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BETTI NUMBERS OF RANDOM NODAL SETS OF ELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS*

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Abstract. Given an elliptic self-adjoint pseudo-differential operator P bounded from below, acting on the sections of a Riemannian line bundle over a smooth closed manifold M equipped with some Lebesgue measure, we estimate from above, as L grows to infinity, the Betti numbers of the vanishing locus of a random section taken in the direct sum of the eigenspaces of P with eigenvalues below L. These upper estimates follow from some equidistribution of the critical points of the restriction of a fixed Morse function to this vanishing locus. We then consider the examples of the Laplace-Beltrami and the Dirichlet-to-Neumann operators associated to some Riemannian metric on M.

Key words. Pseudo-differential operator, random nodal sets, random matrix.

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Introduction. Let M be a smooth closed manifold of positive dimension n, by which we mean a smooth compact n-dimensional manifold without boundary. Let $|dy|$ be a Lebesgue measure on M, that is locally the absolute value of some volume form. Let E be a real line bundle over M equipped with some Riemannian metric h_E. The space $\Gamma(M, E)$ of smooth global sections of E inherits from $|dy|$ and h_E the L^2-scalar product

$$(s,t) \in \Gamma(M,E)^2 \mapsto \langle s,t \rangle = \int_M h_E(s(y),t(y))|dy| \in \mathbb{R}. \tag{0.1}$$

Let then $P : \Gamma(M, E) \to \Gamma(M, E)$ be an elliptic pseudo-differential operator of order $m > 0$ which is self-adjoint with respect to (0.1) and bounded from below, see §5.2. For every $L \in \mathbb{R}$, we denote by

$$U_L = \bigoplus_{\lambda \leq L} \ker(P - \lambda Id) \tag{0.2}$$

and by N_L its dimension. It is equipped with the restriction $\langle\, ,\, \rangle_L$ of (0.1) and thus with the associated Gaussian measure μ_L whose density with respect to the Lebesgue measure $|ds|$ of U_L reads at every $s \in U_L$,

$$d\mu_L(s) = \frac{1}{\sqrt{\pi}^{N_L}} e^{-\langle s,s \rangle} |ds|.$$

What is the expected topology of the vanishing locus $s^{-1}(0) \subset M$ of a section s taken at random in (U_L, μ_L)? When P is the Laplace-Beltrami operator associated to a Riemannian metric on M, the expected value for this number of connected components for pure harmonics on the round two-sphere has been estimated by F. Nazarov and M. Sodin [19], a work partially extended in several directions (see [18],

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[21], [20], [23]). We studied a similar question in real algebraic geometry, where M is replaced by a real projective manifold X and U_L by the space $\mathbb{R}H^0(X, E \otimes L^d)$ of real holomorphic sections of the tensor product of some holomorphic vector bundle E with some ample real line bundle L over X (see [9], [12], [8], [10]). We there could estimate from above and below the expected value of each Betti number of $s^{-1}(0)$. Our aim now is, likewise, to estimate from above the mathematical expectations of all Betti numbers of $s^{-1}(0)$ for a random section $s \in U_L$, as L grows to infinity, see Corollary 0.2 (see [11] for lower estimates). This turns out to involve asymptotic estimates of the derivatives of the Schwartz kernel associated to the orthogonal projection onto U_L which we establish in Appendix 5.3, see Theorem 2.3. The asymptotic value of this kernel has been computed by L. Hörmander in [14], after Carleman [3] and Gärding [7] and for some derivatives, it is given by Safarov and Vassiliev in [22], but we could not find a general result for all derivatives in the literature.

Let us now formulate our main result. When $n \geq 2$, we choose a Morse function $p : M \to \mathbb{R}$ and set

$$\Delta_L = \{s \in U_L \,|\, s \text{ does not vanish transversally or } p_{|s^{-1}(0)} \text{ is not Morse}\}.$$

Then, for every $s \in U_L \setminus \Delta_L$ and every $i \in \{0, \cdots, n-1\}$, we introduce the empirical measure

$$\nu_i(s) = \sum_{x \in Crit_i(p_{|s^{-1}(0)}) \setminus Crit(p)} \delta_x,$$

where $Crit(p)$ denotes the critical locus of p, $Crit_i(p_{|s^{-1}(0)})$ the set of critical points of index i of $p_{|s^{-1}(0)}$ and δ_x the Dirac measure at x. When $n = 1$, we set

$$\nu_0(s) = \sum_{x \in s^{-1}(0)} \delta_x.$$

The mathematical expectation of ν_i is defined as

$$\mathbb{E}(\nu_i) = \int_{U_L \setminus \Delta_L} \nu_i(s) d\mu_L(s).$$

Recall that the pseudo-differential operator P has a (homogenized) principal symbol $\sigma_P : T^*M \to \mathbb{R}$ which is homogeneous of degree m, see Definition 5.6, and we set

$$K = \{\xi \in T^*M \,|\, \sigma_P(\xi) \leq 1\}. \tag{0.3}$$

The volume of K for the Lebesgue measure $|d\xi|$ induced on the fibres of T^*M by $|dy|$ is encoded by the function

$$c_0 : x \in M \mapsto \frac{1}{(2\pi)^n} \int_{K \cap T^*_x M} |d\xi| \in \mathbb{R}_+. \tag{0.4}$$

It turns out that K together with $|d\xi|$ induce a Riemannian metric on M, namely

$$g_P : (u, v) \in T_x M \mapsto \frac{1}{(2\pi)^n} \int_{K \cap T^*_x M} \xi(u)\xi(v)|d\xi| \tag{0.5}$$

and we denote by $|dvol_P|$ the associated Lebesgue measure of M.

THEOREM 0.1. *Let M be a smooth closed manifold of dimension n equipped with a Morse function p and a Lebesgue measure $|dy|$. Let (E, h_E) be a Riemannian real line bundle over M and $P : \Gamma(M,E) \to \Gamma(M,E)$ be an elliptic self-ajdoint pseudo-differential operator of order $m > 0$ which is bounded from below. Then, for every $i \in \{0, \cdots, n-1\}$,*

$$\frac{1}{L^{\frac{n}{m}}}\mathbb{E}(\nu_i) \underset{L\to\infty}{\to} \frac{1}{\sqrt{\pi}^{n+1}\sqrt{c_0}}\mathbb{E}(i, \ker dp)|dvol_P|. \tag{0.6}$$

The convergence given by (0.6) is the weak convergence on the whole M. Also, in Theorem 0.1, $\mathbb{E}(i, \ker dp)$ denotes, for every point $x \in M$, the expected determinant of random symmetric operators of signature $(i, n-1-i)$ on $\ker d_{|x}p$ when $n > 1$, see (0.8), while it equals 1 when $n = 1$. Namely, P together with $|dy|$ induce a Riemannian metric $\langle\ ,\rangle_P$ on the space $Sym^2(TM)$ of symmetric bilinear forms on T^*M, which reads for every $(b_1, b_2) \in Sym^2(TM)^2$,

$$\langle b_1, b_2\rangle_P = \frac{1}{(2\pi)^n}\Big(\int_K b_1(\xi)b_2(\xi)|d\xi| - \frac{1}{\int_K |d\xi|}\iint_{K^2} b_1(\xi)b_2(\xi')|d\xi||d\xi'|\Big), \tag{0.7}$$

where in the right-hand side of (0.7) the quadratic forms associated to b_1 and b_2 are also denoted by b_1 and b_2, by abuse of notation. The first term in the right-hand side of (0.7) already defines a natural Riemannian metric on $Sym^2(TM)$, see §2.2, but the one playing a rôle in Theorem 0.1 is indeed (0.7), where the second term induces some correlations similar to the ones already observed by L. Nicolaescu in [21]. By duality and restriction to $(\ker dp)^*$, (0.7) induces a Riemannian metric on $Sym^2((\ker dp)^*)$ see §2.3.1, with Gaussian measure μ_P. Let $Sym^2_i((\ker dp)^*)$ be the open cone of non-degenerate symmetric bilinear forms of index i on $\ker dp$. We set

$$\mathbb{E}(i, \ker dp) = \int_{Sym^2_i((\ker dp)^*)} |\det\beta| d\mu_P(\beta), \tag{0.8}$$

where $\det\beta$ is computed with respect to the metric g_P restricted to $\ker dp$ and given by (0.5).

From Theorem 0.1 we thus know that the critical points of index i of $p_{|s^{-1}(0)}$ equidistribute in the manifold M with respect to g_P, with a density involving random symmetric endomorphisms of $\ker dp \subset TM$. Let us mention two consequences of Theorem 0.1. First, for every $s \in U_L \setminus \Delta_L$, we denote by $m_i(s)$ the i-th Morse number of $s^{-1}(0)$, that is

$$m_i(s) = \inf_{f \text{ Morse on } s^{-1}(0)} \#Crit_i(f)$$

and set

$$\mathbb{E}(m_i) = \int_{U_L\setminus\Delta_L} m_i(s) d\mu_L(s). \tag{0.9}$$

From Morse theory we know that these Morse numbers bound from above all i-th Betti numbers b_i of $s^{-1}(0)$, whatever the coefficient rings are.

COROLLARY 0.2. *Under the hypotheses of Theorem 0.1, when $n \geq 2$,*

$$\limsup_{L\to\infty} \frac{1}{L^{\frac{n}{m}}}\mathbb{E}(m_i) \leq \frac{1}{\sqrt{\pi}^{n+1}} \inf_{p \text{ Morse function on } M} \int_M \frac{1}{\sqrt{c_0}}\mathbb{E}(i, \ker dp)|dvol_P|,$$

while when $n = 1$, we have the convergence

$$\frac{1}{L^{\frac{1}{m}}}\mathbb{E}(b_0) \underset{L\to\infty}{\to} \frac{1}{\pi}\int_M \frac{1}{\sqrt{c_0}}|dvol_P|.$$

Theorem 0.1 also specializes to the case of the Laplace-Beltrami operator Δ_g associated to some Riemannian metric g on M. In this case, we denote by $|dvol_g|$ the Lebesgue measure associated to g and by $Vol_g(M)$ its total volume $\int_M |dvol_g|$.

Corollary 0.3. *Let (M, g) be a closed Riemannian manifold of positive dimension n equipped with a Morse function $p : M \to \mathbb{R}$. Then, when $n \geq 2$, for every $i \in \{0, \cdots, n-1\}$,*

$$\frac{1}{\sqrt{L}^n}\mathbb{E}(\nu_i) \underset{L\to\infty}{\to} \frac{\mathbb{E}(i, n-1-i)}{\sqrt{\pi}^{n+1}\sqrt{(n+2)(n+4)^{n-1}}}|dvol_g|,$$

where the convergence is weak on M. In particular,

$$\limsup_{L\to\infty} \frac{1}{\sqrt{L}^n}\mathbb{E}(m_i) \leq \frac{\mathbb{E}(i, n-1-i)}{\sqrt{\pi}^{n+1}\sqrt{(n+2)(n+4)^{n-1}}}Vol_g(M).$$

When $n = 1$, $\frac{1}{\sqrt{L}}\mathbb{E}(\nu_0) \underset{L\to\infty}{\to} \frac{1}{\pi\sqrt{3}}|dvol_g|$ so that $\frac{1}{\sqrt{L}}\mathbb{E}(b_0) \underset{L\to\infty}{\to} \frac{1}{\pi\sqrt{3}}Vol_g(M)$.

The case $n = 1$ in Corollary 0.3 turns out also to follow from the volume computations carried out by P. Bérard in [1]. Note that in Corollary 0.3, $\mathbb{E}(\nu_i)$ is defined using $P = \Delta_g$ as a differential operator, so that $m = 2$ with the notations of Theorem 0.1. Moreover,

$$\mathbb{E}(i, n-1-i) = \int_{Sym(i,n-1-i,\mathbb{R})} |\det A| d\mu(A), \tag{0.10}$$

where $Sym(i, n-1-i, \mathbb{R})$ denotes the open cone of non-degenerate symmetric matrices of index i, size $(n-1)\times(n-1)$ and real coefficients, while μ denotes the Gaussian measure on $Sym(n-1, \mathbb{R})$ associated to the scalar product

$$(A, B) \in Sym(n-1, \mathbb{R})^2 \mapsto \frac{1}{2}\operatorname{Tr}(AB) + \frac{1}{6}(\operatorname{Tr} A)(\operatorname{Tr} B) \in \mathbb{R}, \tag{0.11}$$

see §3.1. This measure differs from the standard GOE measure on $Sym(n-1, \mathbb{R})$. When M is a surface for example, Corollary 0.3 implies that for $i \in \{0, 1\}$,

$$\limsup_{L\to\infty} \frac{1}{L}\mathbb{E}(m_i) \leq \frac{1}{8\pi^2}Vol_g(M).$$

For large values of the dimension n, we observe some exponential decrease of the upper estimates given by Corollary 0.3 away from the mid-dimensional Betti numbers. This exponential decrease given by Proposition 0.4 is similar to the one given by Theorem 1.6 of [8].

Proposition 0.4. *For every $\epsilon > 0$, there exist $\delta > 0$ and $C > 0$ such that for every smooth closed Riemannian manifold M of positive dimension n,*

$$\limsup_{L\to\infty} \frac{1}{N_L} \sum_{|\frac{i}{n}-\frac{1}{2}|\geq\epsilon} \mathbb{E}(m_i) \leq C\exp(-\delta n^2).$$

In particular,

$$\limsup_{L\to\infty} \frac{1}{N_L}\mathbb{E}(b_0) \to_{n\to\infty} 0.$$

Again, in Proposition 0.4, $\mathbb{E}(m_i)$ is defined using $P = \Delta_g$ as a differential operator. This proposition may be compared with Courant's Theorem which bounds by N_L the number of nodal domains of any eigenfunction $s \in U_L$, see [4].

