

SCHUBERT DECOMPOSITIONS FOR IND-VARIETIES OF GENERALIZED FLAGS*

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Abstract. Let \mathbf{G} be one of the ind-groups $GL(\infty)$, $O(\infty)$, $Sp(\infty)$ and $\mathbf{P} \subset \mathbf{G}$ be a splitting parabolic ind-subgroup. The ind-variety \mathbf{G}/\mathbf{P} has been identified with an ind-variety of generalized flags in [4]. In the present paper we define a Schubert cell on \mathbf{G}/\mathbf{P} as a \mathbf{B} -orbit on \mathbf{G}/\mathbf{P} , where \mathbf{B} is any Borel ind-subgroup of \mathbf{G} which intersects \mathbf{P} in a maximal ind-torus. A significant difference with the finite-dimensional case is that in general \mathbf{B} is not conjugate to an ind-subgroup of \mathbf{P} , whence \mathbf{G}/\mathbf{P} admits many non-conjugate Schubert decompositions. We study the basic properties of the Schubert cells, proving in particular that they are usual finite-dimensional cells or are isomorphic to affine ind-spaces.

We then define Schubert ind-varieties as closures of Schubert cells and study the smoothness of Schubert ind-varieties. Our approach to Schubert ind-varieties differs from an earlier approach by H. Salmasian [12].

Key words. Classical ind-group, Bruhat decomposition, Schubert decomposition, generalized flag, homogeneous ind-variety.

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1. Introduction. If G is a reductive algebraic group, the flag variety G/B is the most important geometric object attached to G . If \mathbf{G} is a classical ind-group, $\mathbf{G} = GL(\infty), O(\infty), Sp(\infty)$, then there are infinitely many conjugacy classes of splitting Borel subgroups \mathbf{B} (the notion of splitting subgroup is defined in Section 2.2), and hence there are infinitely many flag ind-varieties \mathbf{G}/\mathbf{B} . These smooth ind-varieties have been studied in [3, 4, 5], and in [4] each such ind-variety has been described explicitly as the ind-variety of certain generalized flags in the natural representation V of \mathbf{G} . A generalized flag is a chain of subspaces of V satisfying two conditions (see Definition 1), but notably such a chain is rarely ordered by an ordered subset of \mathbb{Z} .

In this paper we undertake a next step in the study of the generalized flag ind-varieties \mathbf{G}/\mathbf{B} , and more generally any ind-variety of the form \mathbf{G}/\mathbf{P} where \mathbf{P} is a splitting parabolic subgroup of \mathbf{G} . Namely, we define and study the Schubert decompositions of the ind-varieties \mathbf{G}/\mathbf{P} . The classical Schubert decomposition of a finite-dimensional flag variety is a key to important purely geometric theories such as Schubert calculus, as well as to geometric representation theory. In particular, Schubert varieties, i.e. closures of Schubert cells, play a crucial role in the geometric theory of category \mathcal{O} . The challenge to find ind-analogues of those theories motivates our study of Schubert decompositions of ind-varieties of generalized flags.

We define the Schubert cells on \mathbf{G}/\mathbf{P} as the \mathbf{B} -orbits on \mathbf{G}/\mathbf{P} for any Borel ind-subgroup \mathbf{B} which contains a common splitting maximal ind-torus with \mathbf{P} . The essential difference with the finite-dimensional case is that \mathbf{B} is not necessarily conjugate to a Borel subgroup of \mathbf{P} . This leads to the existence of many non-conjugate Schubert decompositions of a given ind-variety of generalized flags \mathbf{G}/\mathbf{P} . We compute the dimensions of the cells of all Schubert decompositions of \mathbf{G}/\mathbf{P} for any splitting

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Borel subgroup $\mathbf{B} \subset \mathbf{G}$. We also point out the Bruhat decomposition into double cosets of the ind-group \mathbf{G} which results from a Schubert decomposition of \mathbf{G}/\mathbf{P} .

In the last part of the paper we study the smoothness of Schubert ind-varieties which we define as closures of Schubert cells. We establish a criterion for smoothness which allows us to conclude that certain known criteria for smoothness of finite-dimensional Schubert varieties pass to the limit at infinity.

In his work [12], H. Salmasian introduced Schubert ind-subvarieties of \mathbf{G}/\mathbf{B} as arbitrary direct limits of Schubert varieties on finite-dimensional flag subvarieties of \mathbf{G}/\mathbf{B} . He showed that such an ind-variety may be singular at all of its points. With our definition, which takes into account the natural action of \mathbf{G} on \mathbf{G}/\mathbf{B} , a Schubert ind-variety has always a smooth big cell.

2. Preliminaries. In what follows \mathbb{K} is an algebraically closed field of characteristic zero. All varieties and algebraic groups are defined over \mathbb{K} . If A is a finite or infinite set, then $|A|$ denotes its cardinality. If A is a subset of the linear space V , then $\langle A \rangle$ denotes the linear subspace spanned by A .

2.1. Ind-varieties. An *ind-variety* is the direct limit $\mathbf{X} = \varinjlim X_n$ of a chain of morphisms of algebraic varieties

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} X_n \xrightarrow{\varphi_n} X_{n+1} \xrightarrow{\varphi_{n+1}} \dots \tag{1}$$

Note that the direct limit of the chain (1) does not change if we replace the sequence $\{X_n\}_{n \geq 1}$ by a subsequence $\{X_{i_n}\}_{n \geq 1}$ and the morphisms φ_n by the compositions $\tilde{\varphi}_{i_n} := \varphi_{i_{n+1}-1} \circ \dots \circ \varphi_{i_n+1} \circ \varphi_{i_n}$. Let \mathbf{X}' be a second ind-variety obtained as the direct limit of a chain

$$X'_1 \xrightarrow{\varphi'_1} X'_2 \xrightarrow{\varphi'_2} \dots \xrightarrow{\varphi'_{n-1}} X'_n \xrightarrow{\varphi'_n} X'_{n+1} \xrightarrow{\varphi'_{n+1}} \dots$$

A *morphism of ind-varieties* $\mathbf{f} : \mathbf{X}' \rightarrow \mathbf{X}$ is a map from $\varinjlim X'_n$ to $\varinjlim X_n$ induced by a collection of morphisms of algebraic varieties $\{f_n : X'_n \rightarrow X_n\}_{n \geq 1}$ such that $\tilde{\varphi}_{i_n} \circ f_n = f_{n+1} \circ \varphi'_n$ for all $n \geq 1$. The *identity morphism* $\text{id}_{\mathbf{X}}$ is a morphism that induces the identity as a set-theoretic map $\mathbf{X} \rightarrow \mathbf{X}$. A morphism $\mathbf{f} : \mathbf{X}' \rightarrow \mathbf{X}$ is an *isomorphism* if there exists a morphism $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{X}'$ such that $\mathbf{g} \circ \mathbf{f} = \text{id}_{\mathbf{X}'}$ and $\mathbf{f} \circ \mathbf{g} = \text{id}_{\mathbf{X}}$.

