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MINIMAL SURFACES OF GENERAL TYPE WITH $p_g = q = 0$ ARISING FROM SHIMURA SURFACES*

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Abstract. Quaternionic Shimura surfaces are quotients of the product of two copies of the upper half plane by irreducible cocompact arithmetic groups. In the present paper we are interested in (smooth) quaternionic Shimura surfaces admitting an automorphism with one-dimensional fixed locus; such automorphisms are involutions. We propose a new construction of surfaces of general type with $q = p_g = 0$ as quotients of quaternionic Shimura surfaces by such involutions. These quotients have finite fundamental group.

Key words. Shimura surfaces, surface automorphisms, quotients by finite groups, surfaces of general type.

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1. Introduction. Among smooth minimal surfaces of general type, the ones with vanishing geometric genus p_g are of main interest (see e.g. [BFP11]). For such surfaces, the Chern number $c_1^2 = K^2$ belongs to the set $\{1, \dots, 9\}$ and $c_2 = 12 - c_1^2$. We are far away from a complete classification, although great advances have been done recently, e.g. for surfaces with $c_1^2 = 9$, the fake projective planes, which have been completely classified, see [PY07], [CS10]. In the other cases, a major task is to construct new examples of such surfaces.

In this paper we give a uniform construction of surfaces with $q = p_g = 0$ and $c_1^2 = 1, \dots, 7$. These surfaces are obtained as quotients of smooth quaternionic Shimura surfaces X by a special kind of involution. Recall that a smooth Shimura surface $X = X_\Gamma$ is the quotient of $\mathbb{H} \times \mathbb{H}$, the product of two copies of the complex upper half plane, by a discrete cocompact torsion free group $\Gamma \subset \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ of holomorphic automorphisms defined by certain quaternion algebra. The invariants of X are $c_1^2(X) = 2c_2(X) = 8(1 + p_g(X))$ and $q(X) = 0$. An automorphism of a Shimura surface either has only isolated fixed points or its fixed locus is purely one-dimensional, in which case the automorphism is an involution and is induced by the involution $\mu \in \mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ exchanging the two factors (see [DR14]). We call the automorphisms of latter type *involutions of the second kind.*

We show (see Theorem 11 and Theorem 15)

THEOREM. *Let X be a smooth Shimura surface admitting an involution of second kind σ. The fixed point set C of σ is a a union of disjoint smooth Shimura curves. The arithmetic genus g of C satisfies $2 \leq g \leq p_g(X)$ and the quotient surface $Z = X/\sigma$ is smooth with finite fundamental group. If moreover $g = p_g(X) \leq 8$, the surface Z is of general type with invariants:*

$$c_1(Z)^2 = 9 - p_g(X),\ c_2(Z) = 3 + p_g(X),\ p_g(Z) = q(Z) = 0.$$

If $p_g = 2$ or 3 then $g = p_g$ and therefore $c_1^2 = 7$ and 6 respectively.

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Concentrating on the construction of such Shimura surfaces with low p_g and admitting an involution of second kind, we find that such surfaces are rather exceptional. For instance, if we restrict our consideration to totally real fields of degree 2, there are at most 14 isomorphism classes of quaternion algebras leading to smooth Shimura surfaces of geometric genus $2 \leq p_g \leq 8$ admitting an involution of second kind (see Theorem 20). Considering Shimura surfaces corresponding to congruence subgroups, we were able to find Shimura surfaces X with $p_g(X) = 5, 6$ (see Sections 4.5 and 4.6). In the light of some open questions concerning fundamental groups of surfaces with geometric genus zero, see [BFP11], an example of a smooth Shimura surface with $p_g = 2$ admitting an involution of second kind would be highly interesting.

This paper mixes two fields: theory of Shimura surfaces and classical algebraic geometry of surfaces. Shimura surfaces are closely related to Hilbert modular surfaces (which were first systematically studied by Hirzebruch, see for instance [vdG88], but are less known and studied). It was also one of our aims to develop that theory of Shimura surfaces.

The paper is organized as follows: In Section 2 we discuss the conditions on Γ under which the Shimura surface X_Γ has an involution of second kind. In Section 3 we study the quotient surface of a smooth Shimura surface by the action of an involution of second kind and we prove that its fundamental group is finite. In Section 4, we investigate examples of Shimura surfaces with low geometric genus admitting an involution of second kind. In particular we develop new tools to create smooth quaternionic Shimura surfaces and we make a systematic study of Shimura surfaces defined over quadratic fields with low geometric genus. We give examples of surfaces with $p_g = 5, 6$ admitting an involution of second kind. Finally, in section 5 we present the method to identify the Shimura curve fixed by an involution of second kind acting on a Shimura surface and we furthermore examine the example with $p_g = 5$.

2. Involutions of second kind acting on Shimura surfaces.

2.1. Quaternionic Shimura surfaces. Let us briefly recall the construction of quaternionic Shimura surfaces and introduce some notations.

Let k be a totally real number field of degree $n = [k : \mathbb{Q}]$. and A be a division quaternion algebra whose center is k. For every place v of k, we denote by k_v the completion of k with respect to v and define $A_v = A \otimes_k k_v$. The algebra A is *ramified* at v if A_v is a division algebra over k_v and *unramified* otherwise, that is, if $A_v \cong M_2(k_v)$. Suppose that

$$A \otimes_{\mathbb{Q}} \mathbb{R} \to M_2(\mathbb{R})^2 \times \mathbf{H}^{n-2}$$

where $\mathbf{H}$ denotes the skew field of Hamiltonian quaternions. Then, A is uniquely determined up to an isomorphism by finitely many prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ where A is ramified. We write $A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_r)$, for such a quaternion algebra in the following. The subgroup A^+ of A consisting of the units of A having totally positive reduced norm can be identified via the isomorphism $A \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{f} M_2(\mathbb{R})^2 \times \mathbf{H}^{n-2}$ with a subgroup of $GL_2^+(\mathbb{R})^2 \times \mathbf{H}^{*n-2}$, and projecting to the first two factors gives an injection of A^+ into $GL_2^+(\mathbb{R})^2$. We denote by

$$\rho = (\rho_1, \rho_2) : A \to M_2(\mathbb{R})^2$$

this representation of A. Note that these ρ_i, $i = 1, 2$ are extensions of two morphisms $\sigma_1, \sigma_2 : k \to \mathbb{R}$ (where $\mathbb{R} \subset M_2(\mathbb{R})$ is identified with diagonal matrices).

Let us denote by $\mathcal{O}$ and $\mathcal{O}_k$ a maximal order of A and the ring of integers of k. Let $\mathcal{O}^*$ denote the group of units of $\mathcal{O}$, $\mathcal{O}^+$ the group of units in $\mathcal{O}$ with totally positive reduced norm and $\mathcal{O}^1 \subset \mathcal{O}$ the group of units of reduced norm 1. The group $\mathcal{O}^1$ is via ρ a discrete subgroup of $SL_2(\mathbb{R})^2$, whereas $\mathcal{O}^+$ is embedded as a discrete subgroup in $GL_2^+(\mathbb{R})^2$.

The group A^+ and any subgroup $G \subset A^+$ acts on the product $\mathbb{H} \times \mathbb{H}$ of two copies of the upper half plane by fractional linear transformations on each factors through the representation ρ. Note that for $g \in A^+$, the entries of the matrix $\rho_1(g)$ are conjugates to those of $\rho_2(g)$ with respect to the places σ_1, σ_2 over the Galois closure of k in $\mathbb{R}$. The action of G is not effective; the center $Z(G)$ acts trivially on $\mathbb{H} \times \mathbb{H}$ and therefore we will consider subgroups $\Gamma = G/Z(G) \subset A^+/k^*$ rather than subgroups $G \subset A^+$. Let us write $\Gamma_{\mathcal{O}}(1)$ or sometimes simply $\Gamma(1)$ for the group $\mathcal{O}^1/\{\pm 1\}$. The group $\Gamma(1)$ and also any subgroup Γ of A^+/k^* commensurable with $\Gamma(1)$ acts properly discontinuously on $\mathbb{H} \times \mathbb{H}$.

The quotient $X_\Gamma = \mathbb{H} \times \mathbb{H}/\Gamma$ is a compact algebraic surface, which will be called *quaternionic Shimura surface* (corresponding to Γ) in the sequel.

2.2. The involution exchanging factors and involutions of the second kind. Let, as above, $A = A(k, \mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ be a quaternion algebra.

Definition 1. An *involution of second kind* on A is a map $\tau : A \to A$ such that $\tau^2(a) = a$, $\tau(a + b) = \tau(a) + \tau(b)$, $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in A$ and such that the restriction of τ to k is a non-trivial automorphism of k.