As a second example, Theorem 0.1 specializes to the case of the Dirichlet-to-Neumann operator on the boundary M of some compact Riemannian manifold (W, g), see §3.2. We then obtain

Corollary 0.5. *Let (W, g) be a smooth compact Riemannian manifold of positive dimension $n + 1$ with boundary M, Λ_g be the Dirichlet-to-Neumann operator on M, and $p : M \to \mathbb{R}$ be a fixed Morse function. Then, when $n \geq 2$, for every $i \in \{0, \cdots, n-1\}$,*

$$\frac{1}{L^n}\mathbb{E}(\nu_i) \underset{L\to\infty}{\to} \frac{\mathbb{E}(i, n-1-i)}{\sqrt{\pi}^{n+1}\sqrt{(n+2)(n+4)^{n-1}}}|dvol_g|,$$

where the convergence is weak on M and $|dvol_g|$ is the volume form on M induced by g. In particular,

$$\limsup_{L\to\infty} \frac{1}{L^n}\mathbb{E}(m_i) \leq \frac{\mathbb{E}(i, n-1-i)}{\sqrt{\pi}^{n+1}\sqrt{(n+2)(n+4)^{n-1}}}Vol_g(M).$$

When $n = 1$, $\frac{1}{L}\mathbb{E}(\nu) \underset{L\to\infty}{\to} \frac{1}{\pi\sqrt{3}}|dvol_g|$ so that $\frac{1}{L}\mathbb{E}(b_0) \underset{L\to\infty}{\to} \frac{1}{\pi\sqrt{3}}Vol_g(M)$.

In the first section we study the general case of an ample finite dimensional subspace U of $\Gamma(M, E)$ equipped with any scalar product, see Definition 1.1. In this case, we prove that the expected empirical measures $\mathbb{E}(\nu_i)$ turn out to be densities on M. Thanks to the coarea formula and a natural change of variables, we express these densities as integrals over the sum of the space of symmetric bilinear forms of signature $(i, n-1-i)$ on the kernel of $dp_{|x}$, and the space of linear forms vanishing on this kernel, see Theorem 1.10. Here, the integrands are functions of the 2-jet of the Schwartz kernel associated to U. In the case of a family $(U_L)_{L\in\mathbb{R}^*_+}$ whose elements rescale naturally with respect to L, see Definition 1.14, we give an asymptotic equivalent of $\mathbb{E}(\nu_i)$ in terms of a power of L as L grows to infinity, see Corollary 1.15. These results are applied in the second section to prove our main theorems, namely Theorem 0.1 and Corollary 0.2, which correspond to the special case where $U = U_L$ for some elliptic self-adjoint pseudo-differential operator bounded from below, see (0.2). We check that the latter family U_L indeed rescales with respect to L in the sense of Definition 1.14, see Theorem 2.3. The third section is devoted to the examples of the Laplace-Beltrami and Dirichlet-to-Neumann operators. The principal symbols of these operators are powers of the norm, making it possible to prove the explicit computations given by Corollary 0.3, Proposition 0.4, and Corollary 0.5. In the last section we discuss some related problems which we plan to consider in a separated paper. We finally give in Appendix 5 several auxiliary results, in particular the proof of Theorem 2.3, which provides estimates of the derivatives of the Schwartz kernel associated to U_L.

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1. Morse numbers of the vanishing locus of random sections. Let M be a smooth manifold of positive dimension n, $E \to M$ be a real line bundle and $p : M \to \mathbb{R}$ be a Morse function. We denote by $\mathcal{H}$ the singular foliation by level sets of p and for every $x \in M \setminus Crit(p)$ we set

$$H_x = T_x\mathcal{H} = \ker d_{|x}p.$$

1.1. Ample linear subspaces and incidence varieties. For every $l \geq 0$, we denote by $\mathcal{J}^l(E)$ the fibre bundle of l-jets of sections of E and for every $m \geq l \geq 0$, we denote by $\pi^{m,l} : \mathcal{J}^m(E) \to \mathcal{J}^l(E)$ the tautological projections which restricts the m-jets to l-jets. The jet maps are denoted by

$$j^l : s \in \Gamma(M, E) \mapsto j^l(s) \in \Gamma(M, \mathcal{J}^l(E)).$$

Recall that the kernel of $\pi^{l+1,l}$ is canonically isomorphic to the bundle $Sym^{l+1}(T^*M) \otimes E$ of symmetric $(l+1)$-linear forms on TM with values in E. In particular, any Riemannian metric on $\mathcal{J}^l(E)$ induces an isomorphism

$$\mathcal{J}^l(E) \cong S^l(T^*M) \otimes E,$$

where $S^l(T^*M) = \bigoplus_{k=0}^{l} Sym^k(T^*M)$.

Let $U \subset \Gamma(M, E)$ be a linear subspace of positive dimension N and $\underline{U} = M \times U$ be the associated rank N trivial bundle over M. The maps j^l define bundle morphisms

$$j^l : (x, s) \in \underline{U} \mapsto (x, j^l(s)_{|x}) \in \mathcal{J}^l(E).$$

Definition 1.1 (compare Def. 2.1 of [21]). *The vector subspace U of $\Gamma(M, E)$ is said to be l-ample if and only if the morphism $j^l : \underline{U} \to \mathcal{J}^l(E)$ is onto. It is said to be ample if and only if it is 1-ample.*

We also need a relative version of this ampleness property. For every $l \geq 0$, we denote by $\mathcal{J}^l(E_{|\mathcal{H}}) \to M \setminus Crit(p)$ the fibre bundle of l-jets of restrictions of sections of E to the leaves of $\mathcal{H}$. If $x \in M \setminus Crit(p)$ and $\mathcal{H}_x = p^{-1}(p(x))$, then the fibre of $\mathcal{J}^l(E_{|\mathcal{H}})$ over x is the space of l-jets at x of sections of the restriction $E_{|\mathcal{H}_x}$. These bundles are likewise equipped with projections

$$\pi^{m,l} : \mathcal{J}^m(E_{|\mathcal{H}}) \to \mathcal{J}^l(E_{|\mathcal{H}}),$$

$m \geq l \geq 0$ and with jet maps

$$j^l_{\mathcal{H}} : s \in \Gamma(M, E) \mapsto j^l_{\mathcal{H}}(s) \in \Gamma(M \setminus Crit(p), \mathcal{J}^l(E_{|\mathcal{H}})).$$

These jet maps induce bundle morphisms

$$j^l_{\mathcal{H}} : (x, s) \in \underline{U}_{|M \setminus Crit(p)} \mapsto (x, j^l_{\mathcal{H}}(s)_{|x}) \in \mathcal{J}^l(E_{|\mathcal{H}}). \tag{1.1}$$

Definition 1.2. *The linear subspace U of $\Gamma(M,E)$ is said to be relatively l-ample if and only if the bundle morphism $j^l_{\mathcal{H}} : \underline{U}_{|M\setminus Crit(p)} \to \mathcal{J}^l(E_{|\mathcal{H}})$ is onto. It is said to be relatively ample if and only if it is relatively 1-ample. The kernel of $j^l_{\mathcal{H}}$ is then called the l-th incidence variety and denoted by $\mathcal{I}^l$.*

The incidence varieties given by Definition 1.2 are equipped with projections

$$\begin{aligned}\pi_M :&(x,s) \in \mathcal{I}^l \mapsto x \in M \setminus Crit(p) \text{ and}\\ \pi_U :&(x,s) \in \mathcal{I}^l \mapsto s \in U,\end{aligned}$$

see 5.1 for further properties. We set

$$\begin{aligned}\Delta_0 &= \{s \in U \mid s \text{ does not vanish transversally}\} \text{ and if } n \geq 2,\\ \Delta_1 &= \Delta_0 \cup \{s \in U \setminus \Delta_0 \mid p_{|s^{-1}(0)} \text{ is not Morse.}\}\end{aligned} \tag{1.2}$$

Then, for every $i \in \{0, \cdots, n-1\}$, we set

$$\mathcal{I}^1_i = \{(x,s) \in (M \setminus Crit(p)) \times (U \setminus \Delta_1) \mid s(x) = 0 \text{ and } x \in Crit_i(p_{|s^{-1}(0)})\},$$

where $Crit_i(p_{|s^{-1}(0)})$ denotes the set of critical points of index i of the restriction of p to $s^{-1}(0)$. The disjoint union $\mathcal{I}^1_0 \cup \cdots \cup \mathcal{I}^1_{n-1}$ provides a partition of $\mathcal{I}^1 \setminus \pi_U^{-1}(\Delta_1)$, see Appendix 5.1.

These incidence varieties equip $\underline{U}_{|M\setminus Crit(p)}$ with some filtration whose first graded maps read

$$gr^0 : (x, s_0) \in \underline{U}/\mathcal{I}^0 \mapsto s_0(x) \in E$$

and

$$gr^1 : (x, s_0, s_1) \in \underline{U}/\mathcal{I}^0 \oplus \mathcal{I}^0/\mathcal{I}^1 \mapsto (s_0(x), \nabla s_{1|H_x}) \in E \oplus (H^* \otimes E).$$

Finally, we set

$$\begin{aligned}H^\circ &= \{\lambda \in T^*M \mid \lambda_{|H} = 0\} \text{ and}\\ j : (x,s) &\in \mathcal{I}^1 \mapsto (x, \nabla s, \nabla^2 s_{|H_x}) \in (H^\circ \oplus Sym^2(H^*)) \otimes E\end{aligned}$$

when $n \geq 2$, while we set

$$j_0 : (x,s) \in \mathcal{I}^0 \mapsto (x, \nabla s) \in T^*M \otimes E$$

when $n = 1$. Note that $\det(gr^1) = \det(j^1_{\mathcal{H}}) : \det(\underline{U}/\mathcal{I}^1) \to \det(H^*) \otimes (\det E)^n$ and that for every $(x,s) \in \mathcal{I}^1$, $j(x,s)$ induces the morphisms

$$\begin{aligned}j(x,s) &: T_xM/H_x \oplus H_x \to E_x \oplus (H^*_x \otimes E_x) \text{ and}\\ \det(j(x,s)) &: \det(T_xM) \to \det(H^*) \otimes (\det E)^n.\end{aligned}$$

1.2. The induced Riemannian metrics.

Lemma 1.3. *Let F, G be two finite dimensional real vector spaces and $A : F \to G$ be an onto linear map. Let $\langle , \rangle_F$ be a scalar product on F and $\# : F^* \to F$ be the associated isomorphism. Then, the composition $(A\#A^*)^{-1} : G \to G^*$ defines a* *scalar product* $\langle\, ,\rangle_G$ *on* G. *Moreover, if* μ_F *(resp.* μ_G*) denotes the Gaussian measure associated to* $\langle\, ,\rangle_F$ *(resp.* $\langle\, ,\rangle_G$*), then* $\mu_G = A_*\mu_F$.

Let $|df|$ (resp. $|dg|$) be the Lebesgue measure associated to $\langle\, ,\rangle_F$ (resp. $\langle\, ,\rangle_G$). Then

$$d\mu_F(f) = \frac{1}{\sqrt{\pi}^{\dim F}} e^{-\|f\|^2}|df|$$

and $d\mu_G(g) = \frac{1}{\sqrt{\pi}^{\dim G}} e^{-\|g\|^2}|dg|$, where $\|f\|^2 = \langle f, f\rangle_F$ and $\|g\|^2 = \langle g, g\rangle_G$.

Proof. Let $g_1^*, g_2^* \in G^*$. Then $\langle g_1^*, g_2^*\rangle_{G^*} = g_2^*(A\#A^*(g_1^*)) = A^*(g_2^*)(\#A^*(g_1^*)) = \langle \#A^*(g_2^*), \#A^*(g_1^*)\rangle_F$. Since A^* is injective, we deduce that $\langle\, ,\rangle_{G^*}$ is a scalar product on G^* and hence that $\langle\, ,\rangle_G$ is a scalar product on G. Moreover, $\#A^* : G^* \to (\ker A)^\perp$ is an isometry, so that $A : (\ker A)^\perp \to G$ is an isometry. Since μ_F is a product measure, we deduce that $\mu_G = A_*\mu_F$. □

Definition 1.4. *Under the hypotheses of Lemma 1.3,* $\langle\, ,\rangle_G$ *(resp.* μ_G *) is called the push-forward of* $\langle\, ,\rangle_F$ *(resp.* μ_F*) under* A.

Definition 1.5. *Let* $U \subset \Gamma(M, E)$ *be an ample finite dimensional linear subspace, which is equipped with a scalar product* $\langle\, ,\rangle$. *The latter induces a Riemannian metric on the trivial bundle* $\underline{U}$ *which restricts to a metric on* $\mathcal{I}^l$, $l \in \mathbb{N}$. *We denote by* $\mu_{\mathcal{I}^l}$ *the associated Gaussian measure and by*

- g^1 *the push-forward on* $E \oplus (H^* \otimes E)$ *of* $\langle\, ,\rangle$ *under* gr^1,
- h^l *the push-forward on* $\mathcal{J}^l(E_{|\mathcal{H}})$ *of* $\langle\, ,\rangle$ *under* $j^l_{\mathcal{H}}$ *and*
- h *the push-forward on* $Im(j) \subset (H^\circ \oplus Sym^2(H^*)) \otimes E$ *of* $\langle\, ,\rangle$ *under* j,

see §1.1 and Lemma 1.3.

When $n = 1$, *we denote by*

- g^0 *the push-forward on* E *of* $\langle\, ,\rangle$ *under* gr^0,
- h_0 *the push-forward on* $Im(j_0) \subset T^*M \otimes E$ *of* $\langle\, ,\rangle$ *under* j_0.

Definition 1.6. *The Schwartz kernel of* $(U, \langle\, ,\rangle)$ *is the section* e *of* $\underline{U} \otimes E$ *satisfying for every* $s \in U$ *and* $x \in M$, $s(x) = \langle e_x, s\rangle$.

Note that if $(s_1, \cdots, s_N)$ denotes an orthonormal basis of U, then for every $x \in M$, $e_x = \sum_{i=1}^N s_i(x)s_i$. The metrics g^1, h^l and h given by Definition 1.5 can be computed in terms of the Schwartz kernel e, as follows from Lemma 1.7 and 1.8, compare [5], [21]

Lemma 1.7. *Let* E *be a real line bundle over a smooth manifold* M *equipped with a Morse function. Let* U *be a finite dimensional linear subspace of* $\Gamma(M, E)$ *which is relatively* l*-ample for* $l \in \mathbb{N}^*$ *and equipped with a scalar product. Let* e *be its Schwartz kernel. Then, the metrics* h^l *and* g^1 *are given by the restriction to the diagonal of* $(j^l_{\mathcal{H}} j^l_{\mathcal{H}} e)^{-1}$ *and* $(gr^1 gr^1 e)^{-1}$.