Any ind-variety \mathbf{X} is endowed with a topology by declaring a subset $\mathbf{U} \subset \mathbf{X}$ open if its inverse image by the natural map $X_m \rightarrow \varinjlim X_n$ is open for all m . Clearly, any open (resp., closed) (resp., locally closed) subset \mathbf{Z} of \mathbf{X} has a structure of ind-variety induced by the ind-variety structure on \mathbf{X} . We call \mathbf{Z} an *ind-subvariety* of \mathbf{X} .

In what follows we only consider chains (1) where the morphisms φ_n are inclusions, so that we can write $\mathbf{X} = \bigcup_{n \geq 1} X_n$. Then the sequence $\{X_n\}_{n \geq 1}$ is called *exhaustion* of \mathbf{X} .

Let $x \in \mathbf{X}$, so that $x \in X_n$ for n large enough. Let $\mathfrak{m}_{n,x} \subset \mathcal{O}_{X_n,x}$ be the maximal ideal of the localization at x of \mathcal{O}_{X_n} . For each $k \geq 1$ we have an epimorphism

$$\alpha_{n,k} : S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) \rightarrow \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1}. \tag{2}$$

Note that the point x is smooth in X_n if and only if $\alpha_{n,k}$ is an isomorphism for all k . By taking the inverse limit, we obtain a map

$$\hat{\alpha}_k := \varprojlim \alpha_{n,k} : \varprojlim S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) \rightarrow \varprojlim \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1}$$

which is an epimorphism for all k . We say that x is a *smooth point* of \mathbf{X} if and only if $\hat{\alpha}_k$ is an isomorphism for all k . We say that x is a *singular point* otherwise. The notion of smoothness of a point is independent of the choice of exhaustion $\{X_n\}_{n \geq 1}$ of \mathbf{X} . We say that \mathbf{X} is *smooth* if every point $x \in \mathbf{X}$ is smooth. As general references on smooth ind-varieties see [8, Chapter 4] and [11].

EXAMPLE 1. (a) Assume that every variety X_n in the chain (1) is an affine space, every image $\varphi_n(X_n)$ is an affine subspace of X_{n+1} , and $\lim_{n \rightarrow \infty} \dim X_n = \infty$. Then, up to isomorphism, $\mathbf{X} = \varinjlim X_n$ is independent of the choice of $\{X_n, \varphi_n\}_{n \geq 1}$ with these properties. We write $\mathbf{X} = \mathbb{A}^\infty$ and call it the *infinite-dimensional affine space*. For instance, \mathbb{A}^∞ admits the exhaustion $\mathbb{A}^\infty = \bigcup_{n \geq 1} \mathbb{A}^n$ where \mathbb{A}^n stands for the n -dimensional affine space. The infinite-dimensional affine space \mathbb{A}^∞ is a smooth ind-variety.

(b) If every variety X_n in the chain (1) is a projective space, every image $\varphi_n(X_n)$ is a projective subspace of X_{n+1} , and $\lim_{n \rightarrow \infty} \dim X_n = \infty$, then $\mathbf{X} = \varinjlim X_n$ is independent of the choice of $\{X_n, \varphi_n\}_{n \geq 1}$ with these properties. We write $\mathbf{X} = \mathbb{P}^\infty = \bigcup_{n \geq 1} \mathbb{P}^n$ and call \mathbb{P}^∞ the *infinite-dimensional projective space*. The infinite-dimensional projective space \mathbb{P}^∞ is also a smooth ind-variety.

A *cell decomposition* of an ind-variety \mathbf{X} is a decomposition $\mathbf{X} = \bigsqcup_{i \in I} \mathbf{X}_i$ into locally closed ind-subvarieties \mathbf{X}_i , each being a finite-dimensional or infinite-dimensional affine space, and such that the closure of each \mathbf{X}_i in \mathbf{X} is a union of some subsets \mathbf{X}_j ($j \in I$).

2.2. Ind-groups. An *ind-group* is an ind-variety \mathbf{G} endowed with a group structure such that the multiplication $\mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$, $(g, h) \mapsto gh$, and the inversion $\mathbf{G} \rightarrow \mathbf{G}$, $g \mapsto g^{-1}$ are morphisms of ind-varieties. A *morphism of ind-groups* $\mathbf{f} : \mathbf{G}' \rightarrow \mathbf{G}$ is by definition a morphism of groups which is also a morphism of ind-varieties. A *closed ind-subgroup* is a subgroup $\mathbf{H} \subset \mathbf{G}$ which is also a closed ind-subvariety.

We only consider *locally linear ind-groups*, i.e., ind-groups admitting an exhaustion $\{G_n\}_{n \geq 1}$ by linear algebraic groups. Moreover, we focus on the *classical ind-groups* $GL(\infty)$, $O(\infty)$, $Sp(\infty)$, which are obtained as subgroups of the group $Aut(V)$ of linear automorphisms of a countable-dimensional vector space V :

- Let E be a basis of V . Define $\mathbf{G}(E)$ as the subgroup of elements $g \in Aut(V)$ such that $g(e) = e$ for all but finitely many basis vectors $e \in E$. Given any filtration $E = \bigcup_{n \geq 1} E_n$ of the basis E by finite subsets, we have

$$\mathbf{G}(E) = \bigcup_{n \geq 1} G(E_n) \tag{3}$$

where $G(E_n)$ stands for $GL(\langle E_n \rangle)$. Thus $\mathbf{G}(E)$ is a locally linear ind-group. We also write $\mathbf{G}(E) = GL(\infty)$.

- Assume that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω . A basis E of V is called ω -isotropic if it is equipped with an involution $i_E : E \rightarrow E$ with at most one fixed point, such that $\omega(e, e') = 0$ for any $e, e' \in E$ unless $e' = i_E(e)$. Given an ω -isotropic basis E of V , define $\mathbf{G}^\omega(E)$ as the subgroup of elements $g \in \mathbf{G}(E)$ which preserve the bilinear form ω . If a filtration $E = \bigcup_{n \geq 1} E_n$ of the basis E by i_E -stable finite subsets is fixed, we have

$$\mathbf{G}^\omega(E) = \bigcup_{n \geq 1} G^\omega(E_n) \tag{4}$$

where $G^\omega(E_n)$ stands for the subgroup of elements $g \in G(E_n)$ preserving the restriction of ω . Thereby $\mathbf{G}^\omega(E)$ has a natural structure of locally linear ind-group. We also write $\mathbf{G}^\omega(E) = Sp(\infty)$ when ω is symplectic, and $\mathbf{G}^\omega(E) = O(\infty)$ when ω is symmetric.