Remark 2. Note that classically, an involution of second kind is anti-commutative i.e. $\tau(ab) = \tau(b)\tau(a)$, however we choose Granath's convention in [Gra02] which is our main reference. In fact, if $a \to a^*$ denotes the canonical anti-involution of A, one can see that the map sending an involution τ as in Definition 1 to the involution $a \to \tau(a)^*$ gives a one-to-one correspondance between involutions in our sense and (anti-commutative) classical involutions.

Let $\ell = k^\tau$ be the fixed field of τ. Then k/ℓ is a quadratic extension and in this case we will say that τ is a *k/ℓ-involution*. Let $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}$ be the involution that exchanges the two factors. Then $Aut(\mathbb{H} \times \mathbb{H}) = Aut(\mathbb{H})^2 \rtimes \langle \mu \rangle$ is a semi-direct product. The following is an immediate consequence of the definition of involution of second kind:

Proposition 3. *Let Γ be a subgroup of A^+ commensurable with $\mathcal{O}^1$ for some maximal order $\mathcal{O}$ in A. Suppose that there exists an involution τ of second kind on A preserving Γ. Then, the automorphism μ of $\mathbb{H} \times \mathbb{H}$ induces an involution σ on the surface $X_\Gamma = \mathbb{H} \times \mathbb{H}/\Gamma$.*

Remark 4. (See also [Gra02, Lemma 4.2]) Let τ and σ be two k/ℓ-involutions of second kind on A. Then there exists $m \in A^*$ such that $\sigma(a) = m^{-1}\tau(a)m$ and $\tau(m)^* = m$, where $a \to a^*$ is the canonical anti-involution on A, that is, the uniquely determined anti-involution $* : A \to A$ such that the reduced norm is $Nrd(x) = xx^*$ and the reduced trace is $Trd(x) = x + x^*$.

We have the following criterion for the existence of such involutions:

Proposition 5. (See [Gra02, Lemma 4.3], [Lan37, Theorem 3]) *Let α be the non-trivial ℓ-automorphism of a quadratic extension k/ℓ. There exists a k/ℓ-involution τ of second kind on $A = A(k, \mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ if and only if*

(1) *r is even and after a suitable renumbering of the $\mathfrak{p}_i$, we have $\mathfrak{p}_{2i-1} = \mathfrak{p}_{2i}^{\alpha}$ and $\mathfrak{p}_i \neq \mathfrak{p}_i^{\alpha}$ $(i = 1, \dots, r/2)$.*
(2) *With the notations of section 2.1, it holds that $\sigma_2 = \sigma_1 \circ \alpha$.*

REMARK 6. Let us observe that the above Proposition implies that we have the relation $\rho_2 = \rho_1 \circ \tau$ in the case of the existence of a k/ℓ-involution τ on A. Note also that the condition (2) is superfluous in the case where k is a real quadratic field, since there is only one choice of $\sigma_2 = \sigma_1 \circ \alpha$.

Let τ be an involution of second kind on A. Let $\Gamma \subset A^+/k^*$ be a subgroup stable by τ, that is, for all $\gamma \in \Gamma$ we have $\tau(\gamma) \in \Gamma$. Since $\rho_2 = \rho_1 \circ \tau$, the images of Γ in $PSL_2(\mathbb{R})$ by ρ_1 and ρ_2 are the same, and since this image is isomorphic to Γ we denote it also by Γ. In order to avoid more long-winded notations, let us write for an involution of second kind τ shortly

$$\tau(a) = \overline{a} \text{ for all } a \in A.$$

In particular, the non-trivial ℓ-automorphism of k is denoted by $x \mapsto \overline{x}$ (for $x \in k$). Identifying k with $\sigma_1(k)$, say, the action of any element $\gamma \in A^+/k^*$ on $\mathbb{H} \times \mathbb{H}$ given by

$$\gamma(z_1, z_2) = (\rho_1(\gamma)z_1, \rho_2(\gamma)z_2)$$

is now written as

$$\gamma(z_1, z_2) = (\gamma z_1, \bar{\gamma} z_2).$$

2.3. Automorphisms of Shimura surfaces. Let $\mu \in \mathrm{Aut}\mathbb{H} \times \mathbb{H}$ be the involution that exchanges the two factors. Let $X_\Gamma = \mathbb{H} \times \mathbb{H}/\Gamma$ be a smooth quaternionic Shimura surface. By [DR14, Theorem 3.12] and its proof, we get:

PROPOSITION 7. *Suppose that the fixed locus C of an automorphism σ of $X = X_\Gamma$ contains an one-dimensional component. Then σ is an involution that lifts to the universal cover to (a conjugate of) μ.*

Let us consider an involution $a \mapsto \bar{a}$ on A of second kind and a torsion-free group Γ commensurable with a group $\Gamma(1)$, and as above stable under $a \mapsto \overline{a}$. The image of $t = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ on $X_\Gamma = \mathbb{H} \times \mathbb{H}/\Gamma$ is a fixed point of the involution σ induced by μ if and only if

$$\Gamma(z_1, z_2) = \Gamma(z_2, z_1),$$

which is the case if and only if there exists γ in Γ such that

$$(z_1, z_2) = \gamma(z_2, z_1) = (\gamma z_2, \bar{\gamma} z_1),$$

that is, if and only if $z_1 = \gamma z_2$ and $z_2 = \bar{\gamma}\gamma z_2$. Since X is smooth, Γ is torsion-free, thus the image of t in X is fixed by σ if and only if $\bar{\gamma}\gamma = 1$.

As in [Gra02], for any $\beta \in A \setminus \{0\}$, let Δ_β be the upper half plane $\Delta_\beta = \{(z, \beta z)/z \in \mathbb{H}\}$. We have $\lambda\Delta_\beta = \Delta_{\bar{\lambda}\beta\lambda^{-1}}$ for any $\lambda \in A \setminus \{0\}$. We denote by F_β the image of Δ_β in X_Γ.

Let $\gamma \in \Gamma$ be such that $\bar{\gamma} = \gamma^{-1}$. Since $\bar{\gamma}\gamma = 1$, we have $(\gamma z, z) = \gamma(z, \gamma z)$, therefore for any point t of Δ_γ, we obtain

$$\mu\Gamma t = \Gamma\mu t = \Gamma\gamma t = \Gamma t,$$

and the image F_γ of Δ_γ on X_Γ is fixed by the involution σ. That implies that F_γ is a smooth irreducible algebraic curve, more precisely a Shimura curve. Since an automorphism of a surface fixes at most a finite number of curves and the union of these curves is smooth, we obtain:

COROLLARY 8. *The fixed point set of σ is the union of the smooth disjoint Shimura curves F_γ, for all $\gamma \in \Gamma$ such that $\gamma\bar{\gamma} = 1$.*

REMARK 9. Since the irreducible components of the fixed locus C are smooth disjoint Shimura curves, we get by the Hirzebruch Proportionality Theorem: $2C^2 = -K_X C = 4(1-g)$ where $g > 1$ is the arithmetic genus of C.

Recall that $\lambda C_\beta = C_{\bar{\lambda}\beta\lambda^{-1}}$ for any $\lambda \in A \setminus \{0\}$, and in particular for every $\lambda \in \Gamma$ we have: $F_1 = F_{\bar{\lambda}\lambda^{-1}}$. Of course, for $\alpha = \bar{\lambda}\lambda^{-1}$ we have $\alpha\bar{\alpha} = 1$.

REMARK 10. One can ask if the fixed locus C of such σ is always irreducible. As we see immediately, the curve C is irreducible if and only if for any $\lambda \in \Gamma$ such that $\lambda\bar{\lambda} = 1$, there exists $\gamma \in \Gamma$ such that $\lambda = \bar{\gamma}\gamma^{-1}$.

3. Quotient of a quaternionic Shimura surface by an involution of second kind.

3.1. Invariants of the quotient. Let Γ be a lattice preserved by an involution of the second kind and let σ be the corresponding involution acting on the Shimura surface $X = X_\Gamma$. Let C be the smooth curve of arithmetic genus g fixed by the involution.

THEOREM 11. *The quotient surface $Z = X/\sigma$ is smooth and has invariants:*

$$K_Z^2 = e(X) + 5(1-g),\ c_2 = \frac{1}{2}e(X) + 1 - g,\ p_g = \frac{e(X) - 4 - 4g}{8},\ q = 0,$$

where $e(X) = c_2(X)$ is the topological Euler number of X.
If $(K_X - C)^2 > 0$, then Z has general type; this condition on the positivity is satisfied if $e(X) \leq 36$.
Suppose $e(X) = 12$, then C is irreducible of genus $g = 2$ and the quotient surface $Z = X/\sigma$ has invariants: $K_Z^2 = 7$, $c_2 = 5$, $p_g = 0$, $q = 0$. Suppose $e(X) = 16$, then C is irreducible of genus $g = 3$ and the quotient surface Z has invariants: $K_Z^2 = 6$, $c_2 = 6$, $p_g = 0$, $q = 0$.

