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002

QUANTISING PROPER ACTIONS ON SPIN^c-MANIFOLDS*

PETER HOCHS† AND VARGHESE MATHAI‡

Abstract. Paradan and Vergne generalised the quantisation commutes with reduction principle of Guillemin and Sternberg from symplectic to Spin^c-manifolds. We extend their result to noncompact groups and manifolds. This leads to a result for cocompact actions, and a result for non-cocompact actions for reduction at zero. The result for cocompact actions is stated in terms of K-theory of group C^*-algebras, and the result for non-cocompact actions is an equality of numerical indices. In the non-cocompact case, the result generalises to Spin^c-Dirac operators twisted by vector bundles. This yields an index formula for Braverman's analytic index of such operators, in terms of characteristic classes on reduced spaces.

Key words. Spinc manifolds, geometric quantisation, quantisation commutes with reduction, proper Lie group actions, noncompact index theorem.

AMS subject classifications. 53C27 (Primary), 53D50, 53D20, 58J20, 81S10.

1. Introduction. Recently, Paradan and Vergne [30, 31] generalised the *quantisation commutes with reduction* principle [13, 25, 26, 28, 36, 39] from the symplectic setting to the Spin^c-setting. In this paper, we extend their result to noncompact groups and manifolds. Whereas Paradan and Vergne use topological methods, we generalise Tian and Zhang's analytic approach [15, 36] to possibly non-cocompact actions on Spin^c-manifolds. This approach generalises to Spin^c-Dirac operators twisted by vector bundles, and implies an index formula for Braverman's analytic index [7] of such operators. For cocompact actions, we generalise and apply the KK-theoretic *quantisation commutes with induction* methods of [17, 18]. Applications of our results include a proof of a Spin^c-version of Landsman's conjecture [21], and various topological properties of the index of twisted Spin^c-Dirac operators for possibly non-cocompact actions.

The compact case. Cannas da Silva, Karshon and Tolman noted in [8] that Spin^c-quantisation is the most general, and possibly natural, notion of geometric quantisation. This version of quantisation has a much greater scope for applications than geometric quantisation in the symplectic setting. It was shown in Theorem 3 of [8] that Spin^c-quantisation commutes with reduction for circle actions on compact Spin^c-manifolds, under a certain assumption on the fixed points of the action. Paradan and Vergne's result generalises this to actions by arbitrary compact, connected Lie groups, without the additional assumption made in [8].

Paradan and Vergne considered a compact, connected Lie group K acting on a compact, connected, even-dimensional manifold M, equipped with a K-equivariant Spin^c-structure. For a Spin^c-Dirac operator D on M, they defined the Spin^c-quantisation of the action as

$$Q_K^{\mathrm{Spin}^c}(M) := K\text{-index}(D),$$

*Received July 2, 2015; accepted for publication December 29, 2015.
†School of Mathematical Sciences, University of Adelaide, South Australia 5005, Australia (peter.hochs@adelaide.edu.au).
‡School of Mathematical Sciences, University of Adelaide, South Australia 5005, Australia (mathai.varghese@adelaide.edu.au).

which lies in the representation ring of K, and computed the multiplicities m_π in

$$Q_K^{\mathrm{Spin}^c}(M) = \bigoplus_{\pi\in\hat{K}} m_\pi \pi.$$

These multiplicities are expressed in terms of indices of Spin^c-Dirac operators on reduced spaces

$$M_\xi := \mu^{-1}\bigl(\mathrm{Ad}^*(K)\xi\bigr)/K,$$

where $\xi \in \mathfrak{k}^*$, and the Spin^c-*momentum map* $\mu\colon M \to \mathfrak{k}^*$ is a generalisation of the momentum map in symplectic geometry.

The cocompact case. We first generalise this result to proper cocompact actions by a Lie group G on a manifold M, i.e. actions for which M/G is compact. This is achieved by applying the quantisation commutes with induction machinery of [17, 18] to it, together with a Spin^c-slice theorem. In the cocompact case, one can define Spin^c-quantisation using the analytic assembly map, denoted by G-index, from the Baum–Connes conjecture [2]:

$$Q_G^{\mathrm{Spin}^c}(M) := G\text{-index}(D) \quad \in K_*(C^*G),$$

where $K_*(C^*G)$ is the K-theory of the maximal group C^*-algebra of G. This notion of quantisation was introduced by Landsman [21] in the symplectic setting. He conjectured that quantisation commutes with reduction at zero in that case.

To obtain a statement for reduction at nonzero values of the momentum map, we apply the natural map

$$r_*\colon K_*(C^*G) \to K_*(C^*_rG),$$

where C^*_rG is the reduced C^*-algebra of G. The group $K_*(C^*_rG)$ has natural generators $[\lambda]$, which have representation theoretic meaning in many cases. The first main result in this paper, Theorem 4.7, yields an expression for the multiplicities m_λ in

$$r_*\bigl(Q_G^{\mathrm{Spin}^c}(M)\bigr) = \sum_\lambda m_\lambda[\lambda]. \tag{1.1}$$

The non-cocompact case. In the symplectic setting, the invariant part of geometric quantisation was defined in [15] for possibly non-cocompact actions. Braverman [7] then combined techniques from [6] and [15] to extend this definition to general Dirac operators, and proved important properties of the resulting index. We generalise the main result from [15] from symplectic to Spin^c-manifolds. In addition, we obtain a generalisation to Spin^c-Dirac operators twisted by arbitrary vector bundles $E \to M$. This allows us to express Braverman's index of such operators in terms of topological data on M_0.

To be more precise, let D_p^E be the Spin^c-Dirac operator on M, twisted by E via a connection on E, for the Spin^c-structure whose determinant line bundle is the p'th tensor power of the determinant line bundle of a fixed Spin^c-structure. Then in Theorem 6.12, we obtain the index formula

$$\mathrm{index}^G D_p^E = \int_{M_0} \mathrm{ch}(E_0) e^{\frac{p}{2}c_1(L_0)} \hat{A}(M_0), \tag{1.2}$$

for $p \in \mathbb{N}$ large enough, where index^G denotes Braverman's index [7]. Here $E_0 := (E|_{(\mu^\nabla)^{-1}(0)})/G$, and $L_0 := (L|_{(\mu^\nabla)^{-1}(0)})/G$. This equality holds if M_0 is smooth, and a generalisation of the Kirwan vector field has a cocompact set of zeros. This implies that M_0 is compact. If M and G are both compact, analogous results were obtained in [32, 37].

If M/G is noncompact, it is not clear a priori how to define a topological couterpart to Braverman's index. Gromov and Lawson [11] face a similar problem in their study of Dirac operators on noncompact manifolds. They define a *relative* topological index, representing the *difference* of the indices of two operators satisfying their criteria, although these indices are not defined for each operator separately. They prove that the relative topological index equals the difference of the analytical indices of the operators in question (Theorem 4.18 in [11]). Localisation to $(\mu^\nabla)^{-1}(0)$ allows us to give the topological expression (1.2) for the index of a single twisted Spin^c-Dirac operator, i.e. an 'absolute' rather than a relative index formula.

Applications and examples. If M/G is compact, Theorem 6.8 implies that the main result of [24], which to a large extent solves Landsman's conjecture mentioned above, generalises to the Spin^c-setting. We give a way to construct examples where our results apply, from cases where the group acting is compact. The main result (1.1) in the cocompact case has a purely geometric consequence, not involving K-theory and C^*-algebras. A special case of this consequence is an expression for the formal degree of a discrete series representation in terms of an $\hat{A}$-type genus of the corresponding coadjoint orbit. Finally, the index formula (1.2) allows us to draw conclusions about topological properties of the index of twisted Spin^c-Dirac operators. These include an excision property, and a twisted version of Hirzebruch's signature theorem in the noncompact case.

Outline of this paper. In Section 2, we first briefly recall the definition of Spin^c-Dirac operators. Then we state the definition of Spin^c-reduction as in [30, 31], and define stabilisation and destabilisation of Spin^c-structures in terms of the two out of three lemma. We give conditions for reduced spaces to have naturally defined Spin^c-structures in Section 3. We also discuss a Spin^c-slice theorem, and its relation to Spin^c-reduction.

Section 4 contains the statements of Paradan and Vergne's result from [30, 31], and our main result on cocompact actions, Theorem 4.6. This result is proved in Section 5.

The main result for untwisted Spin^c-Dirac operators for possibly non-cocompact actions, Theorem 6.8, is stated in Section 6. It is proved in Sections 7 and 8. The index formula for Spin^c-Dirac operators twisted by vector bundles is also stated in Section 6, and is proved in Section 9.

Finally, in Section 10, we mention some applications of the main results, and a way to construct examples where they apply.

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Notation and conventions. We will denote the dimension of a manifold Y by d_Y. If a group H acts on Y, we denote the quotient map $Y \to Y/H$ by q, or by q_H to emphasise which group is acting. For a finite-dimensional representation space V of H, we write V_Y for the trivial vector bundle $Y \times V \to M$, with the diagonal H-action. (So that, for proper, free actions, $V_Y/H \to Y/H$ is the vector bundle associated to the principal fibre bundle $Y \to Y/H$.)

Part I. Preliminaries.

2. Dirac operators and reduced spaces. In this section, we review the constructions of Spin^c-Dirac operators and Spin^c-*momentum maps*. These Spin^c-momentum maps were introduced by Paradan and Vergne [30, 31], and can be used to define reduced spaces, which play a central role in the results in this paper. We mention the two out of three lemma, which we will use to construct Spin^c-structures on these reduced spaces in Section 3.

2.1. Dirac operators. Let G be a Lie group, acting properly on a manifold M. Let $E \to M$ be a real, G-equivariant vector bundle of rank r. Then a G-*equivariant* Spin^c-*structure* on E is a right principal Spin$^c(r)$-bundle $P_E \to M$ with a left action by G, together with a $G\times \mathrm{Spin}^c(r)$-equivariant buncle map $P_E \to \mathrm{GLF}(E)$. Here $\mathrm{GLF}(E)$ is the frame bundle of E, on which Spin$^c(r)$ acts via the natural map $\pi\colon \mathrm{Spin}^c(r) \to \mathrm{SO}(r)$. Then we have a G-equivariant vector bundle isomorphism

$$P_E \times_\pi \mathbb{R}^r \cong E.$$

The standard orientation and Euclidean metric on $\mathbb{R}^r$ induce an orientation and metric on E in this way. We will usually refer to P_E as a Spin^c-structure on E, without making explicit mention of the map $P_E \to \mathrm{GLF}(E)$. A Spin^c-structure on M is by definition a Spin^c-structure on TM. Many, but not all, oriented manifolds have Spin^c-structures, see e.g. Appendix D of [22].

If Δ_r is the standard Spin representation of Spin$^c(r)$, then the *spinor bundle* of the Spin^c-structure P_E is the G-equivariant, Hermitian vector bundle $P_E \times_{\mathrm{Spin}^c(r)} \Delta_r \to M$. Let $\det\colon \mathrm{Spin}^c(r) \to U(1)$ be the map given by $[g, z] \mapsto z^2$, for $g \in \mathrm{Spin}(r)$ and $z \in U(1)$. Then the *determinant line bundle* of the Spin^c-structure is the G-equivariant, Hermitian line bundle $P_E \times_{\det} \mathbb{C} \to M$.

Suppose M is equipped with a G-equivariant Spin^c-structure. Let $L \to M$ be the associated determinant line bundle, and let a G-invariant, Hermitian connection ∇ on L be given. Let $\mathcal{S} \to M$ be the spinor bundle associated to the Spin^c-structure on M. The connection ∇ and the Levi–Civita connection on TM (associated to the Riemannian metric induced by the Spin^c-structure), together induce a connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$, as discussed for example in Proposition D.11 in [22]. The construction of the connection $\nabla^{\mathcal{S}}$ involves local decompositions

$$\mathcal{S}|_U \cong \mathcal{S}_U^{\mathrm{Spin}} \otimes L|_U^{1/2}$$

on open sets $U \subset M$, where $\mathcal{S}_U^{\mathrm{Spin}}$ is the spinor bundle associated to a local Spin-structure, to which the Levi–Civita connection lifts.

Let

$$c\colon TM \to \mathrm{End}(\mathcal{S})$$

be the Clifford action. Identifying $T^*M \cong TM$ via the Riemannian metric, one gets an action

$$c\colon T^*M \otimes \mathcal{S} \to \mathcal{S}.$$

The Spin^c-*Dirac operator* associated to the Spin^c-structure on M and the connection ∇ on L is then defined as the composition

$$D\colon \Gamma^\infty(\mathcal{S}) \xrightarrow{\nabla^{\mathcal{S}}} \Omega^1(M;\mathcal{S}) \xrightarrow{c} \Gamma^\infty(\mathcal{S}).$$

Write $d_M := \dim(M)$. If $\{e_1, \ldots, e_{d_M}\}$ is a local orthonormal frame for TM, then, locally,

$$D = \sum_{j=1}^{d_M} c(e_j)\nabla^{\mathcal{S}}_{e_j}.$$

For certain arguments, we will also need the operator D_p on the vector bundle $\mathcal{S}_p := \mathcal{S} \otimes L^p$, defined in the same way by a connection on $\mathcal{S}_p$ which is induced by the Levi–Civita connection and ∇, via local decompositions

$$\mathcal{S}_p|_U \cong \mathcal{S}_U^{\mathrm{Spin}} \otimes L|_U^{p+1/2}. \tag{2.1}$$

Note that $\mathcal{S}_p$ is the spinor bundle of the Spin^c-structure on M obtained by twisting the original Spin^c-structure by the line bundle L^p (see e.g. (D.15) in [22]).

2.2. Momentum maps and reduction. A Spin^c-momentum map is a generalisation of the momentum map in symplectic geometry. It was used by Paradan and Vergne in [30, 31]. (See also Definition 7.5 in [3].)

For $X \in \mathfrak{g}$, let X^M be the induced vector field on M, and let $\mathcal{L}_X^E$ be the Lie derivative of sections of any G-vector bundle $E \to M$.

Definition 2.1. The Spin^c-*momentum map* associated to the connection ∇ is the map

$$\mu^\nabla\colon M \to \mathfrak{g}^*$$

defined by[1]

$$2\pi i \mu_X^\nabla = \nabla_{X^M} - \mathcal{L}_X^L \quad \in \mathrm{End}(L) = C^\infty(M), \tag{2.2}$$

for any $X \in \mathfrak{g}$. Here μ_X^∇ denotes the pairing of μ^∇ with X.

The notion of a Spin^c-momentum map is a special case of the notion of an *abstract moment map*, as for example in Definition 3.1 of [12]. This is an equivariant map

$$\Phi\colon M \to \mathfrak{g}^*$$

such that for all $X \in \mathfrak{g}$, the pairing Φ_X of Φ with X is locally constant on the set $\mathrm{Crit}(X^M)$ of zeros of the vector field X^M. A Spin^c-momentum map is an abstract moment map in this sense. This was already noted in the introduction to [28], and follows from the following well-known fact, whose proof is a straightforward verification.

[1] In [30, 31], a factor $-i/2$ is used instead of $2\pi i$. Our convention is consistent with [15, 36].

LEMMA 2.2. *For any G-equivariant line bundle $L \to M$ and a G-invariant connection ∇ on L, and any $X \in \mathfrak{g}$, one has*

$$2\pi i d\mu_X^\nabla = R^\nabla(-, X^M),$$

with R^∇ the curvature of ∇.

Analogously to symplectic reduction [23], one can define reduced spaces in the Spin^c-setting.

DEFINITION 2.3. For any $\xi \in \mathfrak{g}^*$, the space

$$M_\xi := (\mu^\nabla)^{-1}(\xi)/G_\xi = (\mu^\nabla)^{-1}\big(\mathrm{Ad}^*(G)\xi\big)/G$$

is the *reduced space* at ξ.

As in the symplectic case, the stabiliser G_ξ acts infinitesimally freely on $\mu^{-1}(\xi)$, if ξ is a regular value of μ^∇. Since $M_\xi \cong (\mu^\nabla)^{-1}(\xi)/G_\xi$, this implies that the reduced space M_ξ is an orbifold if ξ is a regular value of μ^∇, and the action is proper.

LEMMA 2.4. *In the setting of Lemma 2.2, let $\xi \in \mathfrak{g}^*$ be a regular value of μ^∇. Then for all $m \in \mu^{-1}(\xi)$, the infinitesimal stabiliser $\mathfrak{g}_m$ is zero.*

Proof. In the situation of the lemma, let $X \in \mathfrak{g}_m$. Then for all $v \in T_mM$, we saw in Lemma 2.2 that

$$\langle T_m\mu^\nabla(v), X\rangle = v(\mu_X^\nabla)(m) = \frac{1}{2\pi i}R_m^\nabla(v, X_m^M) = 0,$$

since $X_m^M = 0$. Because $T_m\mu^\nabla$ is surjective, it follows that $X = 0$. □

(See Lemma 5.4 in [12] for a version of this lemma where G is a torus and μ^∇ is replaced by any abstract momentum map.)

2.3. Stabilising and destabilising Spin^c-structures. To study Spin^c-structures on reduced spaces, we will use the notions of *stabilisation* and *destabilisation* of Spin^c-structures. These will also be used to obtain a Spin^c-slice theorem in Subsection 3.2.

Stabilisation and destabilisation are based on the *two out of three lemma*.

LEMMA 2.5. *Let $E, F \to M$ be oriented vector bundles with metrics, over a manifold M. Then* Spin^c*-structures on two of the three vector bundles E, F and $E \oplus F$ determine a unique* Spin^c*-structure on the third. The determinant line bundles L_E, L_F and $L_{E\oplus F}$ of the respective* Spin^c*-structures are related by*

$$L_{E\oplus F} = L_E \otimes L_F.$$

Proof. See e.g. Section 3.1 of [35]. The uniqueness part of the statement refers to the constructions given there. □

REMARK 2.6. Suppose a group G acts on the vector bundles E and F in Lemma 2.5, and the two Spin^c-structures initially given in the lemma are G-equivariant. Then the Spin^c-structure on the third bundle, as constructed in Section 3.1 of [35], is also G-equivariant. Here one uses the fact that the actions by G on the spinor bundles associated to the Spin^c-structures on E, F and $E \oplus F$ are compatible, since they are induced by the actions by G on E and F.

Definition 2.7. In the setting of Lemma 2.5, suppose E and F have Spin^c-structures. Let P_E be the Spin^c-structure on E. Then the resulting Spin^c-structure on $E \oplus F$ is the *stabilisation*

$$\mathrm{Stab}_F(P_E) \to M.$$

If F and $E \oplus F$ have Spin^c-structures, and $P_{E\oplus F}$ is the Spin^c-structure on $E \oplus F$, then the resulting Spin^c-structure on E is the *destabilisation*

$$\mathrm{Destab}_F(P_{E\oplus F}) \to M.$$

The terms stabilisation and destabilisation are motivated by the case where F is a trivial vector bundle. See also Section 3.2 in [35], Lemma 2.4 in [8] and Section D.3.2 in [12].

We will use the following properties of stabilisation and destabilisation of Spin^c-structures.

Lemma 2.8. *Let $E, F \to M$ be vector bundles with* Spin^c*-structures over a manifold M. Then*

$$\mathrm{Stab}_E \circ \mathrm{Destab}_E = \mathrm{id}; \tag{2.3}$$
$$\mathrm{Destab}_E \circ \mathrm{Stab}_E = \mathrm{id}; \tag{2.4}$$
$$\mathrm{Stab}_E \circ \mathrm{Stab}_F = \mathrm{Stab}_{E\oplus F}; \tag{2.5}$$
$$\mathrm{Destab}_E \circ \mathrm{Destab}_F = \mathrm{Destab}_{E\oplus F}\,. \tag{2.6}$$

(Here id *means leaving* Spin^c*-structures on the relevant bundles unchanged.)*

Proof. The relations (2.3) and (2.4) follow from the uniqueness part of Lemma 2.5. The explicit constructions in Section 3.1 of [35] imply that (2.5) and (2.6) hold. □

3. Spin^c**-structures on reduced spaces.** Consider the setting of Subsection 2.2. One can define quantisation of smooth or orbifold reduced spaces using Spin^c-structures induced by the Spin^c-structure on M. If G is a torus, these are described in Proposition D.60 of [12]. In general, we will see that the Spin^c-structure on M induces one on reduced spaces at Spin^c*-regular values* of the Spin^c-momentum map μ^∇. In Proposition 3.5, we give a relation between Spin^c-regular values and usual regular values. We then discuss how Abels' slice theorem for proper actions can be used in the Spin^c-context, and how it is related to Spin^c-reduction. The proofs of the main statements in this section will be given Section 5.

3.1. Spin^c**-regular values.** For $\xi \in \mathfrak{g}^*$, we will denote the quotient map $(\mu^\nabla)^{-1}(\xi) \to M_\xi$ by q.

Definition 3.1. A value $\xi \in \mu^\nabla(M)$ of μ^∇ is a Spin^c*-regular value* if
- $(\mu^\nabla)^{-1}(\xi)$ is smooth;
- G_ξ acts locally freely on $(\mu^\nabla)^{-1}(\xi)$; and
- there is a G_ξ-invariant splitting

$$TM|_{(\mu^\nabla)^{-1}(\xi)} = q^*TM_\xi \oplus \mathcal{N}^\xi,$$

for a vector bundle $\mathcal{N}^\xi \to (\mu^\nabla)^{-1}(\xi)$ with a G_ξ-equivariant Spin-structure.

Remark 3.2. The third point in Definition 3.1 appears to have a choice of the bundle $\mathcal{N}^\xi$ in it, but these are all isomorphic; the condition is really that the quotient bundle

$$TM|_{(\mu^\nabla)^{-1}(\xi)}/q^*TM_\xi$$

has a G_ξ-equivariant Spin-structure.

Note that a Spin-structure is equivalent to a Spin^c-structure with a trivial determinant line bundle. In the equivariant setting, an equivariant Spin-structure is equivalent to a Spin^c-structure with an equivariantly trivial determinant line bundle. Indeed, if the determinant line bundle of a Spin^c-structure is equivariantly trivial, then its spinor bundle equals the spinor bundle of the underlying Spin-structure as equivariant vector bundles.