Note that e is a section of $E \boxtimes E$ over $M \times M$, so that $j^l_{\mathcal{H}} j^l_{\mathcal{H}} e$ (resp. $gr^1 gr^1 e$), which applies $j^l_{\mathcal{H}}$ (resp. gr^1) on each variable of e, is a section of $\mathcal{J}^l(E_{|\mathcal{H}})^{\boxtimes 2}$ (resp. $(E \oplus (H^* \otimes E))^{\boxtimes 2}$). Its restriction to the diagonal thus defines a symmetric bilinear form on $\mathcal{J}^l(E_{|\mathcal{H}})^*$ (resp. $(E \oplus (H^* \otimes E))^*$).

Proof. Let $\theta^* \in \mathcal{J}^l(E_{|\mathcal{H}})^*$ and $s \in U$. Then, $s = \langle e, s\rangle$ and $(j^{l*}_{\mathcal{H}}\theta^*)(s) = \langle \theta^*(j^l_{\mathcal{H}} e), s\rangle$. Consequently, $\#(j^l_{\mathcal{H}})^*\theta^* = \theta^*(j^l_{\mathcal{H}} e)$ and $j^l_{\mathcal{H}}\#(j^l_{\mathcal{H}})^* = j^l_{\mathcal{H}} j^l_{\mathcal{H}} e$. Likewise, $gr^1\#gr^{1*} = gr^1 gr^1 e$. □

LEMMA 1.8 (Compare appendix A of [21]). *Let $A : F \to G$ be a linear map between two real finite dimensional vector spaces. Let K_F (resp. K_G) be a subspace of F (resp. G) such that $A(K_F) \subset K_G$ and $a : K_F \to K_G$ be the restriction of A. Let $\langle\, ,\,\rangle_F$ be a scalar product on F and let K_F be equipped with its restriction. Let L_G be a complement subspace of K_G in G and $b : K_F^{\perp} \to K_G$ (resp. $c : K_F^{\perp} \to L_G$) be such that*

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : K_F \oplus K_F^{\perp} \to K_G \oplus L_G.$$

Then,

$$A\#A^* = \begin{bmatrix} a\#a^* + b\#b^* & b\#c^* \\ c\#b^* & c\#c^* \end{bmatrix}.$$

□

REMARK 1.9. *Since $a\#a^* = (a\#a^* + b\#b^*) - b\#c^*(c\#c^*)^{-1}c\#b^*$, we deduce from Lemma 1.8 that the scalar product $(a\#a^*)^{-1}$ can be computed from $(A\#A^*)^{-1}$. Applying Lemma 1.8 to*

$$\begin{cases} F &= \underline{U}, \\ G &= \mathcal{J}^1(E) \times_M \mathcal{J}^2(E, \mathcal{H}), \\ K_F &= \mathcal{I}^1 \text{ and} \\ K_G &= Im(j) \subset (H^\circ \oplus Sym^2(H^*)) \otimes E, \end{cases}$$

we deduce that the metric h can be computed in terms of the Schwartz kernel e of U and the jet maps j^1 and $j^2_{\mathcal{H}}$.

1.3. Distribution of critical points.

1.3.1. The main result. Let $U \subset \Gamma(M, E)$ be a relatively l-ample linear subspace of finite dimension N, see Definition 1.2. We equip U with a scalar product $\langle\, ,\,\rangle$ and denote by μ_U the associated Gaussian measure, so that at every point $s \in U$ its density against the Lebesgue measure $|ds|$ on U equals $\frac{1}{\sqrt{\pi}^N} e^{-\langle s,s\rangle}$. Then, for every $i \in \{0, \cdots n-1\}$ and every $s \in U \setminus \Delta_1$, where Δ_1 is given by (1.2) we set

$$\nu_i(s) = \sum_{x \in Crit_i(p_{|s^{-1}(0)}) \setminus Crit(p)} \delta_x$$
$$\mathbb{E}(\nu_i) = \int_{U \setminus \Delta_1} \nu_i(s) d\mu_U(s)$$

when $n \geq 2$, while when $n = 1$, we set

$$\nu_0(s) = \sum_{x \in s^{-1}(0)} \delta_x$$
$$\mathbb{E}(\nu_0) = \int_{s \in U \setminus \Delta_0} \nu_0(s) d\mu_U(s).$$

Note that we have no control a priori on the number of critical points of the restriction of p to $s^{-1}(0)$, so that $\mathbb{E}(\nu_i)$ may not be well defined.

Theorem 1.10. *Let E be a real line bundle over a smooth n-dimensional manifold M equipped with a Morse function. Let $U \subset \Gamma(M, E)$ be a finite dimensional relatively ample linear subspace equipped with a scalar product. Then, when $n \geq 2$, for every $i \in \{0, \cdots, n-1\}$,*

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{\pi}^n} \iint_{(H^\circ \oplus Sym_i^2(H^*)) \otimes E} |(\alpha, \beta)^* dvol_{g^1}| j_* d\mu_{\mathcal{I}^1}(\alpha, \beta). \tag{1.3}$$

Moreover, this measure has no atom and its density with respect to any Lebesgue measure lies in $C^\infty(M \setminus Crit(p))$. If in addition at every point $x \in Crit(p)$ the jet map $j^1 : U \to \mathcal{J}^1(E)_{|x}$ is onto, then this density lies in $L^1_{loc}(M)$, so that $\mathbb{E}(\nu_i)$ defines a measure on the whole M. When $n = 1$,

$$\mathbb{E}(\nu_0) = \frac{1}{\sqrt{\pi}} \int_{T^*M \otimes E} |\alpha^* dvol_{g^0}| j_{0_*} d\mu_{\mathcal{I}^0}(\alpha).$$

Theorem 1.10 describes the expected distribution of critical points of the restriction $p_{|s^{-1}(0)}$. Every pair $(\alpha, \beta) \in (H^\circ \oplus Sym^2(H^*)) \otimes E$ defines a morphism

$$(\alpha, \beta) : (TM/H) \oplus H \to E \oplus (H^* \otimes E),$$

while the bundle $E \oplus (H^* \otimes E)$ is equipped with the metric g^1 and its associated volume form $dvol_{g^1}$, see Definition 1.5. It follows that $(TM/H) \oplus H$ inherits the n-form $(\alpha, \beta)^* dvol_{g^1}$. The latter induces a n-form on TM, also denoted by $(\alpha, \beta)^* dvol_{g^1}$, since $\det(TM)$ is canonically isomorphic to $\det((TM/H) \oplus H)$. Finally, we have denoted by $Sym_i^2(H^*)$ the open cone of non-degenerate symmetric bilinear forms of index i on H. Recall that the index of a symmetric bilinear form is the maximal dimension of a subspace on which the form restricts to a negative definite one. Note that the form $(\alpha, \beta)^* dvol_{g^1}$ depends polynomially on (α, β), so that it is integrable with respect to the Gaussian measure $j_* \mu_{\mathcal{I}^1}$. Note finally that from Lemma 1.7 and Remark 1.9, both g^1 and $j_* d\mu_{\mathcal{I}^1}$ can be computed in terms of the Schwartz kernel of $(U, \langle \, , \, \rangle)$, see Definition 1.6.

Proof. By definition, $\mathbb{E}(\nu_i) = (\pi_{M|\mathcal{I}_i^1})_* \pi_U^* d\mu_U$ since the measure of Δ_1 vanishes by Lemma 5.1. From the coarea formula, see Theorem 3.2.3 of [6] or Theorem 1 of [24], we get

$$(\pi_{M|\mathcal{I}_i^1})_* \pi_U^* d\mu_U = \frac{1}{\sqrt{\pi}^n} \int_{\mathcal{I}_i^1} |dvol_{((d\pi_M \circ d\pi_U^{-1})\#(d\pi_M \circ d\pi_U^{-1})^*)^{-1}}| d\mu_{\mathcal{I}^1}, \tag{1.4}$$

Note indeed that $\mathcal{I}^1$ has codimension n in $\underline{U}$, so that the normalization in $d\mu_{\mathcal{I}_i^1}$ and $d\mu_U$ differs by a factor $1/\sqrt{\pi}^n$. For every $(x, s) \in \mathcal{I}^1$,

$$T_{(x,s)}\mathcal{I}^1 = \{(\dot{x}, \dot{s}) \in T_{(x,s)}\underline{U} \mid j^1_{\mathcal{H}}(\dot{s}) + \nabla^{\mathcal{J}}_{\dot{x}}(j^1_{\mathcal{H}}(s)) = 0\},$$

see (5.2), so that $d_{|(x,s)}\pi_M \circ d_{|(x,s)}\pi_U^{-1} = -(\nabla^{\mathcal{J}}(j^1_{\mathcal{H}}(s)))^{-1} \circ j^1_{\mathcal{H}}$. The operator $\nabla^{\mathcal{J}}(j^1_{\mathcal{H}}(s))$ is invertible since $s \in U_L \setminus \Delta_1$, see Remark 5.2. It follows that the determinant of the morphism $\underline{U}/\mathcal{I}^1 \to TM$ induced by $d_{|(x,s)}\pi_M \circ d_{|(x,s)}\pi_U^{-1}$ coincides with the one of

$$-j(s)^{-1} \circ gr^1 : \underline{U}/\mathcal{I}^0 \oplus \mathcal{I}^0/\mathcal{I}^1 \to TM/H \oplus H$$

via the canonical isomorphisms $\det(\underline{U}/\mathcal{I}^1) \cong \det(\underline{U}/\mathcal{I}^0 \oplus \mathcal{I}^0/\mathcal{I}^1)$ and $\det(TM) \cong \det(TM/H \oplus H)$. We deduce that

$$dvol_{((d\pi_M \circ d\pi_U^{-1})\#(d\pi_M \circ d\pi_U^{-1})^*)^{-1}} = dvol_{((j(s)^{-1}\circ gr^1)\#(j(s)^{-1}\circ gr^1)^*)^{-1}} = j(s)^* dvol_{g^1}.$$

Using the substitution $(\alpha, \beta) = j(s)$, we conclude that

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{\pi}^n} \int_{(H^\circ \oplus Sym_i^2(H^*))\otimes E} |(\alpha,\beta)^* dvol_{g^1}| j_* \mu_{\mathcal{I}^1}(\alpha, \beta).$$

Note that g^1 is a smooth metric on $E \oplus (H^* \otimes E)$ since $\mu_{\mathcal{I}^1}$ is a smooth family of Gaussian measures on $\mathcal{I}^1$ and j a smooth morphism. We deduce that $\mathbb{E}(\nu_i)$ has no atom and that its density with respect to any Lebesgue measure on M belongs to $C^\infty(M \setminus Crit(p))$.

Now, let us assume in addition that at every critical point x of p, the jet map $j^1 : U \to \mathcal{J}^1(E)_{|x}$ is onto and let us prove that this density then also belongs to $L^1_{loc}(M)$, so that $\mathbb{E}(\nu_i)$ extends to a measure without atom on the whole M. We denote by $\pi : P(T^*M) \to M$ the projectivization of the cotangent bundle and by $\tau \subset \pi^*(T^*M)$ the tautological line bundle over $P(T^*M)$. From the inclusion $\tau \otimes \pi^* E \to \pi^*(T^*M \otimes E)$ we deduce the short exact sequence

$$0 \to \tau \otimes \pi^* E \to \pi^* \mathcal{J}^1(E) \to \pi^* \mathcal{J}^1(E)/\tau \otimes \pi^* E \to 0.$$

With a slight abuse of notation, we denote by $H \subset \pi^*(TM)$ the codimension one subbundle given by the kernels of the elements of $\tau \setminus \{0\}$ and by $\mathcal{J}^1(E, H)$ the quotient bundle $\pi^* \mathcal{J}^1(E)/\tau \otimes \pi^*(E)$. Let V be a compact neighbourhood of $Crit(p)$ such that the restriction of the morphism $j^1 : \underline{U}_{|V} \to \mathcal{J}^1(E)_{|V}$ is onto. We deduce a morphism $j^1 : \pi^* \underline{U} \to \pi^* \mathcal{J}^1(E)$ over $P(T^*M)_{|V}$ which is onto and by composition with the onto map $\pi^* \mathcal{J}^1(E) \to \mathcal{J}^1(E, H)$, an onto morphism $\pi^* \underline{U} \to \mathcal{J}^1(E, H)$. We denote, with an abuse of notation, by $\mathcal{I}$ the kernel of the latter and by g^1 the metric that this morphism induces by push-forward on $\mathcal{J}^1(E, H)$ over $P(T^*M)_{|V}$, see Lemma 1.3. Now, let ∇ be a torsion-free connection on M and let ∇^E be a connection on E. They define a bundle morphism

$$\mathcal{J} : s \in \mathcal{I} \mapsto (\nabla s_{|H}, \nabla(\nabla^E s)_{|H^2}) \in (\tau \oplus Sym^2(H^*)) \otimes \pi^* E.$$