Proof. Let $\pi : X \to Z = X/\sigma$ be the quotient map. Since $K_X^2 = 2e$ (for $e = c_2(X)$), $K_X = \pi^* K_Z + C$ and $C^2 = 2(1-g)$, $K_X C = 4(g-1)$ (see Remark 9),we get

$$K_Z^2 = \frac{1}{2}(K_X - C)^2 = \frac{1}{2}(K_X^2 - 2K_X C + C^2) = e - 5(g-1).$$

Moreover by general formulas on quotient surfaces $e(Z) = \frac{1}{2}e(X) + 1 - g$. As $q(X) = 0$, we get $q(Z) = 0$ and $p_g(Z) = \chi(\mathcal{O}_Z) - 1 = \frac{e(X) - 4(g+1)}{8}$. As $g \geq 2$ because X is hyperbolic, we get that $2 \leq g \leq \frac{e(X)-4}{4}$, thus

$$e(Z) \geq \frac{e(X)}{4} + 2,\ \chi(\mathcal{O}_Z) \geq 1$$

and $K_Z^2 \geq 10 - \frac{e}{4}$. Let us prove that Z has general type if $K_Z^2 > 0$. Since $\pi^* K_Z = K_X - C$, it is enough to prove that powers $\mathcal{L}^m$ of $\mathcal{L} = \mathcal{O}_X(K_X - C)$ have sections growing in $c \cdot m^2$, where $c > 0$ is a constant. Suppose that $K_Z^2 = \frac{1}{2}(K_X - C)^2 > 0$ (this is the case if $e(X) < 40$). By the Riemann-Roch Theorem, we have

$$\chi(\mathcal{L}^m) = \frac{m^2}{2}(K_X - C)^2 - \frac{m}{2} K_X(K_X - C) + \chi(\mathcal{O}_X).$$

Serre duality gives

$$\chi(\mathcal{L}^m) = H^0(X, \mathcal{L}^m) - H^1(X, \mathcal{L}^m) + H^0(X, mC - (m-1)K_X).$$

Suppose that $D = mC - (m-1)K_X$ is effective. As K_X is ample $K_X D > 0$. But

$$K_X D = m(4g - 4 - 2e) + 2e$$

and as $g \leq \frac{e(X)-4}{4}$, we get $4g - 4 \leq e - 8$ and

$$K_X D \leq m(-8 - e) + 2e < 0$$

for $m \geq 3$. Therefore $H^0(X, mC - (m-1)K_X) = 0$ and Z has general type. □

The possibilities for values $12, 16, \dots, 36$ of $e(X)$ and for the genus g of C are listed in the above tables:

$e(X)$	g	K_Z^2	$c_2(Z)$	$p_g(Z)$
$e = 12$	2	7	5	0
$e = 16$	3	6	6	0
$e = 20$	2	15	9	1
	4	5	7	0
$e = 24$	3	14	10	1
	5	4	8	0
$e = 28$	2	23	13	2
	4	13	11	1

$e(X)$	g	K_Z^2	$c_2(Z)$	$p_g(Z)$
$e = 28$	6	3	9	0
	3	22	14	2
$e = 32$	5	12	12	1
	7	2	10	0
	2	31	17	3
$e = 36$	4	21	15	2
	6	11	13	1
	8	1	11	0

3.2. The fundamental group of the quotient. Let us recall some results about fundamental groups. Let G be a discontinuous group of homeomorphisms of a path connected, simply connected, locally compact metric space M, and let G_{tor} be the normal subgroup of G generated by those elements which have fixed points, or equivalently, the torsion elements. Then

Theorem 12. [Arm68] *The fundamental group of the orbit space M/G is isomorphic to the factor group G/G_{tor}.*

Let $X = X_\Gamma$ be our Shimura surface with fundamental group Γ such that the involution μ switching the two factors of $\mathbb{H} \times \mathbb{H}$ acts on X by an involution denoted by σ. The fundamental group of the quotient surface X/σ is isomorphic to Γ'/Γ'_{tor}, where Γ' is the group generated by Γ and μ.

Lemma 13. *The group Γ' is discontinuous.*

Proof. Since $M = \mathbb{H} \times \mathbb{H}$ is a locally compact Hausdorff space, Γ' is discontinuous if and only if it is discrete subgroup of $Aut(\mathbb{H} \times \mathbb{H})$. This is the case since Γ' is an index-2 extension of the discrete group Γ. □

For any $\gamma \in \Gamma$, we have $\mu\gamma = \bar{\gamma}\mu$. Let $g \in \Gamma'_{tor}$ be a non-trivial torsion element. Since Γ is torsion-free, $g \notin \Gamma$ and there exists $\lambda \in \Gamma$ such that $g = \lambda\mu$. The order of g is then divisible by 2 and we have $g^{2n} = (\lambda\bar{\lambda})^n$. Since Γ is torsion-free, $g^{2n} = 1$ if and only if $\lambda\bar{\lambda} = 1$. Therefore a torsion element g of Γ' has order 2 and there exists $\lambda \in \Gamma$ such that $g = \lambda\mu$ and $\lambda\bar{\lambda} = 1$. As an immediate consequence we obtain:

Lemma 14. *The fundamental group Γ'/Γ'_{tor} of X/σ is isomorphic to the group Γ/N where N is the normal group generated by the $\lambda \in \Gamma$ such that $\lambda\bar{\lambda} = 1$.*

Since for any $\gamma \in \Gamma$, the group Γ'_{tor} contains $\bar{\gamma}\mu\bar{\gamma}^{-1} = \bar{\gamma}\gamma^{-1}\mu$, we see that N contains $\bar{\gamma}\gamma^{-1}$ for every $\gamma \in \Gamma$, therefore the quotient Γ/N forces the relation $\gamma = \bar{\gamma}$ for any $\gamma \in \Gamma$.

Let us denote by H the normal subgroup of Γ generated by the elements $\bar{\gamma}\gamma^{-1}$, $\gamma \in \Gamma$. The group $\pi_1(X/\sigma) = \Gamma'/\Gamma'_{tor} \simeq \Gamma/N$ is a quotient of Γ/H. Note that by Remark 10, if the fixed curve under the involution σ is irreducible, then the group H equals to N.

Theorem 15. *Suppose that Γ is a subgroup of $\Gamma(1)$. Then Γ/H is a finite group and the fundamental group of X/σ is finite.*

Proof. Let A' be a quaternion algebra over the field ℓ as above such that $A = A'\otimes k$ and the involution of second kind on A is given by $a' \otimes u \to a' \otimes \bar{u}$. Let k' be a degree 2 extension of ℓ such that $A' \otimes_\ell k' = M_2(k')$ and let K be the compositum of k, k' : $K = k \otimes_\ell k'$. Then the algebra $A \otimes_k K$ is $A' \otimes_\ell K = M_2(k') \otimes_\ell k = M_2(K)$. The involution of second kind $a \to \bar{a}$ extends to $M_2(K)$ and acts on each entry fixing $M_2(k') \subset M_2(K)$. The embedding $j : A \hookrightarrow M_2(K)$ is equivariant for the action of the involution: $\forall a \in A$, $j(\bar{a}) = \overline{j(a)}$, where the action on the left hand side is the conjugation on each entries of the matrix.

The group $j(\Gamma(1))$ is a subgroup of $PSL_2(\mathcal{O}_K)$. Let $I \subset \mathcal{O}_K$ be the (non-trivial) ideal generated by the elements $\bar{a}-a$, $a \in \mathcal{O}_K$. The ring O_K/I is a finite ring therefore the subgroup Γ/H of $PSL_2(\mathcal{O}_K/I)$ is finite.

The fundamental group of X/σ is (isomorphic to) Γ/N which is a quotient of the finite group Γ/H, therefore $\pi_1(X/\sigma)$ is finite. □

4. Examples.

4.1. Aim and terminology. Our goal is to find examples of smooth quaternionic Shimura surfaces X_Γ together with an involution σ on X_Γ having one-dimensional fixed locus. So, we consider an indefinite quaternion algebra A over a totally real field k of degree $n = [k : \mathbb{Q}]$, unramified exactly at two infinite places of k and consider groups Γ commensurable with $\Gamma_{\mathcal{O}}(1)$, the group of norm-1 elements of a maximal order $\mathcal{O} \subset A$ (modulo center).

