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A FUNCTIONAL INEQUALITY ON THE BOUNDARY OF STATIC MANIFOLDS*

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Abstract. On the boundary of a compact Riemannian manifold (Ω, g) whose metric g is static, we establish a functional inequality involving the static potential of (Ω, g), the second fundamental form and the mean curvature of the boundary $\partial\Omega$ respectively.

Key words. Static metrics, functional inequality.

AMS subject classifications. 53C24, 53C21.

1. Introduction and statement of results. The research in this paper is largely motivated by the following result concerning a functional inequality on the boundary of bounded domains in the Euclidean space $\mathbb{R}^n$, proved in [11, Corollary 3.1].

THEOREM 1 ([11]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Σ. Let H and $\mathbb{II}$ be the mean curvature and the second fundamental form of Σ with respect to the outward normal respectively. If $H > 0$, then*

$$\int_\Sigma \left[\frac{(\Delta_\Sigma \eta)^2}{H} - \mathbb{II}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) \right] d\sigma \geq 0 \tag{1.1}$$

for any smooth function η on Σ. Here ∇_Σ, Δ_Σ denote the gradient, the Laplacian on Σ respectively, and $d\sigma$ is the volume form on Σ. Moreover, equality in (1.1) *holds for some η if and only if $\eta = a_0 + \sum_{i=1}^n a_i x_i$ for some constants $a_0, a_1, \ldots, a_n$. Here $\{x_1, \ldots, x_n\}$ are the standard coordinate functions on $\mathbb{R}^n$.*

When $n = 3$ and Σ is convex, it is known ([11]) that the functional on the left side of (1.1) represents the second variation along η of the Wang-Yau quasi-local energy ([16, 17]) at the 2-surface Σ, lying in the time-symmetric slice $\mathbb{R}^3 = \{t = 0\}$, in the Minkowski spacetime $\mathbb{R}^{3,1}$. Thus, (1.1) can be relativistically interpreted as the stability inequality of the Wang-Yau energy at Σ. The general case of such a stability inequality is implied by results in [6, 17] for a closed, embedded, spacelike 2-surface in $\mathbb{R}^{3,1}$ that projects to a convex 2-surface along some timelike direction.

In this paper, adopting a Riemannian geometry point of view, we generalize Theorem 1 to hypersurfaces that are boundaries of bounded domains in a simply connected space form. More generally, we give an analogue of (1.1) on the boundary of compact Riemannian manifolds whose metrics are *static* (see Definition 1).

First, we fix some notations. Given a constant $\kappa > 0$, let $\mathbb{H}^n(\kappa)$ and $\mathbb{S}^n_+(\kappa)$ denote an n-dimensional hyperbolic space of constant sectional curvature $-\kappa$ and an n-dimensional open hemisphere of constant sectional curvature κ respectively.

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Theorem 2. *Suppose (M,g) is one of $\mathbb{R}^n$, $\mathbb{H}^n(\kappa)$ and $\mathbb{S}^n_+(\kappa)$. Let V be the positive function on M given by*

$$V=\begin{cases}1, & \text{if } (M,g)=\mathbb{R}^n,\\ \cosh\sqrt{\kappa}r, & \text{if } (M,g)=\mathbb{H}^n(\kappa),\\ \cos\sqrt{\kappa}r, & \text{if } (M,g)=\mathbb{S}^n_+(\kappa),\end{cases} \tag{1.2}$$

where r is the distance function from a fixed point p on (M,g). When $(M,g)=\mathbb{S}^n_+(\kappa)$, p is chosen to be the center of $\mathbb{S}^n_+(\kappa)$ so that $V>0$ on M. Given a bounded domain $\Omega\subset M$ with smooth boundary Σ, let H and $\mathbb{II}$ be the mean curvature and the second fundamental form of Σ respectively. If $H>0$, then for any smooth function η on Σ,

$$\begin{aligned}&\int_\Sigma V\left[\frac{\left[\Delta_\Sigma\eta+(n-1)k\eta\right]^2}{H}-\mathbb{II}(\nabla_\Sigma\eta,\nabla_\Sigma\eta)\right]d\sigma\\ \geq&\int_\Sigma\frac{\partial V}{\partial\nu}\left[|\nabla_\Sigma\eta|^2-(n-1)k\eta^2\right]d\sigma.\end{aligned} \tag{1.3}$$