Lemma 3.3. *If ξ is a* Spin^c*-regular value of μ^∇, then the* Spin^c*-structure on M induces an orbifold* Spin^c*-structure on M_ξ, with determinant line bundle*

$$L_\xi := (L|_{(\mu^\nabla)^{-1}(\xi)})/G_\xi \to M_\xi$$

Proof. We generalise the proof of Proposition D.60 in [12] to cases where G may not be a torus.

We apply the equivariant version of Lemma 2.5 (see Remark 2.6) to the vector bundles q^*TM_ξ and $\mathcal{N}^\xi$. This yields a G_ξ-equivariant Spin^c-structure on q^*TM_ξ, with determinant line bundle $L|_{(\mu^\nabla)^{-1}(\xi)}$. On the quotient M_ξ, this induces an orbifold Spin^c-structure, with determinant line bundle L_ξ. □

Remark 3.4. If G_ξ acts *freely* on $(\mu^\nabla)^{-1}(\xi)$, then one can also use the Spin^c-structure

$$(P_M|_{(\mu^\nabla)^{-1}(\xi)})/G_\xi \to M_\xi, \tag{3.1}$$

on $(TM|_{(\mu^\nabla)^{-1}(\xi)})/G_\xi$, where $P_M \to M$ is the given Spin^c-structure on M. The determinant line bundle of (3.1) is L_ξ. By the assumption on $\mathcal{N}^\xi$, Lemma 2.5 yields a Spin^c-structure on TM_ξ, with the same determinant line bundle.

If G_ξ only acts locally freely on $(\mu^\nabla)^{-1}(\xi)$, then one would need an orbifold version of Lemma 2.5 to use this argument.

In the language of Definition 2.7, the Spin^c-structure P_{M_ξ} on M_ξ induced by the Spin^c-structure P_M on M equals

$$P_{M_\xi} = \mathrm{Destab}_{\mathcal{N}^\xi}\big(P_M|_{(\mu^\nabla)^{-1}(\xi)}\big)/G_\xi. \tag{3.2}$$

In Definition 3.1, it was not assumed that ξ is a regular value of μ^∇ in the usual sense, since this will not necessarily be the case in the situation considered in Subsection 3.2. If ξ is a regular value, then the first two conditions of Definition 3.1 hold by Lemma 2.4. One can use the following fact to check the third condition.

Proposition 3.5. *Suppose that ξ is a regular value of μ^∇, and that*
- *G and G_ξ are unimodular;*
- *there is an* $\mathrm{Ad}(G_\xi)$*-invariant, nondegenerate bilinear form on $\mathfrak{g}$;*
- *there is an* $\mathrm{Ad}(G_\xi)$*-invariant subspace $V \subset \mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{g}_\xi \oplus V;$$

and

- *there is an* $\mathrm{Ad}(G_\xi)$*-invariant complex structure on* V.

Then ξ *is a* Spin^c*-regular value of* μ^∇.

EXAMPLE 3.6. If $\mathfrak{g}_\xi = \mathfrak{g}$, then the last two conditions in Proposition 3.5 are vacuous. Therefore,

- if G is *Abelian*, any regular value of μ^∇ is a Spin^c-regular value;
- if 0 is a regular value of μ^∇, and G is semisimple, then 0 is a Spin^c-regular value. This holds more generally if G is unimodular and $\mathfrak{g}$ admits an $\mathrm{Ad}(G)$-invariant, nondegenerate metric.

EXAMPLE 3.7. If G is unimodular, and G_ξ is *compact* (i.e. ξ is strongly elliptic), then one can use an $\mathrm{Ad}(G_\xi)$-invariant inner product on $\mathfrak{g}$. Together with the standard symplectic form on

$$V := \mathfrak{g}_\xi^\perp \cong \mathfrak{g}/\mathfrak{g}_\xi \cong T_\xi(G \cdot \xi),$$

this induces an $\mathrm{Ad}(G_\xi)$-invariant complex structure on V (see e.g. Example D.12 in [12]).

For semisimple Lie groups, strongly elliptic elements and coadjoint orbits correspond to discrete series representations, under an integrality condition. (See also [29].)

REMARK 3.8. If the bilinear form in the second point of Proposition 3.5 is positive definite on $\mathfrak{g}_\xi$, then one can take $V = \mathfrak{g}_\xi^\perp$, and the third condition in Proposition 3.5 holds.

If, on the other hand, the bilinear form is positive definite on V, then one has an induced $\mathrm{Ad}(G_\xi)$-invariant complex structure on V (as in Example 3.7), so the fourth condition in Proposition 3.5 holds.

We will prove Proposition 3.5 in Subsection 5.1.

3.2. Spin^c**-slices.** Let G be an almost connected Lie group, and let $K < G$ be a maximal compact subgroup. Let M be any smooth manifold, on which G acts properly. Then Abels showed (see p. 2 of [1]) that there is a K-invariant submanifold (or *slice*) $N \subset M$ such that the map $[g, n] \mapsto g \cdot n$ is a G-equivariant diffeomorphism

$$G \times_K N \cong M.$$

Explicitly, the left hand side is the quotient of $G \times N$ by the K-action given by

$$k \cdot (g, n) = (gk^{-1}, kn),$$

for $k \in K$, $g \in G$ and $n \in N$.

Fix an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$, and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement to $\mathfrak{k}$. After replacing G by a double cover if necessary, we may assume that $\mathrm{Ad}\colon K \to \mathrm{SO}(\mathfrak{p})$ lifts to

$$\widetilde{\mathrm{Ad}}\colon K \to \mathrm{Spin}(\mathfrak{p}). \tag{3.3}$$

Indeed, consider the diagram

$$\begin{array}{ccc} \widetilde{K} & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathrm{Spin}(\mathfrak{p}) \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \pi}\ {\scriptstyle 2:1} \\ K & \xrightarrow{\mathrm{Ad}} & \mathrm{SO}(\mathfrak{p}), \end{array}$$

where

$$\begin{aligned}\widetilde{K} &:= \{(k,a) \in K \times \mathrm{Spin}(\mathfrak{p}); \mathrm{Ad}(k) = \pi(a)\};\\ \pi_K(k,a) &:= k;\\ \widetilde{\mathrm{Ad}}(k,a) &:= a,\end{aligned}$$

for $k \in K$ and $a \in \mathrm{Spin}(\mathfrak{p})$. Then for all $k \in K$,

$$\pi_K^{-1}(k) \cong \pi^{-1}(\mathrm{Ad}(k)) \cong \mathbb{Z}_2,$$

so π_K is a double covering map. In what follows, we will assume the lift (3.3) exists.

It was shown in Section 3.2 of [17] that a K-equivariant Spin^c-structure P_N on N induces a G-equivariant Spin^c-structure P_M on M. In terms of stabilisation of Spin^c-structures (Definition 2.7), one has

$$P_M = G \times_K \mathrm{Stab}_{\mathfrak{p}_N}(P_N). \tag{3.4}$$

Here $\mathfrak{p}_N \to N$ is the trivial vector bundle $N \times \mathfrak{p} \to N$, equipped with the K-action

$$k(n, X) = (kn, \mathrm{Ad}(k)X),$$

for $k \in K$, $n \in N$ and $X \in \mathfrak{p}$. It has the K-equivariant Spin-structure

$$N \times \mathrm{Spin}(\mathfrak{p}) \to N, \tag{3.5}$$

with the diagonal K-action defined via the lift (3.3) of the adjoint action. To show that (3.4) defines a Spin^c-structure on M, one uses the isomorphism

$$TM = G \times_K (TN \oplus \mathfrak{p}_N) \tag{3.6}$$

(see Proposition 2.1 and Lemma 2.2 in [17]).

Analogously to Section 2.4 in [17] in the symplectic setting, the construction (3.4) is invertible. Indeed, given a G-equivariant Spin^c-structure $P_M \to M$ on M, consider the K-equivariant Spin^c-structure

$$P_N := \mathrm{Destab}_{\mathfrak{p}_N}(P_M|_N) \to N \tag{3.7}$$

on N. Here we again use (3.6).

Lemma 3.9. *The constructions* (3.4) *and* (3.7) *are inverse to one another.*

Proof. Starting with a K-equivariant Spin^c-structure $P_N \to N$ on N, we see that (2.4) implies that

$$\mathrm{Destab}_{\mathfrak{p}_N}\big((G \times_K \mathrm{Stab}_{\mathfrak{p}_N}(P_N))|_N\big) = \mathrm{Destab}_{\mathfrak{p}_N}\big(\mathrm{Stab}_{\mathfrak{p}_N}(P_N)\big) = P_N.$$

On the other hand, suppose $P_M \to M$ is a G-equivariant Spin^c-structure on M. Then we have by (2.3),

$$G \times_K \mathrm{Stab}_{\mathfrak{p}_N}\big(\mathrm{Destab}_{\mathfrak{p}_N}(P_M|_N)\big) = G \times_K (P_M|_N),$$

which is ismorphic to P_M via the map $[g, f] \mapsto g \cdot f$, for $g \in G$ and $f \in P_M|_N$. □

Combining Abels' theorem and Lemma 3.9, we obtain the following Spin^c-slice theorem.

Proposition 3.10. *For any G-equivariant Spin^c-structure P_M on a proper G-manifold M, there is a K-invariant submanifold $N \subset M$ and a K-equivariant Spin^c-structure $P_N \to N$ such that $M \cong G \times_K N$, and*

$$P_M = G \times_K \mathrm{Stab}_{\mathfrak{p}_N}(P_N).$$

3.3. Reduction and slices. Consider the situation of Subsection 3.2, and fix N and P_N as in Proposition 3.10. To relate Spin^c-reductions of the actions by G on M and by K on N, we will use a relation between Spin^c-momentum maps for these two actions. Let $L^M \to M$ and $L^N \to N$ be the determinant line bundles of P_M and P_N, respectively. Let ∇^M be a G-invariant Hermitian connection on L^M, let $j\colon N \hookrightarrow M$ be the inclusion map, and consider the connection $\nabla^N := j^*\nabla^M$ on L^N. Let $\mu^{\nabla^M}: M \to \mathfrak{g}^*$ and $\mu^{\nabla^N}: N \to \mathfrak{k}^*$ be the Spin^c-momentum maps associated to these connections. Let $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{k}}\colon \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction map.

LEMMA 3.11. *One has*

1. $L^N = L^M|_N$;
2. $L^M = G \times_K L^N$;
3. $\mu^{\nabla^N} = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{k}} \circ \mu^{\nabla^M}|_N$;
4. *if* $\mu^{\nabla^M}(n) \in \mathfrak{k}^*$ *for all* $n \in N$, *then*

$$\mu^{\nabla^M}([g,n]) = \mathrm{Ad}^*(g)\mu^{\nabla^N}(n), \tag{3.8}$$

for all $g \in G$ *and* $n \in N$.

In the fourth point of this lemma, and in the rest of this paper, we embed $\mathfrak{k}^*$ into $\mathfrak{g}^*$ as the annihilator of $\mathfrak{p}$.

Proof. The Spin-structure (3.5) on $\mathfrak{p}_N$ induces a Spin^c-structure with equivariantly trivial determinant line bundle $L^{\mathfrak{p}_N} \to N$. Since

$$P_N = \mathrm{Destab}_{\mathfrak{p}_N}(P_M|_N),$$

Lemma 2.5 implies that

$$L^N = L^N \otimes L^{\mathfrak{p}_N} = L^M|_N.$$

So the first claim holds, and the second claim follows from this: $L^M = G \cdot L^M|_N = G \times_K L^N$.

To prove the third claim, we use the first claim, and note that for all $X \in \mathfrak{k}$,

$$2\pi i \mu_X^{\nabla^N} = \nabla^N_{X^N} - \mathcal{L}_X^{L^N} = \left(\nabla^M_{X^M} - \mathcal{L}_X^{L^M}\right)\Big|_{\Gamma^\infty(L^N)} = 2\pi i \mu_X^{\nabla^M}|_N.$$

The fourth claim follows from the third. □

In the symplectic case, it was shown in Proposition 2.8 of [17] that one may take $N = (\mu^{\nabla^M})^{-1}(\mathfrak{k}^*)$. Then the condition in the fourth point of Lemma 3.11 holds, so one has (3.8). In the Spin^c-setting, we use an arbitrary slice N. In Subsection 5.2, we show that a K-invariant connection ∇^N on L^N induces a G-invariant connection ∇^M on L^M such that the condition in the fourth point of Lemma 3.11 is satisfied (see Lemma 5.3). From now on, we suppose that ∇^M was chosen in this way, so that (3.8) holds.

In that case, a regular value of μ^{∇^N} is not necessarily a regular value of μ^{∇^M}. Indeed, any tangent vector to M at $[e,n]$, for $n \in N$, is of the form $T_{(e,n)}q(X,v) = X^M_{[e,n]} + v$, for $X \in \mathfrak{g}$ and $v \in T_nN$. Using (3.8) one computes that

$$T_{[e,n]}\mu^{\nabla^M}(X^M_{[e,n]} + v) = \mathrm{ad}^*(X)\big(\mu^{\nabla^N}(n)\big) + T_n\mu^{\nabla^N}(v).$$

If, for example, $\mu^{\nabla^N}(n) = 0$, then $T_{[e,n]}\mu^{\nabla^M}$ can only be surjective if $\mathfrak{g} = \mathfrak{k}$, even if 0 is a regular value of μ^{∇^N}. However, all regular values of μ^{∇^N} are Spin^c-regular values of μ^{∇^M}.

PROPOSITION 3.12. *If ξ is a regular value of μ^{∇^N}, then it is a* Spin^c*-regular value of μ^{∇^M}.*

Note that by the third point of Lemma 3.11, ξ is a regular value of μ^{∇^N} if and only if it is a regular value of $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{k}} \circ \mu^{\nabla^M}$. Fix $\xi \in \mathfrak{k}^*$ satisfying this condition, and let $P_{M_\xi} \to M_\xi$ be the Spin^c-structure on M_ξ as in Lemma 3.3.

There is another way to define a Spin^c-structure on M_ξ, using the following fact.

PROPOSITION 3.13. *Suppose G is reductive. Then for any $\eta \in \mathfrak{k}^*$, the inclusion map map $N \hookrightarrow M$ induces a homeomorphism*

$$N_\eta \cong M_\eta.$$

Since ξ is a regular value of μ^{∇^N}, Proposition 3.5 implies that the Spin^c-structure on N induces a Spin^c-structure on N_ξ, which equals M_ξ. In the proof of Theorem 4.6, we will use the fact that the two Spin^c structures P_{M_ξ} and P_{N_ξ} are the same.

PROPOSITION 3.14. *The* Spin^c*-structures P_{M_ξ} and P_{N_ξ} on $M_\xi \cong N_\xi$ are equal.*

Propositions 3.12–3.14 will be proved in Subsections 5.2–5.4.

We end this section by mentioning a compatibility property of stabilising and destabilising Spin-structures with the fibred product construction that appears in the slice theorem. This property will be used in the proof of Proposition 3.14. Suppose $H < G$ is any closed subgroup, acting on a manifold N, and let $E \to N$ be an H-vector bundle with an H-equivariant Spin^c-structure $P_E \to N$. Then $G \times_H P_E \to G \times_H N$ is a G-equivariant Spin^c-structure for the G-vector bundle $G \times_H E \to G \times_H N$ (see Lemma 3.7 in [17]). In the proof of Proposition 3.14, we will use the fact that this construction is compatible with stabilisation and destabilisation.

LEMMA 3.15. *In the above setting, let $F \to N$ be another H-vector bundle.*

1. *If $P_F \to N$ is an H-equivariant* Spin^c*-structure on P_F, then*

$$G \times_H \mathrm{Stab}_E(P_F) = \mathrm{Stab}_{G\times_H E}(G \times_H P_F).$$

2. *If $P_{E\oplus F} \to N$ is an H-equivariant* Spin^c*-structure on $P_{E\oplus F}$, then*

$$G \times_H \mathrm{Destab}_E(P_{E\oplus F}) = \mathrm{Destab}_{G\times_H E}(G \times_H P_{E\oplus F}).$$

Proof. The first point follows from the explicit constructions in Section 3.1 of [35]. Here one uses the fact that the spinor bundle associated to $G \times_H P_E$ is $G \times_H \mathcal{S}_E$, where $\mathcal{S}_E \to N$ is the spinor bundle associated to P_E. This is compatible with the grading operators.

The second point can be proved in a similar way, or deduced from the first point, by using the fact that destabilisation is the inverse of stabilisation, as in (2.3) and (2.4). ◻

REMARK 3.16. We have only considered the principle $\mathrm{Spin}^c(r)$-bundle part $P_E \to X$ of a Spin^c-structure on a vector bundle $E \to X$ of rank r over a manifold X, not the isomorphism

$$P_E \times_{\mathrm{Spin}^c(r)} \mathbb{R}^r \cong E.$$

If E is the tangent bundle to X, then this isomorphism determines the Riemannian metric on X induced by the Spin^c-structure. For cocompact actions, where we will apply the material in this subsection, the index of the Spin^c-Dirac operator is independent of this metric, however.

Part II. Cocompact actions.

4. The result on cocompact actions. The main result on cocompact actions is Theorem 4.6, which states that that Spin^c-quantisation commutes with reduction at K-theory generators. In this section, we state Paradan and Vergne's result for compact groups and manifolds in [30, 31], and Theorem 4.6 for cocompact actions. We will deduce Theorem 4.6 from Paradan and Vergne's result in Section 5.

We keep using the notation of Section 2.

4.1. The compact case. First of all, we define Spin^c-quantisation of sufficiently regular reduced spaces, which will always be compact in the settings we consider. Let ξ be a Spin^c-regular value of μ^∇. Then by Lemma 3.3, the reduced space M_ξ is a Spin^c-orbifold. Suppose that M_ξ is compact and even-dimensional. Let D_{M_ξ} be the Spin^c-Dirac operator on M_ξ, defined with the connection on the determinant line bundle $L_\xi \to M_\xi$ induced by a given connection on the determinant line bundle $L \to M$.

DEFINITION 4.1. The Spin^c-*quantisation* of M_ξ is the index of D_{M_ξ}:

$$Q^{\mathrm{Spin}^c}(M_\xi) := \mathrm{index}(D_{M_\xi}) \quad \in \mathbb{Z}.$$

This index is the usual one if M_ξ is smooth, and the orbifold index [19] in general.

Now suppose that $G = K$ is compact and connected. Suppose that M is even-dimensional, and also compact and connected. Since M is even-dimensional, the spinor bundle $\mathcal{S}$ splits into even and odd parts, sections of which are interchanged by the Spin^c-Dirac operator D. Because M is compact, this Dirac operator has finite-dimensional kernel, and one can define

$$Q_K^{\mathrm{Spin}^c}(M) := K\text{-index}(D) = [\ker D^+] - [\ker D^-] \quad \in R(K), \tag{4.1}$$

where $D^\pm$ are the restrictions of D to the even and odd parts of $\mathcal{S}$, repectively, and $R(K)$ is the representation ring of K.

Let $T < K$ be a maximal torus, with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a choice of (closed) positive Weyl chamber. Let R be the set of roots of $(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, and let R^+ be the set of positive roots with respect to $\mathfrak{t}^*_+$. Set

$$\rho_K := \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

Let $\mathcal{F}$ be the set of relative interiors of faces of $\mathfrak{t}^*_+$. Then

$$\mathfrak{t}^*_+ = \bigcup_{\sigma \in \mathcal{F}} \sigma,$$

a disjoint union. For $\sigma \in \mathcal{F}$, let $\mathfrak{k}_\sigma$ be the infinitesimal stabiliser of a point in σ. Let R_σ be the set of roots of $\big((\mathfrak{k}_\sigma)_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}\big)$, and let $R_\sigma^+ := R_\sigma \cap R^+$. Set

$$\rho_\sigma := \frac{1}{2} \sum_{\alpha \in R_\sigma^+} \alpha.$$

Note that, if σ is the interior of $\mathfrak{t}_+^*$, then $\rho_\sigma = 0$.

For any subalgebra $\mathfrak{h} \subset \mathfrak{k}$, let $(\mathfrak{h})$ be its conjugacy class. Set

$$\mathcal{H}_{\mathfrak{k}} := \{(\mathfrak{k}_\xi); \xi \in \mathfrak{k}\}.$$

For $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$, write

$$\mathcal{F}(\mathfrak{h}) := \{\sigma \in \mathcal{F}; (\mathfrak{k}_\sigma) = (\mathfrak{h})\}.$$

Let $(\mathfrak{k}^M)$ be the conjugacy class of the generic (i.e. minimal) infinitesimal stabiliser $\mathfrak{k}^M$ of the action by K on M. Note that by Lemma 2.4, one has $(\mathfrak{k}^M) = 0$ if μ^∇ has regular values.

Let $\Lambda_+ \subset i\mathfrak{t}^*$ be the set of dominant integral weights. In the Spin^c-setting, it is natural to parametrise the irreducible representations by their infinitesimal characters, rather than by their highest weights. For $\lambda \in \Lambda_+ + \rho_K$, let π_λ^K be the irreducible representation of K with infinitesimal character λ, i.e. with highest weight $\lambda - \rho_K$. Then one has, for such λ,

$$Q^{\mathrm{Spin}^c}(K \cdot \lambda) = \pi_\lambda^K,$$

see Lemma 2.1 in [30] or Lemma 4.1 in [33].

Write

$$Q_K^{\mathrm{Spin}^c}(M) = \bigoplus_{\lambda \in \Lambda_+ + \rho_K} m_\lambda [\pi_\lambda^K],$$

with $m_\lambda \in \mathbb{Z}$. Then Paradan and Vergne proved the following expression for m_λ in terms of reduced spaces.

THEOREM 4.2 ([30], Theorem 3.4; [31], Theorem 1.4). *Suppose* $([\mathfrak{k}^M, \mathfrak{k}^M]) = ([\mathfrak{h}, \mathfrak{h}])$, *for* $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. *Then*

$$m_\lambda = \sum_{\substack{\sigma \in \mathcal{F}(\mathfrak{h}) \text{ s.t.} \\ \lambda - \rho_\sigma \in \sigma}} Q^{\mathrm{Spin}^c}(M_{\lambda - \rho_\sigma}). \tag{4.2}$$

Here the quantisation $Q^{\mathrm{Spin}^c}(M_{\lambda-\rho_\sigma})$ of the reduced space[2] $M_{\lambda-\rho_\sigma}$ is defined in Section 4 of [30] and Section 5 of [31], which includes cases where Lemma 3.3 does not apply, and reduced spaces are singular.