Definition 16. Let $k, A, \mathcal{O}$ be as above and e an integer. A discrete group Γ in the commensurability class of $\Gamma_{\mathcal{O}}(1)$ is said *admissible of type* e if:

(1) Γ is torsion-free.
(2) $e(X_\Gamma) = e$ where $e(X_\Gamma)$ is denoting the (orbifold-) Euler number.
(3) On A there exists an involution τ of second kind such that Γ is invariant under τ.

Let us remark that according to Proposition 11, the quotient X_Γ/σ will be of general type if $e(X_\Gamma) \leq 36$. Hence, we will focus on admissible groups of type $e = 12, 16, 20, ..., 36$.

4.2. Smoothness and the Euler number. Let $A = A(k, \mathfrak{p}_1, ..., \mathfrak{p}_{2m})$ be as above and assume that there exists a k/ℓ-involution on A with respect to a subfield $\ell \subset k$. According to Proposition 5, we in particular assume that the primes $\mathfrak{p}_i$, $i = 1, \dots, 2m$ in $\mathcal{O}_k$ come in pairs: there exist primes $\mathcal{P}_1$,...,$\mathcal{P}_m$ of $\mathcal{O}_\ell$ such that $\mathfrak{p}_1\mathfrak{p}_2 = (\mathcal{P}_1)$,...,$\mathfrak{p}_{2m-1}\mathfrak{p}_{2m} = (\mathcal{P}_m)$.

If Γ is commensurable with $\Gamma_{\mathcal{O}}(1)$, we have the following general formula for the orbifold Euler number of the Shimura surface X_Γ

Proposition 17. (see [Shi63], [Vig76]) *Let $n = [k : \mathbb{Q}]$, $\zeta_k(\ )$ be the Dedekind zeta function of k and d_k denote the discriminant of k. Then the orbifold Euler number of X_Γ equals*

$$e(X_\Gamma) = [\Gamma_{\mathcal{O}}(1) : \Gamma] \cdot (-1)^n 2^{-n+3} \zeta_k(-1) \prod_{i=1}^{m} (N\mathcal{P}_i - 1)^2$$

where $N\mathcal{P} = |\mathcal{O}_\ell/\mathcal{P}|$ denotes the norm of $\mathcal{P}$ and where $[\Gamma_{\mathcal{O}}(1) : \Gamma] = \frac{[\Gamma_{\mathcal{O}}(1) : \Gamma \cap \Gamma_{\mathcal{O}}(1)]}{[\Gamma : \Gamma \cap \Gamma_{\mathcal{O}}(1)]}$ is the generalized index of two commensurable groups.

Expressing the value $\zeta_k(-1)$ in terms of second generalized Bernoulli numbers $B_{2,k}$ associated with the quadratic Dirichlet character $\chi_k(p) = \left(\frac{d_k}{p}\right)$ of k, we get

Corollary 18. (see [Sha78]) *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. Then,*

$$e(X_\Gamma) = [\Gamma_{\mathcal{O}}(1) : \Gamma] \cdot \frac{B_{2,k}}{12} \prod_{i=1}^{m} (p_i - 1)^2,$$

where $p_1, \dots, p_m$ are rational primes such that $\mathfrak{p}_{2j-1}\mathfrak{p}_{2j} = (p_j)$.

The surface X_Γ is smooth if and only if Γ is torsion-free. Here, we will concentrate on subgroups $\Gamma \subset \Gamma_{\mathcal{O}}(1)$.

Lemma 19. (see [Sha78]) *Let ξ_n be a primitive n-th root of unity. There exists an element of order n in $\Gamma_{\mathcal{O}}(1)$ if and only if:*
(1) $\xi_n + \xi_n^{-1} \in k$
(2) *every ramified prime in A is non-split in $k(\xi_n)$*

The above lemma already gives us a bound for the order of possible torsion elements. Namely, for any $a \in A \setminus k$, the algebra $k(a)$ is commutative subfield of A. Since $\dim_k A = 4$, and $a \notin k$, we get $\dim_k k(a) = 2$ and $k(a)$ is a quadratic extension of k. Assume now that ξ is a primitive n-th root of unity embedded in $\mathcal{O}$, then as $L = k(\xi) \subset A$ is a quadratic extension of k, we have $\varphi(n) \leq 2[k : \mathbb{Q}]$, since $\varphi(n) = [\mathbb{Q}(\xi) : \mathbb{Q}] \leq [L : \mathbb{Q}] \leq 2[k : \mathbb{Q}]$.

The above results provide us with criteria to test the conditions in the definition of admissible groups.

Theorem 20. *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field and consider the totally indefinite quaternion algebra $A = A(k, \mathfrak{p}_1, \overline{\mathfrak{p}}_1 \dots, \mathfrak{p}_m, \overline{\mathfrak{p}}_m)$ over k, ramified at the prime ideals dividing rational primes $p_1, \dots, p_m$ which are split in k. If $\Gamma \subset \Gamma_{\mathcal{O}}(1)$ is an admissible group of type e, then the possibilities are as follows:*

type e	k	$\mathrm{Ram}(A)$	$[\Gamma_{\mathcal{O}}(1):\Gamma]$
12	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	18
16	$\mathbb{Q}(\sqrt{13})$	$\mathfrak{p}_3, \overline{\mathfrak{p}}_3$	12
16	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	24
20	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	30
24	$\mathbb{Q}(\sqrt{13})$	$\mathfrak{p}_3, \overline{\mathfrak{p}}_3$	18
24	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	36
24	$\mathbb{Q}(\sqrt{2})$	$\mathfrak{p}_7, \overline{\mathfrak{p}}_7$	4
24	$\mathbb{Q}(\sqrt{33})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	12
28	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	42
32	$\mathbb{Q}(\sqrt{13})$	$\mathfrak{p}_3, \overline{\mathfrak{p}}_3$	24
32	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	48
32	$\mathbb{Q}(\sqrt{28})$	$\mathfrak{p}_3, \overline{\mathfrak{p}}_3$	6
36	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	54
36	$\mathbb{Q}(\sqrt{33})$	$\mathfrak{p}_2, \overline{\mathfrak{p}}_2$	18

In order to prove this theorem we need the following elementary lemma.

LEMMA 21. *Let G be an arbitrary group and $H \subset G$ a torsion-free subgroup of finite index. If $T \subset G$ is a finite subgroup, then $|T|$ divides $[G : H]$.*

Proof. Let $G/H = \{g_1H, \dots, g_IH\}$ be the set of left cosets of H in G. The group T acts by left multiplication on it. Moreover it is free, otherwise, we would have $t \cdot g_iH = g_iH$ with some non-trivial $t \in T$ and consequently, $g_i^{-1}tg_i \in H$, which is not possible, since H is torsion-free and t as well as $g_i^{-1}tg_i$, is of finite order. The length of a T-orbit on G/H equals to the order of $|T|$. Since G/H is the union of different T-orbits, $|T|$ divides $|G/H|$. □

Proof of Theorem 20. If $\Gamma \subset \Gamma_{\mathcal{O}}(1)$ is admissible of type $e = 12 + 4j \le 36$, then $36 \ge [\Gamma_{\mathcal{O}}(1) : \Gamma] \frac{B_{2,k}}{12} \prod_{i=1}^{m} (p_i - 1)^2$. Since $[\Gamma_{\mathcal{O}}(1) : \Gamma]$ and $(p_i - 1)^2$ are positive, we have the condition $36 \ge \frac{B_{2,k}}{12}$. By [Sha78], Proposition 3.2, $B_{2,k}$ is bounded below by $3d_k^{3/2}/50$ and this implies the upper bound $d_k < 373$ for the discriminant of k. Using the formula in [Sha78], p. 228, we can compute all the values $B_{2,k}$ for $d_k < 373$. With the list of all these values, we check the necessary conditions:

- $36 \ge B_{2,k}/12$
- $B_{2,k} \mid 12 \cdot e = 144, 192, 240, \dots 432$ for integral $B_{2,k}$ ($\Leftrightarrow d_k \neq 5$) and with obvious modification for $d_k = 5$.
- The square part of $12e/B_{2,k}$ is divisible by the product $\prod_p (p-1)^2$ where p runs over subsets of rational primes which are split in k (note that $p = 2$, if split in k, contributes the factor 1 to the product).