REMARK 1. (a) Note that the group $\mathbf{G}(E) = GL(\infty)$ depends on the choice of the basis E . For this reason, in what follows, we prefer the notation $\mathbf{G}(E)$ instead of $GL(\infty)$.

An alternative construction of $GL(\infty)$ is as follows. Note that the dual space V^* is uncountable dimensional. Let $V_* \subset V^*$ be a countable-dimensional subspace such that the pairing $V_* \times V \rightarrow \mathbb{K}$ is nondegenerate. Then the group

$$\mathbf{G}(V, V_*) := \{g \in Aut(V) : g(V_*) = V_* \text{ and there are finite-codimensional subspaces of } V \text{ and } V_* \text{ fixed pointwise by } g\}$$

is an ind-group isomorphic to $GL(\infty)$. Moreover, we have $\mathbf{G}(V, V_*) = \mathbf{G}(E)$ whenever V_* is spanned by the dual family of E .

(b) The form ω induces a countable-dimensional subspace $V_* := \{\omega(v, \cdot) : v \in V\} \subset V^*$ of the dual space. Then the group

$$\mathbf{G}(V, \omega) := \{g \in \mathbf{G}(V, V_*) : g \text{ preserves } \omega\}$$

is an ind-subgroup of $\mathbf{G}(V, V_*)$ isomorphic to $Sp(\infty)$ (if ω is symplectic) or $O(\infty)$ (if ω is symmetric). The equality $\mathbf{G}(V, \omega) = \mathbf{G}^\omega(E)$ holds whenever E is an ω -isotropic basis.

(c) If ω is symplectic, then the involution $i_E : E \rightarrow E$ has no fixed point; the basis E is said to be of *type C* in this case. If ω is symmetric, then the involution $i_E : E \rightarrow E$ can have one fixed point, in which case the basis E is said to be of *type B*; if i_E has no fixed point, the basis E is said to be of *type D*. Bases of both types B and D exist in V (see [4, Lemma 4.2]).

In the rest of the paper, we fix once and for all a basis E of V and a filtration $E = \bigcup_{n \geq 1} E_n$ by finite subsets. We assume that the basis E is ω -isotropic and that the subsets E_n are i_E -stable whenever the bilinear form ω is considered.

Moreover, if the form ω is symmetric, in view of Remark 1 (b)–(c) in what follows we assume that the basis E is of type B and that every subset E_n of the filtration contains the fixed point of the involution i_E . This convention ensures that the variety of isotropic flags of a given type of each finite-dimensional space $\langle E_n \rangle$ is connected and $G^\omega(E_n)$ -homogeneous. Similarly, every i_E -stable finite subset of E considered in the sequel is assumed to contain the fixed point of i_E .

By \mathbf{G} we denote one of the ind-groups $\mathbf{G}(E)$, $\mathbf{G}^\omega(E)$.

Let \mathbf{H} be the subgroup of elements $g \in \mathbf{G}$ which are diagonal in the basis E . Then \mathbf{H} is a closed ind-subgroup of \mathbf{G} called *splitting Cartan subgroup*. A closed ind-subgroup $\mathbf{B} \subset \mathbf{G}$ which contains \mathbf{H} is called *splitting Borel subgroup* if it is locally solvable (i.e., every finite-dimensional ind-subgroup $B \subset \mathbf{B}$ is solvable) and is maximal with this property. A closed ind-subgroup which contains such a splitting Borel subgroup \mathbf{B} is called *splitting parabolic subgroup*. Equivalently, a closed ind-subgroup \mathbf{P} of \mathbf{G} containing \mathbf{H} is a splitting parabolic subgroup of \mathbf{G} if and only if $\mathbf{P} \cap G_n$ is a parabolic subgroup of G_n for all $n \geq 1$, where $\mathbf{G} = \bigcup_{n \geq 1} G_n$ is the natural exhaustion of (3) or (4). The quotient $\mathbf{G}/\mathbf{P} = \bigcup_{n \geq 1} G_n/(\mathbf{P} \cap G_n)$ is a *locally projective* ind-variety (i.e., an ind-variety exhausted by projective varieties); note however that

\mathbf{G}/\mathbf{P} is in general not a *projective* ind-variety (i.e., is not isomorphic to a closed ind-subvariety of the infinite-dimensional projective space \mathbb{P}^∞): see [4, Proposition 7.2] and [5, Proposition 15.1].

In [4] it is shown that the ind-variety \mathbf{G}/\mathbf{P} can be interpreted as an ind-variety of certain generalized flags. This construction is reviewed in the following section.

3. Ind-varieties of generalized flags. In Section 3.1 we recall from [3, 4] the notion of generalized flag and the correspondence between splitting parabolic subgroups \mathbf{P} of $\mathbf{G}(E)$ and E -compatible generalized flags \mathcal{F} . We also recall from [4] the construction of the ind-varieties $\mathbf{Fl}(\mathcal{F}, E)$ of generalized flags and their correspondence with homogeneous ind-spaces of the form $\mathbf{G}(E)/\mathbf{P}$.

In Section 3.2 we recall from [3, 4] the notion of ω -isotropic generalized flags and the construction of the ind-variety $\mathbf{Fl}(\mathcal{F}, \omega, E)$ of ω -isotropic generalized flags, as well as the correspondence with splitting parabolic subgroups of $\mathbf{G}^\omega(E)$ and the corresponding homogeneous ind-spaces.

For later use, some technical aspects of the construction of the ind-varieties $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$ are emphasized in Section 3.3.

3.1. Ind-variety of generalized flags. By *chain* of subspaces of V we mean a set of vector subspaces of V which is totally ordered by inclusion.

DEFINITION 1 ([3, 4]). A *generalized flag* is a chain \mathcal{F} of subspaces of V satisfying the following additional conditions:

- (i) every $F \in \mathcal{F}$ has an immediate predecessor F' in \mathcal{F} or an immediate successor F'' in \mathcal{F} ;
- (ii) for every nonzero vector $v \in V$, there is a pair (F', F'') of consecutive elements of \mathcal{F} such that $v \in F'' \setminus F'$.

