \frac{H}{|H|}\rangle \nu = \frac{|P|^2}{|H|}\nu.$$

The second relation is a direct consequence of the first one. ☐

LEMMA 4.6. *On a submanifold with PPN, the following relations hold:*

$$P_i^k B_{kj} = P_j^k B_{ki} \tag{4.9}$$

$$S_{ikjl} P^{ij} P^{kl} = P_i^k P_{kj} Q^{ij}, \tag{4.10}$$

where $Q_{ij} = \langle B_i^k, B_{kj}\rangle$.

Proof. To show the first relation, we apply the Ricci equation (2.3) to H,

$$0 = \nabla_i^\perp \nabla_j^\perp H - \nabla_j^\perp \nabla_i^\perp H = \langle B_j^k, H\rangle B_{ki} - \langle B_i^k, H\rangle B_{kj} = P_j^k B_{ki} - P_i^k B_{kj},$$

where we use PPN in the first equality. The second relation (4.10) easily follows from the first one. □

Next we compute Simon's identity for a translating soliton. Combining the general Simon's identity (2.5) with (4.7) we calculate

$$\begin{aligned} 0 =& \Delta|B|^2 - 2|\nabla^\perp B|^2 + 2P_i^k\langle B_{kj}, B^{ij}\rangle + 2\varphi^k\langle \nabla_k^\perp B_{ij}, B^{ij}\rangle \\ &+ 2|S|^2 + 2|R^\perp|^2 - 2\langle P, Q\rangle. \end{aligned}$$

Hence we get:

LEMMA 4.7. *On a translating soliton, the following identity holds:*

$$\Delta|B|^2 - 2|\nabla^\perp B|^2 + \langle \nabla|B|^2, \nabla u\rangle + 2|S|^2 + 2|R^\perp|^2 = 0.$$

The following relation (4.11) holds on any submanifold with PPN. We repeat the proof by Smoczyk [13]. Since M^m has a PPN, we can calculate as follows:

$$\begin{aligned} \big\langle \nabla^\perp H, \nabla^\perp B_{ij}\big\rangle &= \big\langle \nabla^k|H|\nu, \nabla_k B_{ij}\big\rangle P^{ij} \\ &= \nabla^k|H|\nabla_k\big(\langle \nu, B_{ij}\rangle\big)P^{ij} \\ &= \nabla^k|H|\nabla_k\Big(\frac{P_{ij}}{|H|}\Big)P_{ij} \\ &= \frac{1}{2|H|}\big\langle \nabla|H|, \nabla|P|^2\big\rangle - \frac{|P|^2}{|H|^2}|\nabla|H||^2. \end{aligned} \tag{4.11}$$

On the other hand, a direct computation yields

$$\frac{2}{|H|}\Big\langle \nabla|H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle = \frac{2}{|H|^5}\Big\langle \nabla|H|, \nabla|P|^2\Big\rangle - \frac{8|P|^2}{|H|^6}|\nabla|H||^2. \tag{4.12}$$

Substitute (4.12) into (4.11), we obtain

$$\frac{4}{|H|^4}\langle \nabla^\perp H, \nabla^\perp B_{ij}\rangle P^{ij} = \frac{2}{|H|}\Big\langle \nabla|H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle + \frac{4|P|^2}{|H|^6}|\nabla|H||^2. \tag{4.13}$$

We can also check the following by using (4.12):

$$\begin{aligned} \frac{2}{|H|^4}\Big|\nabla|H| \otimes \frac{P}{|H|} - |H|\nabla\Big(\frac{P}{|H|}\Big)\Big|^2 =& \frac{2|\nabla P|^2}{|H|^4} - \frac{8|P|^2}{|H|^6}|\nabla|H||^2 \\ &- \frac{4}{|H|}\Big\langle \nabla|H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle. \end{aligned} \tag{4.14}$$