We obtain the following list of tuples satisfying all the conditions

d_k	$B_{2,k}$	e	$12e/B_{2,k} = I \cdot \prod(p-1)^2$
137	192	16	$1 = (2-1)^2$
113	144	12	$1 = (2-1)^2$
109	108	36	$4 = (3-1)^2$
105	144	12	$1 = (2-1)^2$
85	72	24	$1 = (3-1)^2$
40	28	28	$12 = 3 \cdot (3-1)^2$
37	20	20	$12 = 3 \cdot (3-1)^2$
33	24	24	$12 \cdot (2-1)^2$
29	12	16	$16 = (5-1)^2$
29	12	32	$32 = 2 \cdot (5-1)^2$
29	12	36	$36 = (7-1)^2$
28	16	16	$12 = 3 \cdot (3-1)^2$
28	16	32	$24 = 6(3-1)^2$
24	12	16	$16 = (5-1)^2$
24	12	32	$32 = 2 \cdot (5-1)^2$
17	8	$12+4j,\ j=0,\dots,6$	$(12+4j) \cdot (2-1)^2$
13	4	$12+4j,\ j=0,\dots,6$	$(9+3j) \cdot (3-1)^2$
8	2	$12, 24, 36$	$2 \cdot (7-1)^2$,$4 \cdot (7-1)^2$, $6 \cdot (7-1)^2$
5	$\frac{4}{5}$	20	$3 \cdot (11-1)^2$

We observe (keeping also in mind the splitting behavior of 2 in k) that the set of ramified places in A is determined by the value $12e/B_{2,k}$ in the table.

As next, we identify those subgroups $\Gamma \subset \Gamma_{\mathcal{O}}(1)$ which cannot be torsion-free. For this we use the two Lemmas 19 and 21. Namely, note first that $\Gamma_{\mathcal{O}}(1)$ contains at most torsions of order $2, 3$ and 6 for $k \neq \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{2})$ and additionally elements of order 5 for $k = \mathbb{Q}(\sqrt{5})$ and of order 4 for $k = \mathbb{Q}(\sqrt{2})$. Case by case analysis leads to the final statement; to check the splitting behavior in $k(\xi_n)$ one can use the criterion of Shavel (see [Sha78], Theorem 4.8). We double-checked the conditions of Lemma 19 explicitly with PARI/GP.

4.3. Admissible groups defined by congruences. Let k a totally real number field and $A(k, \mathfrak{p}_1, \overline{\mathfrak{p}}_1, \dots, \mathfrak{p}_m, \overline{\mathfrak{p}}_m)$ an indefinite quaternion algebra over k, $\mathcal{O}$ a maximal order in A, $\mathcal{O}$ and $\Gamma_{\mathcal{O}}(1)$ as in the previous sections. If $\mathfrak{a}$ is a two-sided $\mathcal{O}$ ideal in $\mathcal{O}$, the *principal congruence subgroup* in $\mathcal{O}(1)$ associated with $\mathfrak{a}$ is defined as

$$\mathcal{O}(\mathfrak{a}) = \{g \in \mathcal{O} \mid Nrd(g) = 1, g - 1 \in \mathfrak{a}\}.$$

Additionally we define $\Gamma_{\mathcal{O}}(\mathfrak{a}) = \mathcal{O}(\mathfrak{a})/Z$ where Z denotes the center of $\mathcal{O}(\mathfrak{a})$. A *congruence subgroup* in $\mathcal{O}(1)$, resp. $\Gamma_{\mathcal{O}}(1)$, is a subgroup $G \subset \mathcal{O}(1)$, resp. $\Gamma \subset \Gamma_{\mathcal{O}}(1)$, which contains some $\mathcal{O}(\mathfrak{a})$, resp. $\Gamma_{\mathcal{O}}(\mathfrak{a})$. The group $\mathcal{O}(\mathfrak{a})$ is a normal subgroup of finite index in $\mathcal{O}(1)$ and we have $\mathcal{O}(1)/\mathcal{O}(\mathfrak{a}) \cong \Gamma_{\mathcal{O}}(1)/\pm\Gamma_{\mathcal{O}}(\mathfrak{a})$ if $2 \notin \mathfrak{a}$ and $\mathcal{O}(1)/\mathcal{O}(\mathfrak{a}) \cong \Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{a})$ if $2 \in \mathfrak{a}$. The size $|\mathcal{O}(1)/\mathcal{O}(\mathfrak{a})|$ is computed as follows

- Any two-sided ideal $\mathfrak{a}$ in $\mathcal{O}$ has a unique decomposition $\mathfrak{a} = \mathfrak{Q}_1^{e_1} \cdot \ldots \cdot \mathfrak{Q}_r^{e_r}$ as a product of prime ideal powers. Then, $\mathcal{O}(1)/\mathcal{O}(\mathfrak{a})$ is a direct product

$$\mathcal{O}(1)/\mathcal{O}(\mathfrak{a}) = \mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}_1^{e_1}) \times \ldots \times \mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}_r^{e_r}).$$

- Let $\mathfrak{Q}$ be a prime ideal in $\mathcal{O}$. The $\mathcal{O}_k$-ideal $\mathfrak{q} = Nrd(\mathfrak{Q})$ generated by the reduced norms of elements in $\mathfrak{Q}$, which is also the intersection $\mathfrak{q} = \mathfrak{Q} \cap \mathcal{O}_k$, is a prime ideal and there are two possible cases:

– $\mathfrak{q} \notin Ram(A)$. Then, $\mathfrak{Q} = \mathfrak{q}\mathcal{O}$ and $\mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}_1^e) \cong SL_2(\mathcal{O}_k/\mathfrak{q}^e)$.
– $\mathfrak{q} \in Ram(A)$. Then, $\mathfrak{Q}^2 = \mathfrak{q}\mathcal{O}$ and $\mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}^e) \cong (\mathcal{O}/\mathfrak{Q}^e)_1 = \ker\left((\mathcal{O}/\mathfrak{Q}^e)^* \overset{Nr}{\to} (\mathcal{O}_k/\mathfrak{q}^e)^*\right)$, where Nr is the norm map induced by the reduced norm $Nrd : \mathcal{O} \to \mathcal{O}_k$.

Remark 22. If we want to search for admissible groups among the principal congruence subgroups, then we must note the following: for $g \in \mathcal{O}(\mathfrak{Q})$ we have $\overline{g} \in \mathcal{O}(\overline{\mathfrak{Q}})$, so, $\Gamma_{\mathcal{O}}(\mathfrak{Q})$ will be admissible if and only if $\mathfrak{Q} = \overline{\mathfrak{Q}}$ (for more precise statement see Theorem 24 and Lemma 25 below). This is already a strong condition: If k is a quadratic field, the prime $\mathfrak{q}$ under $\mathfrak{Q}$ must be inert or ramified over ℓ. For instance, this in combination with list of possible candidates from Theorem 20 shows there are no admissible principal congruence subgroups of any type e defined over a real quadratic field.

In the following we will make use of the following well-known fact.

Lemma 23. *Let $\mathfrak{q}$ be a prime ideal in $\mathcal{O}_k$, unramified in a k-central quaternion algebra A and $\mathfrak{Q} = \mathfrak{q}\mathcal{O}$ the corresponding $\mathcal{O}$-ideal . Let $q\mathbb{Z} = \mathfrak{q} \cap \mathbb{Z}$ be the rational prime divisible by $\mathfrak{q}$ and finally, let $x \in \mathcal{O}^1(\mathfrak{Q})$ be an element of order p, where p is a prime. Then $p = q$.*

Proof. We have $x \in \mathcal{O}^1(\mathfrak{Q}) \Leftrightarrow Nrd(x-1) \in \mathfrak{q}$. We can assume that x is a primitive p-th root of unity contained in A. Since $k(x) \subset A$, we have $Nrd(x-1) = N_{k(x)/k}(x-1)$. Taking $N_{k/\mathbb{Q}}(\cdot)$ on both sides we obtain

$$N_{k/\mathbb{Q}}(Nrd(x-1)) = N_{k(x)/\mathbb{Q}}(x-1) \in N_{k/\mathbb{Q}}(\mathfrak{q}) = q^f\mathbb{Z}.$$

where $f > 0$ is the inertia degree of $\mathfrak{q}$. On the other hand $x - 1 \in \mathbb{Q}(x)$, and therefore $N_{k(x)/\mathbb{Q}}(x-1) = N_{\mathbb{Q}(x)/\mathbb{Q}}(x-1)^d$, where $d = [k(x) : \mathbb{Q}(x)]$. Altogether, we obtain the relation

$$N_{\mathbb{Q}(x)/\mathbb{Q}}(x-1)^d \in q^f\mathbb{Z}.$$

Finally, it is well-known that $N_{\mathbb{Q}(x)/\mathbb{Q}}(x-1) = \pm p$ and from this the claim follows. □