Let $A_{\mathcal{F}}$ denote the set of pairs (F', F'') of consecutive subspaces $F', F'' \in \mathcal{F}$. The set $A_{\mathcal{F}}$ is totally ordered by the inclusion of pairs. Given a totally ordered set (A, \preceq) , we denote by $\mathbf{Fl}_A(V)$ the set of generalized flags such that $(A_{\mathcal{F}}, \subset)$ is isomorphic to (A, \preceq) . Equivalently, $\mathbf{Fl}_A(V)$ is the set of generalized flags \mathcal{F} which can be written in the form

$$\mathcal{F} = \{F'_\alpha, F''_\alpha : \alpha \in A\}, \tag{5}$$

where F'_α, F''_α are subspaces of V such that

$$\begin{cases} F'_\alpha \subsetneq F''_\alpha \text{ for all } \alpha \in A; \\ F''_\alpha \subset F'_\beta \text{ whenever } \alpha \prec \beta \text{ (possibly } F''_\alpha = F'_\beta); \\ V \setminus \{0\} = \bigsqcup_{\alpha \in A} F''_\alpha \setminus F'_\alpha. \end{cases} \tag{6}$$

DEFINITION 2. Let L be a basis of the space V . A generalized flag $\mathcal{F} = \{F'_\alpha, F''_\alpha : \alpha \in A\} \in \mathbf{Fl}_A(V)$ is said to be *compatible with L* if there is a (necessarily surjective) map $\sigma : L \rightarrow A$ such that

$$F'_\alpha = \langle e \in L : \sigma(e) \prec \alpha \rangle, \quad F''_\alpha = \langle e \in L : \sigma(e) \preceq \alpha \rangle$$

for all $\alpha \in A$.

Every generalized flag admits a compatible basis (see [4, Proposition 4.1]). The group $\mathbf{G}(E)$ acts on generalized flags in a natural way. Let $\mathbf{H}(E) \subset \mathbf{G}(E)$ be the

splitting Cartan subgroup formed by elements diagonal in E . It is easy to see that a generalized flag \mathcal{F} is compatible with E if and only if it is preserved by $\mathbf{H}(E)$. We denote by $\mathbf{P}_{\mathcal{F}} \subset \mathbf{G}(E)$ the subgroup of all elements which preserve \mathcal{F} .

- PROPOSITION 1 ([3, 4]). (a) *If \mathcal{F} is a generalized flag compatible with E , then $\mathbf{P}_{\mathcal{F}}$ is a splitting parabolic subgroup of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$.*
 (b) *The map $\mathcal{F} \mapsto \mathbf{P}_{\mathcal{F}}$ is a bijection between generalized flags compatible with E and splitting parabolic subgroups of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$.*
 (c) *A splitting parabolic subgroup $\mathbf{P}_{\mathcal{F}}$ is a splitting Borel subgroup if and only if the generalized flag \mathcal{F} is maximal (i.e., $\dim F''/F' = 1$ for every pair (F', F'') of consecutive elements of \mathcal{F}).*

REMARK 2. Proposition 1 (c) can be interpreted as a version of Lie’s theorem for the action of any splitting Borel subgroup on the space V . A general version of Lie’s theorem has been proved by J. Hennig in [6].

- DEFINITION 3 ([4]). (a) We say that a generalized flag \mathcal{F} is *weakly compatible with E* if \mathcal{F} is compatible with a basis L of V such that $E \setminus E \cap L$ is a finite set (equivalently $\text{codim}_V \langle E \cap L \rangle$ is finite).
 (b) Two generalized flags \mathcal{F}, \mathcal{G} are said to be *E -commensurable* if both \mathcal{F} and \mathcal{G} are weakly compatible with E , and there are an isomorphism of ordered sets $\phi : \mathcal{F} \rightarrow \mathcal{G}$ and a finite-dimensional subspace $U \subset V$ such that
 (i) $\phi(F) + U = F + U$ for all $F \in \mathcal{F}$,
 (ii) $\dim \phi(F) \cap U = \dim F \cap U$ for all $F \in \mathcal{F}$.

- REMARK 3. (a) Clearly, if \mathcal{F}, \mathcal{G} are E -commensurable with respect to a finite-dimensional subspace U , then \mathcal{F}, \mathcal{G} are E -commensurable with respect to any finite-dimensional subspace $U' \subset V$ such that $U' \supset U$.
 (b) E -commensurability is an equivalence relation on the set of generalized flags weakly compatible with E .

Let \mathcal{F} be a generalized flag compatible with E . We denote by $\mathbf{FI}(\mathcal{F}, E)$ the set of all generalized flags which are E -commensurable with \mathcal{F} .

PROPOSITION 2 ([4]). *The set $\mathbf{FI}(\mathcal{F}, E)$ is endowed with a natural structure of ind-variety. Moreover, this ind-variety is $\mathbf{G}(E)$ -homogeneous and the map $g \mapsto g\mathcal{F}$ induces an isomorphism of ind-varieties $\mathbf{G}(E)/\mathbf{P}_{\mathcal{F}} \cong \mathbf{FI}(\mathcal{F}, E)$.*

3.2. Ind-variety of isotropic generalized flags. In this section we assume that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω . We write U^\perp for the orthogonal subspace of the subspace $U \subset V$ with respect to ω . We assume that the basis E is ω -isotropic, i.e., endowed with an involution $i_E : E \rightarrow E$ with at most one fixed point and such that, for any $e, e' \in E$, $\omega(e, e') = 0$ unless $e' = i_E(e)$.

DEFINITION 4 ([3, 4]). A generalized flag \mathcal{F} is said to be ω -isotropic if $F^\perp \in \mathcal{F}$ whenever $F \in \mathcal{F}$, and if the map $F \mapsto F^\perp$ is an involution of \mathcal{F} .