From the definition $P_{ij} = \langle B_{ij}, H\rangle$, we have the following:

$$\nabla_k P_{ij} = \big\langle \nabla_k^\perp B_{ij}, H\big\rangle + \big\langle B_{ij}, \nabla_k^\perp H\big\rangle, \tag{4.15}$$

$$\Delta P_{ij} = \big\langle \Delta^\perp B_{ij}, H\big\rangle + 2\big\langle \nabla^\perp B_{ij}, \nabla^\perp H\big\rangle + \big\langle B_{ij}, \Delta^\perp H\big\rangle. \tag{4.16}$$

Now we assume that M^m is a translating soliton. From Simon's identity (2.5), with $R_{ij} = P_{ij} - Q_{ij}$ and (4.7), we obtain

$$\Delta^{\perp} B_{ij} + \varphi^k \nabla^{\perp}_k B_{ij} + Q^k_i B_{kj} + Q^k_j B_{ki} + (R_{ikjl} - S_{ikjl}) B^{kl} = 0. \tag{4.17}$$

Substitute (4.8) and (4.17) into (4.16) to compute ΔP_{ij}:

$$\begin{aligned}\Delta P_{ij} =& \big\langle -\varphi^k \nabla^{\perp}_k B_{ij} - Q^k_i B_{kj} - Q^k_j B_{ki} - (R_{ikjl} - S_{ikjl}) B^{kl}, H \big\rangle \\ &+ 2\big\langle \nabla^{\perp} B_{ij}, \nabla^{\perp} H \big\rangle + \big\langle -P^{kl} B_{kl} - \varphi^k \nabla^{\perp}_k H, B_{ij} \big\rangle.\end{aligned}$$

Note that

$$\big\langle \nabla u, \nabla P_{ij} \big\rangle = \big\langle \varphi^k \nabla^{\perp}_k H, B_{ij} \big\rangle + \big\langle H, \varphi^k \nabla^{\perp}_k B_{ij} \big\rangle. \tag{4.18}$$

We continue the computation by using (4.18),

$$\begin{aligned}\Delta P_{ij} =& - Q^k_i P_{kj} - Q^k_j P_{ki} - (R_{ikjl} - S_{ikjl}) P^{kl} + 2\big\langle \nabla^{\perp} B_{ij}, \nabla^{\perp} H \big\rangle \\ &- P^{kl} S_{ijkl} - \big\langle \nabla u, \nabla P_{ij} \big\rangle \\ =& - \big\langle \nabla u, \nabla P_{ij} \big\rangle + 2\big\langle \nabla^{\perp} H, \nabla^{\perp} B_{ij} \big\rangle \\ &- Q^k_i P_{kj} - Q^k_j P_{ki} + 2(S_{ikjl} - S_{ijkl}) P^{kl}.\end{aligned}$$

As a direct consequence of this identity, we obtain

$$\begin{aligned}\Delta |P|^2 =& 2|\nabla P|^2 - \big\langle \nabla u, \nabla |P|^2 \big\rangle + 4\big\langle \nabla^{\perp} H, \nabla^{\perp} B_{ij} \big\rangle P^{ij} \\ &- 4P^k_i P_{kj} Q^{ij} + 4(S_{ikjl} - S_{ijkl}) P^{ij} P^{kl}.\end{aligned} \tag{4.19}$$

Finally we can compute the Laplacian of $|P|^2/|H|^4$ by using Lemma 4.4 and (4.19):

$$\begin{aligned}\Delta\Big(\frac{|P|^2}{|H|^4}\Big) =& \frac{\Delta |P|^2}{|H|^4} - \frac{2|P|^2 \Delta |H|^2}{|H|^6} - \frac{8}{|H|}\Big\langle \nabla |H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle - \frac{8|P|^2 |\nabla |H||^2}{|H|^6} \\ =& \frac{1}{|H|^4}\Bigg(2|\nabla P|^2 - \big\langle \nabla u, \nabla |P|^2 \big\rangle + 4\big\langle \nabla^{\perp} H, \nabla^{\perp} B_{ij} \big\rangle P^{ij} \\ &\qquad - 4P^k_i P_{kj} Q^{ij} + 4(S_{ikjl} - S_{ijkl}) P^{ij} P^{kl}\Bigg) \\ &- \frac{2|P|^2}{|H|^6}\Big(2|\nabla |H||^2 - 2|P|^2 - \varphi^k \nabla_k |H|^2\Big) \\ &- \frac{8}{|H|}\Big\langle \nabla |H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle - \frac{8|P|^2}{|H|^6} |\nabla |H||^2.\end{aligned}$$