If the generic stabiliser $\mathfrak{k}^M$ is *Abelian*, Theorem 4.2 simplifies considerably. As noted above, this occurs in particular if μ^∇ has a regular value. More generally, this simplification holds if $[\mathfrak{k}^M, \mathfrak{k}^M]$ is Abelian, which is true if there is a $\xi \in \mu^\nabla(M) \cap \mathfrak{k}^*$ such that $\mathfrak{k}_\xi$ is Abelian.

[2] for $\xi \in i\mathfrak{k}^*$, we write $M_\xi := M_{\xi/i}$.

COROLLARY 4.3. *If* $[\mathfrak{k}^M, \mathfrak{k}^M]$ *is Abelian, then*

$$m_\lambda = Q^{\mathrm{Spin}^c}(M_\lambda).$$

Proof. If one takes $\mathfrak{h} = \mathfrak{t}$ in Theorem 4.2, then $\mathcal{F}(\mathfrak{h})$ only contains the interior of $\mathfrak{t}^*_+$. Hence $\rho_\sigma = 0$, for the single element $\sigma \in \mathcal{F}(\mathfrak{h})$. □

In particular, if 0 is a regular value of μ^∇, then the invariant part of the Spin^c-quantisation of M is

$$Q_K^{\mathrm{Spin}^c}(M)^K = Q^{\mathrm{Spin}^c}(M_{\rho_K}), \tag{4.3}$$

since $\pi^K_{\rho_K}$ is the trivial representation.

4.2. The cocompact case. Now suppose M and G may be noncompact, but M/G is compact. Then Landsman [14, 21] defined geometric quantisation via the *analytic assembly map* from the Baum–Connes conjecture [2]. This takes values in the K-theory of the maximal or reduced group C^*-algebra C^*G or C^*_rG of G. Landsman's definition extends directly to the Spin^c case.

DEFINITION 4.4. If M/G is compact, the Spin^c-*quantisation* of the action by G on M is

$$Q_G^{\mathrm{Spin}^c}(M) := G\text{-index}(D) \quad \in K_*(C^*G), \tag{4.4}$$

where G-index denotes the analytic assembly map.

In this definition, the maximal C^*-algebra C^*G of G was used. By applying the map

$$r_*\colon K_*(C^*G) \to K_*(C^*_rG)$$

induced by the natural map $r\colon C^*G \to C^*_rG$, one obtains the *reduced*[3] Spin^c-*quantisation*

$$Q_G^{\mathrm{Spin}^c}(M)_r := r_*\big(Q_G^{\mathrm{Spin}^c}(M)\big) \quad \in K_*(C^*_rG).$$

(This is equal to (4.4), if G-index denotes the assembly map for C^*_rG, but we include the map r_* to make the distinction clear.) If G is compact, then $K_*(C^*G)$ and $K_*(C^*_rG)$ equal the representation ring $R(G)$ of G. Then the above definitions of Spin^c-quantisation and reduced Spin^c-quantisation both reduce to (4.1).

Landsman used the reduction map

$$R_0\colon K_*(C^*G) \to \mathbb{Z}$$

induced on K-theory by the continuous map

$$C^*G \to \mathbb{C},$$

which on $C_c(G) \subset C^*G$ is given by integration over G. If G is compact, then $R_0\colon R(G) \to \mathbb{Z}$ is taking the multiplicity of the trivial representation. Landsman conjectured that

$$R_0\big(Q_G(M)\big) = Q(M_0), \tag{4.5}$$

[3]Note that the word 'reduced' and the map r_* used here have nothing to do with reduction; this is just an unfortunate clash of terminology.

in the symplectic case (if M_0 is smooth). Here quantisation is defined as in Definition 4.4, where D is a Dirac operator coupled to a prequantum line bundle.

This conjecture was proved by Hochs and Landsman [14] for a specific class of groups G, and by Mathai and Zhang [24] for general G, where one may need to replace the prequantum line bundle by a tensor power. As a special case of Theorem 6.8, we will obtain a generalisation to the Spin^c-setting of Mathai and Zhang's result on the Landsman conjecture (see Corollary 10.1). This asserts that (4.5) still holds for Spin^c-quantisation, for large enough powers of the determinant line bundle, and a connection on this bundle. (See Subsection 6.3 for questions about ρ-shifts in this context.)

4.3. Reduction at nonzero values of μ^∇. Landsman's conjecture was extended to reduction at K-theory classes corresponding to nontrivial representations in [17, 18]. Here one works with reduced quantisation, with values in $K_*(C^*_rG)$.

Because we will deduce the result in this subsection from Paradan and Vergne's result in [30, 31], we now adopt their convention concerning the definition of the momentum map:

$$-\frac{i}{2}\mu^\nabla_X = \nabla_{X^M} - \mathcal{L}^L_X.$$

I.e., the factor $2\pi i$ in (2.2), which was chosen for consistency with [15, 36], is replaced by $-i/2$. We use this convention in the present subsection, and in Section 5.

Suppose G is almost connected, and let $K < G$ be a maximal compact subgroup. For now we suppose that the lift (3.3) exists. In Subsection 4.4 we explain how to remove this assumption by using a double cover of G. With notation as in Subsection 4.1, one has

$$R(K) = \bigoplus_{\lambda\in\Lambda_+ + \rho_K} \mathbb{Z}[\pi^K_\lambda].$$

Set $d := \dim(G/K)$. By the Connes–Kasparov conjecture, proved in [9] for almost connected groups, the *Dirac induction* map

$$\text{D-Ind}^G_K\colon R(K) \to K_d(C^*_rG)$$

is an isomorphism of Abelian groups, while $K_{d+1}(C^*_rG) = 0$. In other words, the K-theory group $K_*(C^*_rG)$ is the free Abelian group generated by

$$[\lambda] := \text{D-Ind}^G_K[\pi^K_\lambda], \tag{4.6}$$

for $\lambda \in \Lambda_+ + \rho_K$, and these generators have degree d. For G semisimple with discrete series, 'most' of the generators $[\lambda]$ are associated to discrete series representations [20]. If G is complex-semisimple, they are associated to families of principal series representations [34]. See also [18].

Since $K_{d+1}(C^*_rG) = 0$, it follows that $Q^{\mathrm{Spin}^c}_G(M)_r = 0$ if d_M and d have different parities. (Recall that we set $d_M := \dim(M)$.) So assume $d_M - d$ is even. In [18], the case where M carries a (pre)symplectic form was considered. It was conjectured that quantisation commutes with reduction at any $\lambda \in \Lambda_+ + \rho_K$, in the sense that

$$Q^{\mathrm{Spin}^c}_G(M)_r = \sum_{\lambda\in\Lambda_+ + \rho_K} Q(M_\lambda)[\lambda] \quad \in K_d(C^*_rG). \tag{4.7}$$

It was assumed that the momentum map image has nonzero intersection with the interior of a positive Weyl chamber, to simplify the ρ-shifts that occur (analogously to the way Theorem 4.2 simplifies to Corollary 4.3). We will not make this assumption in Theorem 4.6.

In the symplectic setting, a formal version of quantisation, defined as the right hand side of (4.7), was extended to non-cocompact actions and studied in [16].

Replacing G by a double cover if necessary, we may assume the lift (3.3) of the adjoint action by K on $\mathfrak{p}$ exists. Let the slice $N \subset M$ and the Spin^c-structure $P_N \to N$ be as in Proposition 3.10. Since M/G is compact, N is compact in this case. We choose a connection ∇^M on L^M such that (3.8) holds.

To quantise singular reduced spaces for actions by reductive groups, we extend Definition 4.1 by using the homeomorphism of Proposition 3.13 and Paradan and Vergne's definition in the singular case. Recall that μ^{∇^N} is the Spin^c-momentum map for the action by K on N.

Definition 4.5. If G is reductive and $\xi \in \mathfrak{k}^*$ is a singular value of μ^{∇^N}, then

$$Q^{\mathrm{Spin}^c}(M_\xi) := Q^{\mathrm{Spin}^c}(N_\xi),$$

where $Q^{\mathrm{Spin}^c}(N_\xi)$ is defined as in Section 4 of [30] and Section 5 of [30].

Note that different choices of N lead to homeomorphic reduced spaces by Proposition 3.13. If ξ is a *regular* value of μ^{∇^N}, then Definition 4.1 applies by Proposition 3.12. Because of Proposition 3.14, one has $Q^{\mathrm{Spin}^c}(M_\xi) = Q^{\mathrm{Spin}^c}(N_\xi)$ in that case, so Definitions 4.1 and 4.5 are consistent.

Paradan and Vergne's result generalises to the cocompact setting in the following way.

Theorem 4.6 (Spin^c quantisation commutes with reduction; cocompact case). *If M and G are connected, G is reductive, $d_M - d$ is even, and the lift* (3.3) *exists, then*

$$Q_G^{\mathrm{Spin}^c}(M)_r = \sum_{\lambda \in \Lambda_+ + \rho_K} m_\lambda [\lambda], \tag{4.8}$$

with m_λ given by (4.2).

As noted before, if $d_M - d$ is odd, then $Q_G^{\mathrm{Spin}^c}(M)_r = 0$. The case where the lift (3.3) does not exist is treated in Subsection 4.4.

This result will be proved in Section 5. We will use the constructions in Subsections 3.2 and 3.3 and a quantisation commutes with induction result to deduce it from Paradan and Vergne's result. In the symplectic setting, an additional assumption was needed in [17] to apply a similar kind of reasoning. The authors view this as a sign that it is very natural to study the quantisation commutes with reduction problem in the Spin^c-setting.

4.4. Double covers of G. So far, we assumed that the lift (3.3) exists, and noted this is true for a double cover of G. Theorem 4.6 also holds without this assumption, however, as we will explain now.

Let $\pi_G \colon \widetilde{G} \to G$ be a double cover for which (3.3) exists. Write $\widetilde{K} := \pi_G^{-1}(K)$. Let $u \in \ker \pi_G$ be the nontrivial element. Set

$$R_{\mathrm{Spin}}(K) := \{V \in R(\widetilde{K}); u \text{ acts trivially on } V \otimes \Delta_{\mathfrak{p}}\}.$$

Then for $V \in R_{\mathrm{Spin}}(K)$, we can view $V \otimes \Delta_{\mathfrak{p}}$ as a (virtual) representation of K. For such V, we can define the Dirac operator $D^V_{G/K}$ on the vector bundle $G \times_K (V \otimes \Delta_{\mathfrak{p}}) \to G/K$, as in the definition of Dirac induction. Its image under the assembly map is by definition the element

$$\text{D-Ind}_K^G[V] \in K_d(C^*_r G).$$

Let $\Lambda_+ \subset i\mathfrak{t}^*$ be the dominant weight lattice as in Subsection 4.1. Let $\widetilde{\Lambda}_+ \subset i\mathfrak{t}^*$ be the dominant weight lattice for $(\widetilde{K}, \pi_G^{-1}(T))$. Set

$$\Lambda_+^{\mathrm{Spin}} := \{\lambda \in \widetilde{\Lambda}_+ ; \pi_\lambda^{\widetilde{K}} \otimes \Delta_{\mathfrak{p}} \in R_{\mathrm{Spin}}(K)\}.$$

Here, as before, $\pi_\lambda^{\widetilde{K}}$ is the irreducible representation of $\widetilde{K}$ with infinitesimal character λ. If $\lambda \in \Lambda_+^{\mathrm{Spin}} + \rho_K$, then we write

$$[\lambda] := \text{D-Ind}_K^G[\pi_\lambda^{\widetilde{K}}] \in K_d(C^*_r G).$$

Theorem 4.6 generalises as follows.

THEOREM 4.7. *If M and G are connected, G is reductive, and $d_M - d$ is even, then*

$$Q_G^{\mathrm{Spin}^c}(M)_r = \sum_{\lambda \in \Lambda_+^{\mathrm{Spin}} + \rho_K} m_\lambda [\lambda],$$

with m_λ given by (4.2).

To deduce this theorem from Theorem 4.6, we use the map

$$(\pi_G)_* \colon C^*_r \widetilde{G} \to C^*_r G \tag{4.9}$$

given by

$$((\pi_G)_* f)(g) = f(\tilde{g}) + f(u\tilde{g}),$$

for $f \in C_c(\widetilde{G})$, $g \in G$ and $\tilde{g} \in \pi_G^{-1}(g)$. We denote the induced map on K-theory by $(\pi_G)_*$ as well.

On K-homology, we have the map

$$(\pi_G)_* \colon K_*^{\widetilde{G}}(M) \to K_*^G(M/\ker \pi_G),$$

as defined in Section 3.2 of [27] (see Section 3.1 in [14] for the non-discrete case). Without going into this definition, we can say that since $\ker \pi_G$ acts trivially on M and $\mathcal{S}$, the codomain of this map is $K_*^G(M)$, and we have

$$(\pi_G)_*[D_M] = [D_M]. \tag{4.10}$$

It was shown in Section 3.2 of [27] (see Appendix A in [14] for the nondiscrete case) that the following diagram commutes:

$$\begin{array}{ccc} K_*^{\widetilde{G}}(M) & \xrightarrow{\mu_M^{\widetilde{G}}} & K_*(C^*_r \widetilde{G}) \\ \downarrow{\scriptstyle (\pi_G)_*} & & \downarrow{\scriptstyle (\pi_G)_*} \\ K_*^G(M) & \xrightarrow{\mu_M^G} & K_*(C^*_r G). \end{array} \tag{4.11}$$

Here $\mu_M^{\widetilde{G}}$ and μ_M^G denote the analytic assembly maps for $\widetilde{G}$ and G, respectively.

One step in the proof of Theorem 4.7 is that the map (4.9) relates Dirac induction for the groups $\widetilde{G}$ and G to each other.

LEMMA 4.8. *The following diagram commutes:*

$$\begin{array}{ccc} R(\widetilde{K}) & \xrightarrow{\text{D-Ind}_{\widetilde{K}}^{\widetilde{G}}} & K_*(C^*_r\widetilde{G}) \\ \uparrow & & \downarrow (\pi_G)_* \\ R_{\mathrm{Spin}}(K) & \xrightarrow{\text{D-Ind}_K^G} & K_*(C^*_r G). \end{array}$$

Proof. Since $\ker \pi_G \subset \widetilde{K}$, this group acts trivially on $\widetilde{G}/\widetilde{K} = G/K$. Let $V \in R_{\mathrm{Spin}}(K)$. Then, as in (4.10), we have

$$(\pi_G)_*[D^V_{\widetilde{G}/\widetilde{K}}] = [D^V_{G/K}] \quad \in K^G_*(G/K).$$

Using (4.11) with M replaced by G/K, we find that

$$(\pi_G)_*\mu^{\widetilde{G}}_{\widetilde{G}/\widetilde{K}}[D^V_{\widetilde{G}/\widetilde{K}}] = \mu^G_{G/K}[D^V_{G/K}] \quad \in K_*(C^*_r G).$$

□

Proof of Theorem 4.7. By (4.10) and commutativity of (4.11), we have

$$Q^{\mathrm{Spin}^c}_G(M)_r = (\pi_G)_*\big(Q^{\mathrm{Spin}^c}_{\widetilde{G}}(M)_r\big) \tag{4.12}$$

For $\lambda \in \widetilde{\Lambda}_+ + \rho_K$, write

$$\widetilde{[\lambda]} := \text{D-Ind}^{\widetilde{G}}_{\widetilde{K}}[\pi^{\widetilde{K}}_\lambda] \quad \in K_d(C^*_r\widetilde{G}).$$

Theorem 4.6 implies that the right hand side of (4.12) equals

$$\sum_{\lambda\in\widetilde{\Lambda}_+ + \rho_K} \tilde{m}_\lambda(\pi_G)_*\widetilde{[\lambda]}, \tag{4.13}$$

with $\tilde{m}_\lambda$ given by (4.2), for reduced spaces by $\widetilde{G}$. By the Connes–Kasprov conjecture, we have

$$Q^{\mathrm{Spin}^c}_G(M)_r \in K_*(C^*_r G) = \text{D-Ind}^G_K(R_{\mathrm{Spin}}(K)).$$

Therefore, only terms with $\lambda \in \Lambda^{\mathrm{Spin}}_+$ contribute to (4.13). We find that

$$Q^{\mathrm{Spin}^c}_G(M)_r = \sum_{\lambda\in\Lambda^{\mathrm{Spin}}_+ + \rho_K} \tilde{m}_\lambda(\pi_G)_*\widetilde{[\lambda]}.$$

By Lemma 4.8, we have $(\pi_G)_*\widetilde{[\lambda]} = [\lambda] \in K_d(C^*_r G)$. The claim therefore follows from the fact that

$$\tilde{m}_\lambda = m_\lambda.$$

This equality holds because $\ker \pi_G$ acts trivially on M, so the momentum maps and the reduced spaces for the actions by $\widetilde{G}$ and G are the same. Also, the coadjoint action by $\ker \pi_G$ preserves coadjoint orbits of G, so the coadjoint orbits of $\widetilde{G}$ and G are the same. □

5. Spin^c**-structures on reduced spaces and fibred products.** In this section, we prove the statements in Section 3. Together with a generalisation of the *quantisation commutes with induction* results in [17, 18], this allows us to deduce Theorem 4.6 from Paradan and Vergne's result, Theorem 4.2. Note that in Subsections 3.1 and 3.2, group actions were not asumed to be cocompact. So the statements made there apply more generally (and many will also be used in Part 5.5). The cocompactness assumption will only be made in Subsection 5.5.

Propositions 3.5 and 3.12–3.14 are proved in Subsections 5.1–5.4. In Subsection 5.5, we show that quantisation commutes with induction in the Spin^c-setting, and use this to prove Theorem 4.6.

In this section, we assume that the lift (3.3) of the representation $\mathrm{Ad}\colon K \to \mathrm{SO}(\mathfrak{p})$ exists. We saw in Subsection 4.4 how to remove that assumption.

5.1. Spin^c**-reduction at regular values.** We start by proving Proposition 3.5. Suppose $\xi \in \mathfrak{g}^*$ is a regular value of μ^∇. Then by Lemma 2.4, G_ξ acts locally freely on $(\mu^\nabla)^{-1}(\xi)$. Let $q\colon (\mu^\nabla)^{-1}(\xi) \to M_\xi$ be the quotient map. The restriction of TM to $(\mu^\nabla)^{-1}(\xi)$ decomposes as follows.

LEMMA 5.1. *There is a G_ξ-equivariant isomorphism of vector bundles*

$$TM|_{(\mu^\nabla)^{-1}(\xi)} = q^*TM_\xi \oplus \mathfrak{g}^* \oplus \mathfrak{g}_\xi, \tag{5.1}$$

where G_ξ acts on the right hand side by

$$g\big((m, v), \eta, X\big) = \big((gm, v), \mathrm{Ad}^*(g)\eta, \mathrm{Ad}(g)X\big),$$

for $g \in G_\xi$, $m \in (\mu^\nabla)^{-1}(\xi)$, $v \in T_{G_\xi \cdot m}M_\xi$, $\eta \in \mathfrak{g}^*$ and $X \in \mathfrak{g}_\xi$.

Proof. See (5.6) in [12] for the case where G is a torus. In general, since ξ is a regular value of μ^∇, we have the short exact sequence

$$0 \to \ker(T\mu^\nabla) \to TM|_{(\mu^\nabla)^{-1}(\xi)} \xrightarrow{T\mu^\nabla} (\mu^\nabla)^{-1}(\xi) \times \mathfrak{g}^* \to 0. \tag{5.2}$$

Now $\ker(T\mu^\nabla) = T\big((\mu^\nabla)^{-1}(\xi)\big)$ fits into the short exact sequence

$$0 \to \ker(Tq) \to T\big((\mu^\nabla)^{-1}(\xi)\big) \xrightarrow{Tq} TM_\xi \to 0. \tag{5.3}$$

Since $\ker(Tq)$ is the bundle of tangent spaces to G_ξ-orbits, and $\mathfrak{g}_\xi$ acts locally freely on $(\mu^\nabla)^{-1}(\xi)$ by Lemma 2.4, we have

$$\ker(Tq) \cong (\mu^\nabla)^{-1}(\xi) \times \mathfrak{g}_\xi, \tag{5.4}$$

via the map

$$(m, X) \mapsto X^M_m,$$

for $(m, X) \in (\mu^\nabla)^{-1}(\xi) \times \mathfrak{g}_\xi$.

Combining (5.2), (5.3) and (5.4), we obtain the desired vector bundle isomorphism. □

Because of Lemma 5.1, Proposition 3.5 follows from the following fact.

LEMMA 5.2. *If the conditions in Proposition 3.5 hold, then the sub-bundle*

$$(\mu^\nabla)^{-1}(\xi) \times (\mathfrak{g}^* \oplus \mathfrak{g}_\xi) \to (\mu^\nabla)^{-1}(\xi) \tag{5.5}$$

of (5.1) *has a G_ξ-equivariant* Spin-*structure.*

Proof. Using the given $\mathrm{Ad}(G_\xi)$-invariant, nondegenerate bilinear form on $\mathfrak{g}$, and the subspace $V \subset \mathfrak{g}$, we obtain an $\mathrm{Ad}(G_\xi)$-equivariant isomorphism

$$\mathfrak{g}^* \oplus \mathfrak{g}_\xi \cong (\mathfrak{g}_\xi \oplus \mathfrak{g}_\xi) \oplus V. \tag{5.6}$$

Identifying $\mathfrak{g}_\xi \oplus \mathfrak{g}_\xi \cong \mathfrak{g}_\xi + i\mathfrak{g}_\xi = (\mathfrak{g}_\xi)_\mathbb{C}$, and using the given complex structure on V, one gets an $\mathrm{Ad}(G_\xi)$-invariant complex structure on (5.6). This induces a G_ξ-equivariant Spin^c-structure on the vector bundle (5.5), with determinant line bundle

$$(\mu^\nabla)^{-1}(\xi) \times \textstyle\bigwedge_\mathbb{C}^{\mathrm{top}} \big((\mathfrak{g}_\xi)_\mathbb{C} \oplus V\big) \to (\mu^\nabla)^{-1}(\xi). \tag{5.7}$$

Since G and G_ξ are unimodular, the adjoint action by G_ξ on $\mathfrak{g}$, $\mathfrak{g}_\xi$ and hence V, has determinant one. Therefore, G_ξ acts trivially on

$$\textstyle\bigwedge_\mathbb{C}^{\mathrm{top}} (\mathfrak{g}_\xi)_\mathbb{C} \otimes \bigwedge_\mathbb{C}^{\mathrm{top}} V = \bigwedge_\mathbb{C}^{\mathrm{top}} \big((\mathfrak{g}_\xi)_\mathbb{C} \oplus V\big),$$

so that the determinant line bundle (5.7) is equivariantly trivial. Hence the Spin^c-structure on (5.5) is induced by a G-equivariant Spin-structure. (Compare this with the fact that the natural embedding of $\mathrm{U}(n)$ into $\mathrm{Spin}^c(2n)$ maps $\mathrm{SU}(n)$ into $\mathrm{Spin}(2n)$.) □

5.2. Induced connections and momentum maps. In the rest of this section, we fix a slice $N \subset M$ and a K-equivariant Spin^c-structure $P_N \to N$ as in Proposition 3.10.