For \mathcal{F} as in Definition 4, the involution $F \mapsto F^\perp$ is an anti-automorphism of the ordered set (\mathcal{F}, \subset) , i.e., it reverses the inclusion relation. Moreover, it induces an involutive anti-automorphism $(F'_\alpha, F''_\alpha) \mapsto ((F''_\alpha)^\perp, (F'_\alpha)^\perp)$ of the set $(A_{\mathcal{F}}, \subset)$ of pairs of consecutive subspaces of \mathcal{F} . Given a totally ordered set (A, \preceq, i_A) equipped with an

involutive anti-automorphism $i_A : A \rightarrow A$, we denote by $\mathbf{Fl}_A^\omega(V)$ the set of generalized flags $\mathcal{F} \in \mathbf{Fl}_A(V)$ (see (5)–(6)) which are ω -isotropic and satisfy the condition

$$((F''_\alpha)^\perp, (F'_\alpha)^\perp) = (F'_{i_A(\alpha)}, F''_{i_A(\alpha)}) \text{ for all } \alpha \in A. \tag{7}$$

REMARK 4. Note that the set A decomposes as

$$A = A_\ell \sqcup A_0 \sqcup A_r$$

where $A_\ell = \{\alpha \in A : \alpha \prec i_A(\alpha)\}$, $A_0 = \{\alpha \in A : \alpha = i_A(\alpha)\}$ (formed by at most one element), $A_r = \{\alpha \in A : \alpha \succ i_A(\alpha)\}$, and the map i_A restricts to bijections $A_\ell \rightarrow A_r$ and $A_r \rightarrow A_\ell$.

Given any $\mathcal{F} \in \mathbf{Fl}_A^\omega(V)$, we set $\mathcal{T}' = \bigcup_{\alpha \in A_\ell} F''_\alpha$ and $\mathcal{T}'' = \bigcap_{\alpha \in A_r} F'_\alpha$. Clearly, $\mathcal{T}' \subset \mathcal{T}''$, moreover it is easy to see that $(\mathcal{T}')^\perp = \mathcal{T}''$. We have either $\mathcal{T}' = \mathcal{T}''$ or $\mathcal{T}' \subsetneq \mathcal{T}''$. In the former case, the anti-automorphism i_A has no fixed point, hence $A = A_\ell \sqcup A_r$. In the latter case, the subspaces $\mathcal{T}', \mathcal{T}''$ necessarily belong to \mathcal{F} , moreover we have $(\mathcal{T}', \mathcal{T}'') = (F'_{\alpha_0}, F''_{\alpha_0})$ where $\alpha_0 \in A$ is the unique fixed point of i_A ; thus $A = A_\ell \sqcup \{\alpha_0\} \sqcup A_r$ in this case.

The following lemma shows that the notions of compatibility and weak-compatibility with a basis (Definitions 2–3) translate in a natural way to the context of ω -isotropic generalized flags and bases.

LEMMA 1. *Let $\mathcal{F} \in \mathbf{Fl}_A^\omega(V)$, with (A, \preceq, i_A) as above.*

- (a) *Let L be an ω -isotropic basis with corresponding involution $i_L : L \rightarrow L$. Assume that \mathcal{F} is compatible with L in the sense of Definition 2, via a surjective map $\sigma : L \rightarrow A$. Then the map σ satisfies $\sigma \circ i_L = i_A \circ \sigma$.*
- (b) *Assume that \mathcal{F} is weakly compatible with E . Then there is an ω -isotropic basis L such that the set $E \setminus E \cap L$ is finite and \mathcal{F} is compatible with L .*

Proof. (a) For every $e \in L$, we have $e \in F''_{\sigma(e)} \setminus F'_{\sigma(e)}$. Then $i_L(e) \in (F'_{\sigma(e)})^\perp \setminus (F''_{\sigma(e)})^\perp$. The definition of i_A yields $\sigma(i_L(e)) = i_A(\sigma(e))$.

(b) Let L be a basis of V such that $E \setminus E \cap L$ is finite and \mathcal{F} is compatible with L . Take a subset $E' \subset E$ stable by the involution i_E , such that i_E has no fixed point in E' , $E \setminus E'$ is finite, and $E' \subset E \cap L$. Then $V'' := \langle E \setminus E' \rangle$ is a finite-dimensional space and the restriction of ω to V'' is nondegenerate. The intersections $\mathcal{F}|_{V''} := \{F \cap V'' : F \in \mathcal{F}\}$ form an isotropic flag of V'' . Since V'' is finite dimensional, it is routine to find an ω -isotropic basis E'' of V'' such that $\mathcal{F}|_{V''}$ is compatible with E'' . Then $E' \cup E''$ is an ω -isotropic basis of V , and \mathcal{F} is compatible with $E' \cup E''$. \square

The group $\mathbf{G}^\omega(E)$ acts in a natural way on ω -isotropic generalized flags. Let $\mathbf{H}^\omega(E) \subset \mathbf{G}^\omega(E)$ be the splitting Cartan subgroup formed by elements diagonal in E . An ω -isotropic generalized flag is compatible with the basis E if and only if it is preserved by $\mathbf{H}^\omega(E)$. Given an ω -isotropic generalized flag \mathcal{F} compatible with E , we denote by $\mathbf{P}_\mathcal{F}^\omega \subset \mathbf{G}^\omega(E)$ the subgroup of all elements which preserve \mathcal{F} . Moreover, we denote by $\mathbf{Fl}(\mathcal{F}, \omega, E)$ the set of all ω -isotropic generalized flags which are E -commensurable with \mathcal{F} .

PROPOSITION 3 ([3, 4]). (a) *The map $\mathcal{F} \mapsto \mathbf{P}_\mathcal{F}^\omega$ is a bijection between ω -isotropic generalized flags compatible with E and splitting parabolic subgroups of $\mathbf{G}^\omega(E)$ containing $\mathbf{H}^\omega(E)$.*

(b) *A splitting parabolic subgroup $\mathbf{P}_\mathcal{F}^\omega$ is a splitting Borel subgroup of $\mathbf{G}^\omega(E)$ if and*

only if the generalized flag \mathcal{F} is maximal.

(c) The set $\mathbf{Fl}(\mathcal{F}, \omega, E)$ is endowed with a natural structure of ind-variety. This ind-variety is $\mathbf{G}^\omega(E)$ -homogeneous and the map $g \mapsto g\mathcal{F}$ induces an isomorphism of ind-varieties $\mathbf{G}^\omega(E)/\mathbf{P}_{\mathcal{F}}^\omega \cong \mathbf{Fl}(\mathcal{F}, \omega, E)$.

3.3. Structure of ind-variety on $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$. In this section we present the structure of ind-variety on $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$ mentioned in Propositions 2–3.

We assume that \mathcal{F} is a generalized flag compatible with the basis E . Let (A, \preceq) be a totally ordered set such that $\mathcal{F} \in \mathbf{Fl}_A(V)$. Hence we can write $\mathcal{F} = \{F'_\alpha, F''_\alpha : \alpha \in A\}$. Let $\sigma : E \rightarrow A$ be the surjective map corresponding to \mathcal{F} in the sense of Definition 2.