By Lemma 4.5 and Lemma 4.6 we have

$$-P^k_i P_{kj} Q^{ij} + (S_{ikjl} - S_{ijkl}) P^{ij} P^{kl} + \frac{|P|^4}{|H|^2} = 0.$$

Hence we continue:

$$\begin{aligned}\Delta\Big(\frac{|P|^2}{|H|^4}\Big) =&\frac{2|\nabla P|^2}{|H|^4} - \frac{\langle \nabla u, \nabla |P|^2\rangle}{|H|^4} + \frac{4\langle \nabla^\perp H, \nabla^\perp B_{ij}\rangle P^{ij}}{|H|^4} - \frac{4|P|^2|\nabla|H||^2}{|H|^6}\\ &+ \frac{2|P|^2\langle \nabla u, \nabla |H|^2\rangle}{|H|^6} - \frac{8}{|H|}\Big\langle \nabla|H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle - \frac{8|P|^2}{|H|^6}|\nabla|H||^2\\ =&-\frac{4|P|^2|\nabla|H||^2}{|H|^6} + \frac{2}{|H|^4}\Big|\nabla_i|H|\frac{P_{jk}}{|H|} - |H|\nabla_i\Big(\frac{P_{jk}}{|H|}\Big)\Big|^2 - \frac{\langle \nabla u, \nabla |P|^2\rangle}{|H|^4}\\ &+ \frac{4\langle \nabla^\perp H, \nabla^\perp B_{ij}\rangle P^{ij}}{|H|^4} + \frac{2|P|^2\langle \nabla u, \nabla |H|^2\rangle}{|H|^6} - \frac{4}{|H|}\Big\langle \nabla|H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle.\end{aligned}$$

Substitute the relation (4.13) into the last identity to obtain:

Lemma 4.8. *Let $M^m \subset \mathbb{R}^n$ be a translating soliton with $|H| \neq 0$ and PPN, then we have*

$$\begin{aligned}\Delta\Big(\frac{|P|^2}{|H|^4}\Big) =&\frac{2}{|H|^4}\Big|\nabla|H| \otimes \frac{P}{|H|} - |H|\nabla\Big(\frac{P}{|H|}\Big)\Big|^2\\ &- \Big\langle \nabla u, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle - \frac{2}{|H|}\Big\langle \nabla|H|, \nabla\Big(\frac{|P|^2}{|H|^4}\Big)\Big\rangle.\end{aligned} \tag{4.20}$$

5. Splitting theorem for translating solitons with PPN. In this last section, we give a proof of Theorem B. Let $M^m \subset \mathbb{R}^n$ be an m-dimensional complete translating soliton with $|H| \neq 0$ and PPN. Recall the main statement:

Theorem B. *A complete translating soliton M^m with PPN in arbitrary codimension such that $|P^2|/|H|^4$ attains its maximum on M^m can only be the product immersion consisting of the grim reaper $\gamma \subset \mathbb{R}^2$ and a complete minimal submanifold $L^{m-1} \subset \mathbb{R}^{n-2}$.*

We assume that the quantity $|P|^2/|H|^4$ attains its maximum on M^m. Then by the strong maximum principle (see [4], Theorem 3.5) for (4.20), we conclude that $|P|^2/|H|^4$ is nonnegative constant and

$$\Big|\nabla_i|H|\frac{P_{jk}}{|H|} - |H|\nabla_i\Big(\frac{P_{jk}}{|H|}\Big)\Big|^2 \equiv 0. \tag{5.1}$$

Note that the property PPN and the Codazzi equation (2.2) imply

$$\begin{aligned}\nabla_i\Big(\frac{P_{jk}}{|H|}\Big) &= \nabla_i\Big\langle B_{jk}, \frac{H}{|H|}\Big\rangle = \nabla_i\langle B_{jk}, \nu\rangle\\ &= \langle \nabla_i^\perp B_{jk}, \nu\rangle = \langle \nabla_j^\perp B_{ik}, \nu\rangle,\end{aligned}$$

so that

$$\nabla_i\Big(\frac{P_{jk}}{|H|}\Big) = \nabla_j\Big(\frac{P_{ik}}{|H|}\Big). \tag{5.2}$$