We then set

$$\begin{aligned}\Omega &= \frac{1}{\sqrt{\pi}^n} \int_{\mathcal{I}} \mathcal{J}(s)^* |dvol_{g^1}| d\mu_{\mathcal{I}}(s) \\ &= \frac{1}{\sqrt{\pi}^n} \int_{\tau \otimes \pi^* E} \int_{Sym^2(H^*)\otimes \pi^* E} (\alpha, \beta)^* |dvol_{g^1}| (\mathcal{J}_* d\mu_{\mathcal{I}})(\alpha, \beta),\end{aligned}$$

where $\mu_{\mathcal{I}}$ denotes the fiberwise Gaussian measure associated to the restriction of the metric of $\pi^* \underline{U}$ to $\mathcal{I}$. Consequently, Ω provides a section of the fibre bundle $\pi^* \det(T^*M)$ over the compact $P(T^*M)_{|V}$. Let ω be a volume form on V. It trivializes $\det(T^*M)$ over V and $\pi^* \det(T^*M)$ over $P(T^*M)_{|V}$. We deduce that there exists a positive constant $c > 0$ such that $|\Omega| \leq c|\omega|$ over $P(T^*M)_{|V}$. However, from Lemma 5.3, the jet map j on $\mathcal{I}^1$ factors as $j = T \circ \mathcal{J}$, where T denotes the trigonal endomorphism of $(H^\circ \oplus Sym^2(H^*)) \otimes E$ defined by

$$(\alpha, \beta) \mapsto (\alpha, \beta - (\frac{1}{dp} \nabla(dp)_{|H^2})\alpha)$$

and where $\mathcal{I}^1$ is identified with the pull-back $[dp]^*\mathcal{I}$ by the section $[dp]$ of $P(T^*M)_{|M\setminus Crit(p)}$ defined by the differential of p. Finally,

$$\begin{aligned}\mathbb{E}(\nu_i) &= \frac{1}{\sqrt{\pi}^n}\iint_{(H^\circ\oplus Sym^2_i(H^*))\otimes\pi^*E} |T^*(\alpha,\beta)^* dvol_{g^1}|\mathcal{J}_* d\mu_{\mathcal{I}^1}\\ &\leq \frac{C|\omega|}{\sqrt{\pi}^n}\iint_{(H^\circ\oplus Sym^2_i(H^*))\otimes\pi^*E} |\det T(\alpha,\beta)|\mathcal{J}_* d\mu_{\mathcal{I}^1}.\end{aligned}$$

Since the differential dp vanishes transversally on $Crit(p)$, the function $\det T(\alpha,\beta)$ is polynomial in α,β and his coefficients are smooth functions on $M\setminus Crit(p)$ with poles of order at most $n-1$ at $M\setminus Crit(p)$. After integration against the Gaussian measure $\mathcal{J}_* d\mu_{\mathcal{I}^1}$, we deduce that the function

$$\int_{H^\circ\otimes E}\int_{Sym^2_i(H^*)\otimes\pi^*E} |\det T(\alpha,\beta)|\mathcal{J}_* d\mu_{\mathcal{I}^1}$$

is smooth over $M\setminus Crit(p)$ with poles of order at most $n-1$ on $Crit(p)$ (compare Remark 3.3.3 of [10]). Since $\dim M = n$, we deduce that this function belongs to $L^1_{loc}(M)$, so that $\mathbb{E}(\nu_i)$ extends to a measure without atom over the whole M.

In the case $n=1$,

$$T_{(x,s)}\mathcal{I}^0 = \{(\dot{x},\dot{s})\in T_{(x,s)}\underline{U}\,|\,\dot{s}(x)+\nabla_{\dot{x}}s_{|x}=0\},$$

so that $d_{|(x,s)}\pi_M\circ d_{|(x,s)}\pi_U^{-1} = -(j_0(s))^{-1}\circ gr^0$. We deduce that

$$dvol_{((d\pi_M\circ d\pi_U^{-1})\#(d\pi_M\circ d\pi_U^{-1})^*)^{-1}} = dvol_{((j_0(s)^{-1}\circ gr^0)\#(j_0(s)^{-1}\circ gr^0)^*)^{-1}} = j_0(s)^* dvol_{g^0}.$$

Using the substitution $\alpha = j_0(s)$, we conclude that

$$\mathbb{E}(\nu) = \frac{1}{\sqrt{\pi}}\int_{T^*M\otimes E} |\alpha^* dvol_{g^0}|{j_0}_*\mu_{\mathcal{I}^0}(\alpha).$$

□

1.3.2. Mean Morse numbers. Under the hypotheses of Theorem 1.10, assume in addition that M is compact without boundary. Then, for every $s\in U\setminus\Delta_1$, $s^{-1}(0)$ is a smooth compact hypersurface of M and for every $i\in\{0,\cdots,n-1\}$, we set

$$\mathbb{E}(m_i) = \int_{U\setminus\Delta_1} m_i(s)d\mu_U(s),$$

see (0.9).

COROLLARY 1.11. *Under the hypotheses of Theorem 1.10, we assume in addition that M is closed. Then, for every $i\in\{0,\cdots,n-1\}$ and every volume form ω on M,*

$$\mathbb{E}(m_i) \leq \frac{1}{\sqrt{\pi}^n}\int_M\iint_{(H^\circ\oplus Sym^2_i(H^*))\otimes E} |(\alpha,\beta)^* dvol_{g^1}|j_* d\mu_{\mathcal{I}^1}(\alpha,\beta).$$

Proof. Corollary 1.11 is a consequence of Theorem 1.10 after integration of the constant function 1. □

1.3.3. An asymptotic result. Let now $(U_L)_{L\in\mathbb{R}_+^*}$ be a family of finite dimensional linear subspaces of $\Gamma(M,E)$ which are ample for L large enough. We want to estimate the asymptotic of the measure $\mathbb{E}(\nu_i)$ computed by Theorem 1.10 as L grows to infinity. In order to do so, we need to assume that the family $(U_L)_{L\in\mathbb{R}_+^*}$ is tamed in some sense and from Remark 1.9, we know that it is sufficient to tame the Schwartz kernel $(e_L)_{L\in\mathbb{R}_+^*}$, see Definition 1.6. However, we found it convenient to tame directly the induced metrics given by Definition 1.5, see Definition 1.14.

Definition 1.12. *Let p,q be two positive integers. A one-parameter (p,q)-group of endomorphisms of jet bundles is a one-parameter group $(a_L)_{L\in\mathbb{R}_+^*}$ of diagonalizable endomorphisms on the jet bundles $\mathcal{J}^l(E)$, $l\in\mathbb{N}$ such that*

1. *For every $0\le l\le m$, the projection $\pi^{m,l}:\mathcal{J}^m(E)\to\mathcal{J}^l(E)$ is a_L-equivariant.*
2. *For every $l\in\mathbb{N}$, the restriction of a_L to $\ker\pi^{l+1,l}=Sym^{l+1}(TM^*)\otimes E$ is a homothetic transformation of ratio $L^{-p-(l+1)q}$.*

Any such one-parameter (p,q)-group of endomorphisms is obtained in the following way. We choose, for every $l\in\mathbb{N}$, a complement subspace K_{l+1} to $\ker\pi^{l+1,l}$ in $\mathcal{J}^{l+1}(E)$ and then we require that a_L preserves K_{l+1} for every $l\in\mathbb{N}$, $L\in\mathbb{R}_+^*$. The two conditions of Definition 1.12 then determine $(a_L)_{L\in\mathbb{R}_+^*}$ in a unique way. Note that any metric on $\mathcal{J}^{l+1}(E)$ provides such a complement K_{l+1} to $\ker\pi^{l+1,l}$, namely its orthogonal complement and induces then an isomorphism $\mathcal{J}^{l+1}(E)\cong S^{l+1}(T^*M\otimes E)$.

Lemma 1.13. *Let E be a real fibre bundle over a smooth manifold M. Let $(a_L)_{L\in\mathbb{R}_+^*}$ and $(b_L)_{L\in\mathbb{R}_+^*}$ be two one-parameter (p,q)-groups of jet bundle endomorphisms, $p,q>0$. Then, for every $l\in\mathbb{N}$, the composition*

$$a_L\circ b_L^{-1}:\mathcal{J}^l(E)\to\mathcal{J}^l(E)$$

converges to the identity as L grows to ∞.

Proof. We proceed by induction on $l\in\mathbb{N}$. When $l=0$, a_L and b_L are homothetic transformations of ratio L^{-p} on $\mathcal{J}^0(E)$, so that $a_L\circ b_L^{-1}$ equals the identity for every $L\in\mathbb{R}_+^*$. Let us now assume that Lemma 1.13 holds true up to $l\in\mathbb{N}$ and prove it for $l+1$. The endomorphisms a_L and b_L are diagonalizable and hence leave invariant some complement subspaces K_L^a and K_L^b of $\ker\pi^{l+1,l}$ in $\mathcal{J}^{l+1}(E)$. These complement subspaces do not depend on $L\in\mathbb{R}_+^*$ since a_L and $a_{L'}$ (resp. b_L and $b_{L'}$) commute for all L, $L'\in\mathbb{R}_+^*$. We deduce that in a diagonalization basis of a_L, where the eigenvalues are ordered in the decreasing way, $L^{-p},L^{-p-q},L^{-p-2q},\cdots,L^{-p-(l+1)q}$, there exists a lower unipotent endomorphism T such that $b_L=T\circ a_L\circ T^{-1}$. It follows that $a_L\circ b_L^{-1}=(a_L\circ T\circ a_L^{-1})\circ T^{-1}$ is a product of unipotent endomorphisms $(a_L\circ T\circ a_L^{-1})$ and T^{-1}. The coefficients of $a_L\circ T\circ a_L^{-1}$ converge outside the diagonal to 0 and the same holds for those of the product $a_L\circ T\circ a_L^{-1}\circ T^{-1}$. □

Note that every one-parameter (p,q)-group of endomorphisms $(a_L)_{L\in\mathbb{R}_+^*}$ of jet bundle $\mathcal{J}^l(E)$, $l\in\mathbb{N}$, induces a one-parameter group of endomorphisms of the bundle $\mathcal{J}^l(E_{|\mathcal{H}})$ denoted by $(a_L)_{L\in\mathbb{R}_+^*}$ too.

Now, let $(U_L)_{L\in\mathbb{R}_+^*}$ be a family of finite dimensional subspaces of $\Gamma(M,E)$ which are asymptotically ample, meaning ample for L large enough. We equip them with scalar products $\langle\ ,\ \rangle_{L\in\mathbb{R}_+^*}$ For L large enough the latter induces after push-forward by gr^0 and gr^1 respectively, a sequence of Riemannian metrics g_L^0, g_L^1 on E and

$E \oplus (H^* \otimes E)$ respectively, see Definition 1.5. It also induces the sequence of push-forwarded measures $j_{0*}\mu_{\mathcal{I}^0}$ and $j_*\mu_{\mathcal{I}^1_i}$ on $(H^\circ \oplus Sym^2(H)) \otimes E$.

DEFINITION 1.14. *The family $(U_L, \langle\ ,\rangle_L)_{L\in\mathbb{R}^*_+}$ is said to be (p,q)-tamed if and only if there exists a one-parameter (p,q)-group of endomorphisms $(a_L)_{L\in\mathbb{R}^*_+}$ of jet bundles such that*

- *When $n \geq 2$, $(a_L)^{-1*}g^1_L$ converges to a metric g_∞ on $E \oplus (H^* \otimes E)$ and for every $i \in \{0, \cdots, n-1\}$, $(a_L)_* j_* \mu_{\mathcal{I}^1_i}$ converges to a measure μ^i_∞.*
- *When $n = 1$, $(a^*_L)^{-1}g^0_L$ converges to a metric g_∞ on E and $(a_L)_* j_{0*}\mu_{\mathcal{I}^0}$ converges to a measure μ_∞.*

COROLLARY 1.15. *Let E be a real line bundle over a smooth manifold equipped with a Morse function. Let $(U_L, \langle\ ,\ \rangle_L)_{L\in\mathbb{R}^*_+}$ be a family of asymptotically ample finite dimensional linear subspaces of $\Gamma(M,E)$, which are (p,q)-tamed for some $p, q > 0$. Then, for every $i \in \{0, \cdots, n-1\}$,*

$$\frac{1}{L^{qn}}\mathbb{E}(\nu_i) \underset{L\to\infty}{\to} \frac{1}{\sqrt{\pi}^n} \iint_{(H^\circ \oplus Sym^2_i(H))\otimes E} |(\alpha,\beta)^* dvol_{g_\infty}| d\mu^i_\infty(\alpha,\beta)$$

weakly on M when $n \geq 2$. When $n = 1$, $\frac{1}{L^q}\mathbb{E}(\nu) \underset{L\to\infty}{\to} \frac{1}{\sqrt{\pi}} \int_{T^*M\otimes E} |\alpha^* dvol_{g_\infty}| d\mu_\infty(\alpha)$.

Proof. From Theorem 1.10, for every $L \in \mathbb{R}^*_+$,

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{\pi}^n} \iint_{(H^\circ \oplus Sym^2_i(H))\otimes E} |(\alpha,\beta)^* dvol_{g^1}| j_*\mu_{\mathcal{I}^1_i}(\alpha,\beta).$$

Let $(a_L)_{L\in\mathbb{R}^*_+}$ be the one-parameter (p,q)-group of endomorphisms of jet bundles such that $(a_L^{-1})^* g^1_L$ converges to g_∞ as L grows to infinity and $(a_L)_* j_* \mu_{\mathcal{I}^1_i}$ converges to μ^i_∞. Then,

$$dvol_{a_L^{-1*}g^1_L} = a_L^{-1*} dvol_{g^1_L} = L^{p+(n-1)(p+q)} dvol_{g^1_L},$$

so that $dvol_{a_L^{-1*}g^1_L} \underset{L\to\infty}{\sim} L^{-p-(n-1)(p+q)} dvol_{g_\infty}$. We perform the substitution $a_L\alpha = \tilde{\alpha}$ and $a_L\beta = \tilde{\beta}$, so that

$$\mathbb{E}(\nu_i) \underset{L\to\infty}{\sim} L^{-p-(n-1)(p+q)} L^{p+q+(n-1)(p+2q)} \dots$$
$$\dots \frac{1}{\sqrt{\pi}^n} \iint_{(H^\circ \oplus Sym^2_i(H))\otimes E} |(\tilde{\alpha},\tilde{\beta})^* dvol_{g_\infty}| d\mu^i_\infty(\tilde{\alpha},\tilde{\beta})$$

since $(a_L \circ j)_* \mu_{\mathcal{I}^1_i} \underset{L\to\infty}{\to} \mu^i_\infty$. The proof in the case $n = 1$ is similar. ☐

2. Random eigensections of a self-adjoint elliptic operator. The aim of this section is to prove Theorem 0.1 and Corollary 0.2, see §2.3.2. We first recall in §2.1 the asymptotic estimates of the derivatives of the spectral function along the diagonal, which are needed to get these results from Remark 1.9. A proof of these estimates is given in Appendix 5.3 while several basic definitions on pseudo-differential operators are recalled in Appendix 5.2.