To prove Proposition 3.13, we will choose the connection ∇^M in such a way that the Spin^c-momentum maps are related as in (3.8). Let ∇^N be a K-equivariant Hermitian connection on the determinant line bundle $L^N \to N$. We will use the connection ∇^M on $L^M = G \times_K L^N$ induced by ∇^N, as discussed in Section 3.1 in [17]. We briefly review the construction of this connection.

Let $p_N \colon G \times N \to N$ be projection onto the second factor. For a K-invariant section $s \in \Gamma^\infty(G \times N, p_N^* L^N)^K$, one has the section $\sigma \in \Gamma^\infty(L^M)$ given by

$$\sigma[g, n] = [g, s(g, n)]. \tag{5.8}$$

(Here s is viewed as a map $G \times N \to L^N$.) For such an s, and for $g \in G$ and $n \in N$, write

$$s_g(n) := s(g, n) =: s^n(g) \quad \in L^N_n.$$

This defines $s_g \in \Gamma^\infty(L^N)$ and $s_n \in C^\infty(G, L^N_n) \cong C^\infty(G)$.

Let $q \colon G \times N \to M$ be the quotient map. Note that

$$q^* L^M \cong p_N^* L^N \cong G \times L^N \to G \times N,$$

and that under this isomorphism, $q^*\sigma$ corresponds to s. For $X \in \mathfrak{g}$, $n \in N$ and $v \in T_n N$, one has

$$Tq(X, v) \in T_{[g,n]} M.$$

Write $X = X_\mathfrak{k} + X_\mathfrak{p}$ according to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the connection ∇^M is defined by the properties that it is G-invariant, and satisfies

$$\big(\nabla^M_{Tq(X,v)}\sigma\big)[e, n] = \big[e, (\nabla^N_v s_e)(n) + X(s^n)(e) + 2\pi i \mu^{\nabla^N}_{X_\mathfrak{k}}(n) s(e, n)\big], \tag{5.9}$$

for $X \in \mathfrak{g}$, $n \in N$, $v \in T_nN$, and σ and s as above.

Let $\mu^{\nabla^N} : N \to \mathfrak{k}^*$ be the Spin^c-momentum map associated to ∇^N, and let $\mu^{\nabla^M} : M \to \mathfrak{g}^*$ be the Spin^c-momentum map for the induced connection ∇^M.

LEMMA 5.3. *For all $n \in N$, one has $\mu^{\nabla^M}(n) \in \mathfrak{k}^*$.*

Recall that we consider $\mathfrak{k}^*$ as a subspace of $\mathfrak{g}^*$ by identifying it with the annihilator of $\mathfrak{p}$.

Proof. As in (5.8), let $s \in \Gamma^\infty(G \times N, p_N^* L^N)^K$, and let $\sigma \in \Gamma^\infty(L^M)$ be the associated section of L^M. Let $X \in \mathfrak{g}$, and $n \in N$. Then one has

$$
\begin{aligned}
(\mathcal{L}_X^{L^M}\sigma)[e,n] &= \left.\frac{d}{dt}\right|_{t=0} \exp(tX)[\exp(-tX), s(\exp(-tX), n)] \\
&= \left.\frac{d}{dt}\right|_{t=0} [e, s(\exp(-tX), n)] \\
&= [e, X(s^n)(e)] \quad \in L^M_{[e,n]}.
\end{aligned}
$$

Since $Tq(X, 0) = X^M$ in (5.9), one therefore has

$$(\nabla^M_{X^M}\sigma)[e,n] = (\mathcal{L}_X^{L^M}\sigma)[e,n] + 2\pi i \mu^{\nabla^N}_{X_\mathfrak{k}}(n)\sigma[e,n].$$

Here $X = X_\mathfrak{k} + X_\mathfrak{p}$ according to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The claim follows. $\square$

Ther assumption that G is reductive is used in the following step in the proof of Proposition 3.13.

LEMMA 5.4. *If G is reductive and $\eta \in \mathfrak{k}^*$, then*

$$(G \cdot \eta) \cap \mathfrak{k}^* = K \cdot \eta.$$

Proof. If G is reductive, we may assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition. Fix $X \in \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal commutative subalgebra, containing X. For a restricted root α for $(\mathfrak{g}, \mathfrak{a})$, we denote the corresponding restricted root space by $\mathfrak{g}_\alpha$. We set $\mathfrak{p}_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$. Consider the function

$$f(x) = \frac{\sinh(h)}{x} = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j+1)!}.$$

Since it is even, we have for all $Y_\alpha \in \mathfrak{p}_\alpha$,

$$f(\mathrm{ad}(X))Y_\alpha = f(\langle \alpha, X\rangle)Y_\alpha. \tag{5.10}$$

Now suppose that $\mathrm{Ad}^*(\exp X)\eta \in \mathfrak{k}^*$. This element equals

$$e^{\mathrm{ad}^*(X)}\eta = \cosh(\mathrm{ad}^*(X))\eta + \sinh(\mathrm{ad}^*(X))\eta.$$

The first term on the right hand side lies in $\mathfrak{k}^*$, while the second lies in $\mathfrak{p}^*$. The assumption that $\mathrm{Ad}^*(\exp X)\eta \in \mathfrak{k}^*$ is therefore equivalent to

$$f(\mathrm{ad}^*(X))\,\mathrm{ad}^*(X)\eta = \sinh(\mathrm{ad}^*(X))\eta = 0.$$

Note that $\xi := \mathrm{ad}^*(X)\eta \in \mathfrak{p}^*$. Identifying $\mathfrak{p} \cong \mathfrak{p}^*$ via the Killing form, we write

$$\xi = \sum_{\alpha > 0} \xi_\alpha$$

where $\xi_\alpha \in \mathfrak{p}_\alpha$, and the sum runs over a choice of a positive restricted root system. Then by (5.10),

$$\sum_\alpha f(\langle \alpha, X\rangle)\xi_\alpha = 0.$$

Since $f(x) \geq 1$ for all $x \in \mathbb{R}$, this implies that $\xi = 0$. Hence $\mathrm{Ad}^*(\exp(X))\eta = \eta$. The claim follows. □

Proof of Proposition 3.13. Let $\eta \in \mathfrak{k}^*$ be given. By Lemma 5.3 and the fourth point in Lemma 3.11, we have the relation (3.8) between μ^{∇^N} and μ^{∇^M}. This implies that the map

$$(\mu^{\nabla^N})^{-1}(K\cdot\eta)/K \to (\mu^{\nabla^M})^{-1}(G\cdot\eta)/G$$

mapping $K\cdot n$ to $G\cdot n$, for $n \in (\mu^{\nabla^N})^{-1}(K\cdot\eta)$, is well-defined. To see that it is injective, let $n, n' \in (\mu^{\nabla^N})^{-1}(K\cdot\eta)$, and suppose $G\cdot n = G\cdot n'$. Since the only elements in G preserving N lie in K, we then must have $K\cdot n = K\cdot n'$, so the map is indeed injective.

For surjectivity, we use the fact that, since G is reductive, Lemma 5.4 implies that

$$G\cdot\eta\cap\mathfrak{k}^* = K\cdot\eta. \tag{5.11}$$

Let $m \in (\mu^{\nabla^M})^{-1}(G\cdot\eta)$ be given; fix a $g\in G$ such that $\mu^\nabla(m) = g\cdot\eta$. Write $m = g'n$, for $g' \in G$ and $n \in N$. Then

$$g\cdot\eta = g'\mu^{\nabla^N}(n),$$

so $g'^{-1}g\eta$ lies in the left hand side of (5.11). Since this equals the right hand side, we have $n \in (\mu^{\nabla^N})^{-1}(K\cdot\eta)$. □

5.3. Spin^c**-reduction for fibred products.** We now turn to a proof of Proposition 3.12. For any group H acting on a manifold Y, we use the notation q_H for the quotient map $Y \to Y/H$. If $H < K$, we will write $\mathfrak{p}_Y$ for the trivial bundle $Y\times\mathfrak{p}\to Y$, on which H acts via the adjoint representation on $\mathfrak{p}$.

Proof of Proposition 3.12. Let $\xi \in \mathfrak{k}^*$ be a regular value of μ^{∇^N}. Since (3.8) holds, we have

$$(\mu^{\nabla^M})^{-1}(G\xi) = G\times_K (\mu^{\nabla^N})^{-1}(K\xi). \tag{5.12}$$

Because of this relation, it will be convenient to initially consider the restriction of TM to $(\mu^{\nabla^M})^{-1}(G\xi)$, rather than to $(\mu^{\nabla^M})^{-1}(\xi)$. Let

$$TN|_{(\mu^{\nabla^N})^{-1}(K\xi)} = q_K^*TN_\xi \oplus \mathcal{N}_N^{K\xi} \tag{5.13}$$

be a K-invariant splitting. By Lemma 5.5 below, we have a G-invariant splitting

$$TM|_{(\mu^\nabla)^{-1}(G\xi)} = q_G^*TM_\xi \oplus \mathcal{N}_M^{G\xi},$$

with

$$\mathcal{N}_M^{G\xi} = \left(G\times_K \mathcal{N}_N^{K\xi}\right) \oplus \left(G\times_K \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}\right).$$

By Lemma 5.6, the vector bundles

$$G \times_K \mathcal{N}_N^{K\xi}$$

and

$$G \times_K \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}$$

over $(\mu^{\nabla^M})^{-1}(G\xi) = G \times_K (\mu^{\nabla^N})^{-1}(K\xi)$ have G-equivariant Spin-structures. By Lemma 2.5 and Remark 2.6, these induce a G-equivariant Spin^c-structure on $\mathcal{N}_M^{G\xi}$ with equivariantly trivial determinant line bundle, i.e. a G-equivariant Spin-structure. Restricting all bundles from $(\mu^{\nabla^M})^{-1}(G\xi)$ to $(\mu^{\nabla^M})^{-1}(\xi)$, and group actions from G to G_ξ, we obtain a G_ξ-equivariant splitting

$$TM|_{(\mu^\nabla)^{-1}(\xi)} = q_{G_\xi}^* TM_\xi \oplus \mathcal{N}_M^\xi,$$

where $\mathcal{N}_M^\xi$ has a G_ξ-equivariant Spin-structure. □

It remains to prove Lemmas 5.5 and 5.6, used in the proof of Proposition 3.12.

LEMMA 5.5. *One has*

$$TM|_{(\mu^\nabla)^{-1}(G\xi)} = q_G^* TM_\xi \oplus \mathcal{N}_M^{G\xi}, \tag{5.14}$$

with

$$\mathcal{N}_M^{G\xi} = \big(G \times_K \mathcal{N}_N^{K\xi}\big) \oplus \big(G \times_K \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}\big),$$

and $\mathcal{N}_N^{K\xi}$ *as in* (5.13).

Proof. Because of (3.6) and (5.12), we see that

$$\begin{aligned} TM|_{(\mu^{\nabla^M})^{-1}(G\xi)} &= G \times_K \big(TN|_{(\mu^{\nabla^N})^{-1}(K\xi)} \oplus \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}\big) \\ &= G \times_K \big(q_K^* TN_\xi \oplus \mathcal{N}_N^{K\xi} \oplus \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}\big) \\ &= q_G^* TM_\xi \oplus \big(G \times_K \mathcal{N}_N^{K\xi}\big) \oplus \big(G \times_K \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}\big). \end{aligned}$$

□

LEMMA 5.6. *For a choice of the bundle* $\mathcal{N}_N^{K\xi}$ *as in* (5.13), *and hence for any such bundle, the vector bundles*

$$G \times_K \mathcal{N}_N^{K\xi}$$

and

$$G \times_K \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}$$

over $(\mu^{\nabla^M})^{-1}(G\xi) = G \times_K (\mu^{\nabla^N})^{-1}(K\xi)$ *have* G*-equivariant* Spin*-structures.*

Proof. Since K is compact, and ξ is a regular value of μ^{∇^N}, Proposition 3.5 and Example 3.7 imply that

$$TN|_{(\mu^{\nabla^N})^{-1}(\xi)} = q_{K_\xi}^* TN_\xi \oplus \mathcal{N}_N^\xi,$$

where $\mathcal{N}_N^\xi$ has a K_ξ-equivariant Spin-structure P_N^ξ. Set

$$\mathcal{N}_N^{K\xi} := K \cdot \mathcal{N}^\xi.$$

Then we have a K-equivariant vector bundle isomorphism

$$K \times_{K_\xi} \mathcal{N}_N^\xi \cong \mathcal{N}_N^{K\xi},$$

given by $[k, v] \mapsto T_n k(v)$, for $n \in (\mu^{\nabla^N})^{-1}(\xi)$, $v \in (\mathcal{N}_N^\xi)_n$ and $k \in K$. This extends to a G-equivariant isomorphism

$$G \times_{K_\xi} \mathcal{N}_N^\xi \cong G \times_K \mathcal{N}_N^{K\xi} \tag{5.15}$$

Now

$$P_N^{G\xi} := G \times_{K_\xi} P_N^\xi \to G \times_{K_\xi} (\mu^{\nabla^N})^{-1}(\xi) \cong (\mu^{\nabla^M})^{-1}(G\xi)$$

defines a Spin-structure on (5.15).

Furthermore, since the adjoint action by K on $\mathfrak{p}$ lifts to $\mathrm{Spin}(\mathfrak{p})$, the vector bundle $\mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)}$ has a K-equivariant Spin-structure

$$(\mu^{\nabla^N})^{-1}(K\xi) \times \mathrm{Spin}(\mathfrak{p}).$$

As above, this induces a G-equivariant Spin-structure on

$$G \times_K \mathfrak{p}_{(\mu^{\nabla^N})^{-1}(K\xi)} \to (\mu^{\nabla^M})^{-1}(G\xi).$$

□

5.4. Spin^c**-structures on** N_ξ **and** M_ξ**.** The last statement from Section 3 we prove is Proposition 3.14. As before, let $\xi \in \mathfrak{k}^*$ be a regular value of μ^{∇^N}, and let let the Spin^c-structure $P_N \to N$ be as in Proposition 3.10. To prove Proposition 3.14, we must show that the Spin^c-structures induced on N_ξ and M_ξ, induced by P_N and P_M respectively, via Propositions 3.5 and 3.12, coincide.

We first give a slightly different description of Spin^c-structures induced on reduced spaces from the expression (3.2).

Lemma 5.7. *In the setting of Lemma 3.3, the* Spin^c*-structure* P_{M_ξ} *induced on* M_ξ *equals*

$$P_{M_\xi} = \mathrm{Destab}_{\mathcal{N}^{G\xi}}\big(P_M|_{(\mu^\nabla)^{-1}(G\xi)}\big)/G,$$

where $\mathcal{N}^{G\xi} \to (\mu^\nabla)^{-1}(G\xi)$ *is a vector bundle with the property of* $\mathcal{N}_M^{G\xi}$ *in* (5.14), *and with a* G*-equivariant* Spin*-structure.*

Proof. By (3.2) and Lemma 3.15, we have

$$\begin{aligned} P_{M_\xi} &= \mathrm{Destab}_{\mathcal{N}^\xi}\big(P_M|_{(\mu^\nabla)^{-1}(\xi)}\big)/G_\xi \\ &= \big(G \times_{G_\xi} \mathrm{Destab}_{\mathcal{N}^\xi}\big(P_M|_{(\mu^\nabla)^{-1}(\xi)}\big)\big)/G \\ &= \mathrm{Destab}_{G\times_{G_\xi}\mathcal{N}^\xi}\big(G \times_{G_\xi} \big(P_M|_{(\mu^\nabla)^{-1}(\xi)}\big)\big)/G. \end{aligned}$$

Here $\mathcal{N}^\xi \to (\mu^\nabla)^{-1}(\xi)$ has a G_ξ-equivariant Spin-structure $P_{\mathcal{N}^\xi}$.

Similarly to the proof of Lemma 5.6, set $\mathcal{N}^{G\xi} := G \cdot \mathcal{N}^{\xi}$. Then

$$G \times_{G_\xi} \mathcal{N}^{\xi} \cong \mathcal{N}^{G\xi}.$$

The left hand side has the G-equivariant Spin-structure $G \times_{G_\xi} P_{\mathcal{N}^\xi}$. Since also

$$G \times_{G_\xi} \left(P_M|_{(\mu^\nabla)^{-1}(\xi)}\right) \cong P_M|_{(\mu^\nabla)^{-1}(G\xi)}.$$

the claim follows. □

Proof of Proposition 3.14. Let $P_{N_\xi} \to N_\xi$ be the Spin^c-structure on N_ξ induced by P_N because of Proposition 3.5, and let $P_{M_\xi} \to M_\xi$ be the Spin^c-structure on M_ξ induced by P_M because of Proposition 3.12. We saw in Proposition 3.10 that

$$P_M = G \times_K \mathrm{Stab}_{\mathfrak{p}_N}(P_N).$$

Let $\mathcal{N}_M^{G\xi}$ and $\mathcal{N}_N^{K\xi}$ be as in Lemma 5.5. Then, by Lemma 5.7,

$$\begin{aligned}
P_{M_\xi} &= \mathrm{Destab}_{\mathcal{N}_M^{G\xi}}\left(P_M|_{(\mu^{\nabla^M})^{-1}(G\xi)}\right)/G \\
&= \mathrm{Destab}_{\mathcal{N}_M^{G\xi}}\left((G \times_K \mathrm{Stab}_{\mathfrak{p}_N}(P_N))|_{(\mu^{\nabla^M})^{-1}(G\xi)}\right)/G \\
&= \mathrm{Destab}_{\mathcal{N}_M^{G\xi}}\left(\mathrm{Stab}_{G\times_K\mathfrak{p}_N}\left(G \times_K (P_N|_{(\mu^{\nabla^N})^{-1}(K\xi)})\right)\right)/G \\
&= \mathrm{Destab}_{G\times_K\mathcal{N}_N^{K\xi}}\left(G \times_K (P_N|_{(\mu^{\nabla^N})^{-1}(K\xi)})\right)/G.
\end{aligned}$$

In the third equality, we have used the first point of Lemma 3.15 and (5.12). In the last equality, we applied Lemmas 2.8 and 5.5. By the second point of Lemma 3.15, we conclude that

$$\begin{aligned}
P_{M_\xi} &= G \times_K \left(\mathrm{Destab}_{\mathcal{N}_N^{K\xi}}(P_N|_{(\mu^{\nabla^N})^{-1}(K\xi)})\right)/G \\
&= \left(\mathrm{Destab}_{\mathcal{N}_N^{K\xi}}(P_N|_{(\mu^{\nabla^N})^{-1}(K\xi)})\right)/K \\
&= P_{N_\xi},
\end{aligned}$$

by Lemma 5.7 (applied to the action by K on N). □

5.5. Quantisation commutes with induction. Together with the constructions of Spin^c-structures proved so far in this section, the quantisation commutes with induction techniques of [17, 18] allow us to deduce Theorem 4.6 from Paradan and Vergne's result, Theorem 4.2.

We now suppose that M/G, and hence N is *compact.* The connections ∇^N and ∇^M induce Dirac operators on N and M, which can be used to define the quantisations of these manifolds. After the quantisation commutes with induction results of [17] (in the symplectic setting) and [18] (in the presymplectic setting), the following Spin^c-version of this principle is perhaps the most natural and general.

THEOREM 5.8 (Spin^c-quantisation commutes with induction). *In the setting of Proposition 3.10, the Dirac induction map* D-Ind$_K^G$ *maps the* Spin^c*-quantisation of* N *to the* Spin^c*-quantisation of* M*:*

$$\text{D-Ind}_K^G\left(Q_K^{\mathrm{Spin}^c}(N)\right) = Q_G^{\mathrm{Spin}^c}(M)_r \quad \in K_*(C_r^*G).$$

Proof. Let $K_*^K(N)$ and $K_*^G(M)$ be the equivariant K-homology groups [2] of N and M, respectively. In Theorem 4.6 in [17] and Theorem 4.5 in [18], a map

$$\text{K-Ind}_K^G \colon K_*^K(N) \to K_*^G(M)$$

is constructed, such that the following diagram commutes:

$$\begin{array}{ccc} K_*^G(M) & \xrightarrow{r_* \circ G\text{-index}} & K_*(C_r^*G) \\ \uparrow {\scriptstyle \text{K-Ind}_K^G} & & \uparrow {\scriptstyle \text{D-Ind}_K^G} \\ K_*^K(N) & \xrightarrow{K\text{-index}} & R(K). \end{array}$$

Here, as before, G-index is the analytic assembly map. The map K-index is the analytic assembly map for the action by K on N, which coincides with the usual equivariant index.

In Section 6 of [17], it is shown that the map K-Ind_K^G maps the class

$$[D_N] \in K_0^K(N)$$

of to the Spin^c-Dirac operator D_N on N, to the class

$$[D_M] \in K_d^G(M)$$

of the Spin^c-Dirac operator D_M on M. Although in [17] the symplectic setting is considered, the arguments in Section 6 of that paper are stated purely in terms of Spin^c-structures. Hence they apply in this more general setting, and we conclude that

$$\begin{aligned} \text{D-Ind}_K^G\big(Q_K^{\text{Spin}^c}(N)\big) &= \text{D-Ind}_K^G\big(K\text{-index}[D_N]\big) \\ &= r_* \circ G\text{-index}\big(\text{K-Ind}_K^G[D_N]\big) \\ &= r_* \circ G\text{-index}[D_M] \\ &= Q_G^{\text{Spin}^c}(M)_r. \end{aligned}$$

□

Theorem 4.6 follows by combining Theorem 5.8, Proposition 3.10, Proposition 3.14, and Paradan and Vergne's Theorem 4.2.