Here $k=0$ or $\pm\kappa$ is the sectional curvature of (M,g). Moreover, equality in (1.3) *holds if and only if η is the restriction of a function*

$$u=\begin{cases}a_0+\sum_{i=1}^n a_ix_i, & \text{if } (M,g)=\mathbb{R}^n,\\ a_0t+\sum_{i=1}^n a_ix_i, & \text{if } (M,g)=\mathbb{H}^n(\kappa),\\ a_0x_0+\sum_{i=1}^n a_ix_i, & \text{if } (M,g)=\mathbb{S}^n_+(\kappa).\end{cases} \tag{1.4}$$

Here $a_0,\ldots,a_n$ are arbitrary constants, $\mathbb{H}^n(\kappa)$ is identified with

$$\left\{(t,x_1,\ldots,x_n)\in\mathbb{R}^{n,1}\ \Big|\ -t^2+\sum_{i=1}^n x_i^2=-\frac{1}{\kappa},\ t>0\right\}$$

in the $(n+1)$-dimensional Minkowski space $\mathbb{R}^{n,1}$ and $\mathbb{S}^n_+(\kappa)$ is identified with

$$\left\{(x_0,x_1,\ldots,x_n)\in\mathbb{R}^{n+1}\ \Big|\sum_{i=0}^n x_i^2=\frac{1}{\kappa},\ x_0>0\right\}$$

in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

The standard metrics on $\mathbb{R}^n$, $\mathbb{H}^n(\kappa)$, $\mathbb{S}^n_+(\kappa)$ are all examples of static metrics which admit a *positive* static potential. We recall the following definition from [9]:

Definition 1 ([9]). *A Riemannian metric g on a manifold M is called static if the linearized scalar curvature map at g has a nontrivial cokernel, i.e. if there exists a nontrivial function f on M such that*

$$-(\Delta f)g+\nabla^2 f-f\mathrm{Ric}=0. \tag{1.5}$$

Here ∇^2, Δ and Ric *denote the Hessian, the Laplacian and the Ricci curvature of g respectively.*

On a connected (M,g) of dimension n, the space of functions f satisfying (1.5) has dimension at most $n+1$ (cf. [9, Corollary 2.4]). When g is static on M, a nontrivial solution f to (1.5) is called a *static potential* of (M,g).

It is known that a static metric necessarily has constant scalar curvature (cf. [9, Proposition 2.3]). Indeed, direct calculation shows that (M, g) is static with a positive static potential f if and only if the Lorentz warped product $\bar{g} = -f^2 dt^2 + g$ satisfies $\mathrm{Ric}(\bar{g}) = \frac{R}{n-1}\bar{g}$ where R is the scalar curvature of g (cf. [9, Proposition 2.7]). This interpretation explains why static metrics have been widely studied in the field of mathematical relativity (see e.g. [3, 1, 9, 7, 2, 8, 10]).

Our next theorem generalizes Theorem 1 to the boundary of a compact Riemannian manifold whose metric is static.

THEOREM 3. *Suppose g is a static metric on an n-dimensional compact manifold Ω with boundary Σ and V is a positive static potential on (Ω, g). Let H, $\mathbb{II}$ be the mean curvature, the second fundamental form of Σ in (Ω, g) respectively. If $H > 0$, then*

$$\begin{aligned} &\int_\Sigma V \left[\frac{[\Delta_\Sigma \eta + (n-1)k\eta]^2}{H} - \mathbb{II}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) \right] d\sigma \\ \geq &\int_\Sigma \frac{\partial V}{\partial \nu} \left[|\nabla_\Sigma \eta|^2 - (n-1)k\eta^2 \right] d\sigma \end{aligned} \tag{1.6}$$

for any function η on Σ. Here $k \leq 0$ is a nonpositive constant satisfying $\mathrm{Ric} \geq (n-1)kg$. *Moreover, equality holds only if*

(i) $k = 0$ and η is the boundary value of a function u on (Ω, g) satisfying $\nabla^2 u = 0$.

or

(ii) $k < 0$, g is Einstein, i.e. $\mathrm{Ric} = (n-1)kg$, *and η is the boundary value of a function u on (Ω, g) satisfying $\nabla^2 u + kug = 0$.*

In Theorem 3, the fact that k is taken as a nonpositive lower bound of the Ricci curvature of g is restricted by the method of our proof (cf. Remark 2.2). Thus, if g has positive Ricci curvature, (1.6) is always a strict inequality. However, in this case, if in addition that g is Einstein, then k can be chosen to be positive and (1.6) is sharp (cf. Remark 2.3).