2.1. Asymptotic derivatives of the spectral function along the diagonal. Under the hypotheses of Theorem 0.1, we assume P to be positive, see Remark 5.9 and for every $L \in \mathbb{R}_+^*$, we denote by $e_L \in \Gamma(M \times M, E \boxtimes E)$ the spectral function of U_L, so that

$$\forall s \in U_L, \forall x \in M, s(x) = \int_M h_E(e_L(x,y), s(y))|dy|,$$

compare Definition 1.6. In particular, if $(s_1, \cdots, s_{N_L})$ denotes an orthonormal basis of U_L, then for every $x, y \in M, e_L(x,y) = \sum_{i=1}^{N_L} s_i(x)s_i(y)$. The metric h_E induces an isomorphism between the restriction of $E \boxtimes E$ to the diagonal of $M \times M$ and the trivial line bundle over M and under this isomorphism, for every $x \in M, e_L(x,x) = \sum_{i=1}^{N_L} h_E(s_i(x), s_i(x)) > 0$. The dimension N_L of U_L then reads $N_L = \int_M e_L(y,y)|dy|$. The asymptotic behaviour of the spectral function e_L along the diagonal is given by Theorem 2.1, due to Carleman [3] when $m = 2$ and to Gårding [7] in general.

Theorem 2.1 ([3], [7]). *Let P be an elliptic pseudo-differential operator of order $m > 0$, which is self-adjoint and bounded from below, acting on a real Riemannian line bundle over a smooth closed manifold $(M, |dy|)$ of positive dimension n. Let σ_P be the principal symbol of P and e_L be its spectral function, $L \in \mathbb{R}_+$. Then, for every $x \in M$,*

$$e_L(x,x) \underset{L \to \infty}{\sim} \frac{1}{(2\pi)^n} \int_{K_L} |d\xi|,$$

where $|d\xi|$ denotes the measure on T_x^*M induced by $|dy|$ and

$$K_L = \{\xi \in T^*M \mid \sigma_P(\xi) \leq L\}. \tag{2.1}$$

Note that $K_1 = K$, see (0.3). In particular, the asymptotic given by Theorem 2.1 neither depends on the Riemannian metric of E, nor on the global geometry of M, it only depends on the measure $|dy|$ of M at x and on the symbol of P.

Remark 2.2. *Recall that Theorem 2.1 recovers Weyl's theorem, which computes the dimension*

$$\frac{1}{L^{\frac{n}{m}}} N_L \underset{L \to \infty}{\to} \int_M c_0(y)|dy|,$$

see (0.4). For example, when P stands for the Laplace-Beltrami operator associated to some Riemannian metric on M, this formula reads

$$\frac{1}{\sqrt{L}^n} N_L \underset{L \to \infty}{\to} \frac{1}{(2\pi)^n} Vol(\mathbb{B}_n) Vol_g M,$$

where $Vol(\mathbb{B}_n)$ denotes the volume of the unit ball in $\mathbb{R}^n$, see §3.1.

In order to apply the results of §1, we have to know in addition the asymptotic of the partial derivatives of the spectral function e_L along the diagonal. This is the subject of Theorem 2.3.

Theorem 2.3. *Under the hypotheses of Theorem 2.1, let Q_1 and Q_2 be two differential operators on E with principal symbols σ_{Q_1} and σ_{Q_2}, of order $|\sigma_{Q_1}|$ and*

$|\sigma_{Q_2}|$, acting on the first and second variables of e_L respectively. Then, for every $x \in M$,

$$Q_1 Q_2 e_{L|(x,x)} = \frac{1}{(2\pi)^n} \int_{K_L} \sigma_{Q_1}(i\xi)\overline{\sigma_{Q_2}(i\xi)}|d\xi| + O(L^{\frac{n+|\sigma_{Q_1}|+|\sigma_{Q_2}|-1}{m}}), \tag{2.2}$$

see (2.1).

Theorem 2.3 is proved by L. Hörmander in [14] when Q_1 and Q_2 are trivial, providing the order of the error term in Theorem 2.1. It is written in [22] when Q_1 and Q_2 are of the same order, see Theorem 1.8.5 of [22], but we did not find a reference for the general case, which we need here. In the particular case where P is the Laplace-Beltrami operator, Theorem 2.3 is proved in [2], see also [21]. We give in Appendix 5.3 a proof of Theorem 2.3 which follows closely [14]. Note that when $|\sigma_{Q_1}|$ and $|\sigma_{Q_2}|$ are not of the same parity, the main term of the right-hand side of (2.2) vanishes since for every $\xi \in T^*M$, $\sigma_P(-\xi) = \sigma_P(\xi)$ while the principal symbols σ_{Q_1} and σ_{Q_2} are homogeneous. When $|\sigma_{Q_1}| = |\sigma_{Q_2}|\, mod(2)$, (2.2) reads

$$Q_1 Q_2 e_{L|(x,x)} \underset{L\to\infty}{\sim} \frac{1}{(2\pi)^n}(-1)^{\frac{|\sigma_{Q_1}|-|\sigma_{Q_2}|}{2}} \int_{K_L} \sigma_{Q_1}(\xi)\overline{\sigma_{Q_2}(\xi)}|d\xi|.$$

2.2. Metrics on symmetric tensor algebras. Let V be a real vector space and V^* be its dual. For every $k \in \mathbb{N}$, we denote by $Sym^k(V)$ the space of symmetric k-linear forms on V^*. For every $q \in Sym^k(V)$ and every $\xi \in V^*$, we set $q(\xi) = q(\xi, \cdots, \xi)$ and $q(i\xi) = i^k q(\xi)$. For every $l \in \mathbb{N}$, we set

$$\begin{aligned} S^l(V) &= \bigoplus_{0\le k\le l} Sym^k(V), \\ S^l_+(V) &= \{q \in S^l(V) \mid q(\xi) = q(-\xi)\}, \\ S^l_-(V) &= \{q \in S^l(V) \mid q(\xi) = -q(-\xi)\}. \end{aligned}$$

Lemma 2.4. *Let V be a real vector space and $l \in \mathbb{N}$. Let $K \subset V^*$ and μ be a positive finite measure on K such that*

1. $-id$ preserves K and μ

2. The support of μ is not included in any degree l algebraic hypersurface of V. Then, the bilinear form

$$\begin{aligned} \kappa^l : S^l(V) \times S^l(V) &\to \mathbb{C} \\ (q_1, q_2) &\mapsto \frac{1}{\mu(K)} \int_K q_1(i\xi)\overline{q_2(i\xi)} d\mu(\xi) \in \mathbb{C} \end{aligned}$$

associated to (K, μ) only takes real values and defines a scalar product on $S^l(V)$. Moreover, $S^l_+(V)$ and $S^l_-(V)$ are orthogonal to each other with respect to κ^l.

Proof. The form κ^l is bilinear and the change of variables $\xi \in K \mapsto -\xi \in K$ yields that $S^l_+(V)$ and $S^l_-(V)$ are orthogonal to each other. Moreover, the restrictions of κ^l to $S^l_+(V)$ and $S^l_-(V)$ are real and symmetric, so that κ^l itself is symmetric and takes only real values. Lastly, if $q = \sum_{j=0}^{\lfloor l/2\rfloor} q_j \in S^l_+(V)$, where for every $j \in \{0, \cdots, \lfloor l/2\rfloor\}$, $q_j \in Sym^{2j}(V^l)$, then

$$\kappa^l(q, q) = \frac{1}{\mu(K)} \int_K (\sum_{j=0}^{\lfloor l/2\rfloor} (-1)^j q_j(\xi))^2 d\mu(\xi),$$

so that the restriction of κ^l to $S^l_+(V)$ is non negative and the second hypothesis implies that it is positive definite. The same conclusion holds for the restriction of κ^l to $S^l_-(V)$, hence the result. □

REMARK 2.5.

1. *Under the hypotheses of Lemma 2.4, the restriction of κ^1 to $Sym^1(V) = V$ defines a scalar product on V.*
2. *If the measure μ can be chosen to be the absolute value of an alternated $\dim V$-linear form on V, then the scalar products κ^l given by Lemma 2.4 do not depend on the choice of this form and only depend on K. This is the case when K is bounded and has a non-empty interior.*

2.3. Proof of Theorem 0.1 and Corollary 0.2.

2.3.1. Induced metric on the symmetric tensor bundle. Since P is real and self-adjoint, the set $K_L = \{\xi \in T^*M \,|\, \sigma_P(\xi) \leq L\}$ is invariant under $-Id$ and induces thus a Riemannian metric on M and even on all symmetric tensor powers $S^l(TM)$, $l \in \mathbb{N}$, see Lemma 2.4 and Remark 2.5.

DEFINITION 2.6. *For every $L \in \mathbb{R}^*_+$ and $l \in \mathbb{N}$, we denote by κ^l_L the Riemannian metrics induced by K_L on $S^l(TM)$, see Lemma 2.4.*

Together with the metric h_E, κ^l_L induces a metric on $S^l(TM) \otimes E^*$ and by duality a metric on $S^l(T^*M) \otimes E$, still denoted by κ^l_L.

PROPOSITION 2.7. *Under the hypotheses of Theorem 2.1, for every $l \in \mathbb{N}$ and every large enough $L \in \mathbb{R}^*_+$, $(U_L, \langle\, ,\, \rangle_L)$ is l-ample and $(\frac{n}{2m}, \frac{1}{m})$-tamed. Moreover, the push-forward of $\langle\, ,\, \rangle_L$ under $j^l : \underline{U}_L \to \mathcal{J}^l(E)$ satisfies*

$$j^l_* \langle\, ,\, \rangle_L \underset{L\to\infty}{\sim} L^{\frac{n}{m}} c_0 \kappa^l_L,$$

see §1.3.3.

Proof. From Lemma 1.7, the push-forward h_L of $\langle\, ,\, \rangle_L$ under j^l induces on $\mathcal{J}^l(E)^*$ the metric $j^l j^l e_L$. Let us fix a torsion-free connection ∇ on TM and a connection ∇^E on E. They induce a decomposition $\mathcal{J}^l(E) \cong S^l(T^*M) \otimes E$ which equips $\mathcal{J}^l(E)^*$ with the metric κ^l_L. From Theorem 2.3 follows that the metrics h^1_L and $L^{\frac{n}{m}} c_0 \kappa^l_L$ are equivalent as L grows to infinity. In particular the asymptotic value of the induced metric $L^{\frac{n}{m}} c_0 \kappa^l_L$ on $\mathcal{J}^l(E)^*$ does not depend on the chosen decomposition $\mathcal{J}^l(E) \cong S^l(T^*M) \otimes E$, see Lemma 1.13. Now, κ^l_L is (p,q)-tamed with $p = n/(2m)$ and $q = 1/m$. Indeed, the one-parameter (p,q)-group of fibre bundles endomorphisms

$$a_L : \bigoplus_{k=0}^{l} Sym^k(T^*M) \otimes E \to \bigoplus_{k=0}^{l} Sym^k(T^*M) \otimes E$$
$$(q_k)_{k\in\{0,\cdots,l\}} \mapsto (L^{-\frac{n}{2m} - \frac{k}{m}} q_k)_{k\in\{0,\cdots,l\}}.$$

is such that $L^{n/m} a_L^{-1*} \kappa^l_L$ converges to the metric associated to $(K, d\xi)$ given by Lemma 2.4. □

COROLLARY 2.8. *Under the hypotheses of Theorem 0.1, the push-forward of $\langle\, ,\, \rangle_L$*

under j gets equivalent, as L grows to infinity and when $n \geq 2$, to

$$\big((H^\perp \times Sym^2(H^*)) \otimes E^*\big)^2 \to \mathbb{R}$$
$$((a_1,b_1),(a_2,b_2)) \mapsto \frac{1}{(2\pi)^n}\Big(\int_{K_L} h_E(a_1(\xi),a_2(\xi)) + h_E(b_1(\xi),b_2(\xi))|d\xi| - \cdots$$
$$\cdots \frac{1}{\int_{K_L}|d\xi|}\iint_{K_L^2} h_E(b_1(\xi),b_2(\xi'))|d\xi||d\xi'|\Big).$$

When $n = 1$, the push-forward of $\langle\ ,\ \rangle_L$ under j_0 gets equivalent, as L grows to infinity, to $(a_1,a_2) \in (T^*M \otimes E^*)^2 \mapsto \frac{1}{2\pi}\int_{K_L} h_E(a_1(\xi),a_2(\xi))|d\xi|$.

In Corollary 2.8, $H^\perp$ denotes the orthogonal of H with respect to the Riemannian metric of M associated to K_L, given by Definition 2.6. The distribution H is defined in §1 and j in §1.1.