Assume that the prime ideal $\mathfrak{q} \subset \mathcal{O}_k$ is unramified in A, then $\mathfrak{q}\mathcal{O}$ is a prime ideal in $\mathcal{O} \subset A$. Since in this case $\mathfrak{Q} = \mathfrak{q}\mathcal{O}$ we will write $\Gamma_{\mathcal{O}}(\mathfrak{q}) = \Gamma_{\mathcal{O}}(\mathfrak{Q})$. Let $s = q^f$ be the absolute norm $N_{k/\mathbb{Q}}(\mathfrak{q}) = |\mathcal{O}_k/\mathfrak{q}|$ of $\mathfrak{q}$. Then $\Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{q}) \cong PSL_2(\mathcal{O}_k/\mathfrak{q}) \cong PSL_2(s) = PSL_2(\mathbb{F}_s)$. By the classification theorem of Dickson, we know all the subgroups of $PSL_2(s)$. Let us mention two particular subgroups which we will use later on:

(1) Borel subgroup $B \subset PSL_2(s)$ consisting of all upper triangular matrices in $PSL_2(s)$. The group B is a maximal subgroup of $PSL_2(s)$ of index $s+1$ and order $s(s-1)/t$, with $t = gcd(s-1, 2)$.
(2) Unipotent subgroup $U \subset PSL_2(s)$ consisting of all elements in B of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. The group U is a subgroup of index $(s^2-1)/t$ and order s.

With above notations let $\pi : \Gamma_{\mathcal{O}}(1) \longrightarrow PSL_2(s)$ denote the epimorphism induced by the canonical projection $\Gamma_{\mathcal{O}}(1) \longrightarrow Q = \Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{q})$. Let $\Gamma^B(\mathfrak{q}) = \pi^{-1}(B)$ and $\Gamma^U(\mathfrak{q}) = \pi^{-1}(U)$ be the inverse images of B and U respectively. These are subgroups of $\Gamma_{\mathcal{O}}(1)$ of index equal to the index of its image in $PSL_2(s)$ under π. It is important to mention that $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ is also constructed as the group of the norm-1 elements (modulo center) in an appropriate Eichler order $\mathcal{E}$. The construction is as follows: Let $\mathcal{O}$ be a maximal order and denote $\mathcal{O}_v = \mathcal{O} \otimes_{\mathcal{O}_k} R_v$ the localizations of $\mathcal{O}$

at finite places v of k. There R_v denotes the valuation ring in the localization k_v of k at v. Note that $\mathcal{O} = \bigcap_v \mathcal{O}_v$. Let $\mathcal{O}'$ be another maximal order with the property that for all finite places $v \neq \mathfrak{q}$ we have $\mathcal{O}_v = \mathcal{O}'_v$ and additionally $\mathcal{O}_\mathfrak{q} \cap \mathcal{O}'_\mathfrak{q}$ has index $N(\mathfrak{q}) = \# R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}$ in both, $\mathcal{O}_\mathfrak{q}$ and $\mathcal{O}'_\mathfrak{q}$. Put $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$. Then, by definition, $\mathcal{E}$ is an Eichler order of level $\mathfrak{q}$. If $\mathfrak{q}$ is ramified in A, we have $\mathcal{E} = \mathcal{O}$ and in the case where $\mathfrak{q}$ is unramified, after a possible conjugation, we can assume that $\mathcal{O}_\mathfrak{q} = M_2(R_\mathfrak{q})$ and we can choose $\mathcal{O}'_\mathfrak{q} = PM_2(R_\mathfrak{q})P^{-1}$ with $P = diag(1, \varpi)$, where ϖ is a generator of the valuation ideal $\mathfrak{q}R_\mathfrak{q}$. The reduction modulo $\mathfrak{q}$ maps $\mathcal{E}_\mathfrak{q} = \mathcal{O}_\mathfrak{q} \cap \mathcal{O}'_\mathfrak{q}$ surjectively to the subalgebra of upper triangular matrices in $M_2(R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q})$. Therefore the group $\mathcal{E}^1$ of norm-1 elements in $\mathcal{E}$ corresponds to exactly those elements in $\mathcal{O}^1$ which modulo $\mathfrak{q}\mathcal{O}$ are upper triangular.

In general, a k/ℓ-involution on a quaternion algebra A does not preserve a maximal order. But under certain conditions on A we can ensure the existence of such an order:

Theorem 24. (Scharlau, [Sch84, Theorem 4.6]) *Let A be a quaternion algebra over k admitting a k/ℓ-involution τ. Then there exists a maximal order $\mathcal{O}$ invariant under τ unless the following exceptional situation is given:*

- *the extension k/ℓ is unramified and*
- *the number of places $v \in Ram(A)$ is $\equiv 2 \bmod 4$.*

Corollary 25. *Assume that A admits a k/ℓ-involution τ which preserves a maximal order $\mathcal{O}$ and let $\mathfrak{q} \subset \mathcal{O}_k$ be a prime ideal which is unramified in A and $\mathfrak{q}\mathcal{O}$ the corresponding prime ideal in $\mathcal{O}$. Assume that the non-trivial k/ℓ-automorphism $x \mapsto \overline{x}$ maps $\mathfrak{q}$ to itself, that is, $\overline{\mathfrak{q}} = \mathfrak{q}$. Then, for each of the groups $\Gamma = \Gamma_\mathcal{O}(1)$, $\Gamma^B_\mathcal{O}(\mathfrak{q})$, $\Gamma^U_\mathcal{O}(\mathfrak{q})$ and $\Gamma_\mathcal{O}(\mathfrak{q})$ there exists a k/ℓ-involution on A leaving Γ invariant.*

Proof. The group $\Gamma_\mathcal{O}(1)$ is τ-invariant since $\mathcal{O}$ is τ-invariant and for any $a \in A$ we have $Nrd(\overline{a}) = \overline{Nrd(a)}$. Also, $\Gamma(\mathfrak{q})$ is invariant, since for every $x \in \Gamma_\mathcal{O}(\mathfrak{q})$ there is a representative $x' \in \mathcal{O}$ of the class x satisfying $x - 1 \in \mathfrak{q}\mathcal{O}$, thus $\overline{(x'-1)} \in \overline{\mathfrak{q}\mathcal{O}} = \overline{\mathfrak{q}}\mathcal{O} = \mathfrak{q}\mathcal{O}$, by assumption, hence $\overline{x'} \equiv 1 \bmod \mathfrak{q}\mathcal{O}$.

In order to prove the invariance of other groups, we first localize at $\mathfrak{q}$; since $\mathfrak{q} \notin Ram(A)$, the local algebra $A_\mathfrak{q} = A \otimes_k k_\mathfrak{q}$ is isomorphic to $M_2(k_\mathfrak{q})$ and since $\mathcal{O}$ is maximal $\mathcal{O}_\mathfrak{q} = \mathcal{O} \otimes_{\mathcal{O}_k} R_\mathfrak{q}$ is isomorphic to $M_2(R_\mathfrak{q})$ where $R_\mathfrak{q}$ is the valuation ring in $k_\mathfrak{q}$. Choosing the appropriate isomorphism $A_\mathfrak{q} \cong M_2(k_\mathfrak{q})$ we can assume that $\mathcal{O}_\mathfrak{q} = M_2(R_\mathfrak{q})$. Consider the order $\mathcal{E}_\mathfrak{q} = M_2(R_\mathfrak{q}) \cap PM_2(R_\mathfrak{q})P^{-1}$ with $P = diag(1, \varpi)$, where ϖ is a generator of the valuation ideal $\mathfrak{q}R_\mathfrak{q}$. This is the localization of the global Eichler order $\mathcal{E}$ corresponding to a group $\Gamma^B_\mathcal{O}(\mathfrak{q})$. The involution τ which leaves $\mathcal{O}$ invariant extends to an involution $\hat{\tau}$ on $\mathcal{O}_\mathfrak{q} = \mathcal{O} \otimes_{\mathcal{O}_k} R_\mathfrak{q}$ in an obvious way by defining $\hat{\tau}(x \otimes r) = \tau(x) \otimes \overline{r}$ where $r \mapsto \overline{r}$ is the generator of $Gal(k_\mathfrak{q}/\ell_\mathcal{Q})$ and $\mathcal{Q} = \mathfrak{q} \cap \ell$ is the prime of ℓ lying under $\mathfrak{q}$ and $\ell_\mathcal{Q}$ its localization. By this the involution $\hat{\tau}$ maps the matrix P to $\pm P$ depending on whether $k_\mathfrak{q}/\ell_\mathcal{Q}$ is unramified or ramified but in any case $\hat{\tau}$ preserves $\mathcal{E}_\mathfrak{q}$. From the construction of $\hat{\tau}$ we see that $\hat{\tau}$ also preserves $\mathcal{O} \cap \mathcal{E}_\mathfrak{q} = \mathcal{E}$ and the norm-1 group $\mathcal{E}^1$ whose quotient by the center is $\Gamma^B_\mathcal{O}(\mathfrak{q})$. The group $\Gamma^U_\mathcal{O}(\mathfrak{q})$ consists of those elements in $\Gamma^B_\mathcal{O}(\mathfrak{q})$ which reduce modulo $\mathfrak{q}$ to upper triangular matrices with only 1 on the diagonal. The preimage of such matrices in $\mathcal{E}_\mathfrak{q}$ is preserved by $\hat{\tau}$ and hence τ preserves the preimage of these matrices in $\mathcal{O} \cap \mathcal{E}_\mathfrak{q}$. This implies that with $\Gamma^B_\mathcal{O}(\mathfrak{q})$ also $\Gamma^U_\mathcal{O}(\mathfrak{q})$ is preserved. □