Let $I \subset E$ be a finite subset. The generalized flag \mathcal{F} gives rise to a (finite) flag $\mathcal{F}|_I$ of the finite-dimensional vector space $\langle I \rangle$ by letting

$$\mathcal{F}|_I := \{F \cap \langle I \rangle : F \in \mathcal{F}\} = \{F'_\alpha \cap \langle I \rangle, F''_\alpha \cap \langle I \rangle : \alpha \in A\}.$$

Let

$$d'_\alpha := \dim F'_\alpha \cap \langle I \rangle = |\{e \in I : \sigma(e) \prec \alpha\}| \quad \text{and} \quad d''_\alpha := \dim F''_\alpha \cap \langle I \rangle = |\{e \in I : \sigma(e) \preceq \alpha\}|.$$

We denote by $\mathbf{Fl}(\mathcal{F}, I)$ the projective variety of flags in the space $\langle I \rangle$ of the form $\{M'_\alpha, M''_\alpha : \alpha \in A\}$ where $M'_\alpha, M''_\alpha \subset \langle I \rangle$ are linear subspaces such that

$$\dim M'_\alpha = d'_\alpha, \dim M''_\alpha = d''_\alpha, M'_\alpha \subset M''_\alpha \text{ for all } \alpha \in A, \text{ and } M''_\alpha \subset M'_\beta \text{ whenever } \alpha \prec \beta.$$

If $J \subset E$ is another finite subset such that $I \subset J$, we define an embedding $\iota_{I,J} : \mathbf{Fl}(\mathcal{F}, I) \hookrightarrow \mathbf{Fl}(\mathcal{F}, J)$, $\{M'_\alpha, M''_\alpha : \alpha \in A\} \mapsto \{N'_\alpha, N''_\alpha : \alpha \in A\}$ by letting

$$N'_\alpha = M'_\alpha \oplus \langle e \in J \setminus I : \sigma(e) \prec \alpha \rangle \quad \text{and} \quad N''_\alpha = M''_\alpha \oplus \langle e \in J \setminus I : \sigma(e) \preceq \alpha \rangle \quad \text{for all } \alpha \in A.$$

If we consider a filtration $E = \bigcup_{n \geq 1} E_n$ of the basis E by finite subsets, then we obtain a chain of morphisms of projective varieties

$$\mathbf{Fl}(\mathcal{F}, E_1) \xrightarrow{\iota_1} \mathbf{Fl}(\mathcal{F}, E_2) \xrightarrow{\iota_2} \dots \xrightarrow{\iota_{n-1}} \mathbf{Fl}(\mathcal{F}, E_n) \xrightarrow{\iota_n} \mathbf{Fl}(\mathcal{F}, E_{n+1}) \xrightarrow{\iota_{n+1}} \dots \quad (8)$$

where $\iota_n := \iota_{E_n, E_{n+1}}$.

PROPOSITION 4 ([4]). *The set $\mathbf{Fl}(\mathcal{F}, E)$ is the direct limit of the chain of morphisms (8). Hence $\mathbf{Fl}(\mathcal{F}, E)$ is endowed with a structure of ind-variety. Moreover, this structure is independent of the filtration $\{E_n\}_{n \geq 1}$ of the basis E .*

We assume next that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω , that the basis E is ω -isotropic with corresponding involution $i_E : E \rightarrow E$, that the ordered set (A, \preceq) is equipped with an anti-automorphism $i_A : A \rightarrow A$, and that the surjection $\sigma : E \rightarrow A$ satisfies $\sigma \circ i_E = i_A \circ \sigma$ so that \mathcal{F} is an ω -isotropic generalized flag.

Consider an i_E -stable finite subset $I \subset E$. Then the restriction of ω to the space $\langle I \rangle$ is nondegenerate. Let $\mathbf{Fl}(\mathcal{F}, \omega, I) \subset \mathbf{Fl}(\mathcal{F}, I)$ be the (closed) subvariety formed by flags $\{M'_\alpha, M''_\alpha : \alpha \in A\}$ such that

$$((M''_\alpha)^{\perp_I}, (M'_\alpha)^{\perp_I}) = (M'_{i_A(\alpha)}, M''_{i_A(\alpha)}) \quad \text{for all } \alpha \in A,$$

where the notation \perp_I stands for orthogonal subspace in the space $(\langle I \rangle, \omega)$. If $J \subset E$ is another i_E -stable finite subset, then the embedding $\iota_{I,J}$ restricts to an embedding

$\iota_{I,J}^\omega : \text{Fl}(\mathcal{F}, \omega, I) \hookrightarrow \text{Fl}(\mathcal{F}, \omega, J)$. Consequently, for a filtration $E = \bigcup_{n \geq 1} E_n$ by i_E -stable finite subsets, we obtain a chain of morphisms of projective varieties

$$\text{Fl}(\mathcal{F}, \omega, E_1) \xrightarrow{\iota_1^\omega} \text{Fl}(\mathcal{F}, \omega, E_2) \xrightarrow{\iota_2^\omega} \dots \xrightarrow{\iota_{n-1}^\omega} \text{Fl}(\mathcal{F}, \omega, E_n) \xrightarrow{\iota_n^\omega} \text{Fl}(\mathcal{F}, \omega, E_{n+1}) \xrightarrow{\iota_{n+1}^\omega} \dots \quad (9)$$

where $\iota_n^\omega := \iota_{E_n, E_{n+1}}^\omega$.

PROPOSITION 5 ([4]). *The set $\mathbf{Fl}(\mathcal{F}, \omega, E)$ is the direct limit of the chain of morphisms (9). Hence $\mathbf{Fl}(\mathcal{F}, \omega, E)$ is endowed with a structure of ind-variety, independent of the filtration $\{E_n\}_{n \geq 1}$. Moreover, $\mathbf{Fl}(\mathcal{F}, \omega, E)$ is a closed ind-subvariety of $\mathbf{Fl}(\mathcal{F}, E)$.*

4. Schubert decomposition of $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$. Let \mathbf{G} be one of the groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$. Let \mathbf{P} and \mathbf{B} be respectively a splitting parabolic subgroup and a splitting Borel subgroup of \mathbf{G} , both containing the splitting Cartan subgroup $\mathbf{H} = \mathbf{H}(E)$ or $\mathbf{H}^\omega(E)$. From the previous section we know that the homogeneous space \mathbf{G}/\mathbf{P} can be viewed as an ind-variety of generalized flags of the form $\mathbf{Fl}(\mathcal{F}, E)$ or $\mathbf{Fl}(\mathcal{F}, \omega, E)$. In this section we describe the decomposition of \mathbf{G}/\mathbf{P} into \mathbf{B} -orbits. The main results are stated in Theorem 1 in the case of $\mathbf{G} = \mathbf{G}(E)$ and in Theorem 2 in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$. In both cases it is shown that the \mathbf{B} -orbits form a cell decomposition of \mathbf{G}/\mathbf{P} , and their dimensions and closures are expressed in combinatorial terms. In Section 4.3 we derive the decomposition of the ind-group \mathbf{G} into double cosets. Unlike the case of Kac–Moody groups, the \mathbf{B} -orbits of \mathbf{G}/\mathbf{P} can be infinite dimensional. The cases where all orbits are finite dimensional (resp., infinite dimensional) are characterized in Section 4.4. In Section 4.5 we focus on the situation where \mathbf{G}/\mathbf{P} is an ind-grassmannian.