If the metric g is not static, we also give an inequality similar to that in Theorem 3 but under more stringent assumptions on the boundary and the interior curvature (see Theorem 4).

2. Proof of Theorems 2 and 3. Theorem 1 was derived in [11] as an application of Reilly's formula [13]. (A different generalization of Theorem 1 was given in [12], again by making use of Reilly's formula.) To prove Theorem 2 and 3, we make use of the following weighted Reilly's formula, recently derived by Qiu and Xia in [14, Theorem 1.1].

PROPOSITION 1 ([14]). *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Given two functions f, V on Ω and a constant K, one* *has*

$$
\begin{aligned}
&\int_\Omega V\left[(\Delta f+Knf)^2-\left|\nabla^2 f+Kfg\right|^2\right]dv\\
=&\int_\Omega\left[\nabla^2V-(\Delta V)g-2(n-1)KVg+V\mathrm{Ric}\right](\nabla f,\nabla f)dv\\
&+(n-1)K\int_\Omega(\Delta V+nKV)f^2dv+\int_\Sigma\frac{\partial V}{\partial\nu}\left[|\nabla_\Sigma f|^2-(n-1)Kf^2\right]d\sigma\\
&+\int_\Sigma V\left[2\left(\frac{\partial f}{\partial\nu}\right)\Delta_\Sigma f+H\left(\frac{\partial f}{\partial\nu}\right)^2+\mathrm{I\!I\!I}(\nabla_\Sigma f,\nabla_\Sigma f)+2(n-1)K\left(\frac{\partial f}{\partial\nu}\right)f\right]d\sigma.
\end{aligned}
\tag{2.1}
$$

For readers' convenience, we include a proof of (2.1) below.

Proof. Direct calculation gives

$$
\frac{1}{2}\Delta(V|\nabla f|^2)=\frac{1}{2}(\Delta V)|\nabla f|^2+\frac{1}{2}V\Delta|\nabla f|^2+\langle\nabla V,\nabla|\nabla f|^2\rangle. \tag{2.2}
$$

The integral of $\langle\nabla V,\nabla|\nabla f|^2\rangle$ can be written as

$$
\begin{aligned}
&\int_\Omega\langle\nabla V,\nabla|\nabla f|^2\rangle dv\\
=&\frac{3}{2}\int_\Omega\langle\nabla V,\nabla|\nabla f|^2\rangle-\int_\Omega\nabla^2 f(\nabla V,\nabla f)dv\\
=&-\frac{3}{2}\int_\Omega(\Delta V)|\nabla f|^2dv+\frac{3}{2}\int_\Sigma\frac{\partial V}{\partial\nu}|\nabla f|^2d\sigma-\int_\Sigma\langle\nabla V,\nabla f\rangle\frac{\partial f}{\partial\nu}d\sigma\\
&+\int_\Omega\nabla^2V(\nabla f,\nabla f)dv+\int_\Omega\langle\nabla V,\nabla f\rangle\Delta f dv.
\end{aligned}
\tag{2.3}
$$

It follows from (2.2), (2.3) and the Bochner formula that

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma\frac{\partial}{\partial\nu}\left(V|\nabla f|^2\right)d\sigma-\int_\Omega V\left[|\nabla^2 f|^2+\mathrm{Ric}(\nabla f,\nabla f)+\langle\nabla\Delta f,\nabla f\rangle\right]dv\\
=&-\int_\Omega(\Delta V)|\nabla f|^2+\frac{3}{2}\int_\Sigma\frac{\partial V}{\partial\nu}|\nabla f|^2-\int_\Sigma\langle\nabla V,\nabla f\rangle\frac{\partial f}{\partial\nu}d\sigma\\
&+\int_\Omega\nabla^2V(\nabla f,\nabla f)dv+\int_\Omega\langle\nabla V,\nabla f\rangle\Delta f dv.
\end{aligned}
$$

Using the fact

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla f|^2=&\langle\nabla_\Sigma f,\nabla_\Sigma\left(\frac{\partial f}{\partial\nu}\right)\rangle-\mathrm{I\!I\!I}(\nabla_\Sigma f,\nabla_\Sigma f)\\
&+\frac{\partial f}{\partial\nu}\left(\Delta f-\Delta_\Sigma f-H\frac{\partial f}{\partial\nu}\right)
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
\int_\Omega V\langle\nabla\Delta f,\nabla f\rangle dv=&-\int_\Omega V(\Delta f)^2dv-\int_\Omega\langle\nabla V,\nabla f\rangle\Delta f dv\\
&+\int_\Sigma V(\Delta f)\frac{\partial f}{\partial\nu}d\sigma,
\end{aligned}
\tag{2.5}
$$