4.4. Construction with the Borel subgroup. Let $A(k, \mathfrak{p}_1, \overline{\mathfrak{p}}_1, \ldots, \mathfrak{p}_m, \overline{\mathfrak{p}}_m)$ be as before. Let $\mathfrak{q}$ be a prime ideal of k such that $\mathfrak{q} \neq \mathfrak{p}_i, \overline{\mathfrak{p}}_i$ for $i = 1, \ldots, m$ and consider the group $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$, the inverse image $\pi^{-1}(B)$ of a Borel subgroup $B \subset \Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{q}) \cong PSL_2(\mathcal{O}_k/\mathfrak{q})$. The group $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ is a subgroup of index $N_{k/\mathbb{Q}}(\mathfrak{q})+1$ in $\Gamma_{\mathcal{O}}(1)$. In order to discuss the torsion elements in $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ it will again be useful to interpret $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ as the norm-1 group of an Eichler order as explained in previous section. Let us give conditions under which $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ is torsion-free.

Lemma 26. *Let k, A, $\mathcal{O}$ and $\mathfrak{q}$ be as above. Then, $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ contains a torsion element if and only if a primitive n-th root of unity ξ can be embedded in $\mathcal{E}(\mathfrak{q})$. This happens if and only if every prime $\mathfrak{p} \in Ram(A)$ is either ramified or inert in $k(\xi)$ and $\mathfrak{q}$ is split in $k(\xi)$.*

Proof. Let γ be a torsion element in $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$. Then there is a minimal N such that $\gamma^N = \pm 1$, which implies that γ is an N-th (or $2N$-nth) root of unity contained in $\mathcal{E}(\mathfrak{q})$. Conversely, let ξ be a root of unity, let $L = k(\xi)$ and assume that there exists an embedding $\sigma : L \hookrightarrow A$ such that $\sigma(L) \cap \mathcal{E}(\mathfrak{q}) = \mathcal{O}_k(\xi)$ (that is, $\mathcal{O}_k(\xi)$ is embedded in $\mathcal{E}(\mathfrak{q})$). Then $\sigma(\xi)$ is a torsion element in $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$. By a theorem of Eichler (see [Eic55, Satz 6]) such an embedding is possible if and only if the splitting condition mentioned in the statement of Lemma is satisfied. □

4.5. A Shimura surface with an involution of second kind and $p_g = 5$. Let $k = \mathbb{Q}(\sqrt{33})$ and $A = A(k, \mathfrak{p}_2\overline{\mathfrak{p}}_2)$ the indefinite quaternion algebra ramified exactly at the two places over 2 (note that 2 is split in k, since $33 \equiv 1 \bmod 8$). By Theorem 5, A admits a $k/\mathbb{Q}$-involution and since $k/\mathbb{Q}$ is not totally ramified, Theorem 24 ensures the existence of an involution invariant order $\mathcal{O}$. Let $\mathfrak{q} = \mathfrak{q}_{11}$ be the prime over 11. Since 11 is ramified in k, we have $N_{k/\mathbb{Q}}(\mathfrak{q}) = 11$ and $\Gamma^B_{\mathcal{O}}(\mathfrak{q}_{11})$ is of index 12 in $\Gamma_{\mathcal{O}}$. By the volume formula from Theorem 17, we have $e(\Gamma_{\mathcal{O}}(1)) = 2$, hence $e(\Gamma^B_{\mathcal{O}}(\mathfrak{q})) = 24$. Let us show that $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ is torsion-free. For this we need to exclude the existence of elements of order 2 and 3 only, since these are the only primes for which an embedding of ξ_p in A is possible. Elements of order 2 come from embeddings of $k(\xi_4) = k(\sqrt{-1})$ in A and those of order 3 from embeddings of $k(\xi_6) = k(\sqrt{-3})$. We use Lemma 26: $k(\xi_4) = \mathbb{Q}(\sqrt{33})(\sqrt{-1})$ and we find that $11\mathcal{O}_{k(\zeta_4)} = \mathfrak{Q}^2$ with $(\mathcal{O}_{k(\xi_4)}/\mathfrak{Q} : \mathbb{F}_{11}) = 2$. It follows that $\mathfrak{q}_{11}$ is inert in $k(\xi_4)$ and by Lemma 26, $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ contains no elements of order 2. Similar argument excludes the existence of elements of order 3. Namely, $k(\xi_6) \cong \mathbb{Q}[x]/\langle x^4 - 60x^2 + 1296\rangle$ and in $k(\xi_6)$ we again have $11\mathcal{O}_{k(\zeta_6)} = \mathfrak{Q}^2$ with a prime ideal $\mathfrak{Q}$ in $\mathcal{O}_{k(\zeta_6)}$ whose inertia degree is $(\mathcal{O}_{k(\xi_6)}/\mathfrak{Q} : \mathbb{F}_{11}) = 2$. Again this implies that $\mathfrak{q}_{11}$ is inert in $\mathcal{O}_{k(\xi_6)}$ and by Lemma 26, there are no elements of order 3 in $\Gamma^B_{\mathcal{M}}(\mathfrak{q}_{11})$. Finally by Corollary 25, $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ is invariant under the involution on $\mathcal{O}$ and we get:

Theorem 27. *The group $\Gamma^B_{\mathcal{O}}(\mathfrak{q}_{11})$ is admissible of type* 24.

4.6. A Shimura surface with an involution of second kind and $p_g = 6$. In this example we consider the unique totally real number field k of degree 4 and discriminant $d_k = 725$. Its defining polynomial is $x^4 - x^3 - 3x^2 + x + 1$ and k contains $\ell = \mathbb{Q}(\sqrt{5})$ as a subfield of degree 2. Let us consider the k-central quaternion algebra $A(k, \emptyset)$ ramified exactly at two infinite places v_1 and v_2 of k such that $v_2 = v_1 \circ \sigma$, where $\langle\sigma\rangle = Gal(k/\ell)$. We remark that k is not a Galois extension of $\mathbb{Q}$. The algebra A admits an involution of second kind τ and by Theorem 24, there is a maximal order $\mathcal{O}$ invariant under τ. Consider now the prime $q = 29$. In $\ell = \mathbb{Q}(\sqrt{5})$, $29\mathcal{O}_\ell = \mathcal{Q}_{29}\mathcal{Q}'_{29}$ is a product of two primes. On the other hand, a computation with PARI/GP shows that the ideal $29\mathcal{O}_k = \mathfrak{q}^2_{29}\mathfrak{q}'_{29}$ is also a product of two prime ideals $\mathfrak{q}_{29}$ (with multiplicity 2) and $\mathfrak{q}'_{29}$, hence neither $\mathcal{Q}_{29}$ nor $\mathcal{Q}'_{29}$ is split in k.

Moreover we deduce that $\mathfrak{q}_{29}^2 = \mathcal{Q}_{29}\mathcal{O}_k$ and $\mathfrak{q}'_{29} = \mathcal{Q}'_{29}\mathcal{O}_k$ as well as $\mathcal{O}_k/\mathfrak{q}_{29} \cong \mathbb{F}_{29}$ and $\mathcal{O}_k/\mathfrak{q}'_{29} \cong \mathbb{F}_{29^2}$. By Theorem 17 we have $e(X_{\Gamma_{\mathcal{O}}(1)}) = 1/15$ (we compute $\zeta_k(-1)$ with PARI/GP command zetak). Consider the congruence subgroup $\Gamma^U_{\mathcal{O}}(\mathfrak{q}_{29})$. We obtain $[\Gamma_{\mathcal{O}}(1) : \Gamma^U_{\mathcal{O}}(\mathfrak{q}_{29})] = 420$ and $c_2(X_{\Gamma^U_{\mathcal{O}}(\mathfrak{q}_{29})}) = 28$. By corollary 25, $\Gamma^U_{\mathcal{O}}(\mathfrak{q}_{29})$ is stable under τ. Also $\Gamma^U_{\mathcal{O}}(\mathfrak{q}_{29})$ is torsion-free. Namely, as the order of U is s, any non-trivial torsion element in $\Gamma^U_{\mathcal{O}}(\mathfrak{q}_{29})$ has order 29 (which is impossible by lemma 19) or lies already in $\Gamma_{\mathcal{O}}(\mathfrak{q}_{29})$. But this latter group is torsion-free by Lemma 23.