In this section the results are stated without proofs. The proofs are given in Section 5.

4.1. Decomposition of $\mathbf{Fl}(\mathcal{F}, E)$. Let $\mathbf{G} = \mathbf{G}(E)$, $\mathbf{H} = \mathbf{H}(E)$, and \mathbf{P}, \mathbf{B} be as above. By Propositions 1–2 there is a generalized flag \mathcal{F} compatible with E such that $\mathbf{P} = \mathbf{P}_{\mathcal{F}}$ is the subgroup of elements $g \in \mathbf{G}(E)$ preserving \mathcal{F} , and the homogeneous space $\mathbf{G}(E)/\mathbf{P}$ is isomorphic to the ind-variety of generalized flags $\mathbf{Fl}(\mathcal{F}, E)$. The precise description of the decomposition of $\mathbf{Fl}(\mathcal{F}, E)$ into \mathbf{B} -orbits is the object of this section. It requires some preliminaries and notation.

We denote by $\mathbf{W}(E)$ the group of permutations $w : E \rightarrow E$ such that $w(e) = e$ for all but finitely many $e \in E$. In particular, $\mathbf{W}(E)$ is isomorphic to the infinite symmetric group \mathfrak{S}_∞ . Note that we have

$$\mathbf{W}(E) = \bigcup_{n \geq 1} W(E_n)$$

where $W(E_n)$ is the Weyl group of $G(E_n)$.

Let $(A, \preceq_A) := (A_{\mathcal{F}}, \subset)$ be the set of pairs of consecutive elements of \mathcal{F} , so that $\mathcal{F} \in \mathbf{Fl}_A(V)$ and in fact $\mathbf{Fl}(\mathcal{F}, E) \subset \mathbf{Fl}_A(V)$. Let $\mathfrak{S}(E, A)$ be the set of surjective maps $\sigma : E \rightarrow A$. For $\sigma \in \mathfrak{S}(E, A)$, we denote by \mathcal{F}_σ the generalized flag $\mathcal{F}_\sigma = \{F'_{\sigma,\alpha}, F''_{\sigma,\alpha} : \alpha \in A\}$ given by

$$F'_{\sigma,\alpha} = \langle e \in E : \sigma(e) \prec_A \alpha \rangle \quad \text{and} \quad F''_{\sigma,\alpha} = \langle e \in E : \sigma(e) \preceq_A \alpha \rangle. \quad (10)$$

Thus $\{\mathcal{F}_\sigma : \sigma \in \mathfrak{S}(E, A)\}$ are exactly the generalized flags of $\mathbf{Fl}_A(V)$ compatible with the basis E (see Definition 2). Let $\sigma_0 : E \rightarrow A$ be the surjective map such that $\mathcal{F} = \mathcal{F}_{\sigma_0}$.

REMARK 5. The totally ordered set (A, \preceq_A) and the surjective map $\sigma_0 : E \rightarrow A$ give rise to a partial order $\preceq_{\mathbf{P}}$ on E , defined by letting $e \prec_{\mathbf{P}} e'$ if $\sigma_0(e) \prec_A \sigma_0(e')$. Note that the partial order $\preceq_{\mathbf{P}}$ has the property

the relation “ e is not comparable with e' ” (i.e., neither $e \prec_{\mathbf{P}} e'$ nor $e' \prec_{\mathbf{P}} e$) is (11) an equivalence relation.

In fact, fixing a splitting parabolic subgroup $\mathbf{P} \subset \mathbf{G}(E)$ containing $\mathbf{H}(E)$ is equivalent to fixing a partial order $\preceq_{\mathbf{P}}$ on E satisfying property (11). Moreover, \mathbf{P} is a splitting Borel subgroup if and only if the order $\preceq_{\mathbf{P}}$ is total.

The group $\mathbf{W}(E)$ acts on the set $\mathfrak{S}(E, A)$, hence on E -compatible generalized flags of $\mathbf{Fl}_A(V)$, by the procedure $(w, \sigma) \mapsto \sigma \circ w^{-1}$. Let $\mathbf{W}_{\mathbf{P}}(E) \subset \mathbf{W}(E)$ be the subgroup of permutations such that $\sigma_0 \circ w^{-1} = \sigma_0$. Equivalently, $\mathbf{W}_{\mathbf{P}}(E)$ is the subgroup of permutations $w \in \mathbf{W}(E)$ which preserve the fibers $\sigma_0^{-1}(\alpha)$ ($\alpha \in A$) of the map σ_0 .

LEMMA 2. *The map $w \mapsto \mathcal{F}_{\sigma_0 \circ w^{-1}}$ induces a bijection between the quotient $\mathbf{W}(E)/\mathbf{W}_{\mathbf{P}}(E)$ and the set of E -compatible generalized flags of the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$.*

Let $\mathbf{W}(E) \cdot \sigma_0 = \{\sigma_0 \circ w^{-1} : w \in \mathbf{W}(E)\}$ denote the $\mathbf{W}(E)$ -orbit of σ_0 .

The splitting Borel subgroup \mathbf{B} is the subgroup $\mathbf{B} = \mathbf{P}_{\mathcal{F}_0}$ of elements $g \in \mathbf{G}(E)$ preserving a maximal generalized flag \mathcal{F}_0 compatible with E (see Proposition 1). Equivalently \mathbf{B} corresponds to a total order $\preceq_{\mathbf{B}}$ on the basis E (see Remark 5). Then, the generalized flag $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ is given by $F'_{0,e} = \langle e' \in E : e' \prec_{\mathbf{B}} e \rangle$ and $F''_{0,e} = \langle e' \in E : e' \preceq_{\mathbf{B}} e \rangle$ for all $e \in E$.

Relying on the total order $\preceq_{\mathbf{B}}$, we define a notion of inversion number and an analogue of the Bruhat order on the set $\mathfrak{S}(E, A)$.