Proof of Theorem 4.6. By Proposition 3.10, Theorem 5.8 and Theorem 4.2, we have

$$Q_G^{\text{Spin}^c}(M)_r = \text{D-Ind}_K^G\big(Q_K^{\text{Spin}^c}(N)\big) = \sum_{\lambda \in \Lambda_+ + \rho_K} m_\lambda[\lambda],$$

with m_λ as in (4.2), where $Q^{\text{Spin}^c}(M_\xi)$ is replaced by $Q^{\text{Spin}^c}(N_\xi)$ for all ξ that occur. By Definition 4.5, these two quantisations are equal if ξ is a singular value of μ^{∇^N}. If ξ is a regular value of this map, they are equal by Proposition 3.14, and the claim follows. □

Part III. Non-cocompact actions.

6. The results on non-cocompact actions. The main result in this paper for untwisted Spin^c-Dirac operators, for possibly non-cocompact actions and reduction at zero, is Theorem 6.8. We state it in Subsection 6.2, and prove it in Sections 7 and 8. The generalisation of this result to Spin^c-Dirac operators twisted by vector bundles, Theorem 6.12, is stated in Subsection 6.4. It is proved in Section 9.

While the proof of Theorem 4.6 in Section 5 was based on Paradan and Vergne's result in [30], our proofs of Theorems 6.8 and 6.12 are independent of their result.

To state a Spin^c-quantisation commutes with reduction result without assuming that M/G is compact, we recall some facts about the G-invariant, transversally L^2-index introduced in Section 4 of [15]. We now suppose that G is *unimodular*, and fix a left- and right-invariant Haar measure dg on G.

6.1. The invariant, transversally L^2-index. The definition of the invariant, transversally L^2-index involves *cutoff functions*.

Definition 6.1. Let G be a unimodular locally compact group acting properly on a locally compact Hausdorff space X. A *cutoff function* is a continuous function f on X such that the support of f intersects every G-orbit in a compact set, and for all $x \in X$, one has

$$\int_G f(gx)^2 \, dg = 1,$$

with respect to a Haar measure dg on G.

It is shown in Proposition 8 in Section 2.4 of Chapter 7 in [5] that cutoff functions exist.

Let $E \to M$ be a G-equivariant vector bundle, equipped with a G-invariant metric. Let $L^2(E)$ be the L^2-space of sections of E, with respect to this metric, and the density on M associated to the Riemannian metric induced by the Spin^c-structure.

Definition 6.2. The space $L^2_T(E)$ of *transversally L^2-sections* of E is the space of measurable sections s of E such that $fs \in L^2(E)$ for all cutoff functions f on M, up to equality almost everywhere.

One can show that for a G-invariant transversally L^2-section $s \in L^2_T(E)^G$, the L^2-norm of fs does not depend on the cutoff function f (see Lemma 4.4 in [15]). This turns the G-invariant part $L^2_T(E)^G$ of $L^2_T(E)$ into a Hilbert space.

Let D be a G-equivariant (differential) operator on $\Gamma^\infty(E)$. Suppose E is $\mathbb{Z}_2$-graded, and that D is odd with respect to this grading.

Definition 6.3. The *transversally L^2-kernel* of D is

$$\ker_{L^2_T}(D) := \ker(D) \cap L^2_T(E).$$

If the G-invariant part $\ker_{L^2_T}(D)^G$ of $\ker_{L^2_T}(D)$ is finite-dimensional, then the *G-invariant, transversally L^2-index* of D is the integer

$$\operatorname{index}^G_{L^2_T}(D) := \dim\bigl(\ker_{L^2_T}(D^+)^G\bigr) - \dim\bigl(\ker_{L^2_T}(D^-)^G\bigr),$$

where $D^\pm$ is the restriction of D to the even or odd part of $\Gamma^\infty(E)$.

Remark 6.4. If G is compact, then the transversally L^2-index of D is the G-invariant part of its L^2-index. If M/G is compact, then the transversally L^2-index of D is the index of D restricted to G-invariant smooth sections.

6.2. Invariant quantisation. As shown in [15], the transversally L^2-index of Definition 6.3 allows one to make sense of quantisation and reduction without assuming M, G or M/G to be compact. There will only be a cocompactness assumption on the set of zeros of a vector field on M. This vector field is defined in terms of the momentum map and a *family* of inner products on $\mathfrak{g}^*$, by which we mean a metric on the vector bundle

$$\mathfrak{g}^*_M := M \times \mathfrak{g}^* \to M,$$

with a certain G-invariance property. Using such a family of inner products, rather than a single one, allows us to define a suitable G-invariant vector field, despite the fact that $\mathfrak{g}$ does not admit an $\mathrm{Ad}(G)$-invariant inner product in general.

Let $\{(-,-)_m\}_{m\in M}$ be a G-invariant metric on the vector bundle $\mathfrak{g}^*_M$, with respect to the G-action given by

$$g\cdot(m,\xi) = (g\cdot m, \mathrm{Ad}^*(g)\xi),$$

for $g \in G$, $m \in M$ and $\xi \in \mathfrak{g}^*$. Such a metric exists by Lemma 2.1 in [15]. Consider the map

$$(\mu^\nabla)^*\colon M \to \mathfrak{g}$$

defined by

$$\langle \xi, (\mu^\nabla)^*(m)\rangle = \big(\xi, \mu^\nabla(m)\big)_m, \tag{6.1}$$

for all $\xi \in \mathfrak{g}^*$ and $m \in M$. This induces a G-invariant vector field v^∇ on M, given by

$$v^\nabla_m := 2\big((\mu^\nabla)^*(m)\big)^M_m = 2\left.\frac{d}{dt}\right|_{t=0} \exp\big(t(\mu^\nabla)^*(m)\big)m, \tag{6.2}$$

for $m \in M$. (The factor 2 was included for consistency with [15, 36].) A central assumption we make is that the critical set $\mathrm{Crit}(v^\nabla)$ of zeros of v^∇ is *cocompact*. This implies that M_0 is compact.

Recall the definition of the Dirac operator D_p in Subsection 2.1, for a $p \in \mathbb{N}$. We will apply the invariant, transversally L^2-index to a Witten-type deformation of D_p.

Definition 6.5. For $p \in \mathbb{N}$ and $t \in \mathbb{R}$, the *deformed Dirac operator* $D_{p,t}$ is the operator

$$D_{p,t} := D_p + \frac{it}{2}c(v^\nabla)$$

on $\Gamma^\infty(\mathcal{S}_p)$.

Note that

$$D_{1,1} = D + \frac{i}{2}c(v^\nabla).$$

In general, $D_{p,t}$ is G-equivariant, by G-invariance of v^∇. Suppose that M is even-dimensional. Then $\mathcal{S}_p$ is $\mathbb{Z}_2$-graded, and $D_{p,t}$ is odd with respect to this grading.

Suppose M is complete in the Riemannian metric induced by the Spin^c-structure. It turns out that in this non-cocompact setting, the invariant, transversally L^2-index of $D_{p,t}$ is well-defined for large enough t.

Theorem 6.6. *One can choose the metric on $\mathfrak{g}^*_M$ in such a way that for all $t \geq 1$, the G-invariant part of $\ker_{L^2_T}(D_{p,t})$ is finite-dimensional, for all $p \in \mathbb{N}$.*

This allows us to define the G-invariant part of Spin^c-quantisation.

Definition 6.7. The *G-invariant* Spin^c*-quantisation of M* with respect to the given Spin^c-structure, and the connection ∇ on L, is

$$Q^{\mathrm{Spin}^c}(M)^G := \mathrm{index}^G_{L^2_T}(D_{1,1}).$$

Suppose 0 is a Spin^c-regular value of μ^∇. By Proposition 3.5 and Example 3.6, this is true for example if 0 is a regular value of μ^∇ and G is semisimple or Abelian. Alternatively, by Proposition 3.12, it is enough that 0 is a regular value of a Spin^c-moment map $\mu^{\nabla^N} : N \to \mathfrak{k}^*$ on a Spin^c-slice N. Since M_0 is compact by cocompactness of $\mathrm{Crit}(v^\nabla)$, Definition 4.1 applies, and one has

$$Q^{\mathrm{Spin}^c}(M_0) = \mathrm{index}(D_{M_0}).$$

Analogously to the symplectic case [15] and the compact case (4.3), one expects Spin^c-quantisation to commute with reduction in this non-cocompact setting. We will prove the following version of this statement.

THEOREM 6.8 (Spin^c-quantisation commutes with reduction; non-cocompact case). *Suppose G acts freely*[4] *on $(\mu^\nabla)^{-1}(0)$ (rather than just locally freely). Then there exists a G-equivariant* Spin^c*-structure on M and a connection on the corresponding determinant line bundle, such that, for these choices,*

$$Q^{\mathrm{Spin}^c}(M)^G = Q^{\mathrm{Spin}^c}(M_0) \quad \in \mathbb{Z}. \tag{6.3}$$

REMARK 6.9. The choice of Spin^c-structure in Theorem 6.8 amounts to taking large enough tensor powers of the determinant line bundle of a given Spin^c-structure. I.e. one starts with an initial Spin^c-structure $P \to M$ with determinant line bundle $L \to M$, and the result holds for Spin^c-structures with determinant line bundle $L^p \to M$, for p large enough. So if L is not a torsion class in $H^2(M;\mathbb{Z})$, then the result holds for infinitely many Spin^c-structures.

The connection on the determinant line bundle L^p used can be any connection induced by a connection on L (and the minimal value of p depends on this inital connection on L).

REMARK 6.10. We could prove Theorem 6.6 by referring to [7] and using the elliptic regularity arguments in [15]. We will give an independent proof of finite-dimentionality of $\ker_{L^2_T}(D_{p,t})^G$, however, as a by-product of the localisation arguments needed to prove Theorem 6.8.

6.3. ρ-shifts and asymptotic results. If M and G are compact, one may take $t_0 = 0$ in Definition 6.7. Then $Q^{\mathrm{Spin}^c}(M)^G$ is the invariant part of (4.1), which by (4.3) equals $Q(M_{\rho_K})$. On the other hand, Theorem 6.8 states that, for a certain G-equivariant Spin^c-structure on M and a connection on its determinant line bundle,

$$Q^{\mathrm{Spin}^c}(M)^G = Q^{\mathrm{Spin}^c}(M_0).$$

Hence, apparently, one has

$$Q(M_0) = Q(M_{\rho_K}) \tag{6.4}$$

for this choice of Spin^c-structure and connection.

[4]It will turn out that, for a natural choice of ∇' on the determinant line bundle of the Spin^c-structure used, the Spin^c-momentum maps for ∇ and ∇' differ by a nonzero factor, so that the condition that G acts freely on $(\mu^\nabla)^{-1}(0)$ is the same for the two connections.

This potential contradiction can be resolved, by noting that, for the Spin^c-structure and the connection ∇' used, one has

$$\mu^{\nabla'} = p\mu^{\nabla},$$

for a connection ∇ on the determinant line bundle of a Spin^c-structure initially given, and a large enough integer p. (See (8.9) in the proof of Proposition 8.6.) For any $\xi \in \mathfrak{g}^*$, let M_ξ and M'_ξ be the reduced spaces at ξ for the momentum maps μ^∇ and $\mu^{\nabla'}$, respectively. Then

$$M'_\xi = M_{\xi/p}.$$

In particular, $M'_0 = M_0$, and $M'_{\rho_K} = M_{\rho_K/p}$.

The statement (6.4) is therefore that

$$Q(M_{\rho_K/p}) = Q(M_0),$$

for p large enough. In the symplectic setting, this follows from the fact that $Q(M_\xi)$ is independent of small variations of ξ (see Theorem 2.5 in [26] if the action is free on $(\mu^\nabla)^{-1}(\xi)$, or [41] for a holomorphic version). More generally, if M is of the form $M = G \times_K N$ as in Subsection 4.3, then by Proposition 3.14, one has

$$Q(M_\xi) = Q(N_\xi),$$

which is independent of small variations of ξ if N is a compact Hamiltonian K-manifold (but M is not necessarily symplectic).

In the general non-cocompact setting of Subsection 6.2, this leads one to expect that, if μ^∇ is G-proper (in the sense that the preimage of any cocompact set is cocompact), there is an open neighbourhood U of 0 in $\mathfrak{g}^*$, such that for all Spin^c-regular values $\xi \in U$ of μ^∇,

$$Q(M_\xi) = Q(M_0).$$

The above arguments show that, for 'asymptotic' quantisation commutes with reduction results, reduction at zero (or possibly a nearby regular value of the momentum map) is really the only natural case to consider.

6.4. An index formula for twisted Spin^c**-Dirac operators.** The main results on Spin^c-Dirac operators in the non-cocompact case, Theorems 6.6 and 6.8, generalise to Spin^c-Dirac operators twisted by arbitrary vector bundles. We use this to obtain an index formula for Braverman's analytic index of such operators, Theorem 6.12, expressing it in terms of characteristic classes on M_0. A potentially interesting feature of this formula is that it involves localisation to $(\mu^\nabla)^{-1}(0)$. In the setting we consider, where the manifold M, the group G acting on it, and the quotient M/G may all be noncompact, it is unlikely that there is a topological expression for the index of (twisted) Spin^c-Dirac operators in terms of characteric classes on M. However, localisation to $(\mu^\nabla)^{-1}(0)$ allows us to still define a meaningful topological index, as an integral over the compact space M_0.

In the compact setting, the index of any elliptic operator on a Spin^c-manifold equals the index of a twisted Spin^c-Dirac operator. Hence index formulas for the latter kind of operators immediately generalise to the former. In the noncompact setting we consider here, such a principle is not (yet) available. Still, the index formula we obtain for twisted Spin^c-Dirac operators strongly suggests a more general underlying equality of topological and analytic indices.

Fix $p \in \mathbb{N}$. We retain all other notation used previously. In particular, we have the connection $\nabla^{\mathcal{S}_p}$ on $\mathcal{S}_p$, and the Spin^c-moment map $\mu^\nabla \colon M \to \mathfrak{g}^*$ induced by a connection ∇ on the determinant line bundle $L \to M$. In addition, consider a Hermitian, G-equivariant vector bundle $E \to M$. Let ∇^E be a Hermitian, G-invariant connection on E. Consider the connection

$$\nabla^{\mathcal{S}_p \otimes E} := \nabla^{\mathcal{S}_p} \otimes 1_E + 1_{\mathcal{S}_p} \otimes \nabla^E$$

on $\mathcal{S}_p \otimes E$.

Definition 6.11. The *twisted* Spin^c*-Dirac operator* associated to ∇ and ∇^E is the composition

$$D_p^E \colon \Gamma^\infty(\mathcal{S}_p \otimes E) \xrightarrow{\nabla^{\mathcal{S}_p \otimes E}} \Omega^1(M; \mathcal{S}_p \otimes E) \xrightarrow{c \otimes 1_E} \Gamma^\infty(\mathcal{S}_p \otimes E).$$

For $t \in \mathbb{R}$, the *deformed* Spin^c*-Dirac operator* twisted by E via ∇^E is the operator

$$D_{p,t}^E := D_p^E + \frac{it}{2} c(v^\nabla) \otimes 1_E,$$

Theorems 6.6 and 6.8 generalise to the operator D_p^E as follows.

Theorem 6.12. *Suppose that* 0 *is a* Spin^c*-regular value of* μ^∇*, and that* G *acts freely on* $(\mu^\nabla)^{-1}(0)$*. Then there are a* G*-invariant metric on* $\mathfrak{g}^*_M$ *and a* $p_E \in \mathbb{N}$ *such that if* $p \geq p_E$*, then*

$$\left(\ker_{L^2_T} D_{p,1}^E\right)^G$$

is finite-dimensional, and one has

$$\operatorname{index}^G_{L^2_T} D_{p,1}^E = \operatorname{index} D_{M_0}^{E_0} = \int_{M_0} \operatorname{ch}(E_0) e^{\frac{p}{2} c_1(L_0)} \hat{A}(M_0).$$

Here $E_0 := (E|_{(\mu^\nabla)^{-1}(0)})/G$ *and* $L_0 := (L|_{(\mu^\nabla)^{-1}(0)})/G$*.*

In the compact case, results analogous to Theorem 6.12 were obtained in [32, 37]. Theorem 6.12 will be proved in Section 9. Some applications are given in Subsection 10.4.

7. The square of the deformed Dirac operator. We now turn to proving Theorems 6.6 and 6.8. As in [15, 36], the starting point is an explicit formula, given in Theorem 7.1, for the square of the deformed Dirac operator $D_{p,t}$ of Definition 6.5. This is the basis of the localisation estimates, Propositions 8.1 and 8.2, that will be used to prove Theorems 6.6 and 6.8.

We continue using the notation of Section 2 and Subsection 6.2. We will also write d_M and d_G for the dimensions of M and G, respectively. We denote the Riemannian metric on M induced by the given Spin^c-structure by $(-,-)$. The associated Levi–Civita connection on TM will be denoted by ∇^{TM}.

7.1. A Bochner formula. Let us fix some notation that will be used in the expression for $D_{p,t}^2$. Let $\{h_1, \ldots, h_{d_G}\}$ be an orthonormal frame for $\mathfrak{g}_M^*$ with respect to a given G-invariant metric. (Such a frame can be obtained for example by applying the Gram-Schmidt procedure to a constant frame.) Let $\{h_1^*, \ldots, h_{d_G}^*\}$ be the dual frame of $M \times \mathfrak{g} \to M$. Let $\mu_1^\nabla, \ldots, \mu_{d_G}^\nabla \in C^\infty(M)$ be the functions such that

$$\mu^\nabla = \sum_{j=1}^{d_G} \mu_j^\nabla h_j, \tag{7.1}$$

so that

$$(\mu^\nabla)^* = \sum_{j=1}^{d_G} \mu_j^\nabla h_j^*,$$

and

$$v^\nabla = 2 \sum_{j=1}^{d_G} \mu_j^\nabla V_j, \tag{7.2}$$

where V_j is the vector field given by

$$V_j(m) = \big(h_j^*(m)\big)_m^M, \tag{7.3}$$

at a point $m \in M$. Consider the norm-squared function $\mathcal{H}^\nabla$ of μ^∇, given by

$$\mathcal{H}^\nabla(m) = \|\mu^\nabla(m)\|_m^2 = \sum_{j=1}^{d_G} \mu_j^\nabla(m)^2. \tag{7.4}$$

Here $\|\cdot\|_m$ is the norm on $\mathfrak{g}^*$ induced by $(-,-)_m$.

We will use the operators $\mathcal{L}_{h_j^*}^{\mathcal{S}_p}$ on $\Gamma^\infty(\mathcal{S}_p)$ given by

$$\big(\mathcal{L}_{h_j^*}^{\mathcal{S}_p} s\big)(m) = \big(\mathcal{L}_{h_j^*(m)}^{\mathcal{S}_p} s\big)(m).$$

Finally, for any vector field u on M, consider the commutator vector field $[u, (h_j^*)^M]$, given by

$$[u, (h_j^*)^M](m) = \big[u, h_j^*(m)^M\big](m).$$

Here $h_j^*(m)^M$ is the vector field induced by $h_j^*(m) \in \mathfrak{g}$, and $[-,-]$ is the Lie bracket of vector fields. Importantly, for fixed m, the vector fields V_j and $h_j^*(m)^M$ are equal at the point m, but not necessarily at other points.

The square of $D_{p,t}$ has the following form.

THEOREM 7.1. *One has*

$$D_{p,t}^2 = D_p^2 + tA + (2p+1)2\pi t\mathcal{H}^\nabla + \frac{t^2}{4}\|v^\nabla\|^2 - 2it\sum_{j=1}^{d_G} \mu_j^\nabla \mathcal{L}_{h_j^*}^{\mathcal{S}_p},$$

where A is a vector bundle endomorphism of $\mathcal{S}_p$, given in terms of a local orthonormal frame $\{e_1, \ldots, e_{d_M}\}$ of TM by

$$A := \frac{i}{4}\sum_{k=1}^{d_M} c(e_k)c\big(\nabla^{TM}_{e_k} v^{\nabla}\big) + \frac{i}{2}\sum_{j=1}^{d_G} c(\operatorname{grad}\mu^{\nabla}_j)c(V_j)$$
$$-\frac{i}{2}\sum_{j=1}^{d_G}\sum_{k=1}^{d_M} \mu^{\nabla}_j c(e_k)c\big([e_k, (h_j^*)^M - V_j]\big). \tag{7.5}$$

7.2. Lie derivatives of spinors. An important ingredient of the proof of Theorem 7.1 is an expression for the Lie derivative of sections of $\mathcal{S}_p$.