Proof. From Proposition 2.7, the metric $j^2\#(j^2)^*$ of $\mathcal{J}^2(E)^*$ gets equivalent to $L^{\frac{n}{m}}c_0\kappa_L^2$ as L grows to infinity. By restriction to the fibre product $\big(\mathcal{J}^1(E) \times_{\mathcal{J}^1(E_{|\mathcal{H}})} \mathcal{J}^2(E_{|\mathcal{H}})\big)^*$, we deduce that the metric induced on this space gets equivalent to

$$((\mathbb{R} \oplus TM \oplus Sym^2(H)) \otimes E^*)^2 \to \mathbb{R}$$
$$((c_1,a_1,b_1),(c_2,a_2,b_2)) \mapsto \frac{1}{(2\pi)^n}\int_{K_L} h_E(c_1,c_2)(\xi) - h_E(c_1,b_2)(\xi) - \cdots$$
$$\cdots h_E(b_1,c_2)(\xi) + h_E(b_1,b_2)(\xi) + h_E(a_1,a_2)(\xi)|d\xi|.$$

We apply then Lemma 1.8 and Remark 1.9 to $F = \underline{U}_L$, $G = (\mathcal{J}^1(E) \times_{\mathcal{J}^1(E_{|\mathcal{H}})} \mathcal{J}^2(E_{|\mathcal{H}}))^*$, $K_F = \mathcal{I}^1$ and $K_G = (H^\perp \oplus Sym^2(H)) \otimes E^*$, where the middle term TM splits as $H \oplus H^\perp$. We deduce that the factors $H^\perp \otimes E^*$ and $Sym^2(H) \otimes E^*$ get asymptotically orthogonal, that the metric induced on $H^\perp \otimes E^*$ is asymptotically equivalent to $\frac{\mu(K_L)}{(2\pi)^n}$ times the one induced by K_L and finally that the one induced on $Sym^2(H) \otimes E^*$ is equivalent to

$$(b_1,b_2) \mapsto \frac{1}{(2\pi)^n}\Big(\int_{K_L} h_E(b_1,b_2)(\xi)|d\xi| - \frac{1}{\int_{K_L}|d\xi|}\iint_{K_L \times K_L} h_E(b_1(\xi),b_2(\xi'))|d\xi||d\xi'|\Big).$$

Indeed, with the notations of Lemma 1.8, $L_G = (\mathbb{R} \oplus H) \otimes E^*$ gets a metric $c\#c^*$ for which the factors $\mathbb{R} \otimes E^*$ and $H \otimes E^*$ are asymptotically orthogonal to each other and the metric on $\mathbb{R} \otimes E^*$ is $\frac{1}{(2\pi)^n}\mu(K_L)h_E$. Moreover, the correlation $b\#c^*$ only involves the factors $\mathbb{R} \otimes E^*$ and $Sym^2(H) \otimes E^*$ and reads

$$(c_1,b_2) \in E^* \oplus (Sym^2(H) \otimes E^*) \mapsto -\frac{1}{(2\pi)^n}\int_{K_L} h_E(c_1,b_2)(\xi)|d\xi|.$$

Finally $a\#a^* + b\#b^*$ is a metric on $(H^\perp \oplus Sym^2(H)) \otimes E^*$ for which both factors are asymptotically orthogonal, the metric induced on $H^\perp \otimes E^*$ is asymptotically equivalent to $\frac{\mu(K_L)}{(2\pi)^n}$ times the one induced by K_L, and the one induced on $Sym^2(H) \otimes E^*$ is

$$(b_1,b_2) \in Sym^2(H) \otimes E^* \mapsto \frac{1}{(2\pi)^n}\int_{K_L} h_E(b_1,b_2)(\xi)|d\xi|.$$

We deduce now that the correlation term $b\#c^*(c\#c^*)^{-1}c\#b^*$ just reads

$$\frac{1}{(2\pi)^n\int_{K_L}|d\xi|}\iint_{K_L \times K_L} h_E(b_1(\xi),b_2(\xi'))|d\xi||d\xi'|.$$

Hence the result. □

2.3.2. Proof of Theorem 0.1 and Corollary 0.2. We know from Proposition 2.7 that $U_L = \bigoplus_{\lambda \leq L} \ker(P - \lambda Id)$ equipped with the L^2-scalar product $\langle\, ,\, \rangle_L$ gets ample for L large enough and $(\frac{n}{2m}, \frac{1}{m})$-tamed, see Definition 1.14. From Corollary 1.15, we deduce that $\frac{1}{L^{\frac{n}{m}}}\mathbb{E}(\nu_i)$ weakly converges on the whole M to the measure

$$\frac{1}{\sqrt{\pi}^n} \iint_{(H^\perp \times Sym_i^2(H^*))\otimes E} |(\alpha, \beta)^* dvol_{g_\infty}| d\mu_\infty^i(\alpha, \beta), \tag{2.3}$$

where the metric g_∞ and the measure μ_∞^i are given by Definition 1.14. From Proposition 2.7 and Corollary 2.8, the factors E and $H^* \otimes E$ are orthogonal to each other with respect to g_∞, and g_∞ restricts to $c_0 h_E$ on E and to the metric $g_P \otimes h_E$ on $H^* \otimes E$, see (0.5). Likewise, from Corollary 2.8 the measure μ_∞^i is a product of the measure on $H^\circ \otimes E$ induced by g_P and h_E, and the measure on $Sym_i^2(H) \otimes E$ induced by (0.7) and h_E. We deduce that $dvol_{g_\infty} = \frac{1}{\sqrt{c_0}} dvol_{h_E}$ and that (2.3) becomes

$$\frac{1}{\sqrt{\pi}^n \sqrt{c_0}} \mathbb{E}(i, \ker dp) \Big(\int_{H^\perp \otimes E} |\alpha| d\mu_P(\alpha) \Big) |dvol_P|.$$

We conclude thanks to the equality

$$\int_{H^\perp \otimes E} |\alpha| d\mu_P(\alpha) = \int_{\mathbb{R}} |a| e^{-a^2} \frac{da}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}.$$

When $n = 1$, $\frac{1}{L^{\frac{1}{m}}}\mathbb{E}(\nu)$ weakly converges to the measure

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_{T^*M \otimes E} |\alpha^* dvol_{g_\infty^0}| d\mu_\infty(\alpha) &= \frac{1}{\sqrt{\pi}\sqrt{c_0}} \int_{T^*M \otimes E} |\alpha| d\mu_K(\alpha) |dvol_P| \\ &= \frac{1}{\pi \sqrt{c_0}} |dvol_P|. \ \square \end{aligned}$$

Proof of Corollary 0.2. It is a consequence of Theorem 0.1 after integration of the constant function 1, compare Corollary 1.11. □

3. Examples. We investigate in this third section two examples, the Laplace-Beltrami operator in §3.1, where we prove Corollary 0.3 and Proposition 0.4, and the Dirichlet-to-Neumann operator in §3.2, where we prove Corollary 0.5.

3.1. The Laplace-Beltrami operator.

3.1.1. Proof of Corollary 0.3. The principal symbol of the Laplace-Beltrami operator Δ_g reads $\sigma_{\Delta_g} : \xi \in T^*M \mapsto g(\xi, \xi) \in \mathbb{R}$, so that the compact K defined by (0.3) reads

$$K = \{\xi \in T^*M \,|\, g(\xi, \xi) \leq 1\}.$$

The Riemannian metric g_{Δ_g} induced on M by the pair $(K, |d\xi|)$ reads at every point $x \in M$, $(u, v) \in T_x M^2 \mapsto \frac{1}{(2\pi)^n} \int_K \xi(u)\xi(v)|d\xi|$ by (0.5), so that

$$g_{\Delta_g} = c_1 g \tag{3.1}$$

and

$$|dvol_{\Delta_g}| = \sqrt{c_1}^n |d\xi|, \tag{3.2}$$

where

$$c_1 = \frac{1}{(2\pi)^n} \int_K \xi_1^2 |d\xi|. \tag{3.3}$$

Let us choose an orthonormal basis $(\partial/\partial_{x_1}, \cdots, \partial/\partial_{x_n})$ of T_xM such that $(\partial/\partial_{x_1}, \cdots, \partial/\partial_{x_{n-1}})$ spans H_x and let us denote by $(\xi_1, \cdots, \xi_n)$ its dual basis. They induce isomorphisms $Sym^2(H) \cong Sym(n-1, \mathbb{R})$ and $Sym^2(H)^* \cong Sym(n-1, \mathbb{R})^*$. From Corollary 2.8, when $n > 2$ the metric induced by $(K, |d\xi|)$ on $Sym(n-1, \mathbb{R})^*$ then reads

$$\forall (A, B) = ((a_{ij})_{1\leq i,j\leq n-1}, (b_{ij})_{1\leq i,j\leq n-1}) \in (Sym^*(n-1, \mathbb{R}))^2,$$

$$\langle A, B\rangle_{\Delta_g} = \frac{1}{(2\pi)^n}\Big(\int_K A(\xi)B(\xi)|d\xi| - \frac{1}{\int_K |d\xi|} \int_K A(\xi)|d\xi| \int_K B(\xi)|d\xi|\Big)$$

where

$$\begin{aligned} \int_K A(\xi)B(\xi)|d\xi| = \int_K &(\sum_{i=1}^{n-1} a_{ii}\xi_i^2 + 2 \sum_{1\leq i<j\leq n-1} a_{ij}\xi_i\xi_j) \cdots \\ &\cdots (\sum_{i=1}^{n-1} b_{ii}\xi_i^2 + 2 \sum_{1\leq i<j\leq n-1} b_{ij}\xi_i\xi_j)|d\xi| \end{aligned}$$

and

$$\begin{aligned} \int_K A(\xi)|d\xi| \int_K B(\xi)|d\xi| = \int_K &(\sum_{i=1}^{n-1} a_{ii}\xi_i^2 + 2 \sum_{1\leq i<j\leq n-1} a_{ij}\xi_i\xi_j)|d\xi| \cdots \\ &\cdots \int_K (\sum_{i=1}^{n-1} b_{ii}\xi_i^2 + 2 \sum_{1\leq i<j\leq n-1} b_{ij}\xi_i\xi_j)|d\xi|, \end{aligned}$$

so that

$$\begin{aligned} \langle A, B\rangle_K &= (c_4 - \frac{c_1^2}{c_0}) \sum_{i=1}^{n-1} a_{ii}b_{ii} + (c_2 - \frac{c_1^2}{c_0}) \sum_{1\leq i\neq j\leq n-1} a_{ii}b_{jj} + 4c_2 \sum_{1\leq i<j\leq n-1} a_{ij}b_{ij} \\ &= 2c_2 \operatorname{Tr}(AB) + (c_2 - \frac{c_1^2}{c_0})(\operatorname{Tr} A)(\operatorname{Tr} B), \end{aligned}$$

where

$$\begin{aligned} c_4 &= \tfrac{1}{(2\pi)^n} \textstyle\int_K \xi_1^4 |d\xi|, \\ c_2 &= \tfrac{1}{(2\pi)^n} \textstyle\int_K \xi_1^2\xi_2^2 |d\xi| \text{ and} \\ c_0 &= \tfrac{1}{(2\pi)^n} \textstyle\int_K |d\xi|. \end{aligned}$$

This indeed follows from the relation $c_4 = 3c_2$, see [2], [21] and from the fact that $\int_K \xi_1^k \xi_2^l |d\xi| = 0$ whenever k or l is odd. Note that

$$\begin{aligned} c_2 &= \tfrac{c_0}{(n+4)(n+2)}, \\ c_1 &= \tfrac{c_0}{n+2} \text{ and} \\ c_2 - \tfrac{c_1^2}{c_0} &= \tfrac{-2c_2}{n+2}, \end{aligned} \tag{3.4}$$

see [2] and [21]. Hence, the scalar product induced by $(K, |d\xi|)$ on $Sym(n-1,\mathbb{R})^*$ is given, with the notations of the appendix B of [21], by the symmetric endomorphism $2c_2Q(a,b,c)$ with $a = \frac{n+1}{n+2}$, $b = \frac{-1}{n+2}$ and $c = 1$. As a consequence, the induced scalar product on $Sym(n-1,\mathbb{R})$ is given by the symmetric endomorphism $\frac{1}{2c_2}Q(a',b',c')$ with $a' = \frac{4}{3}$, $b' = \frac{1}{3}$ and $c' = 1$, see [21]. Hence, for every $(A,B) \in Sym(n-1,\mathbb{R})^2$,

$$\langle A, B\rangle_{\Delta_g} = \frac{1}{2c_2}(\mathrm{Tr}(AB) + \frac{1}{3}(\mathrm{Tr}\, A)(\mathrm{Tr}\, B)).$$

Finally,

$$\begin{aligned}
\mathbb{E}(i, \ker dp) &= \int_{Sym^2(H)} |\det \beta| d\mu_{\Delta_g}(\beta) \\
&= \frac{1}{c_1^{n-1}} \int_{Sym(i,n-1-i,\mathbb{R})} |\det B| e^{-\frac{1}{2c_2}(\mathrm{Tr}(B^2)+\frac{1}{3}(\mathrm{Tr}\, B)^2)} d\mu_{\Delta_g}(B) \\
&= \frac{\sqrt{c_2}^{n-1}}{c_1^{n-1}} \mathbb{E}(i, n-1-i),
\end{aligned}$$

see (0.10), since from (3.1), $|\det B| = c_1^{n-1}|\det \beta|$ under the substitution $B = \beta$. We deduce from Theorem 0.1 and (3.2) the weak convergence on M

$$\frac{1}{\sqrt{L}^n}\mathbb{E}(\nu_i) \underset{L\to\infty}{\to} \frac{1}{\sqrt{\pi}^{n+1}\sqrt{c_0}} \frac{\sqrt{c_2}^{n-1}}{c_1^{n-1}} \mathbb{E}(i, n-1-i)\sqrt{c_1^n}|dvol_g|.$$

The result follows now from (3.4) and Corollary 0.2. The proof goes along the same lines when $1 \le n \le 2$ and the result remains true in these cases. □

EXAMPLE 3.1. *When $n = 2$, $\mathbb{E}(0,1) = \mathbb{E}(1,0) = \int_0^{+\infty} ae^{-\frac{2}{3}a^2} d\mu(a) = \frac{\sqrt{3}}{2\sqrt{2}\sqrt{\pi}}$, so that from Corollary 0.3, for every $j \in \{0,1\}$,*

$$\frac{1}{L}\mathbb{E}(\nu_j) \underset{L\to\infty}{\to} \frac{1}{8\pi^2}|dvol_g| \tag{3.5}$$

$$\text{and } \limsup_{L\to\infty} \frac{1}{L}\mathbb{E}(m_j) \le \frac{1}{8\pi^2}Vol_g(M). \tag{3.6}$$

3.1.2. Proof of Proposition 0.4. By Corollary 0.3 and Weyl's Theorem, see Remark 2.2, it is enough to prove that there exist $C > 0$ and $\delta > 0$ such that

$$\forall n \in \mathbb{N}, \sum_{|\frac{i}{n}-\frac{1}{2}|\ge\epsilon} \mathbb{E}(i, n-i) \le C\exp(-\delta n^2),$$

since $\log Vol(\mathbb{B}^n) \sim_{n\to\infty} -\frac{n}{2}\log n$. Now, if $d\mu_{GOE}$ denotes the Gaussian probability measure on $Sym(n,\mathbb{R})$ associated to the scalar product $\langle A, B\rangle = Tr(AB)$, then the Gaussian probability measure μ associated to (0.11) satisfies the bound $\mu \le c_n\mu_{GOE}$ with $c_n = O(n)$. Indeed, $\frac{1}{2}\mathrm{Tr}\, A^2 + \frac{1}{6}(TrA)^2 \ge \frac{1}{2}\mathrm{Tr}\, A^2$, whereas the ratio between the determinants of these scalar product is a $O(n)$, see (B.6) in [21]. Now, Theorem 1.6 of [12] provides the result.