we have

$$\begin{aligned}&\int_\Sigma V\left[-\mathrm{I\!I\!I}(\nabla_\Sigma f,\nabla_\Sigma f)+\frac{\partial f}{\partial\nu}\left(-2\Delta_\Sigma f-H\frac{\partial f}{\partial\nu}\right)\right]d\sigma-\int_\Sigma\frac{\partial V}{\partial\nu}|\nabla_\Sigma f|^2d\sigma\\&=\int_\Omega V\left[|\nabla^2f|^2-(\Delta f)^2\right]+\left[V\mathrm{Ric}-(\Delta V)g+\nabla^2V\right](\nabla f,\nabla f)dv,\end{aligned}\tag{2.6}$$

where we also made the use of

$$\int_\Sigma V\langle\nabla_\Sigma f,\nabla_\Sigma\left(\frac{\partial f}{\partial\nu}\right)\rangle+\langle\nabla_\Sigma V,\nabla_\Sigma f\rangle\frac{\partial f}{\partial\nu}d\sigma=-\int_\Sigma V(\Delta_\Sigma f)\frac{\partial f}{\partial\nu}d\sigma$$

and $|\nabla f|^2=\left(\frac{\partial f}{\partial\nu}\right)^2+|\nabla_\Sigma f|^2$ along Σ. Now (2.1) follows from (2.6) and the fact

$$\begin{aligned}&\int_\Omega V\left[|\nabla^2f|^2-(\Delta f)^2\right]dv\\&=\int_\Omega V\left[|\nabla^2f+Kfg|^2-(\Delta f+nKf)^2\right]dv+(n-1)K\int_\Omega nKVf^2dv\\&\quad+(n-1)K\left[\int_\Sigma\left(2Vf\frac{\partial f}{\partial\nu}-f^2\frac{\partial V}{\partial\nu}\right)d\sigma+\left(\int_\Omega(\Delta V)f^2-2V|\nabla f|^2\right)dv\right].\end{aligned}$$

This completes the proof. □

Remark 2.1. Formula (2.1) reduces to Reilly's formula ([13, equation(14)]) when $V=1$ and $K=0$.

Motivated by equation (1.5) in Definition 1 of static metrics, we can rewrite formula (2.1) as

$$\begin{aligned}&\int_\Omega V\left[(\Delta f+Knf)^2-\left|\nabla^2f+Kfg\right|^2\right]dv\\&=\int_\Omega\left[\nabla^2V-(\Delta V)g-V\mathrm{Ric}\right](\nabla f,\nabla f)dv+2\int_\Omega V\left[\mathrm{Ric}-(n-1)Kg\right](\nabla f,\nabla f)dv\\&\quad+(n-1)K\int_\Omega(\Delta V+nKV)f^2dv+\int_\Sigma\frac{\partial V}{\partial\nu}\left[|\nabla_\Sigma f|^2-(n-1)Kf^2\right]d\sigma\\&\quad+\int_\Sigma V\left[2\left(\frac{\partial f}{\partial\nu}\right)\Delta_\Sigma f+H\left(\frac{\partial f}{\partial\nu}\right)^2+\mathrm{I\!I\!I}(\nabla_\Sigma f,\nabla_\Sigma f)+2(n-1)K\left(\frac{\partial f}{\partial\nu}\right)f\right]d\sigma.\end{aligned}\tag{2.7}$$

It is the second line in (2.7) that prompts one to apply Proposition 1 to domains in a static manifold.

Proof of Theorem 3. As $k\le0$, given any nontrivial η on Σ, there exists a unique solution u to

$$\begin{cases}\Delta u+nku&=\ 0\quad\text{on }\Omega\\ \qquad\quad u&=\ \eta\quad\text{at }\Sigma.\end{cases}\tag{2.8}$$