LEMMA 7.2. *Let $X \in \mathfrak{g}$. Then, as operators on $\Gamma^\infty(\mathcal{S}_p)$, one has*

$$\mathcal{L}^{\mathcal{S}_p}_X = \nabla^{\mathcal{S}_p}_{X^M} - B_X - (2p+1)\pi i \mu^{\nabla}_X,$$

where, in terms of a local orthonormal frame $\{e_1, \ldots, e_{d_M}\}$ of TM,

$$B_X := \frac{1}{4}\sum_{k,l=1}^{d_M} \big(\nabla_{e_k} X^M, e_l\big) c(e_k)c(e_l).$$

Proof. Let $X \in \mathfrak{g}$ be given. We give a local argument on an open subset $U \subset M$, using the decomposition (2.1) of $\mathcal{S}_P|_U$. Let $\nabla^{L|_U^{1/2}}$ be the connection on $L|_U^{1/2} \to U$ induced by ∇. We first note that

$$\mathcal{L}^{L|_U^{1/2}}_X = \nabla^{L|_U^{1/2}}_{X^M} - i\pi\mu^{\nabla}_X|_U. \tag{7.6}$$

Indeed, if $t_1, t_2 \in \Gamma^\infty\big(L|_U^{1/2}\big)$, then by definition of μ^∇,

$$\begin{aligned}
&\big(\mathcal{L}^{L|_U^{1/2}} t_1\big) \otimes t_2 + t_1 \otimes \big(\mathcal{L}^{L|_U^{1/2}} t_2\big)\\
&= \mathcal{L}^{L|_U}_X (t_1 \otimes t_2)\\
&= \big(\nabla_{X^M} - 2\pi i \mu^{\nabla}_X\big)(t_1 \otimes t_2)\\
&= \Big(\big(\nabla^{L|_U^{1/2}}_{X^M} - i\pi\mu^{\nabla}_X\big)t_1\Big) \otimes t_2 + t_1 \otimes \Big(\big(\nabla^{L|_U^{1/2}}_{X^M} - i\pi\mu^{\nabla}_X\big)t_2\Big).
\end{aligned}$$

Let $s \in \Gamma^\infty(\mathcal{S}^{\mathrm{Spin}}_U)$. Then

$$\mathcal{L}^{\mathcal{S}^{\mathrm{Spin}}_U}_X s = \nabla^{\mathcal{S}^{\mathrm{Spin}}_U}_{X^M} s - B_X s. \tag{7.7}$$

Let $t_1, \ldots, t_{2p+1} \in \Gamma^\infty\big(L|_U^{1/2}\big)$. Then

$$s \otimes t_1 \otimes \cdots \otimes t_{2p+1} \in \Gamma^\infty\big(\mathcal{S}^{\mathrm{Spin}}_U \otimes L|_U^{p+1/2}\big) = \Gamma^\infty(\mathcal{S}_p|_U).$$

Because of (7.6) and (7.7), one has

$$\begin{aligned}
&\mathcal{L}_X^{\mathcal{S}_p}(s\otimes t_1\otimes\cdots\otimes t_{2p+1})\\
&= \big(\mathcal{L}_X^{\mathcal{S}_U^{\mathrm{Spin}}} s\big)\otimes t_1\otimes\cdots\otimes t_{2p+1} + s\otimes\left(\sum_{j=1}^{2p+1} t_1\otimes\cdots\otimes\big(\mathcal{L}_X^{L|_U^{1/2}} t_j\big)\otimes\cdots\otimes t_{2p+1}\right)\\
&= \big(\nabla_{X^M}^{\mathcal{S}_U^{\mathrm{Spin}}} s\big)\otimes t_1\otimes\cdots\otimes t_{2p+1} + s\otimes\left(\sum_{j=1}^{2p+1} t_1\otimes\cdots\otimes\big(\nabla_{X^M}^{L|_U^{1/2}} t_j\big)\otimes\cdots\otimes t_{2p+1}\right)\\
&\quad -\big(B_X+(2p+1)\pi i\mu_X\big)s\otimes t_1\otimes\cdots\otimes t_{2p+1}\\
&= \Big(\nabla_{X^M}^{\mathcal{S}_p} - B_X - (2p+1)\pi i\mu_X^{\nabla}\big)\Big)\, s\otimes t_1\otimes\cdots\otimes t_{2p+1}.
\end{aligned}$$

□

7.3. Proof of the Bochner formula. Using Lemma 7.2, we can prove Theorem 7.1.

As in the equality (1.26) in [36], the fact that $\nabla^{\mathcal{S}_p}$ satisfies a Leibniz rule with respect to the Clifford action (see e.g. Proposition 4.11 in [22]) implies that

$$D_{p,t}^2 = D_p^2 + \frac{it}{2}\sum_{k=1}^{d_M} c(e_k)c(\nabla_{e_k}^{TM} v^{\nabla}) - it\nabla_{v^{\nabla}}^{\mathcal{S}_p} + \frac{t^2}{4}\|v^{\nabla}\|^2. \tag{7.8}$$

The main part of the proof of Theorem 7.1 is a computation of an expression for the first-order term $\nabla_{v^{\nabla}}^{\mathcal{S}_p}$.

By (7.2), we have

$$\nabla_{v^{\nabla}}^{\mathcal{S}_p} = 2\sum_{j=1}^{d_G}\mu_j^{\nabla}\nabla_{V_j}^{\mathcal{S}_p}.$$

By Lemma 7.2, one has for all $s\in\Gamma^\infty(\mathcal{S}_p)$, all $m\in M$ and all j,

$$\big(\nabla_{V_j}^{\mathcal{S}_p}s\big)(m) = \big(\nabla_{h_j^*(m)^M}^{\mathcal{S}_p}s\big)(m) = \Big(\Big(\mathcal{L}_{h_j^*(m)}^{\mathcal{S}_p} + B_{h_j^*(m)} + (2p+1)\pi i\mu_j^{\nabla}\Big)\, s\Big)(m).$$

Multiplying this identity by $2\mu_j^{\nabla}(m)$ and summing over j, we obtain

$$\begin{aligned}
\big(\nabla_{v^{\nabla}}^{\mathcal{S}_p}s\big)(m) = {}&\left(\left(2\sum_{j=1}^{d_G}\mu_j^{\nabla}\mathcal{L}_{h_j^*}^{\mathcal{S}_p}\right)s\right)(m)\\
&+\left(\left(2\sum_{j=1}^{d_G}\mu_j^{\nabla}B_{h_j^*(m)}\right)s\right)(m) + \big((2p+1)2\pi i\mathcal{H}^{\nabla}s\big)(m).
\end{aligned}\tag{7.9}$$

Lemma B.2 in [15] allows us to compute

$$\begin{aligned}\left(\left(2\sum_{j=1}^{d_G}\mu_j^\nabla B_{h_j^*(m)}\right)s\right)(m) &= \frac{1}{2}\sum_{j=1}^{d_G}\mu_j^\nabla\sum_{k,l=1}^{d_M}(\nabla_{e_k}h_j^*(m)^M, e_l)c(e_k)c(e_l)\\ &= \left(\left(\frac{1}{4}\sum_{k=1}^{d_M}c(e_k)c\big(\nabla^{TM}_{e_k}v^\nabla\big) - \frac{1}{2}\sum_{j=1}^{d_G}c(\operatorname{grad}\mu_j^\nabla)c(V_j)\right.\right.\\ &\qquad\left.\left.+\frac{1}{2}\sum_{j=1}^{d_G}\sum_{k=1}^{d_M}\mu_j^\nabla c(e_k)c\big([e_k,(h_j^*)^M - V_j]\big)\right)s\right)(m)\\ &= i\left(\left(A - \frac{it}{2}\sum_{k=1}^{d_M}c(e_k)c\big(\nabla^{TM}_{e_k}v^\nabla\big)\right)s\right)(m).\end{aligned}$$

Theorem 7.1 follows from this equality and (7.8) and (7.9).

REMARK 7.3. Lemma B.3 in [15] does not apply in the general Spin^c-case, so that $\operatorname{grad}\mu_j^\nabla$, which appears in the expression for the operator A, cannot be worked out further in the present setting.

7.4. An estimate for the operator A**.** To prepare for the localisation estimates in Section 8, we show that the operator A in Theorem 7.1 satisfies a certain estimate with respect to a rescaling of the metric on $\mathfrak{g}^*_M$ by a function.

For any positive, G-invariant smooth function $\psi \in C^\infty(M)^G$, consider the metric

$$\{\psi(m)(-,-)_m\}_{m\in M} \tag{7.10}$$

on $\mathfrak{g}^*_M$. Let A^ψ be the operator in Theorem 7.1, defined with respect to this metric. In the choice of the metric on $\mathfrak{g}^*_M$ in Proposition 8.3, we will use the following property of the dependence of the operator A^ψ on ψ.

LEMMA 7.4. *There are G-invariant, positive, continuous functions $F_1, F_2 \in C(M)^G$ such that for all G-invariant, positive smooth functions $\psi \in C^\infty(M)$, one has the pointwise estimate*

$$\|A^\psi\| \leq F_1\psi + F_2\|d\psi\|. \tag{7.11}$$

Proof. Let $\psi \in C^\infty(M)^G$ be a G-invariant, positive smooth function. With respect to the metric (7.10) rescaled by ψ, we use the orthonormal frame of $\mathfrak{g}^*_M$ made up of the functions

$$h_j^\psi := \frac{1}{\psi^{1/2}}h_j.$$

The dual frame of $M \times \mathfrak{g} \to M$ consists of the functions

$$(h_j^\psi)^* = \psi^{1/2}h_j^*.$$

Let $(\mu_j^\nabla)^\psi$ be defined like the functions μ_j^∇ in (7.1), with h_j replaced by h_j^ψ. Analogously, let V_j^ψ be the vector field defined like V_j in (7.3), with the same replacement. Then

$$\begin{aligned}(\mu_j^\nabla)^\psi &= \psi^{1/2}\mu_j^\nabla;\\ V_j^\psi &= \psi^{1/2}V_j.\end{aligned} \tag{7.12}$$

It follows for example from the latter two equalities and (7.2) that the vector field $(v^\nabla)^\psi$, defined like v^∇ with the metric on $\mathfrak{g}^*_M$ rescaled by ψ, equals

$$(v^\nabla)^\psi = \psi v^\nabla. \tag{7.13}$$

We start with some local computations for each term in the definition (7.5) of the operator A^ψ. Let $\{e_1, \ldots, e_{d_M}\}$ be a local orthonormal frame for TM. By (7.13), we have for all k,

$$\nabla^{TM}_{e_k}(v^\nabla)^\psi = \psi \nabla^{TM}_{e_k} v^\nabla + e_k(\psi) v^\nabla.$$

Hence

$$\begin{aligned}\left\| \frac{i}{4} \sum_{k=1}^{d_M} c(e_k) c\big(\nabla^{TM}_{e_k}(v^\nabla)^\psi\big) \right\| &\leq \frac{1}{4} \sum_{k=1}^{d_M} \big(\psi \|\nabla^{TM}_{e_k} v^\nabla\| + \|e_k(\psi)\| \|v^\nabla\|\big) \\ &\leq a_1 \psi + a_2 \|d\psi\|,\end{aligned}$$

with

$$\begin{aligned} a_1 &:= \frac{1}{4} \sum_{k=1}^{d_M} \|\nabla^{TM}_{e_k} v^\nabla\|; \\ a_2 &:= \frac{1}{4} d_M \|v^\nabla\|. \end{aligned}$$

Note that the function a_1 is not defined globally, and is not G-invariant on its domain in general. We will come back to this later.

Secondly, because of (7.12), we have

$$\begin{aligned} &\left\| \frac{i}{2} \sum_{j=1}^{d_G} c\big(\operatorname{grad} (\mu_j^\nabla)^\psi\big) c(V_j^\psi) \right\| \\ &\leq \frac{1}{2} \sum_{j=1}^{d_G} \Big(\psi \|\operatorname{grad} \mu_j^\nabla\| \, \|V_j\| + |\mu_j^\nabla| \, \psi^{1/2} \|\operatorname{grad} \psi^{1/2}\| \, \|V_j\| \Big). \end{aligned} \tag{7.14}$$

Since $\psi^{1/2} \|\operatorname{grad} \psi^{1/2}\| = \frac{1}{2}\|d\psi\|$, (7.14) is at most equal to

$$b_1 \psi + b_2 \|d\psi\|,$$

with

$$\begin{aligned} b_1 &:= \frac{1}{2} \sum_{j=1}^{d_G} \|\operatorname{grad} \mu_j^\nabla\| \, \|V_j\|; \\ b_2 &:= \frac{1}{4} \sum_{j=1}^{d_G} |\mu_j^\nabla| \, \|V_j\|. \end{aligned}$$

Finally, Lemma C.8 in [15] implies that

$$\big[e_k, \big((h_j^*)^\psi\big)^M - V_j^\psi\big] = \psi^{1/2} [e_k, (h_j^*)^M - V_j] - e_k(\psi^{1/2}) V_j.$$

Therefore,

$$\left\| -\frac{i}{2}\sum_{j=1}^{d_G}\sum_{k=1}^{d_M}(\mu_j^\nabla)^\psi c(e_k)c\bigl([e_k,\bigl((h_j^*)^\psi\bigr)^M - V_j^\psi]\bigr)\right\|$$
$$\leq \frac{1}{2}\sum_{j=1}^{d_G}\sum_{k=1}^{d_M}\Bigl(\psi|\mu_j^\nabla|\,\bigl\|[e_k,(h_j^*)^M - V_j]\bigr\| + \psi^{1/2}\|e_k(\psi^{1/2})\|\,|\mu_j^\nabla|\,\|V_j\|.\Bigr) \qquad (7.15)$$

Since

$$\psi^{1/2}\|e_k(\psi^{1/2})\| = \frac{1}{2}\|e_k(\psi)\| \leq \frac{1}{2}\|d\psi\|,$$

we find that (7.15) is at most equal to

$$c_1\psi + c_2\|d\psi\|,$$

with

$$c_1 := \frac{1}{2}\sum_{j=1}^{d_G}\sum_{k=1}^{d_M}|\mu_j^\nabla|\,\bigl\|[e_k,(h_j^*)^M - V_j]\bigr\|;$$
$$c_2 := \frac{d_M}{2}\sum_{j=1}^{d_G}|\mu_j^\nabla|\,\|V_j\|.$$

The functions a_j, b_j and c_j are not all defined globally and/or G-invariant. To get a global estimate for A, let $W \subset M$ be an open subset that intersects all G-orbits in nonempty, relatively compact sets. By Lemmas C.1 and C.2 in [15], there are G-invariant, positive, continuous functions F_1 and F_2 on M, and local orthonormal frames of TM around each point in W, such that on W, with respect to these frames, one has

$$a_1 + b_1 + c_1 \leq F_1;$$
$$a_2 + b_2 + c_2 \leq F_2.$$

Then the estimate (7.11) holds on W. Since both sides of (7.11) are G-invariant, and the definition of A is independent of the local orthonormal frame chosen, we get the desired estimate on all of M. □

8. Localisation estimates. Two localisation estimates are at the cores of the proofs of Theorems 6.6 and 6.8. These are Propositions 8.1 and 8.2 below. In the proofs of these estimates, we will not use the assumption that 0 is a Spin^c-regular value of μ^∇. They therefore also hold in the singular case. The regularity assumption is only needed to apply the arguments near $(\mu^\nabla)^{-1}(0)$ to obtain Theorem 6.8.

The localisation estimates are stated in terms of certain Sobolev norms.

8.1. Sobolev norms and estimates for $D_{p,t}$. Theorem 6.6 follows from the fact that for large t, the operator $D_{p,t}$ induces a Fredholm operator between certain Sobolev spaces. By an elliptic regularity argument, the index of this operator is precisely the G-invariant transversally L^2-index $\mathrm{index}^G_{L^2_T}$ of $D_{p,t}$. These Sobolev spaces and the index theory on them that we will use, were introduced in Section 4 of [15].

We will not need to go into the details of these spaces, but will refer to the relevant results in [15]. We do need certain ingredients of the definition of these spaces.

One of these is a smooth cutoff function f on M (see Definition 6.1). We will also consider *transversally compactly supported* sections of vector bundles, by which we mean sections whose support is mapped to a compact set by the quotient map $M \to M/G$. Let $\Gamma^\infty_{tc}(\mathcal{S}_p)^G$ be the space of G-invariant, smooth, transversally compactly supported sections of $\mathcal{S}_p$. For $k \in \mathbb{N}$, and $s, s' \in \Gamma^\infty_{tc}(\mathcal{S}_p)^G$, we set

$$(fs, fs')_k := \sum_{j=0}^{k} (fD_p^j s, fD_p^j s')_{L^2(\mathcal{S}_p)}. \tag{8.1}$$

(Note that $fD_p^j s$ and $fD_p^j s'$ are compactly supported for all j.) By Lemma 4.4 in [15], this inner product is independent of f, since s and s' are G-invariant. We will write $\|\cdot\|_k$ for the induced norm on $f\Gamma^\infty_{tc}(\mathcal{S}_p)^G$.

These Sobolev norms allow us to state the localisation estimates we will use. Fix a G-invariant open neighbourhood V of the set $\mathrm{Crit}(v^\nabla)$ of zeros of v^∇. We assumed that $\mathrm{Crit}(v^\nabla)$ is cocompact, so we may assume that V is relatively cocompact, in the sense that V/G is a relatively compact subset of M/G.

Proposition 8.1. *There is a G-invariant metric on $\mathfrak{g}^*_M$, and there are $t_0, C, b > 0$, such that for all $t \geq t_0$, all $p \in \mathbb{N}$, and all G-invariant $s \in \Gamma^\infty_{tc}(\mathcal{S}_p)^G$ with support disjoint from V, one has*

$$\|fD_{p,t}s\|_0^2 \geq C\big(\|fs\|_1^2 + (t-b)\|fs\|_0^2\big). \tag{8.2}$$

Proposition 8.2. *The metric on $\mathfrak{g}^*_M$ used in Proposition 8.1 can be chosen such that, in addition to the conclusions of that proposition, for every G-invariant open neighbourhood U of $(\mu^\nabla)^{-1}(0)$, there are $p_0 \in \mathbb{N}$ and $t_0, C, b > 0$, such that for all $t \geq t_0$ and $p \geq p_0$, and all G-invariant $s \in \Gamma^\infty_{tc}(\mathcal{S}_p)^G$ with support disjoint from U, the estimate* (8.2) *holds.*

So the estimate holds for all s supported outside V for all p, and for all s supported outside the smaller set U for large p.

It is important that the metric on $\mathfrak{g}^*_M$ used in Propositions 8.1 and 8.2 is the same. They therefore actually form one result, with two conclusions.

In addition, note that the condition that $t \geq t_0$ can be absorbed into the choice of the metric on $\mathfrak{g}^*_M$, since multiplying this metric by a constant results on multiplying the vector field v_∇ by the same constant. The parameter t was just introduced to make the arguments that follow clearer.

8.2. Choosing the metric on $M \times \mathfrak{g}^*$. One advantage of using a family of inner products on $\mathfrak{g}^*$, i.e. a metric on $\mathfrak{g}^*_M$, is that this allows us to define the G-invariant vector field v^∇ and the G-invariant function $\mathcal{H}^\nabla$. Another advantage that is very important for our arguments is that choosing this metric in a suitable way allows us to control the terms that appear in the Bochner formula in Theorem 7.1.

To make this precise, consider the G-invariant, positive, continuous function η on M defined by[5]

$$\eta(m) = \int_G f(gm)\|df\|(gm)\, dg, \tag{8.3}$$

[5] What follows holds for any G-invariant, positive, continuous function η.

for $m \in M$.

PROPOSITION 8.3. *The G-invariant metric on the bundle $\mathfrak{g}^*_M$ can be chosen in such a way that for all $m \in M \setminus V$,*

$$\mathcal{H}^\nabla(m) \geq 1; \tag{8.4}$$

$$\|v^\nabla_m\| \geq 1 + \eta(m), \tag{8.5}$$

and there is a positive constant C, such that for all $m \in M$, the operator A_m on $(\mathcal{S}_p)_m$ is bounded below by

$$A_m \geq -\|v^\nabla_m\|^2 - C. \tag{8.6}$$

Proof. Fix any G-invariant metric $\{(-,-)_m\}_{m\in M}$ on $\mathfrak{g}^*_M$. Let the G-invariant, positive, continuous functions F_1 and F_2 be as in Lemma 7.4. Set

$$\varphi_1 := \min\left(\mathcal{H}^\nabla, \frac{\|v^\nabla\|}{1+\eta}, \frac{\|v^\nabla\|^2}{2F_1}\right)$$
$$\varphi_2 := \frac{\|v^\nabla\|^2}{2F_2}.$$

This defines G-invariant, continuous functions φ_1 and φ_2 on M, which are positive outside $\mathrm{Crit}(v^\nabla)$. Since $\mathrm{Crit}(v^\nabla)/G$ is compact, the functions φ_j have uniform lower bounds outside the neighbourhood V of $\mathrm{Crit}(v^\nabla)$. Hence there are positive, G-invariant, continuous functions $\tilde{\varphi}_j$ on M, such that

$$\tilde{\varphi}_j|_{M\setminus V} = \varphi_j|_{M\setminus V},$$

for $j = 1, 2$. By Lemma C.3 in [15], there is a G-invariant, positive, smooth function ψ on M, such that

$$\psi^{-1} \leq \tilde{\varphi}_1;$$
$$\|d(\psi^{-1})\| \leq \tilde{\varphi}_2.$$

Consider the metric $\{\psi(m)(-,-)_m\}_{m\in M}$ on $\mathfrak{g}^*_M$, obtained by rescaling the given metric by ψ. We claim that this metric has the desired properties.

First of all, the function $\mathcal{H}^\nabla_\psi$ and the vector field $(v^\nabla)^\psi$ associated to this metric satisfy, outside V,

$$\mathcal{H}^\nabla_\psi = \psi\mathcal{H}^\nabla \geq \varphi_1^{-1}\mathcal{H}^\nabla \geq 1;$$
$$\|(v^\nabla)^\psi\| = \psi\|v^\nabla\| \geq \varphi_1^{-1}\|v^\nabla\| \geq 1 + \eta.$$

Furthermore, by Lemma 7.4, the operator A^ψ in Theorem 7.1, associated to the metric on $\mathfrak{g}^*_M$ rescaled by ψ, satisfies, outside V,

$$\begin{aligned}\frac{\|A^\psi\|}{\|(v^\nabla)^\psi\|^2} &\leq \frac{F_1\psi + F_2\|d\psi\|}{\psi^2\|v^\nabla\|^2}\\ &= \frac{F_1}{\|v^\nabla\|^2}\psi^{-1} + \frac{F_2}{\|v^\nabla\|^2}\|d(\psi^{-1})\|\\ &\leq 1.\end{aligned}$$

Hence $\|A^\psi\| \leq \|(v^\nabla)^\psi\|^2$, on $M \setminus V$. Since V is relatively cocompact and A^ψ is G-equivarant, it is bounded on V. So

$$A^\psi \geq -C$$

on V, for a certain $C > 0$. We conclude that

$$A^\psi \geq -\|(v^\nabla)^\psi\|^2 - C$$

on all of M. □

REMARK 8.4. A priori, the choice of metric on $\mathfrak{g}^*_M$ could influence $\mathrm{index}^G_{L^2_T}(D_{p,t})$, if $\mathrm{Crit}(v^\nabla)$ changes (while staying cocompact). Multiplying a metric by a function ψ as in Proposition 8.3 does not change $\mathrm{Crit}(v^\nabla)$, however, and the second point in Theorem 2.15 in [7] implies that $\mathrm{index}^G_{L^2_T}(D_{p,t})$ is independent of ψ. It follows from Theorem 6.8 that this index is independent of the metric in general, as long as $\mathrm{Crit}(v^\nabla)$ is cocompact, for large enough p.