Hence we obtain

$$\Big|\nabla_i|H|P_{jk} - \nabla_j|H|P_{ik}\Big|^2 \equiv 0. \tag{5.3}$$

Now assume $|H|$ is constant. Then Lemma 4.4 implies $|P|^2 \equiv 0$, that is

$$P_{ij} = 0, \quad 1 \le i, j \le m.$$

On the other hand by (4.4), we have

$$0 = P_{ij} = \nabla^2_{ij} u, \quad 1 \le i, j \le m,$$

so that,

$$|H|^2 = \Delta u = g^{ij} \nabla^2_{ij} u = 0.$$

Therefore in this case, the translating soliton must be a minimal submanifold, i.e., $H \equiv 0$. This contradicts our assumption $|H| \neq 0$.

Now we consider the case when $|H|$ is not constant. Then there is a point $x \in M$ such that $\nabla |H|(x) \neq 0$ and this is still true on a neighborhood of $x \in M$. Then we take an orthonormal frame $\{e_1, \cdots, e_m\}$ of TM such that

$$e_1 := \frac{\nabla |H|}{|\nabla |H||}.$$

In the following, we compute with respect to this orthonormal frame $\{e_1, \cdots, e_m\}$. Note that

$$\nabla_i |H| = \begin{cases} |\nabla |H||, & i = 1, \\ 0, & 2 \le i \le m. \end{cases} \tag{5.4}$$

Therefore by (5.3) we see

$$\begin{aligned} 0 &= \Big| \nabla_i |H| P_{jk} - \nabla_j |H| P_{ik} \Big|^2 \\ &= 2 |\nabla |H||^2 |P|^2 - 2 P^i_k P^{kj} \nabla_i |H| \nabla_j |H| \\ &= 2 |\nabla |H||^2 |P|^2 - 2 P^1_k P^k_1 |\nabla |H||^2, \end{aligned}$$

so that

$$|P|^2 = P^1_k P^k_1.$$

This implies

$$P_{ij} = \begin{cases} \lambda \neq 0, & i = j = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{5.5}$$

Hence we have

$$|P|^2 = (\mathrm{Trace} P)^2 = \langle H, \mathrm{Trace} B \rangle^2 = |H|^4.$$

Since $|P|^2/|H|^4$ is constant, we conclude

$$\frac{|P|^2}{|H|^4} \equiv 1,$$

on whole M. Now we know that P_{ij} has only one nonzero eigenvalue $\lambda = |H|^2$, and the corresponding eigenspace is spanned by $e_1 = \nabla |H| / |\nabla |H||$.

We define the distributions $\mathcal{D}^{\perp}(x)$ and $\mathcal{D}(x)$ of TM by

$$\mathcal{D}^{\perp}(x) := \{X \in T_xM | PX = |H|^2 X\},$$
$$\mathcal{D}(x) := \{X \in T_xM | PX = 0\},$$

so that $T_xM = \mathcal{D}^{\perp}(x) \oplus \mathcal{D}(x)$.

LEMMA 5.1. *The distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}$ are parallel along any curve on M^m.*

Proof. We take tangent vectors $v_x \in \mathcal{D}^{\perp}(x), w_x \in \mathcal{D}(x)$. Let $c(t) : (-\epsilon, \epsilon) \to M$ be a curve with $c(0) = x$. Then we extend v_x to $v(t)$ and w_x to $w(t)$ along the curve $c(t)$ by the parallel transport. By (5.1), we see

$$\nabla_i |H| \frac{P_{jk}}{|H|} = |H| \nabla_i \Big(\frac{P_{jk}}{|H|} \Big) = \nabla_i P_{jk} - \nabla_i |H| \frac{P_{jk}}{|H|},$$

and hence

$$\nabla_i P_{jk} = 2 \nabla_i |H| \frac{P_{jk}}{|H|}.$$

From (5.4) and (5.5), we obtain the following:

$$(\nabla_i P)X = \begin{cases} \nabla_1 |H|^2 \mathrm{X}, & i = 1, \\ 0, & 2 \le i \le m. \end{cases} \tag{5.6}$$