On the other hand, taking trace of (1.5) gives

$$\Delta V+\frac{R}{n-1}V=0,\tag{2.9}$$

where R is the scalar curvature of g (which is a constant). Plug this V, together with $f = u$ and $K = k$ in (2.1), using (1.5), (2.7) and (2.9), we have

$$
\begin{aligned}
&-\int_\Omega V \left|\nabla^2 u + kug\right|^2 dv \\
&= 2\int_\Omega V\left[\mathrm{Ric} - (n-1)kg\right](\nabla u, \nabla u) dv + k\left[n(n-1)k - R\right]\int_\Omega V u^2 dv \\
&\quad + \int_\Sigma \frac{\partial V}{\partial \nu}\left[|\nabla_\Sigma \eta|^2 - (n-1)k\eta^2\right] d\sigma \\
&\quad + \int_\Sigma V\left[2\left(\frac{\partial u}{\partial \nu}\right)\Delta_\Sigma \eta + H\left(\frac{\partial u}{\partial \nu}\right)^2 + \mathrm{I\!I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) + 2(n-1)k\left(\frac{\partial u}{\partial \nu}\right)\eta\right] d\sigma.
\end{aligned}
\tag{2.10}
$$

Since $V > 0$, $\mathrm{Ric} \geq (n-1)kg$, $R \geq n(n-1)k$ and $k \leq 0$, (2.10) implies

$$
\begin{aligned}
&\int_\Sigma V\left\{\frac{\left[\Delta_\Sigma \eta + (n-1)k\eta\right]^2}{H} - \mathrm{I\!I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta)\right\} d\sigma \\
&\geq \int_\Omega V\left|\nabla^2 u + kug\right|^2 dv + \int_\Sigma \frac{\partial V}{\partial \nu}\left[|\nabla_\Sigma \eta|^2 - (n-1)K\eta^2\right] d\sigma \\
&\quad + \int_\Sigma V\left[\sqrt{H}\left(\frac{\partial u}{\partial \nu}\right) + \frac{\Delta_\Sigma \eta + (n-1)k\eta}{\sqrt{H}}\right]^2 d\sigma.
\end{aligned}
\tag{2.11}
$$

It follows from (2.11) that

$$
\begin{aligned}
&\int_\Sigma V\left\{\frac{\left[\Delta_\Sigma \eta + (n-1)k\eta\right]^2}{H} - \mathrm{I\!I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta)\right\} d\sigma \\
&\geq \int_\Sigma \frac{\partial V}{\partial \nu}\left[|\nabla_\Sigma \eta|^2 - (n-1)K\eta^2\right] d\sigma.
\end{aligned}
\tag{2.12}
$$

Moreover, by (2.10), equality in (2.12) holds only if

$$k\left[n(n-1)k - R\right] = 0, \tag{2.13}$$

$$\nabla^2 u + kug = 0, \tag{2.14}$$

$$H\left(\frac{\partial u}{\partial \nu}\right) + \Delta_\Sigma \eta + (n-1)k\eta = 0. \tag{2.15}$$

Condition (2.13) implies either $k = 0$ or $R = n(n-1)k$. In the later case, it follows from $\mathrm{Ric} \geq (n-1)kg$ that $\mathrm{Ric} = (n-1)kg$, i.e. g is Einstein. We also note that (2.15) in fact follows from (2.14). This is because, if (2.14) holds, then at Σ,

$$\Delta u = \Delta_\Sigma u + H\frac{\partial u}{\partial \nu} + \nabla^2 u(\nu, \nu) = \Delta_\Sigma u + H\frac{\partial u}{\partial \nu} - ku \tag{2.16}$$

which implies (2.15) since $\Delta u = -nku$. This proves Theorem 3. □

Remark 2.2. In the above proof, the assumption $k \leq 0$ is essentially used in only one place, i.e. to ensure

$$k[n(n-1)k - R] \geq 0. \tag{2.17}$$

The other use of $k \le 0$ in the construction of u is not essential because, by another theorem of Reilly ([13, Theorem 4]), one can still solve (2.8) in the case of $k > 0$, provided (Ω, g) is not isometric to $\mathbb{S}^n_+(k)$.

Remark 2.3. If $\mathrm{Ric} = (n-1)kg$, then

$$k[n(n-1)k - R] = 0$$

regardless of the sign of k. Therefore, the above proof also shows that inequality (1.6) still holds if the assumption "$\mathrm{Ric} \ge (n-1)kg$ and $k \le 0$" is replaced by that g is Einstein. In this case, equality holds if and only if η is the boundary value of some function u that satisfies $\nabla^2 u + kug = 0$ on (Ω, g).

Theorem 2 now follows from Theorem 3 and Remark 2.3.

Proof of Theorem 2. Each positive function V in (1.2) is a solution to (1.5) when $(M, g) = \mathbb{R}^n$, $\mathbb{H}^n(\kappa)$ or $\mathbb{S}^n_+(\kappa)$. Hence, inequality (1.3) follows from (1.6) in Theorem 3 and Remark 2.3.