Also note that one may take $t_0 = 1$ in Theorem 6.6, since, in the notation of the proof of Proposition 8.3,

$$\frac{it}{2} c\big((v^\nabla)^\psi\big) = \frac{i}{2} c\big((v^\nabla)^{t\psi}\big).$$

8.3. Proofs of the localisation estimates. Proposition 8.3 allows us to prove Propositions 8.1 and 8.2. Fix a G-invariant metric on $\mathfrak{g}^*_M$ as in Proposition 8.3, and a smooth cutoff function f. It will be useful to consider the operator

$$\widetilde{D}_{p,t} \colon f\Gamma^\infty_{tc}(\mathcal{S}_p)^G \to f\Gamma^\infty_{tc}(\mathcal{S}_p)^G,$$

defined by

$$\widetilde{D}_{p,t} fs = f D_{p,t} s, \tag{8.7}$$

for $s \in \Gamma^\infty_{tc}(\mathcal{S}_p)^G$. We will write $\widetilde{D}_p := \widetilde{D}_{p,0}$.

We need some arguments to account for the fact that, unlike $D_{p,t}$, the operator $\widetilde{D}_{p,t}$ is not symmetric with respect to the L^2-inner product. Let $\widetilde{D}^*_{p,t}$ be its formal adjoint. Combining Theorem 7.1 and Proposition 8.3, one obtains the following key estimate for the operator $\widetilde{D}^*_{p,t}\widetilde{D}_{p,t}$.

COROLLARY 8.5. *One has*

$$\widetilde{D}^*_{p,t}\widetilde{D}_{p,t} = \widetilde{D_p}^*\widetilde{D_p} + tB + (2p+1)2\pi t\mathcal{H}^\nabla + \frac{t^2}{4}\|v^\nabla\|^2,$$

where B is a vector bundle endomorphism of $\mathcal{S}_p$ for which there is a constant $C > 0$ such that one has the pointwise estimate

$$B \geq -C\big(\|v^\nabla\|^2 + 1\big).$$

Proof. This was proved in the symplectic setting in Proposition 6.7 in [15]. The arguments remain the same, however. References to Theorem 5.1 and to Proposition 6.6 in the proof of Proposition 6.7 in [15] should be replaced by references to Theorem 7.1 and Proposition 8.3 in the present paper, respectively. Note that the last term in the Bochner formula of Theorem 7.1 vanishes on G-invariant sections. □

The proofs of Propositions 8.1 and 8.2 are now the same as the proofs of Propositions 6.1 and 6.3 in [15], with Corollary 8.5 playing the role of Proposition 6.7 in [15].

8.4. Proofs of Theorems 6.6 and 6.8. Theorem 6.6 follows from Proposition 8.1, in the way that Theorem 3.4 in [15] follows from Proposition 6.1 in [15]. Indeed, for $t \geq b+1$ and any p in Proposition 8.1, one has

$$\|fD_{p,t}\|_0^2 \geq C\|fs\|_0^2,$$

for G-invariant sections $s \in \Gamma^\infty_{tc}(\mathcal{S}_p)^G$ with support disjoint from the set V. By Proposition 4.7 in [15], the operator $\widetilde{D}_{p,t}$ therefore extends to a Fredholm operator between Sobolev spaces. By Proposition 4.8 in [15], $\ker_{L^2_T}(D_{p,t})^G$ is finite-dimensional, and the index of the Fredholm operator induced by $\widetilde{D}_{p,t}$ equals $\operatorname{index}^G_{L^2_T}(D_{p,t})$. It is noted in part 2 of Theorem 2.15 in [7] that this index is independent of t, so that Theorem 6.6 follows.

To prove Theorem 6.8, we apply Proposition 8.2. This proposition shows that the arguments in Sections 6.5 and 7 of [15] apply to the operator $D_{p,t}$, for large enough p and t. Therefore, the techniques from Sections 8 and 9 in [4] can be used as in [15, 24, 36]. It follows that, for large enough p and t,

$$\operatorname{index}^G_{L^2_T}(D_{p,t}) = \operatorname{index}(D^{\nabla^0}_{M_0}), \tag{8.8}$$

where $D^{\nabla^0}_{M_0}$ is the Spin^c-Dirac operator on the reduced space M_0 associated to the Spin^c-structure of Lemma 3.3, and the connection ∇^0 on the line bundle $L_0^{2p+1} \to M_0$ induced by the connection ∇ on L. Theorem 6.8 therefore follows from the proposition below.

Proposition 8.6. *For all $p \in \mathbb{N}$, there exists a G-equivariant* Spin^c*-structure on M, and a connection on the associated determinant line bundle, such that the corresponding invariant* Spin^c*-quantisation is*

$$Q^{\mathrm{Spin}^c}(M)^G = \operatorname{index}^G_{L^2_T}(D_{p,t}),$$

for t large enough, and

$$Q^{\mathrm{Spin}^c}(M_0) = \operatorname{index}(D^{\nabla^0}_{M_0}).$$

Proof. Let $P \to M$ be the given G-equivariant principal Spin^c-structure on M. Let $P' \to M$ be the G-equivariant Spin^c-structure with determinant line bundle $L' = L^{2p+1}$. Explicitly,

$$P' = P \times_{\mathrm{U}(1)} \mathrm{UF}(L^p),$$

where UF denotes the unitary frame bundle. (See e.g. part (2) of Proposition D.43 in [12].) Let ∇' be the connection on L' induced by ∇.

Let $\mathcal{S}' \to M$ be the spinor bundle associated to P'. Then $\mathcal{S}' = \mathcal{S}_p$ (see e.g. (D.15) in [22]). Hence the connection $\nabla^{\mathcal{S}'}$ on $\mathcal{S}'$ induced by ∇' and the Levi–Civita connection on TM equals the connection on $\mathcal{S}_p$ used to define the Dirac operator D_p. Therefore, the Spin^c-Dirac operator D' on $\mathcal{S}'$ equals the operator D_p. Furthermore, the Spin^c-momentum map $\mu^{\nabla'}: M \to \mathfrak{g}^*$ associated to ∇' is given by

$$2\pi i \mu^{\nabla'}_X = \nabla'_{X^M} - \mathcal{L}^{L^{2p+1}}_X = 2\pi i(2p+1)\mu^\nabla_X, \tag{8.9}$$

for all $X \in \mathfrak{g}$. It follows that the induced vector field $v^{\nabla'}$ equals

$$v^{\nabla'} = (2p+1)v^{\nabla}.$$

We conclude that the deformed Dirac operator on $\mathcal{S}'$ associated to ∇' is

$$D'_{1,t} = D' + \frac{it}{2}c(v^{\nabla'}) = D_p + \frac{(2p+1)it}{2}c(v^{\nabla}) = D_{p,(2p+1)t}.$$

Let $t_0, t'_0 \in \mathbb{R}$ be as in Theorem 6.6, for he operators $D_{p,t}$ and $D'_{p,t}$, respectively. This theorem states that $\operatorname{index}^G_{L^2_T}(D_{p,t})$ does not depend on $t \geq t_0$. Hence, if

$$\begin{aligned} t &\geq t_0; \\ t' &\geq t'_0; \text{ and} \\ (2p+1)t' &\geq t_0, \end{aligned}$$

then, with respect to the Spin^c-structure P' and the connection ∇',

$$Q^{\mathrm{Spin}^c}(M)^G = \operatorname{index}^G_{L^2_T}(D'_{1,t'}) = \operatorname{index}^G_{L^2_T}(D_{p,(2p+1)t'}) = \operatorname{index}^G_{L^2_T}(D_{p,t}).$$

Finally, by (8.9), one has

$$M_0 = (\mu^{\nabla'})^{-1}(0)/G = (\mu^{\nabla})^{-1}(0)/G.$$

And the connection $(\nabla')^0$ on $L'_0 = L_0^{2p+1}$ is the one induced by the connection ∇^0 on L_0, so the second claim follows as well. □

9. Twisted Spin^c**-Dirac operators.** Theorem 6.12 can be proved by generalising the steps in the proofs of Theorems 6.6 and 6.8 to twisted Spin^c-Dirac operators.

9.1. A Bochner formula for twisted Dirac operators. As in the case for untwisted Dirac operators, the proof of Theorem 6.12 starts with an expression for the square of the deformed Dirac operator $D^E_{p,t}$. This expression will be deduced from Theorem 7.1 by comparing the square of $D^E_{p,t}$ to the square of $D_{p,t}$. The main difference between these two involves the *generalised moment map*

$$\mu^E \in \mathfrak{g}^* \otimes \operatorname{End}(E),$$

defined by

$$2\pi i \mu^E_X = \mathcal{L}^E_X - \nabla^E_{X^M} \quad \in \operatorname{End}(E),$$

for all $X \in \mathfrak{g}$, where $\mathcal{L}^E_X$ is the Lie derivative of sections of E with respect to X. Using the metric on $\mathfrak{g}^*_M$, we obtain

$$(\mu^{\nabla}, \mu^E) \in \operatorname{End}(E).$$

PROPOSITION 9.1. *On G-invariant sections of $\mathcal{S}_p \otimes E$, one has*

$$(D^E_{p,t})^2 = (D^E_p)^2 + tA \otimes 1_E + (2p+1)2\pi t\mathcal{H}^{\nabla} + \frac{t^2}{4}\|v^{\nabla}\|^2 + 4\pi t 1_{\mathcal{S}} \otimes (\mu^{\nabla}, \mu^E),$$

with $A \in \mathrm{End}(\mathcal{S}_p)$ as in Theorem 7.1.

The first step in the proof of Proposition 9.1 is a simple relation between the operators D_p^E and D_p. Fix a local orthonormal frame $\{e_j\}_{j=1}^{d_M}$ of TM. The operator

$$P_E := \sum_{j=1}^{d_M} c(e_j) \otimes \nabla^E_{e_j}$$

on $\Gamma^\infty(\mathcal{S}_p \otimes E)$ is independent of this frame, and hence globally defined.

LEMMA 9.2. *One has*

$$D_p^E = D_p + P_E.$$

Proof. In terms of the frame $\{e_j\}_{j=1}^{d_M}$, we have

$$\begin{aligned} D_p^E &= \sum_{j=1}^{d_M} (c(e_j) \otimes 1_E)\big(\nabla^{\mathcal{S}_p}_{e_j} \otimes 1_E + 1_{\mathcal{S}_p} \otimes \nabla^E_{e_j}\big) \\ &= \sum_{j=1}^{d_M} c(e_j)\nabla^{\mathcal{S}_p}_{e_j} \otimes 1_E + \sum_{j=1}^{d_M} c(e_j) \otimes \nabla^E_{e_j} \\ &= D_p + P_E. \end{aligned}$$

□

LEMMA 9.3. *For all vector fields v on M,*

$$(c(v) \otimes 1_E) \circ P_E + P_E \otimes (c(v) \otimes 1_E) = -2(1_{\mathcal{S}_p} \otimes \nabla^E_v).$$

Proof. Since

$$c(v)c(e_j) + c(v)c(e_j) = -2(v, e_j)$$

for all j, we see that

$$\begin{aligned} (c(v) \otimes 1_E) \circ P_E + P_E \otimes (c(v) \otimes 1_E) &= \sum_{j=1}^{d_M} \big(c(v)c(e_j) + c(e_j)c(v)\big) \otimes \nabla^E_{e_j} \\ &= -2\sum_{j=1}^{d_M} (v, e_j) 1_{\mathcal{S}_p} \otimes \nabla^E_{e_j} \\ &= -2(1_{\mathcal{S}_p} \otimes \nabla^E_v). \end{aligned}$$

□

Let $(\mu^\nabla)^* \colon M \to \mathfrak{g}$ be dual to μ^∇ with respect to a given metric on $\mathfrak{g}^*_M$. For any G-equivariant vector bundle $F \to M$, consider the Lie derivative operator $\mathcal{L}^F_{(\mu^\nabla)^*}$ on $\Gamma^\infty(F)$, defined by

$$(\mathcal{L}^F_{(\mu^\nabla)^*} s)(m) := (\mathcal{L}^F_{(\mu^\nabla)^*(m)} s)(m)$$

for $s \in \Gamma^\infty(F)$ and $m \in M$.

PROPOSITION 9.4. *One has*

$$(D_{p,t}^E)^2 = (D_p^E)^2 + tA \otimes 1_E + (2p+1)2\pi t\mathcal{H}^\nabla + \frac{t^2}{4}\|v^\nabla\|^2 + 4\pi t 1_{\mathcal{S}} \otimes (\mu^\nabla, \mu^E)$$
$$-2it\big(\mathcal{L}^{\mathcal{S}_p}_{(\mu^\nabla)^*} \otimes 1_E + 1_{\mathcal{S}_p} \otimes \mathcal{L}^E_{(\mu^\nabla)^*}\big),$$

with $A \in \mathrm{End}(\mathcal{S}_p)$ as in Theorem 7.1.

Since for all $X \in \mathfrak{g}$,

$$\mathcal{L}^{\mathcal{S}_p}_X \otimes 1_E + 1_{\mathcal{S}_p} \otimes \mathcal{L}^E_X$$

is the Lie derivative on $\mathcal{S}_p \otimes E$ with respect to X, the operator $\mathcal{L}^{\mathcal{S}_p}_{(\mu^\nabla)^*} \otimes 1_E + 1_{\mathcal{S}_p} \otimes \mathcal{L}^E_{(\mu^\nabla)^*}$ equals zero on G-invariant sections. Hence Proposition 9.4 implies Proposition 9.1.
Proof of Proposition 9.4. First note that

$$(D_{p,t}^E)^2 = (D_p^E)^2 + \big(\frac{it}{2}c(v^\nabla) \otimes 1_E\big)^2 + \frac{it}{2}\big(D_p^E \circ (c(v^\nabla) \otimes 1_E) + (c(v^\nabla) \otimes 1_E) \circ D_p^E\big).$$

Because of Lemma 9.2 and 9.3, we have

$$\begin{aligned} &D_p^E \circ (c(v^\nabla) \otimes 1_E) + (c(v^\nabla) \otimes 1_E) \circ D_p^E \\ &= \big(D_p \circ c(v^\nabla) + c(v^\nabla) \circ D_p\big) \otimes 1_E + (c(v^\nabla) \otimes 1_E) \circ P_E + P_E \otimes (c(v^\nabla) \otimes 1_E) \\ &= \big(D_p \circ c(v^\nabla) + c(v^\nabla) \circ D_p\big) \otimes 1_E - 2(1_{\mathcal{S}_p} \otimes \nabla^E_{v^\nabla}). \end{aligned}$$

Furthermore,

$$\big(\frac{it}{2}c(v^\nabla)\big)^2 + \frac{it}{2}\big(D_p \circ c(v^\nabla) + c(v^\nabla) \circ D_p\big) = D_{p,t}^2 - D_p^2.$$

The right hand side of this equality was computed in Theorem 7.1. Using the expression obtained there and the above computations, we find that

$$(D_{p,t}^E)^2 = (D_p^E)^2 + tA \otimes 1_E + (2p+1)2\pi t\mathcal{H}^\nabla + \frac{t^2}{4}\|v^\nabla\|^2 - 2it\mathcal{L}^{\mathcal{S}_p}_{(\mu^\nabla)^*} \otimes 1_E - it 1_{\mathcal{S}_p} \otimes \nabla^E_{v^\nabla}.$$

Now for all $m \in M$,

$$\nabla^E_{v^\nabla_m} = 2\nabla^E_{(\mu^\nabla)^*(m)^M_m} = 2\big(\mathcal{L}^E_{(\mu^\nabla)^*(m)} - 2\pi i \mu^E_{(\mu^\nabla)^*(m)}(m)\big).$$

Since $\mu^E_{(\mu^\nabla)^*(m)}(m) = (\mu^\nabla, \mu^E)(m)$, the claim follows. $\square$

9.2. Localisation. Proposition 8.2, which is the key step in the proof of Theorem 6.8, generalises to twisted Dirac operators in the following way.

PROPOSITION 9.5. *There is a metric on $\mathfrak{g}^*_M$ such that for every G-invariant open neighbourhood U of $(\mu^\nabla)^{-1}(0)$, there are $p_E \in \mathbb{N}$ and $t_0, C, b > 0$, such that for all $t \geq t_0$ and $p \geq p_E$, and all G-invariant $s \in \Gamma^\infty_{tc}(\mathcal{S}_p \otimes E)^G$ with support disjoint from U,*

$$\|fD^E_{p,t}s\|_0^2 \geq C\big(\|fs\|_1^2 + (t-b)\|fs\|_0^2\big).$$

Here $\|\cdot\|_k$ denotes the Sobolev norm defined by the operator D^E_p, as in (8.1).

Theorem 6.12 follows from Proposition 9.5 in the same way that Theorems 6.6 and 6.8 follows from Propositions 8.1 and 8.2, as described in Subsection 8.4. The topological expression for the index of D_{M_0} then follows from the Atiyah–Singer index theorem. We now do not use an analogue of Proposition 8.1 (localisation to neighbourhoods of $\mathrm{Crit}(v^\nabla)$ for $p = 1$), because for twisted Dirac operators we always use large enough powers of L.

It therefore remains to prove Proposition 9.5. This proof is based on a generalisation of Proposition 8.3.

LEMMA 9.6. *There is a G-invariant metric on $\mathfrak{g}^*_M$ such that, in addition to the properties in Proposition 8.3, there is a $C' > 0$ such that the operator (μ^∇, μ^E) satisfies the pointwise estimate*

$$1_{\mathcal{S}_p} \otimes (\mu^\nabla, \mu^E) \geq -\|v^\nabla\|^2 - C' \tag{9.1}$$

(for any $p \in \mathbb{N}$).

Proof. As in Section 8, choose a relatively cocompact, G-invariant neighbourhood V of $\mathrm{Crit}(v^\nabla)$. Choose a G-invariant, positive function $\psi_E \in C^\infty(M)^G$ such that, outside V,

$$\|1_{\mathcal{S}_p} \otimes (\mu^\nabla, \mu^E)\| \leq \psi_E \|v^\nabla\|^2.$$

Fix any G-invariant metric $\{(-,-)_m\}_{m\in M}$ on $\mathfrak{g}^*_M$. Consider the metric $\{\psi_E(m)(-,-)_m\}_{m\in M}$ rescaled by ψ_E, and let $(v^\nabla)^{\psi_E} = \psi_E v^\nabla$ be the vector field associated to this metric. Then, outside V,

$$\|1_{\mathcal{S}_p} \otimes \psi_E(\mu^\nabla, \mu^E)\| \leq \|(v^\nabla)^{\psi_E}\|^2$$

Furthermore, the function $\|1_{\mathcal{S}_p} \otimes \psi_E(\mu^\nabla, \mu^E)\|$ is G-invariant, and hence bounded on V. So there is a $C' > 0$ such that, on all of M,

$$\|1_{\mathcal{S}_p} \otimes \psi_E(\mu^\nabla, \mu^E)\| \leq \|(v^\nabla)^{\psi_E}\|^2 + C'$$

Let $\psi \in C^\infty(M)^G$ be as in the proof of Proposition 8.3. Choose a positive function $\tilde{\psi} \in C^\infty(M)^G$ such that

$$\begin{aligned} \tilde{\psi}^{-1} &\leq \min(\psi^{-1}, \psi_E^{-1}); \\ \|d\tilde{\psi}^{-1}\| &\leq \|d\psi^{-1}\|. \end{aligned}$$

(This is possible by Lemma C.3 in [15].) Then the metric $\{\tilde{\psi}(m)(-,-)_m\}_{m\in M}$ has the properties in Proposition 8.3, and also satisfies (9.1). □

Let $f \in C^\infty(M)$ be a cutoff function. Analogously to (8.7), we define the operator $\widetilde{D}^E_{p,t}$ on $f\Gamma^\infty_{tc}(\mathcal{S}_p \otimes E)^G$ by

$$\widetilde{D}^E_{p,t} f s = f D^E_{p,t} s$$

for all $s \in \Gamma^\infty_{tc}(\mathcal{S}_p \otimes E)^G$. Corollary 8.5 now generalises as follows.

COROLLARY 9.7. *One has*

$$(\widetilde{D}^E_{p,t})^* \widetilde{D}^E_{p,t} = (\widetilde{D}^E_p)^* \widetilde{D}^E_p + tB + (2p+1)2\pi t \mathcal{H}^\nabla + \frac{t^2}{4}\|v^\nabla\|^2,$$

where B is a vector bundle endomorphism of $\mathcal{S}_p \otimes E$ for which there is a constant $C > 0$ such that one has the pointwise estimate

$$B \geq -C\big(\|v^\nabla\|^2 + 1\big).$$

Proof. As in Lemma 6.8 of [15], one has for $s \in \Gamma^\infty_{tc}(\mathcal{S}_p \otimes E)^G$,

$$(\widetilde{D}^E_{p,t})^* fs = \widetilde{D}^E_{p,t} fs + 2(c(df) \otimes 1_E)s.$$

Hence, as in Lemma 6.9 of [15], one deduces from Proposition 9.1 that for such s,

$$\begin{aligned}(\widetilde{D}^E_{p,t})^* \widetilde{D}^E_{p,t} fs = (\widetilde{D}^E_p)^* \widetilde{D}^E_p fs + t\left(A \otimes 1_E + (2p+1)2\pi t \mathcal{H}^\nabla + 4\pi 1_{\mathcal{S}} \otimes (\mu^\nabla, \mu^E)\right) fs \\ + \frac{t^2}{4}\|v^\nabla\|^2 fs + it(c(df)c(v^\nabla) \otimes 1_E)s\end{aligned}$$

with $A \in \mathrm{End}(\mathcal{S}_p)$ as in Theorem 7.1.