Denote the velocity vector of $c(t)$ by $\dot{c}(t) = \dot{c}^1(t) e_1 + \cdots + \dot{c}^m(t) e_m$. From (5.6) we have,

$$\begin{aligned} \frac{d}{dt} |Pv - |H|^2 v|^2 &= 2 \big\langle (\nabla_{\dot{c}} P) v - (\nabla_{\dot{c}} |H|^2) v, Pv - |H|^2 v \big\rangle \\ &= 2 \dot{c}^1 \big\langle (\nabla_1 P) v - (\nabla_1 |H|^2) v, Pv - |H|^2 v \big\rangle \\ &= 0. \end{aligned}$$

On the other hand, we compute

$$\frac{d}{dt} |Pw|^2 = 2 \big\langle (\nabla_{\dot{c}} P) w, Pw \big\rangle = 2 \dot{c}^1 \big\langle (\nabla_1 P) w, Pw \big\rangle = \frac{4 \dot{c}^1}{|H|} \nabla_1 |H| |Pw|^2.$$

The solution to this linear ordinary differential equation with the initial value $|Pw|^2(0) = 0$ must be $|Pw|^2 \equiv 0$ along the curve $c(t)$. Therefore we obtain

$$\frac{d}{dt} |Pv - |H|^2 v|^2 = 0 = \frac{d}{dt} |Pw|^2 \tag{5.7}$$

along any curve $c(t)$. From this fact, we conclude

$$v(t) \in \mathcal{D}^{\perp}(c(t)), \quad w(t) \in \mathcal{D}(c(t))$$

along any curve on M^n, that is, distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}$ are invariant under a parallel transport. □

By taking the definition of a covariant derivative into account, we have the following:

$$\nabla_v \widetilde{v}, \quad [v, \widetilde{v}] \in \mathcal{D}^{\perp}, \quad \text{for} \ \ v, \widetilde{v} \in \mathcal{D}^{\perp},$$
$$\nabla_w \widetilde{w}, \quad [w, \widetilde{w}] \in \mathcal{D}, \quad \text{for} \ \ w, \widetilde{w} \in \mathcal{D}.$$

This means that the distributions $\mathcal{D}^\perp$ and $\mathcal{D}$ are integrable. Apply the de Rham decomposition theorem to M^m. Then M^m is isometric to a product manifold, that is,

$$M^m \cong \gamma \times L^{m-1}.$$

Note that these two manifolds are both totally geodesic in M^m.

Now we want to show that the immersion is actually a product immersion. We can easily check the following:

$$P_1^k B_{k1} = P_1^1 B_{11} = P^{ij} B_{ij} = \frac{|P|^2}{|H|}\nu = |H||H|^2\nu = |H|P_{11}\nu,$$

otherwise we have $P_i^k B_{kj} = 0$, so that

$$P_i^k B_{kj} = |H|P_{ij}\nu. \tag{5.8}$$

From this fact we obtain

$$0 = |H|P_{1j}\nu = P_1^k B_{kj} = P_1^1 B_{1j} = |H|^2 B_{1j}, \quad 2 \le j \le m.$$

Since $|H| \neq 0$ we have

$$B_{1j} = 0, \quad 2 \le j \le m. \tag{5.9}$$

Applying the lemma by Moore ([10], Section 2, Lemma, p.163), we know that F is a product immersion

$$F = F_1 \times F_2 : \gamma \times L^{m-1} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n, \tag{5.10}$$

where $F_1 : \gamma \to \mathbb{R}^{n_1}$ and $F_2 : L^{m-1} \to \mathbb{R}^{n_2}$ are isometric immersions. From (5.8), we obtain

$$B_{11} = |H|\nu = H = H_{M;\mathbb{R}^n}. \tag{5.11}$$

We compute the mean curvature vector of $L \subset \mathbb{R}^n$ by using Lemma 3.8 with (5.11):

$$\begin{aligned} H_{L;\mathbb{R}^n} &= H_{M;\mathbb{R}^n} - H_{\gamma;\mathbb{R}^n} = H_{M;\mathbb{R}^n} - B^{\gamma;\mathbb{R}^n}(e_1, e_1) \\ &= H_{M;\mathbb{R}^n} - B^{M;\mathbb{R}^n}(e_1, e_1) = H_{M;\mathbb{R}^n} - H_{M;\mathbb{R}^n} = 0. \end{aligned}$$