3.2. The Dirichlet-to-Neumann operator. Let (W, g) be a smooth compact Riemannian manifold with boundary and Δ_g be its Laplace-Beltrami operator. Let us denote by M the boundary of W and for every smooth function $f : M \to \mathbb{R}$, we denote by $u \in C^\infty(M, \mathbb{R})$ the solution of the Dirichlet problem

$$\begin{cases} \Delta_g u &= 0 \\ u_{|M} &= f. \end{cases}$$

We then denote by $\partial_n u : M \to \mathbb{R}$ the outward normal derivative of u along M. Then, the Dirichlet-to-Neuman operator Λ_g reads

$$\Lambda_g : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$
$$f \mapsto \partial_n u.$$

THEOREM 3.2 ([17]). *Let (W, g) be a smooth compact Riemannian manifold with boundary M. The Dirichlet-to-Neumann operator Λ_g is an elliptic pseudo-differential operator of order one on M. Its principal symbol equals $\xi \in T^*M \mapsto \|\xi\|_g$.*

Proof. [of Corollary 0.5] The compact K_Λ defined by (0.3) coincides with K_{Δ_g}, where K_{Δ_g} is the compact associated to the Laplace-Beltrami operator on M induced by the restriction of g to M. The proof of Corollary 0.5 thus goes along the same lines as the one of Corollary 0.3. □

4. Some related problems. Let us mention several related problems which we plan to discuss in a separate paper. First, we may consider, as our probability space, the span of eigensections with eigenvalues belonging to a window $[a(L)L, L]$ instead of $[0, L]$, where a is some function of L, compare [18], [23]. That is, we may set

$$U_L^a = \bigoplus_{\lambda \in [a(L)L, L]} \ker(P - \lambda Id).$$

When $\lim_{L\to\infty} a(L) = \gamma \in [0, 1]$, Theorem 0.1 still holds true, with the following modifications: K given by (0.3) should be replaced by the annulus $K^\gamma = \{\xi \in T_x^*M \,|\, \gamma \le \sigma_P(\xi) \le 1\}$ and when $\gamma = 1$, we should assume that $L^{-\frac{1}{m}} = o(1 - a(L))$ and replace $|d\xi|$ by some Lebesgue measure on the sphere K^1. In the latter case for example, when P stands for the Laplace-Beltrami operator associated to some Riemannian metric g on the closed n-dimensional manifold M, we get the weak convergence

$$\frac{1}{\sqrt{L}^n} \mathbb{E}(\nu_i) \underset{l\to\infty}{\to} \frac{1}{\sqrt{\pi}^{n+1}} \frac{1}{\sqrt{n(n+2)^{n-1}}} \mathbb{E}_S(i, n-1-i) |dvol_g|,$$

where $\mathbb{E}_S(i, n-1-i) = \int_{Sym(i,n-1-i),\mathbb{R}} |\det A| d\mu_S(A)$, and μ_S is the Gaussian measure on $Sym(n-1, \mathbb{R})$ associated to the scalar product

$$(A, B) \in Sym(n-1, \mathbb{R})^2 \mapsto \frac{1}{2} \operatorname{Tr}(AB) + \frac{1}{2} (\operatorname{Tr} A)(\operatorname{Tr} B) \in \mathbb{R}. \tag{4.1}$$

Finally, a manifold of special interest is the round unit sphere, where we may consider the space of pure harmonics $U_L^1 = \ker(P - LId)$ as a probability space, compare [19], [18]. Recall that the spectrum of the Laplace-Beltrami operator on the round unit n-dimensional sphere is the set $\{l(l+n-1) \,|\, l \in \mathbb{N}\}$ and that the eigenspace associated to the eigenvalue $\lambda_l = l(l+n-1)$ has dimension $\binom{n+l}{n} - \binom{n+l-2}{n}$. This case of pure spherical harmonics is unfortunately not a special case of the previous one, because $\gamma = 1$ but $L^{-1/m}$ cannot be a $o(1-a(L))$. However, the result remains valid and we also get the weak convergence

$$\frac{1}{\sqrt{L}^n}\mathbb{E}(\nu_i) \underset{l\to\infty}{\to} \frac{\mathbb{E}_S(i, n-1-i)}{\sqrt{\pi}^{n+1}\sqrt{n(n+2)^{n-1}}}|dvol_g|$$

on the whole M. In the case $n = 2$, this provides the upper estimate

$$\limsup_{l\to\infty} \frac{1}{L}\mathbb{E}(b_0) \leq \frac{1}{\pi\sqrt{2}}, \tag{4.2}$$

for the expected number b_0 of connected component of pure spherical harmonics, compare relation (2.41) of [21].

5. Appendix.

5.1. The incidence varieties. We recall that for every subspace U of $\Gamma(M, E)$,

$$\Delta_0 = \{s \in U \mid s \text{ does not vanish transversally}\} \text{ and}$$
$$\Delta_1 = \Delta_0 \cup \{s \in U \setminus \Delta_0 \mid p_{|s^{-1}(0)} \text{ is not Morse,}\}$$

see §1.1 (1.2).

LEMMA 5.1 (compare Proposition 2.8 of [8]). *Let E be a real line bundle over a smooth manifold M equipped with a Morse function $p : M \to \mathbb{R}$ and let U be a relatively l-ample linear subspace of $\Gamma(M, E)$, $l \in \{0, 1\}$. Then, $\mathcal{I}^l$ is a submanifold of $\underline{U}_{|M\setminus Crit(p)} = (M \setminus Crit(p)) \times U$ of codimension $rank(\mathcal{J}^l(E_{|\mathcal{H}}))$. Moreover, Δ_0 coincides with the critical locus of $\pi_U : \mathcal{I}^0 \to U$, whereas $\Delta_1 \setminus \Delta_0$ coincides with the critical locus of the restriction $\pi_{U|(\mathcal{I}^1\setminus\pi_U^{-1}(\Delta_0))} : \mathcal{I}^1 \setminus \pi_U^{-1}(\Delta_0) \to U$.*

From Lemma 5.1 and Sard's Lemma, when U is relatively l-ample, $l \in \{0, 1\}$, Δ_l has measure zero.

Proof. Let us first assume that $l = 0$ and let $(x, s) \in \mathcal{I}^0$. We fix some connection ∇^E on E. Then, the differential of j^0 at (x, s) reads

$$\begin{aligned} d_{|(x,s)}j^0 : T_{(x,s)}\underline{U} &\to T_{(x,0)}E \\ (\dot{x}, \dot{s}) &\mapsto (\dot{x}, \dot{s}(x) + \nabla^E_{\dot{x}} s). \end{aligned}$$

Since j^0 is onto, $d_{|(x,s)}j^0$ is onto as well and it follows from the implicit function theorem that $\mathcal{I}^0$ is a codimension one submanifold of $\underline{U}_{|M\setminus Crit(p)}$ with tangent space

$$T_{(x,s)}\mathcal{I}^0 = \{(\dot{x}, \dot{s}) \in T_{(x,s)}\underline{U} \mid \dot{s}(x) + \nabla^E_{\dot{x}} s = 0\}. \tag{5.1}$$

Moreover, the differential $d_{|(x,s)}\pi_U : (\dot{x}, \dot{s}) \in T_{(x,s)}\mathcal{I}^0 \mapsto \dot{s} \in T_s U = U$ is onto if and only if $\nabla^E s$ is, since j^0 is onto. Hence, Δ_0 coincides with the locus of the singular values of $\pi_U : \mathcal{I}^0 \to U$.

Now, assume that $l = 1$ and let $(x, s) \in \mathcal{I}^1$. The differential of $j^1_{\mathcal{H}}$ at (x, s) reads

$$\begin{aligned} d_{|(x,s)}j^1_{\mathcal{H}} : T_{(x,s)}\underline{U} &\to T_{(x,0)}\mathcal{J}^1(E_{|\mathcal{H}}) \\ (\dot{x}, \dot{s}) &\mapsto (\dot{x}, j^1_{\mathcal{H}}(\dot{s}) + \nabla^{\mathcal{J}}_{\dot{x}}(j^1_{\mathcal{H}}(s))), \end{aligned}$$

where $\nabla^{\mathcal{J}}$ denotes a connection on the bundle $\mathcal{J}^1(E_{|\mathcal{H}})$. Since $j^1_{\mathcal{H}}$ is onto, $d_{|(x,s)} j^1_{\mathcal{H}}$ is onto as well and it follows from the implicit function theorem that $\mathcal{I}^1$ is a submanifold of $\underline{U}_{|M\setminus Crit(p)}$ of codimension $rank(\mathcal{J}^1(E_{|\mathcal{H}})) = n$, with tangent space

$$T_{(x,s)}\mathcal{I}^1 = \{(\dot{x}, \dot{s}) \in T_{(x,s)}\underline{U} \mid j^1_{\mathcal{H}}(\dot{s}) + \nabla^{\mathcal{J}}_{\dot{x}}(j^1_{\mathcal{H}}(s)) = 0\}. \tag{5.2}$$

Let us assume that $s \notin \Delta_0$ and let $(\dot{x}, \dot{s}) \in \ker d_{|(x,s)}\pi_U$. Then $\dot{s} = 0$, which implies that $\nabla^E_{\dot{x}} s = 0$, so that $\dot{x} \in \ker \nabla s_{|x} = H_x$. Then, $0 = \nabla^{\mathcal{H}}_{\dot{x}}(j^1_{\mathcal{H}}(s)) = j^2_{\mathcal{H}}(\dot{x}, \cdot)$, so that $\dot{x} \in \ker j^2_{\mathcal{H}}(s)$. We deduce that the kernel of $d_{|(x,s)}\pi_U$ is reduced to $\{0\}$ if and only if $j^2_{\mathcal{H}}$ is non-degenerate. From Lemma 5.3, $j^2_{\mathcal{H}}(s)$ is non-degenerate if and only if $s \notin \Delta_1$. □

Remark 5.2. *It follows from the proof of Lemma 5.1 that for every $s \in \mathcal{I}^1 \setminus \Delta_1$, the operator $\nabla^{\mathcal{J}}(j^1_{\mathcal{H}}(s))$ which appears in (5.2) is invertible.*

Lemma 5.3. *(compare Lemma 2.9 of [8]) Let E be a real fibre bundle over a smooth manifold M equipped with a Morse function $p : M \to \mathbb{R}$. Let s be a section of E which vanishes transversally and $x \in M \setminus Crit(p)$ be a critical point of $p_{|s^{-1}(0)}$. Let $\lambda \in E^*_x$ such that $\lambda \circ \nabla^E s_{|x} = d_{|x}p$. Then,*

$$\lambda \circ \nabla^p(\nabla^E s_{|\mathcal{H}_x})_{|x} = \lambda \circ \nabla(\nabla^E s)_{|x} - \nabla(dp) = -\nabla^s(dp_{|s^{-1}(0)}).$$

In Lemma 5.3, ∇^E, ∇^p, ∇^s and ∇ denote connections on, respectively, the fibre bundles E, H, $T(s^{-1}(0))$ and TM. These connections induce connections on, respectively, $H^* \otimes E$, $T^*(s^{-1}(0)) \otimes E$ and $T^*M \otimes E$, denoted in the same way by ∇^p, ∇^s and ∇. Note that $\nabla^E s$, $\nabla^p(\nabla^E s_{|\mathcal{H}})$ and $\nabla^s(dp_{|s^{-1}(0)})_{|x}$ do not depend on the choices of ∇^E, ∇^p, ∇^s, whereas $\nabla(\nabla^E s)$ and ∇dp depend on the choice of ∇.

Proof. Let v, w be two vector fields on $s^{-1}(0)$ defined in the neighbourhood of x. Then,

$$0 = \nabla^E_v(\nabla^E_w s)_{|x} = \nabla(\nabla^E s)(v, w) + \nabla^E_{\nabla_v w} s$$

and likewise $\nabla^s(dp)_{|x}(v, w) = d_{|x}(dp(w))(v) = \nabla(dp)(v, w) + d_{|x}p(\nabla_v w)$. We deduce the relation $\nabla_{|x}(dp_{|s^{-1}(0)})(v, w) = \nabla(dp)_{|x}(v, w) - \lambda \circ \nabla(\nabla^E s)(v, w)$. Likewise, if v' and w' are two vector fields of $\mathcal{H}_x$ defined in the neighbourhood of x, we have

$$0 = d_{|x}(dp(w'))(v') = \nabla(dp)(v', w') + dp(\nabla_{v'} w')$$

and $\nabla^p(\nabla^E s)(v', w') = \nabla^E_{v'}(\nabla^E_{w'} s) = \nabla(\nabla^E s)(v', w') + \nabla^E_{\nabla_{v'} w'} s$. Finally,

$$\lambda \circ \nabla^p(\nabla^E s)_{|x} = \lambda \circ \nabla(\nabla^E s)_{|x} - \nabla(dp)_{|x} = -\nabla^s(dp_{|s^{-1}(0)}).$$

□

5.2. Pseudo-differential operators. Let M be a smooth manifold of positive dimension n and E be a real line bundle over M. We denote by $\Gamma(M, E)$ the space of smooth global sections of E.

Definition 5.4 (compare Definition 18.1.32 of [16]). *A linear operator $P : \Gamma(M, E) \to \Gamma(M, E)$ is called pseudo-differential of order $m \in \mathbb{R}$ if and only if there exist an atlas $(U_i)_{i \in I}$ of M and local trivializations $\Phi_i : E_{|U_i} \to V_i \times \mathbb{R}$, where V_i denotes a bounded open subset of $\mathbb{R}^n$, such that*

1. *$\forall i \in I$, there exist smooth kernels $k_i \in \Gamma(M \times M, E^* \boxtimes E)$ such that for every $s_i \in \Gamma(M, E)$ with support in U_i and every $x \in M \setminus U_i$,*

$$P(s_i)(x) = \int_M k_i(x, y) s_i(y) |dy|,$$

where $|dy|$ denotes a Lebesgue measure on M.

2. *$\forall i \in I$, there exist smooth symbols $p_i : V_i \times \mathbb{R}^n \cong T^*M_{|U_i} \to \mathbb{C}$ such that for every $s_i \in \Gamma(M, E)$ with support in U_i and every $x \in V_i$,*

$$\Phi_i(P(s_i))(x) = \iint_{V_i \times \mathbb{R}^n} p_i(x, \xi) e^{i\langle x-y, \xi\rangle} \Phi_i(s_i)(y) d\xi dy,$$

where $dyd\xi$ si the standard Lebesgue measure on $V_i \times \mathbb{R}^n$.