Write

$$Bfs := \left(A \otimes 1_E + 4\pi 1_{\mathcal{S}} \otimes (\mu^\nabla, \mu^E)\right) fs + i(c(df)c(v^\nabla) \otimes 1_E)s,$$

for s as above. By Lemma 9.6, the metric on $\mathfrak{g}^*_M$ can be chosen such that there is a $C' > 0$ for which

$$A \otimes 1_E + 4\pi 1_{\mathcal{S}} \otimes (\mu^\nabla, \mu^E) \geq -(1 + 4\pi)\|v^\nabla\|^2 - C'.$$

By Lemma 6.10 in [15], there is a $C'' > 0$ such that for all $s \in \Gamma^\infty_{tc}(\mathcal{S}_p \otimes E)^G$

$$\mathrm{Re}\big(i(c(df)c(v^\nabla) \otimes 1_E)s, fs\big)_0 \geq -C''\big((\|v^\nabla\|^2 + 1)fs, fs\big)_0.$$

This implies that

$$B \geq -(C' + C'' + 1 + 4\pi)(\|v^\nabla\|^2 + 1).$$

∎

The proof of Proposition 9.5 (and hence of Theorem 6.12) can now be finished as in the proof of Proposition 6.3 in Section 6.4 of [15], with Corollary 9.7 playing the role of Proposition 6.7 in [15].

10. Applications and examples. Let us mention some applications and examples of Theorems 4.6, 6.8 and 6.12. We will see that Theorem 6.8 reduces to a Spin^c-version of the result in [24] in the cocompact case, and discuss how to generate examples of Theorems 4.6 and 6.12. We show how formal degrees of classes in $K_*(C^*_r G)$, generalising formal degrees of discrete series representations, and related to certain charcteristic classes on M. Finally, we use the index formula for twisted Spin^c-Dirac operators in Theorem 6.12 to draw conclusions about Braverman's analytic index of such operators.

As before, we assume G is unimodular.

10.1. Generalising Landsman's conjecture to Spin^c**-manifolds.** As noted in Subsection 4.2, Theorem 6.8 implies that the main result in [24] generalises to the Spin^c-setting.

COROLLARY 10.1. *In the situation of Theorem 6.8, suppose that M/G is* compact. *Then, in the notation of Subsection 4.2,*

$$R_0\big(Q_G^{\mathrm{Spin}^c}(M)\big) = Q(M_0),$$

for the Spin^c*-structures on M and a connections on their determinant line bundles for which Theorem 6.8 holds.*

Proof. If M/G is compact, one may take $t_0 = 0$ in Theorem 6.6. (By using $V = M$ in Proposition 8.1.) As noted in Remark 6.4, the fact that all smooth sections are transversally L^2 in this case implies that

$$Q^{\mathrm{Spin}^c}(M)^G = \dim(\ker D^+)^G - \dim(\ker D^-)^G.$$

Bunke shows in the appendix to [24] that this equals $R_0\big(Q_G^{\mathrm{Spin}^c}(M)\big)$. □

In other words, an extension of Landsman's conjecture (4.5) to the Spin^c-case holds for suitable choices of Spin^c-structures and connections. In fact, Theorem 6.12 can be used to generalise this result to twisted Spin^c-Dirac operators.

10.2. Generating examples. Using the constructions in Subsection 3.2, one can generate a large class of examples of Theorems 4.6 and 6.12 from cases where the group acting is compact. Indeed, let K be a compact, connected Lie group, and let N be a manifold equipped with an action by K and a K-equivariant Spin^c-structure. Let $\mu^{\nabla^N}: N \to \mathfrak{k}^*$ be the Spin^c-momentum map associated to a K-invariant Hermitian connection ∇^N on the determinant line bundle $L^N \to N$ of the Spin^c-structure on N. Let v^{∇^N} be the vector field on N associated to μ^{∇^N} as in (6.2), with respect to a single $\mathrm{Ad}^*(K)$-invariant inner product on $\mathfrak{k}^*$. Suppose it has a compact set $\mathrm{Crit}(v^{\nabla^N})$ of zeros. As noted in Lemma 3.24 in [29], and on page 4 of [38], this is true if N is real-algebraic and μ^{∇^N} is algebraic and proper. (And also, of course, if N is compact.)

Let G be a connected, unimodular Lie group containing K as a maximal compact subgroup. Suppose the lift $\widetilde{\mathrm{Ad}}$ in (3.3) exists, which is true if one replaces G by a double cover if necessary. We saw in Subsections 3.2 and 3.3 that the manifold $M := G \times_K N$ has a G-equivariant Spin^c-structure with determinant line bundle $L^M = G \times_K L^N$. Furthermore, by Proposition 3.10, *all* G-equivariant Spin^c-manifolds arise in this way (though possibly not all Riemannian metrics on such manifolds). In Subsection 5.2, a connection ∇^M on L^M was constructed, such that the associated Spin^c-momentum map μ^{∇^M} is given by (3.8).

If N is compact and even-dimensional, then Theorem 4.6 applies, and yields a decomposition of $Q_G^{\mathrm{Spin}^c}(M)_r \in K_*(C^*_r G)$. If N is possibly noncompact, then Theorem 6.12 applies for a suitable metric on $\mathfrak{g}^*_M$.

COROLLARY 10.2. *Suppose the dimension of M is even. Let $E \to M$ be a G-equivariant, Hermitian vector bundle, equipped with a G-invariant, Hermitian connection. If $0 \in \mathfrak{k}^*$ is a regular value of μ^{∇^N}, and K acts freely on $(\mu^{\nabla^N})^{-1}(0)$, then there are a metric on $\mathfrak{g}^*_M$ and a $p_E \in \mathbb{N}$ such that for all $p \geq p_E$,*

$$\mathrm{index}^G_{L^2_T} D^E_{p,1} = \int_{M_0} \mathrm{ch}(E_0) e^{\frac{p}{2}c_1(L_0)} \hat{A}(M_0).$$

Proof. By Proposition 3.12, zero is a Spin^c-regular value of μ^{∇^M}. By (5.12), G acts freely on $(\mu^{\nabla^M})^{-1}(0)$. To apply Theorem 6.12, it therefore only remains to show that the vector field v^{∇^M} on M, induced by the momentum map μ^{∇^M} as in (6.2), has a cocompact set $\mathrm{Crit}(v^{\nabla^M})$ of zeros. This follows from the fact that

$$\mathrm{Crit}(v^{\nabla^M}) = G \times_K \mathrm{Crit}(v^{\nabla^N}),$$

for a suitable metric on $\mathfrak{g}^*_M$. This is proved in Lemma 10.4 below. Therefore, Theorem 6.12 implies the claim. □

REMARK 10.3. In the setting of Corollary 10.2, Proposition 3.14 implies that

$$Q^{\mathrm{Spin}^c}(M)^G = Q^{\mathrm{Spin}^c}(M_0) = Q^{\mathrm{Spin}^c}(N_0) = Q^{\mathrm{Spin}^c}(N)^K.$$

LEMMA 10.4. *There is a G-invariant metric on $\mathfrak{g}^*_M$ such that the set of zeros of the vector field v^{∇^M} on M, used in the proof of Corollary 10.2, equals*

$$\mathrm{Crit}(v^{\nabla^M}) = G \times_K \mathrm{Crit}(v^{\nabla^N}). \tag{10.1}$$

Proof. Let $(-,-)_K$ be an $\mathrm{Ad}^*(K)$-invariant inner product on $\mathfrak{g}^*$ that extends the inner product on $\mathfrak{k}^*$ used to define v^{∇^N}. Consider the G-invariant metric on $\mathfrak{g}^*_M$ defined by

$$(\xi, \xi')_{[g,n]} := \big(\mathrm{Ad}^*(g)^{-1}\xi, \mathrm{Ad}^*(g)^{-1}\xi'\big)_K,$$

for $\xi, \xi' \in \mathfrak{g}^*$, $g \in G$ and $n \in N$. Let v^{∇^M} be defined via this metric. We will show that $v^{\nabla^M}|_N = v^{\nabla^N}$, where we embed N into M via the map $n \mapsto [e, n]$. Then (10.1) follows by G-invariance of both sides.

The dual map $(\mu^{\nabla^M})^*\colon M \to \mathfrak{g}$, defined with respect to the above metric on $\mathfrak{g}^*_M$, satisfies

$$(\mu^{\nabla^M})^*[e, n] = (\mu^{\nabla^N})^*(n),$$

for all $n \in N$, where $(\mu^{\nabla^N})^*$ is the map dual to μ^{∇^N} with respect to the restriction of $(-,-)_K$ to $\mathfrak{k}^*$. Here $\mathfrak{k}^*$ is embedded into $\mathfrak{g}^*$ via the inner product $(-,-)_K$ (i.e. $\mathfrak{p} \subset \mathfrak{g}$ is defined as the orthogonal complement to $\mathfrak{k}$ with respect to the induced inner product on $\mathfrak{g}$). Hence

$$\begin{aligned} v^{\nabla^M}_{[e,n]} &= 2\big((\mu^{\nabla^N})^*(n)\big)^M_{[e,n]} \\ &= 2\left.\frac{d}{dt}\right|_{t=0} \Big[\exp\big(t(\mu^{\nabla^N})^*(n)\big), n\Big] \\ &= 2\left.\frac{d}{dt}\right|_{t=0} \Big[e, \exp\big(t(\mu^{\nabla^N})^*(n)\big)n\Big] \\ &= v^{\nabla^N}_n, \end{aligned}$$

so the claim follows. □

10.3. Characteristic classes and formal degrees. Theorem 4.6 is stated in terms of K-theory of C^*-algebras, but it has purely geometric consequences. In particular, it yields an expression for the formal degrees of discrete series representations of semisimple groups in trems of characteristic classes on coadjoint orbits.

Let $\tau\colon C^*_rG \to \mathbb{C}$ be the von Neumann trace, determined by

$$\tau\big(R(\varphi)^*R(\varphi)\big) = \int_G |\varphi(g)|^2 dg,$$

for $\varphi \in L^1(G)\cap L^2(G)$, where R denotes the right regular representation. This induces a morphism $\tau_*\colon K_*(C^*_rG) \to \mathbb{R}$. Wang showed in Proposition 4.4 and Theorem 6.12 in [40] that

$$\tau_*\big(Q_G^{\mathrm{Spin}^c}(M)_r\big) = \int_M f e^{\frac{1}{2}c_1(L)}\hat{A}(M).$$

Here f is a cutoff function as in Definition 6.1. For $\lambda \in \Lambda_+ + \rho_K$, let $[\lambda] \in K_d(C^*_rG)$ be as in (4.6). (As before, d is the dimension of G/K.) We define the *formal degree* of $[\lambda]$ as

$$d_\lambda := \tau_*[\lambda] \quad \in \mathbb{R}.$$

Theorem 4.6 has the following consequence.

COROLLARY 10.5. *In the setting of Theorem 4.6, we have*

$$\int_M f e^{\frac{1}{2}c_1(L^M)}\hat{A}(M) = \sum_{\lambda\in\Lambda_+ + \rho_K} m_\lambda d_\lambda,$$

where $m_\lambda \in \mathbb{Z}$ is given by the quantisation commutes with reduction relation (4.2).

This corollary is a noncompact generalisation of the equality

$$\int_N e^{\frac{1}{2}c_1(L^N)}\hat{A}(N) = \sum_{\lambda\in\Lambda_+ + \rho_K} m_\lambda \dim(V_\lambda),$$

in the compact case. Here V_λ is the representation space of π^K_λ.

Now suppose G is semisimple with discrete series, and let $\lambda \in \Lambda_+ + \rho_K$. Let d_π be the formal degree of the discrete series representation π with Harish–Chandra parameter λ. Then by (5.3) in [18] and the remarks in Section 2.3 in [20], we have

$$d_\pi = (-1)^{d/2} d_\lambda.$$

(This motivates the term ‘formal degree’ for the number d_λ in general.) In part (iii) of Proposition 7.3.A in [10], Connes and Moscovici gave a decomposition of the L^2-index of the Spin^c-Dirac operator on a homogeneous space of G into the formal degrees d_π. The left-hand side of the equality in Corollary 10.5 is the L^2-index of the Spin^c-Dirac operator on M by Theorem 6.12 in [40]. Therefore, Corollary 10.5 is a version of quantisation commutes with reduction for an index as in Connes and Moscovici's result, if M is a homogeneous space.

For specific choices of such homogeneous spaces, one actually only picks up a single formal degree. Using Proposition 4.4 in [40] along with Theorem 6.12 in that paper, or Connes and Moscovici's index theorem, Theorem 5.3 in [10], one obtains

$$d_\pi = (-1)^{d/2} \int_{G/K} f \,\mathrm{ch}(G\times_K V_\lambda)\hat{A}(G/K).$$

(Also compare this with Corollary 7.3.B in [10].) In a similar way, Corollary 2.8 in [18] implies that

$$d_\pi = (-1)^{d/2} \int_{G\cdot\lambda} f e^{\frac{1}{2}c_1(L)} \hat{A}\bigl(G\cdot\lambda\bigr).$$

Corollary 10.5 is a generalisation of the latter equality from strongly elliptic coadjoint orbits to arbitrary manifolds (satisfying the hypotheses of Theorem 4.6).

10.4. Consequences of the index formula for twisted Dirac operators. The index formula for twisted Spin^c-Dirac operators in Theorem 6.12 implies some properties of the index of such operators, which are not a priori clear from Braverman's analytic definition of this index.

Braverman's cobordism invariance result, Theorem 3.6 in [7], implies the excision property that the index only depends on data near $\mathrm{Crit}(v^\nabla)$, as in Lemma 3.12 in [7]. Because of Theorem 6.12, the index has a more refined excision property for twisted Spin^c-Dirac operators, namely that it only depends on data near the subset $(\mu^\nabla)^{-1}(0)$ of $\mathrm{Crit}(v^\nabla)$.

Corollary 10.6 (Excision). *For $j = 1, 2$, let M_j be a G-equivariant* Spin^c-*manifold, with spinor bundle $\mathcal{S}_{M_j} \to M_j$. Let ∇^{L_j} be a G-invariant Hermitian connection on the determinant line bundle $L_j \to M_j$. Let $E_j \to M_j$ be a G-equivariant Hermitian vector bundle, equipped with a G-invariant Hermitian connection. Suppose these data satisfy the conditions of Theorem 6.12 for $j = 1, 2$.*

In addition, suppose there are G-invariant neighbourhoods U_j of $(\mu^{\nabla^{L_j}})^{-1}(0)$, and a G-equivariant, isometric diffeomorphism

$$\varphi\colon U_1 \to U_2,$$

such that

$$\begin{aligned} \varphi\bigl((\mu^{\nabla^{L_1}})^{-1}(0)\bigr) &= (\mu^{\nabla^{L_2}})^{-1}(0);\\ \varphi^*\bigl(\mathcal{S}_{M_2}|_{U_2}\bigr) &= \mathcal{S}_{M_1}|_{U_1};\\ \varphi^*(\nabla^{L_2}|_{U_2}) &= \nabla^{L_1}|_{U_1};\\ \varphi^*(E_2|_{U_2}) &= E_1|_{U_1}. \end{aligned}$$

Then there are G-invariant metrics on $\mathfrak{g}^*_{M_1}$ and $\mathfrak{g}^*_{M_2}$, and there is a $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$,

$$\mathrm{index}^G_{L^2_T} D^{E_1}_{p,1} = \mathrm{index}^G_{L^2_T} D^{E_2}_{p,1}. \tag{10.2}$$

Proof. Under the conditions stated, one has

$$\begin{aligned} (\mu^{\nabla^{L_1}})^{-1}(0)/G &\cong (\mu^{\nabla^{L_2}})^{-1}(0)/G =: M_0;\\ \bigl(L_1|_{(\mu^{\nabla^{L_1}})^{-1}(0)}\bigr)/G &\cong \bigl(L_2|_{(\mu^{\nabla^{L_2}})^{-1}(0)}\bigr)/G =: L_0;\\ \bigl(E_1|_{(\mu^{\nabla^{L_1}})^{-1}(0)}\bigr)/G &\cong \bigl(E_2|_{(\mu^{\nabla^{L_2}})^{-1}(0)}\bigr)/G =: E_0. \end{aligned}$$

Furthermore, because $\mathcal{S}_{M_1}$ and $\mathcal{S}_{M_2}$ coincide on a neighbourhood of $(\mu^{\nabla^{L_1}})^{-1}(0) = (\mu^{\nabla^{L_2}})^{-1}(0)$, the Spin^c-structures on M_0 defined by these spinor bundles are equal. Hence by Theorem 6.12, if $p \geq \max(p_{E_1}, p_{E_2})$, both sides of (10.2) equal

$$\int_{M_0} \mathrm{ch}(E_0) e^{\frac{p}{2}c_1(L_0)} \hat{A}(M_0). \qquad \square$$

A direct consequence of Theorem 6.12 is that $\mathrm{index}^G_{L^2_T} D^E_{p,1}$, when defined, depends polynomially on p.

COROLLARY 10.7. *In the setting of Theorem 6.12, there is a $p_E \in \mathbb{N}$ such that for all $p \geq p_E$,*

$$\mathrm{index}^G_{L^2_T} D^E_{p,1} = \sum_{k=0}^{(\dim M_0)/2} a_k p^k,$$

with rational coefficients

$$a_k := \frac{1}{2^k k!} \int_{M_0} \mathrm{ch}(E_0) c_1(L_0)^k \hat{A}(M_0).$$

In particular,

$$\mathrm{index}^G_{L^2_T} D^E_{p,1} - \sum_{k=1}^{(\dim M_0)/2} a_k p^k = \int_{M_0} \mathrm{ch}(E_0)\hat{A}(M_0)$$

is independent of p.

Finally, certain topological invariants of M_0 can be recovered as indices on M. We illustrate this for a twisted version of the signature.

Let γ be the involution of $\bigwedge T^*M \otimes \mathbb{C}$ equal to

$$\gamma := i^{(\dim M + j(j-1))/2} *$$

on $\bigwedge^j T^*M \otimes \mathbb{C}$, where $*$ is the Hodge operator. Consider the de Rham operator

$$B := d + d^*$$

on $\Gamma^\infty(\bigwedge T^*M \otimes \mathbb{C})$. It satisfies $B\gamma = -\gamma B$, and hence defines the *signature operator*

$$B\colon \Gamma^\infty\big(\textstyle\bigwedge^+ T^*M \otimes \mathbb{C}\big) \to \Gamma^\infty\big(\textstyle\bigwedge^- T^*M \otimes \mathbb{C}\big)$$

where the $+$ and $-$ signs denote the $+1$ and -1 eigenspaces of γ. (See e.g. Example 6.2 in [22].)

For any integer p, let B^{L^p} be the signature operator B, twisted by L^p via the given connection on L. Write

$$B^{L_p}_{v^\nabla} := B^{L^p} + \frac{i}{2} c(v^\nabla).$$

Let $\mathcal{N} \to (\mu^\nabla)^{-1}(0)$ be the normal bundle to q^*TM_0 in $TM|_{(\mu^\nabla)^{-1}(0)}$. If 0 is a Spin^c-regular value of μ^∇, then $\mathcal{N}$ has a G-equivariant Spin-structure, with spinor bundle $\mathcal{S}^{\mathcal{N}} \to (\mu^\nabla)^{-1}(0)$. Let $\mathcal{S}^{\mathcal{N}}_0 = (\mathcal{S}^{\mathcal{N}}|_{(\mu^\nabla)^{-1}(0)})/G \to M_0$ be the induced vector bundle over M_0. Then Theorem 6.12 implies a version of Hirzebruch's signature theorem in this setting.

COROLLARY 10.8 (Twisted signature theorem). *Suppose that* 0 *is a* Spin^c*-regular value of μ^∇, and that G acts freely on $(\mu^\nabla)^{-1}(0)$. Then there is a G-invariant metric on $\mathfrak{g}^*_M$ such that for large enough integers p,*

$$\mathrm{index}^G_{L^2_T} B^{L^p}_{v^\nabla} = \int_{M_0} \mathrm{ch}(\mathcal{S}^{\mathcal{N}}_0) e^{(p-\frac{1}{2})c_1(L_0)} L(M_0).$$

Here $L(M_0)$ is the L-class of M_0.

Proof. For all Spin^c-manifolds U, with spinor bundles $\mathcal{S}_U \to U$, one has

$$\bigwedge T^*U \cong \mathrm{Cl}(TU) \cong \mathrm{End}(\mathcal{S}_U) \cong \mathcal{S}_U \otimes \mathcal{S}_U^* \cong \mathcal{S}_U \otimes \mathcal{S}_U.$$

If U is Spin, then under this identification, the signature operator B_U equals the Spin-Dirac operator twisted by $\mathcal{S}_U$:

$$B_U = D_U^{\mathcal{S}_U}.$$

(See e.g. below Proposition 3.62 in [3].) In our setting, M is only Spin^c. But as in (2.1), we have on small enough open sets $U \subset M$,

$$\mathcal{S}_p|_U = \mathcal{S}_U^{\mathrm{Spin}} \otimes L|_U^{p/2},$$

where $\mathcal{S}_U^{\mathrm{Spin}}$ is the spinor bundle of a local Spin-structure. Hence, locally, we have for all $p \in \mathbb{N}$,

$$D_p|_U = (D_U^{\mathrm{Spin}})^{L|_U^{p/2}},$$

the local Spin-Dirac operator D_U^{Spin} coupled to $L|_U^{p/2}$ via the given connection. Twisting D_p by $\mathcal{S}$, we therefore obtain

$$D_p^{\mathcal{S}}|_U = (D_U^{\mathrm{Spin}})^{\mathcal{S}|_U \otimes L|_U^{p/2}} = (D_U^{\mathrm{Spin}})^{\mathcal{S}_U^{\mathrm{Spin}} \otimes L|_U^{(p+1)/2}} = (B|_U)^{L|_U^{(p+1)/2}}.$$

If $p+1$ is even, then $B^{L^{(p+1)/2}}$ is defined globally, and by the above local argument, it equals $D_p^{\mathcal{S}}$. This means that for all $k \in \mathbb{N}$, in the notation of Definition 6.11,

$$B_{v^\nabla}^{L^k} = D_{2k-1,1}^{\mathcal{S}}.$$

Under the conditions stated, Theorem 6.12 therefore yields the equality

$$\mathrm{index}_{L^2_T}^G B_{v^\nabla}^{L^k} = \int_{M_0} \mathrm{ch}(\mathcal{S}_0) e^{(k-\frac{1}{2})c_1(L_0)} \hat{A}(M_0),$$

for k large enough. Since $\mathcal{S}_0 = \mathcal{S}_{M_0} \otimes \mathcal{S}_0^{\mathcal{N}}$ and $\mathrm{ch}(\mathcal{S}_{M_0})\hat{A}(M_0) = L(M_0)$, the claim follows. ∎

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