Hence L is a minimal submanifold in $\mathbb{R}^n$. We apply Lemma 3.7 to conclude that L is a minimal submanifold in $\mathbb{R}^{n_2}$. Now since M^m is a translating soliton, we have

$$H_{\gamma;\mathbb{R}^n} = H_{M;\mathbb{R}^n} = T^\perp.$$

This is an equation of a one dimensional translating soliton in the space $\mathbb{R}^n$. However all one dimensional translating soliton in a space must be planar, so that the curve γ is a part of the grim reaper. Therefore $F = F_1 \times F_2$ is a product immersion of $F_1 : \gamma \to \mathbb{R}^2$ and $F_2 : L^{m-1} \to \mathbb{R}^{n-2}$.

Our argument so far is valid only in the neighborhood of a point $x \in M^m$ such that $\nabla|H| \neq 0$. According to the analytic continuation theorem by Sampson ([11], Theorem 1, p.213), and because of the smoothness of M^m and the fact in Example 3.6, $F(M^m)$ should coincide with a generalized grim reaper cylinder. This completes the proof.

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REFERENCES

[1] D. Altschuler, S. Altschuler, S. Angenent, and L. Wu, *The zoo of solitons for curve shortening in* $\mathbb{R}^n$, Nonlinearity, Vol. 26:5 (2013), pp. 1189–1226.

[2] J. Choe and J. Hoppe, *Higher dimensional minimal submanifolds generalizing the catenoid and helicoid*, Tohoku Math. J., 65 (2013), pp. 43–55.

[3] J. Clutterbuck, O. C. Schnürer, and F. Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial differential Equations, 29:3 (2007), pp. 281–293.

[4] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer (1983).

[5] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, Calif, 1990), Proc. Sympos. Pure Math., Vol. 54, American Mathematical Society, Rhode Island (1993), pp. 175–191.

[6] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math., 183:1 (1999), pp. 45–70.

[7] K. Kunikawa, *Bernstein-type theorem of translating solitons in arbitrary codimension with flat normal bundle*, Calc. Var. Partial Differential Equations, 54:2 (2015), pp. 1331–1344.

[8] K. Kunikawa, *A Bernstein type theorem of ancient solutions to the mean curvature flow*, Proc. Amer. Math. Soc., 144 (2016), pp. 1325–1333.

[9] F. Martin, A. Savas-Halilaj, and K. Smoczyk, *On the topology of translating solitons of the mean curvature flow*, Calc. Var. Partial Differential Equations, 54:3 (2015), pp. 2853–2882.

[10] J. Moore, *Isometric immersions of Riemannian products*, J. Differential Geom., 5 (1971), No. 1-2, pp. 159–168.

[11] J. H. Sampson, *Some properties and applications of harmonic mappings*, Ann. Sci. École Norm. Sup. (4), 11 (1978), pp. 211–228.

[12] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2), 88:1 (1968), pp. 62–105.

[13] K. Smoczyk, *Self-shrinkers of the mean curvature flow in arbitrary codimension*, Int. Math. Res. Not. IMRN, 48 (2005), pp. 2983–3004.

[14] K. Smoczyk, *Mean curvature flow in higher codimension -Introduction and survey-*, Global Differential Geometry (Ed. C. Bär, J. Lohkamp, M. Schwarz), Springer Proc. Math., Vol. 17, pp. 231–274, Springer, Heidelberg, (2012).

[15] Y. L. Xin, *Translating solitons of the mean curvature flow*, Calc. Var. Partial Differential Equations, 54:2 (2015), pp. 1995–2016.