Suppose the equality in (1.3) holds from a nontrivial η. By Theorem 3 and Remark 2.3, η is the boundary value of a function u on (Ω, g) satisfying

$$\nabla^2 u + kug = 0. \tag{2.18}$$

Since the standard metric g on $\mathbb{R}^n$, $\mathbb{H}^n(\kappa)$ and $\mathbb{S}^n_+(\kappa)$ is also Einstein, the static equation (1.5) is equivalent to

$$\nabla^2 f + kfg = 0. \tag{2.19}$$

Therefore, u is the restriction of a static potential of (M, g) to (Ω, g). Theorem 2 now follows from the fact that the space of solutions to (1.5) on (M, g) is spanned by

$$\begin{aligned} \{1, x_1, \ldots, x_n\}, &\ \text{ when } (M, g) = \mathbb{R}^n \\ \{t, x_1, \ldots, x_n\}, &\ \text{ when } (M, g) = \mathbb{H}^n(\kappa) \\ \{x_0, x_1, \ldots, x_n\}, &\ \text{ when } (M, g) = \mathbb{S}^n_+(\kappa). \end{aligned}$$

□

Remark 2.4. By [4] (p. 192-194) (cf. [15] Theorem 2 for a related result), it is known that if (Ω, g) possesses a function u with $\nabla^2 u = -kug$, then g is locally a warped product metric in the sense that there exists a Riemannian manifold (N^{n-1}, g_N) such that g can be locally expressed as $dr^2 + s(r)^2 g_N$ where $s(r)$ is a function on an interval I. In fact, their argument (which is local) shows that u can be expressed as a function of r and $u(r)$ satisfies the linear ODE $u'' = -ku$, and that $s(r) = u'(r)$. Also, $s = u'$ and g_N are unique up to multiplicative constants. Once these have been fixed, u is determined by an additive constant. For example, when $k = 0$, g is locally a product metric $dr^2 + g_N$.

3. A similar inequality. When the metric is not static, there is an inequality similar to that in Theorem 3 but under more stringent conditions on the boundary and the interior curvature.

For a compact Riemannian manifold Ω with boundary Σ, we say it is star-shaped with respect to an interior point $p \in \Omega$ if every point in Ω can be joined by a minimal geodesic starting from p.

THEOREM 4. *Let (Ω, g) be an n-dimensional compact Riemannian manifold with boundary Σ. Suppose Σ has positive mean curvature and is star-shaped with respect to an interior point $p \in \Omega$. Let $\kappa > 0$ be a constant such that $-\kappa$ is a lower bound of the sectional curvature of g. Let $r = d(p, \cdot)$ and $V = \cosh \sqrt{\kappa} r$. Here $d(\cdot, \cdot)$ denotes the distance function on (Ω, g). Then for any function η on Σ,*

$$\begin{aligned} &\int_\Sigma V \left[\frac{[\Delta_\Sigma \eta - (n-1)\kappa\eta]^2}{H} - \mathbb{II}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) \right] d\sigma \\ \geq &\int_\Sigma \frac{\partial V}{\partial \nu} \left[|\nabla_\Sigma \eta|^2 + (n-1)\kappa \eta^2 \right] d\sigma. \end{aligned} \tag{3.1}$$

Moreover, the equality holds only if Ω has constant curvature $-\kappa$. Here H, $\mathbb{II}$ are the mean curvature and the second fundamental form of Σ respectively.

Proof. By Hessian comparison, we have

$$\nabla^2 r \leq \sqrt{\kappa} \coth(\sqrt{\kappa} r)(g - dr^2).$$

This implies

$$\nabla^2 V = \sqrt{\kappa} \sinh(\sqrt{\kappa} r) \nabla^2 r + \kappa \cosh(\sqrt{\kappa} r) dr^2 \leq \kappa \cosh(\sqrt{\kappa} r) g = \kappa V g.$$

By diagonalizing $\nabla^2 V$, we see that

$$\Delta V g - \nabla^2 V \leq (n-1)\kappa V g$$

and $\Delta V \leq n\kappa V$. This implies that, for any function u on Ω,

$$\int_\Omega \left(V \mathrm{Ric} + 2(n-1)\kappa V g + \nabla^2 V - \Delta V g \right) (\nabla u, \nabla u) dv \geq 0.$$

The proof then proceeds as in Theorem 3.

If the equality case holds, then as in the argument of Theorem 3, we have $R = -n(n-1)\kappa$, which implies Ω has constant curvature $-\kappa$ as we assume its curvature $\geq -\kappa$. $\square$

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