Remark 28. Of course, the strategy of Section 4.2 leading to Theorem 20 could be applied also in the case of quaternion algebras over totally real fields k of degree > 2. Restricting ourselves to totally real quartic fields of discriminant $\leq 10^4$ and groups of type $\Gamma^B_{\mathcal{O}}(\mathfrak{q})$ or $\Gamma^U_{\mathcal{O}}(\mathfrak{q})$ we find that example 4.6 is the only example of an admissible group (of any type $e = 12 + 4k$, $0 \leq k \leq 6$).

5. determination of the fixed curve. Let $X_\Gamma = \mathbb{H}^2_{\mathbb{C}}/\Gamma$ be a smooth Shimura surface such that the involution μ on $\mathbb{H}^2_{\mathbb{C}}$ exchanging the factors descends to an involution σ on the quotient X_Γ. The image of the diagonal $\Delta \subset \mathbb{H} \times \mathbb{H}$ is a smooth Shimura curve C_Γ fixed by σ. The aim of this section is to determine that curve in examples we investigated in Sections 4.5 and 4.6.

The analogous problem for Hilbert modular surfaces is well-known, see for instance [Hir73] or [Hau82]. The quaternion algebra $A = A(k, \mathfrak{p}_1, \ldots, \mathfrak{p}_{2r})$ over k defining the group Γ has a non trivial involution τ of second kind with the invariant subfield ℓ. The involution τ determines the involution $\sigma : X_\Gamma \longrightarrow X_\Gamma$. The fixed point set of σ is associated with the invariant ℓ-subalgebra $A' = \{a \in A \mid \tau(a) = a\}$ which is a quaternion algebra over ℓ such that $A' \otimes k \cong A$ and is ramified at every prime $\mathcal{P}_i$ of ℓ satisfying $\mathcal{P}_i\mathcal{O}_k = \mathfrak{p}_{2i-1}\mathfrak{p}_{2i}$ for $i = 1, \ldots, r$ ([Gra02]).

Lemma 29. *Let $A = A(k, \mathfrak{p}_1, \ldots, \mathfrak{p}_{2r})$ be a quaternion algebra admitting an involution of second kind τ and A' the elementwise τ-invariant subalgebra. Let $\mathcal{O}$ be a order in A, then $\mathcal{O}' = \mathcal{O} \cap A'$ is an order in A'. Conversely, assume that $A' = A'(\ell, \mathcal{P}_1, \ldots, \mathcal{P}_s)$ is a quaternion algebra over ℓ and $\mathcal{O}'$ an order in A' then $\mathcal{O} = \mathcal{O}' \otimes_{\mathcal{O}_\ell} \mathcal{O}_k$ is an order in A. Assume that $\mathcal{O}'$ is maximal and each $\mathcal{P}_i$ is split in k then $\mathcal{O}' \otimes_{\mathcal{O}_\ell} \mathcal{O}_k$ is a maximal order in $A = A' \otimes_\ell k$.*

Proof. The first part of the Lemma concerning the correspondence between orders $\mathcal{O}'$ in A' and orders in A is obvious and we shall therefore prove only the second part. Assume that $\mathcal{O}' \subset A'$ is a maximal order and let $\mathcal{O} = \mathcal{O}' \otimes_{\mathcal{O}_\ell} \mathcal{O}_k$. The order $\mathcal{O}$ is maximal if and only if each of its localizations $\mathcal{O}_\mathfrak{p}$ is maximal in the local algebra $A_\mathfrak{p}$. Here, $\mathcal{O}_\mathfrak{p} = \mathcal{O}'_\mathcal{P} \otimes \mathcal{O}_{k_\mathfrak{p}}$ arises from the local maximal order $\mathcal{O}'_\mathcal{P}$ corresponding to a finite place $\mathcal{P}$ of ℓ lying under $\mathfrak{p}$. Assume now that $\mathcal{P} \neq \mathcal{P}_i$ is a finite place at which B is unramified. Then $\mathcal{O}'_\mathcal{P} \cong M_2(\mathcal{O}_{\ell_\mathcal{P}})$ and clearly $\mathcal{O}_\mathfrak{p} \cong M_2(\mathcal{O}_{k_\mathfrak{p}})$ is also maximal. If $\mathcal{P} = \mathcal{P}_i$ is a place such that $A'_{\mathcal{P}_i}$ is a division algebra, as by assumption $k_{\mathfrak{p}_i} = k_{\bar{\mathfrak{p}_i}} = \ell_{\mathcal{P}_i}$, $\mathcal{O}_{\mathfrak{p}_i}$ and $\mathcal{O}_{\bar{\mathfrak{p}}_i}$ are maximal. □

Remark 30. We shall note that in the case where $A' = A'(\ell, \mathcal{P}_1, \ldots, \mathcal{P}_1, \mathcal{Q})$ ramifies also at some prime $\mathcal{Q}$ that is non-split in k the order $\mathcal{O} = \mathcal{O}' \otimes \mathcal{O}_k$ is not maximal even if $\mathcal{O}'$ is maximal.

Example 31. Let $A' = \left(\frac{2,5}{\mathbb{Q}}\right)$ be the quaternion algebra over $\mathbb{Q}$ generated by elements $1, i, j, ij$ such that $i^2 = 2$, $j^2 = 5$ (and $ij = -ji$). The algebra A' is ramified exactly at the primes 2 and 5. Let $\mathcal{O}'$ be a maximal order in A'. The group $\Gamma_{\mathcal{O}'}(1)$ is a Fuchsian group. Let $\Gamma^B_{\mathcal{O}'}(11)$ be the subgroup corresponding to the Borel subgroup of $PSL_2(\mathbb{F}_{11})$. This subgroup can be interpreted as the group of elements of reduced norm 1 of an Eichler order $\mathcal{E}(11)$ of level 11. The group $\Gamma_{\mathcal{O}'}(1)$ is of index 12 in $\Gamma_{\mathcal{O}'}(1)$ and is torsion-free by Lemma 26, as 5 is split in $\mathbb{Q}(i)$ and 11 is non-split in $\mathbb{Q}(\sqrt{3})$. The genus of the curve $C = \Gamma_{\mathcal{O}'}(11)\backslash\mathbb{H}$ can be easily computed with the already used general volume formula from [Shi63] (see also [Vig80, III, Prop. 2.10]) by which $2 - 2g(C) = -8$. Let $k = \mathbb{Q}(\sqrt{33})$. Then $A = A' \otimes k = \left(\frac{2,5}{k}\right)$ is a quaternion algebra over k and is ramified exactly at the two primes lying over 2. The Eichler order $\mathcal{E}(11)$ in A' is naturally contained in the order $\mathcal{E}(11) \otimes_{\mathbb{Z}} \mathcal{O}_k$ of A and the latter one is contained in $\mathcal{E}(\mathfrak{q}_{11})$ the Eichler order of A corresponding to the prime $\mathfrak{q}_{11}$ of k lying over 11, since the elements $\mathcal{E}(11) \otimes_{\mathbb{Z}} \mathcal{O}_k$ become upper triangular modulo 11, hence modulo $\mathfrak{q}_{11}$. This gives an embedding of $\Gamma^B_{\mathcal{O}'}(11)$ in $\Gamma^B_{\mathcal{O}}(\mathfrak{q}_{11})$ and hence an embedding of a Shimura curve of genus 5 into the Shimura surface $X = \Gamma^B_{\mathcal{O}}(\mathfrak{q}_{11})\backslash\mathbb{H} \times \mathbb{H}$ (see Section 4.5) which by construction must be fixed by the involution of second kind on X. This gives a precise characterization of the Shimura surface $Z = X/\sigma$, the quotient of X by the involution on X induced by the involution of second kind:

PROPOSITION 32. *The surface Z is a smooth surface of general type with $p_g = 0$, $K_Z^2 = 4$ and $e(Z) = 8$.*

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