3. *For every compact subset $K_i \subset V_i$ and every $\alpha, \beta \in \mathbb{N}^n$, there exist positive constants $c_{K_i,\alpha,\beta}$ such that*

$$\forall (x, \xi) \in K_i \times \mathbb{R}^n, |\frac{\partial}{\partial x^\beta} \frac{\partial}{\partial \xi^\alpha} p_i(x, \xi)| \leq c_{K_i,\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}.$$

Now, let h_E be a Riemannian metric on E and $|dy|$ be a Lebesgue measure on M, which we assume to be compact and without boundary. Then, $\Gamma(M, E)$ inherits the L^2-scalar product (0.1).

DEFINITION 5.5. *The adjoint of the pseudo-differential operator P is the operator ${}^tP$ satisfying for every $s, t \in \Gamma(M, E), \langle P(s), t\rangle = \langle s, {}^tP(t)\rangle$. When ${}^tP = P$, the operator is said to be self-adjoint.*

DEFINITION 5.6. *(see [13], [14], [15]) A self-adjoint pseudo-differential operator of order $m \in \mathbb{R}$ given by Definitions 5.4, 5.5 is said to be elliptic if and only if for every $i \in I$ and every $(x, \xi) \in T^*M_{|U_i}$ such that $\xi \neq 0$, the limit*

$$\sigma_P(x, \xi) = \lim_{t \to +\infty} \frac{1}{t^m} p_i(x, t\xi)$$

exists and is positive. This limit then does not depend on the choice of $i \in I$ and defines a positive homogeneous function $\sigma_p : T^*M \to \mathbb{R}$ of order m and class C^∞.

The function σ_P given by Definition 5.6 will be called the homogenized principal symbol of P. It is symmetric in the sense that for every $(x, \xi) \in T^*M$, $\sigma_P(x, -\xi) = \sigma_P(x, \xi)$.

EXAMPLE 5.7. *Recall that if in a local trivialization of E the differential operator Q of order m reads $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \tilde{Q}(\partial/\partial x_1, \cdots, \partial/\partial x_n)(f) \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$, where $\tilde{Q} \in C^\infty(\mathbb{R}^n)[X_1, \cdots, X_n]$, and if $\tilde{Q}_m$ is the homogeneous part of order m of $\tilde{Q}$, then the principal symbol of Q is the homogeneous function of order m $\sigma_Q : (\mathbb{R}^n)^* \to \mathbb{C}$ satisfying $\sigma_Q(\xi_1 dx_1 + \cdots + \xi_n dx_n) = \tilde{Q}_m(i\xi_1, \cdots, i\xi_n)$.*

DEFINITION 5.8. *An elliptic self-adjoint pseudo-differential operator P on $\Gamma(M, E)$ is said to be bounded from below if and only if there exists a constant $c \in \mathbb{R}$ such that for every $s \in \Gamma(M, E), \langle P(s), s\rangle \geq c\langle s, s\rangle$. It is said to be positive when $c > 0$.*

Remark 5.9. *The transformation $P \to P - cId$ turns any elliptic self-adjoint pseudo-differential operator bounded from below into a positive one. Since our results are not sensitive to this transformation, they hold for any operator bounded from below even if we sometimes assume it to be positive for simplicity. Recall finally that these operators have discrete spectrum with finite dimensional eigenspaces.*

5.3. Proof of Theorem 2.3 . Set $L = \lambda^m$ and $\tilde{e}_\lambda = e_L$. The strategy followed by Hörmander is the following. The derivative of $\tilde{e}_\lambda$ with respect to λ is a distribution whose support is the set of eigenvalues of P. Its Fourier transform with respect to λ is the kernel of the hyperbolic equation $\partial_t u + iP^{1/m} = 0$, where $P^{1/m}$ stands for the operator with the same eigenfunctions as P and whose eigenvalues are the m-th root of the corresponding ones of P. Hörmander proves that in a neighbourhood V of the diagonal of $M \times M$ and for small values of the time t, this kernel takes the form of a Fourier integral operator, modulo an operator with smooth kernel. Consequently, if $\rho : \mathbb{R} \to \mathbb{R}$ is a non negative function in the Schwartz space such that its Fourier transform $\hat{\rho}$ satisfies $\hat{\rho}(0) = 1$ and $Supp(\hat{\rho}) \subset [-\epsilon, \epsilon]$, then for every $x, y \in V$,

$$\int_{-\infty}^{+\infty} \rho(\lambda - \mu)\partial_\mu \tilde{e}_\mu(x, y)d\mu - \int_{T_y^* M} R(x, \lambda - p'_{|y}(\xi'), y, \xi)e^{i\psi(x,y,\xi)}d\xi$$

is a rapidly decreasing function as $\lambda \to +\infty$, where

- $\psi(x, y, \xi) = \langle x - y, \xi\rangle + O(|x - y|^2|\xi|)$ when $x \to y$, for a scalar product $\langle\, ,\, \rangle$ in a chart of M that contains x and y.
- $p'(\xi) = \sigma_P(\xi)^{1/m} + O(1)$
- $R(x, \lambda, y, \xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\rho}(t)q(x, t, y, \xi)e^{it\lambda}dt$ with $q(x, 0, y, \xi) = (\frac{1}{2\pi})^n + O(1/|\xi|)$, see Lemma 4.1 of [14].

This function R is rapidly decreasing as λ grows to infinity. After differentiation we deduce likewise that

$$\int_{-\infty}^{+\infty} \rho(\lambda-\mu)\partial_\mu Q_1 Q_2 \tilde{e}_\mu(x, y)d\mu - \int_{T_y^* M} Q_1 Q_2(R(x, \lambda - p'_{|y}(\xi), y, \xi)e^{i\psi(x,y,\xi)})d\xi \quad (5.3)$$

is a rapidly decreasing function as λ grows to infinity.

Lemma 5.10. *(Compare Lemma 4.3 of [14]) Under the hypotheses of Theorem 2.3, there exists a constant $C > 0$ such that for every (x, y) in a neighbourhood V of the diagonal of $M \times M$, for every $\lambda \geq 0$ and every $0 \leq \mu \leq 1$,*

$$\|Q_1 Q_2 \tilde{e}_{\lambda+\mu}(x, y) - Q_1 Q_2 \tilde{e}_\lambda(x, y)\|_{h_E} \leq C(1 + |\lambda|)^{n-1+|\sigma_{Q_1}|+|\sigma_{Q_2}|}.$$

Proof. Let us assume first that $Q_1 = Q_2$ and $x = y$. We proceed as in the proof of Lemma 4.3 of [14]. The function

$$\partial_\mu Q_1 Q_1 \tilde{e}_\mu(x, x) = \sum_k \delta_{\lambda_k} h_E(Q_1 s_k(x), Q_1 s_k(x))$$

is positive, where s_k is an eigenfunction with eigenvalue λ_k^m. We deduce the existence of a constant $C_1 > 0$ such that

$$\|Q_1 Q_1 \tilde{e}_{\lambda+\mu}(x, y) - Q_1 Q_1 \tilde{e}_\lambda(x, y)\|_{h_E} \leq C_1 \int_{\mathbb{R}} \rho(\lambda - \mu)\partial_\mu Q_1 Q_1 \tilde{e}_\mu(x, x)d\mu.$$

From (5.3), it is enough to bound from above the integral

$$\int_{T_x^*M} Q_1 Q_1\big(R(x,\lambda - p'(\xi), y, \xi)e^{i\psi(x,y,\xi)}\big)d\xi.$$

From the ellipticity of P we deduce the existence of $C_2 > 0$ such that

$$\forall \xi \in T_x^*M, |Q_1 Q_1 \psi(x,y,\xi)|_{|(x,x)} = |\sigma_{Q_1}(\xi)|^2 \leq C_2(1+p'(\xi))^{2|\sigma_{Q_1}|}.$$

Following [14, p 210], we deduce that

$$\begin{aligned}
&|\int_{T_x^*M} Q_1 Q_{1|(x,x)}(R(x,\lambda - p'(\xi), y, \xi))e^{i\psi(x,y,\xi)}d\xi| \\
&\leq C_3 \int_{\mathbb{R}} (1+|\lambda-\sigma|^{-N})(1+|\sigma|)^{2|\sigma_{Q_1}|}dm(y,\sigma) \\
&\leq O(\lambda^{-\infty}) + C_4(1+|\lambda|)^{n-1+2|\sigma_{Q_1}|},
\end{aligned}$$

where C_3, C_4 are positive constants, N denotes a large enough integer and

$$m(x,\sigma) = \int_{\{\xi \in T_x^*M|\ \sigma_P(\xi)\leq \sigma\}} d\xi.$$

We deduce the result when $Q_1 = Q_2$ and $x = y$, then likewise when (x, y) lies in a neighbourhood V of the diagonal, see Lemma 3.1 of [14]. The general case is now a consequence of the Cauchy-Schwarz inequality and there exists a positive constant c such that $\forall x, y \in N, \forall \lambda > 0, \forall \mu \in [0, 1]$,

$$\begin{aligned}
\|Q_1Q_2\tilde{e}_{\lambda+\mu}(x,y) - Q_1Q_2\tilde{e}_\lambda(x,y)\|_{h_E} &= \|\sum_{k\ |\ \lambda\leq\lambda_k\leq\lambda+\mu} Q_1(s_k(x))Q_2(s_k(y))\| \\
&\leq \Big(\sum_{k\ |\ \lambda\leq\lambda_k\leq\lambda+\mu} \|Q_1(s_k(x))\|^2\Big)^{1/2}\cdots \\
&\quad \cdots\Big(\sum_{k\ |\ \lambda\leq\lambda_k\leq\lambda+\mu} \|Q_2(s_k(y))\|^2\Big)^{1/2} \\
&\leq (\|Q_1Q_{1|(x,x)}\tilde{e}_{\lambda+\mu} - Q_1Q_{1|(x,x)}\tilde{e}_\lambda\|^2)^{1/2}\cdots \\
&\quad \cdots(\|Q_2Q_{2|(y,y)}\tilde{e}_{\lambda+\mu} - Q_2Q_{2|(y,y)}\tilde{e}_\lambda\|^2)^{1/2} \\
&\leq C(1+|\lambda|)^{n-1+|\sigma_{Q_1}|+|\sigma_{Q_2}|}.
\end{aligned}$$

□

Proof of Theorem 2.3. We proceed as in [14], p. 211. We deduce from Lemma 5.10 that $\forall x, y \in U, \forall \lambda \geq 0, \forall \mu \geq 0$,

$$\|Q_1Q_2\tilde{e}_{\lambda+\mu}(x,y) - Q_1Q_2\tilde{e}_\lambda(x,y)\|_{h_E} \leq C(1+\lambda+\mu)^{n-1+|\sigma_{Q_1}|+|\sigma_{Q_2}|}(1+\mu).$$

Thus, there exists $C' > 0$ such that

$$\|\int_{\mathbb{R}} \rho(\lambda-\mu)Q_1Q_2\tilde{e}_\mu(x,y)d\mu - Q_1Q_2\tilde{e}_\lambda(x,y)\|_{h_E} \leq C'(1+\lambda)^{n-1+|\sigma_{Q_1}|+|\sigma_{Q_2}|}.$$

However, by integration of (5.10) over the interval $]-\infty, \lambda]$, we deduce the existence of $C'' > 0$ such that

$$\|Q_1Q_2\tilde{e}_{\lambda+\mu}(x,y) - \int_{T_y^*M}\int_{-\infty}^{\lambda} Q_1Q_2(R(x,\sigma - p'_{|y}, y, \xi)e^{i\psi(x,y,\xi)})d\xi d\sigma\|_{h_E} \leq C''.$$

Moreover, by definition of ψ and R, $\int_{T_y^*M}\int_{-\infty}^{\lambda} Q_1Q_2(R(x,\sigma - p'_{|y}(\xi), y, \xi)e^{i\psi(x,y,\xi)})d\xi d\sigma$ equals

$$\frac{1}{(2\pi)^n}\int_{\{\xi\in T_y^*M\,|\,p'(\xi)\le\lambda\}}(1+O(1/|\xi|))Q_1Q_2e^{i\psi(x,y,\xi)}d\xi+\cdots$$
$$\cdots\int_{T_y^*M}Q_1Q_2(R_1(x,\lambda-p'_{|y}(\xi),y,\xi)e^{i\psi(x,y,\xi)})d\xi,$$

where

$$R_1=\begin{cases}\int_{-\infty}^{\tau}R(x,\sigma,y,\xi)d\sigma & \text{if } \tau\le 0\\ \int_{-\infty}^{\tau}R(x,\sigma,y,\xi)d\sigma-q(x,0,y,\xi) & \text{if } \tau>0\end{cases}$$

is a function which decreases faster than any polynomial, see [14, p. 211]. Thus, there exists a constant $C''' > 0$ such that

$$\|\int_{T_y^*M\times]-\infty,\lambda]}Q_1Q_2(R(x,\sigma-p'_{|y}(\xi),y,\xi)e^{i\psi(x,y,\xi)})d\xi d\sigma-\cdots$$
$$\cdots\frac{1}{(2\pi)^n}\int_{\xi\in T_y^*M\,|\,p'(\xi)\le\lambda\}}\sigma_{Q_1}(\xi)\overline{\sigma_{Q_2}(\xi)}d\xi\|\le C'''(1+\lambda)^{n-1+|\sigma_{Q_1}|+|\sigma_{Q_2}|}.$$

From the triangle inequality, we finally deduce that there exists $C'''' > 0$ such that for every $(x,y)\in V$,

$$\|Q_1Q_2e_L(x,y)-\frac{1}{(2\pi)^n}\int_{\{\xi\in T_y^*M\,|\ \sigma_P(\xi)\le L\}}\sigma_{Q_1}(\xi)\overline{\sigma_{Q_2}(\xi)}d\xi\|$$
$$\le C''''(1+\lambda)^{n-1+|\sigma_{Q_1}|+|\sigma_{Q_2}|}.$$

Hence the result. □

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