Number of inversions $n_{\text{inv}}(\sigma)$. We say that a pair $(e, e') \in E \times E$ is an *inversion* of $\sigma \in \mathfrak{S}(E, A)$ if $e \prec_{\mathbf{B}} e'$ and $\sigma(e) \succ_A \sigma(e')$. Then

$$n_{\text{inv}}(\sigma) := |\{(e, e') \in E \times E : (e, e') \text{ is an inversion of } \sigma\}|$$

is the *inversion number* of σ . Note that the inversion number $n_{\text{inv}}(\sigma)$ may be infinite.

Partial order \leq on $\mathfrak{S}(E, A)$. We now define a partial order on the set $\mathfrak{S}(E, A)$, analogous to the Bruhat order. For $(e, e') \in E \times E$ with $e \neq e'$, we denote by $t_{e,e'}$ the element of $\mathbf{W}(E)$ which exchanges e with e' and fixes every other element $e'' \in E$. Let $\sigma, \tau \in \mathfrak{S}(E, A)$. We set $\sigma \hat{<} \tau$ if $\tau = \sigma \circ t_{e,e'}$ for a pair $(e, e') \in E \times E$ satisfying $e \prec_{\mathbf{B}} e'$ and $\sigma(e) \prec_A \sigma(e')$. We set $\sigma < \tau$ if there is a chain $\tau_0 = \sigma \hat{<} \tau_1 \hat{<} \tau_2 \hat{<} \dots \hat{<} \tau_k = \tau$ of elements of $\mathfrak{S}(E, A)$ (with $k \geq 1$).

Element $\sigma_{\mathcal{G}} \in \mathfrak{S}(E, A)$. Given a generalized flag $\mathcal{G} = \{G'_{\alpha}, G''_{\alpha} : \alpha \in A\} \in \mathbf{Fl}_A(V)$ weakly compatible with E , we define an element $\sigma_{\mathcal{G}} \in \mathfrak{S}(E, A)$ which measures the relative position of \mathcal{G} to the maximal generalized flag \mathcal{F}_0 . Set

$$\sigma_{\mathcal{G}}(e) = \min\{\alpha \in A : G''_{\alpha} \cap F''_{0,e} \neq G'_{\alpha} \cap F'_{0,e}\} \text{ for all } e \in E. \tag{12}$$

[It can be checked directly that the so obtained map $\sigma_{\mathcal{G}} : E \rightarrow A$ is indeed surjective, hence an element of $\mathfrak{S}(E, A)$. This fact is also shown in Section 5.2 in the proof of Theorem 2.]

We are now in position to formulate the statement which describes the decomposition of $\mathbf{Fl}(\mathcal{F}, E)$ into \mathbf{B} -orbits.

THEOREM 1. *Let $\mathbf{P}_{\mathcal{F}}$ be the splitting parabolic subgroup of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$, and corresponding to a generalized flag $\mathcal{F} = \mathcal{F}_{\sigma_0} \in \mathbf{Fl}_A(V)$ (with $\sigma_0 \in \mathfrak{S}(E, A)$) compatible with E . Let \mathbf{B} be any splitting Borel subgroup of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$.*

(a) *We have the decomposition*

$$\mathbf{G}(E)/\mathbf{P}_{\mathcal{F}} = \mathbf{Fl}(\mathcal{F}, E) = \bigsqcup_{\sigma \in \mathbf{W}(E) \cdot \sigma_0} \mathbf{B}\mathcal{F}_{\sigma} = \bigsqcup_{w \in \mathbf{W}(E)/\mathbf{W}_{\mathbf{P}}(E)} \mathbf{B}\mathcal{F}_{\sigma_0 \circ w^{-1}}.$$

(b) *A generalized flag $\mathcal{G} \in \mathbf{Fl}(\mathcal{F}, E)$ belongs to the \mathbf{B} -orbit $\mathbf{B}\mathcal{F}_{\sigma}$ ($\sigma \in \mathbf{W}(E) \cdot \sigma_0$) if and only if $\sigma_{\mathcal{G}} = \sigma$.*

(c) *The orbit $\mathbf{B}\mathcal{F}_{\sigma}$ ($\sigma \in \mathbf{W}(E) \cdot \sigma_0$) is a locally closed ind-subvariety of $\mathbf{Fl}(\mathcal{F}, E)$ isomorphic to the affine space $\mathbb{A}^{n_{\text{inv}}(\sigma)}$ (which is infinite dimensional if $n_{\text{inv}}(\sigma)$ is infinite).*

(d) *For $\sigma, \tau \in \mathbf{W}(E) \cdot \sigma_0$, the inclusion $\mathbf{B}\mathcal{F}_{\sigma} \subset \overline{\mathbf{B}\mathcal{F}_{\tau}}$ holds if and only if $\sigma \leq \tau$.*

REMARK 6. If $\sigma \in \mathbf{W}(E) \cdot \sigma_0$, say $\sigma = \sigma_0 \circ w$ with $w \in \mathbf{W}(E)$, then the inversion number of σ is also given by the formula

$$n_{\text{inv}}(\sigma) = |\{(e, e') \in E \times E : e \prec_{\mathbf{B}} e' \text{ and } w(e) \succ_{\mathbf{P}} w(e')\}|$$

(see Remark 5). Note that the inversion number $n_{\text{inv}}(\sigma)$ cannot be directly interpreted as a Bruhat length because we do not assume \mathbf{B} to be conjugate to a subgroup of \mathbf{P} .

4.2. Decomposition of $\mathbf{Fl}(\mathcal{F}, \omega, E)$. In this section the basis E is ω -isotropic with corresponding involution $i_E : E \rightarrow E$ (see Section 3.2). Let $\mathbf{P} \subset \mathbf{G}^{\omega}(E)$ be a splitting parabolic subgroup containing $\mathbf{H}^{\omega}(E)$, or equivalently let \mathcal{F} be an ω -isotropic generalized flag compatible with E (see Proposition 3). Let $\mathbf{B} \subset \mathbf{G}^{\omega}(E)$ be a splitting Borel subgroup containing $\mathbf{H}^{\omega}(E)$. We study the decomposition of the ind-variety $\mathbf{G}^{\omega}(E)/\mathbf{P} \cong \mathbf{Fl}(\mathcal{F}, \omega, E)$ into \mathbf{B} -orbits.

Let (A, \preceq_A, i_A) be a totally ordered set with involutive anti-automorphism i_A , such that $\mathcal{F} \in \mathbf{Fl}_A^{\omega}(V)$. We denote by $\mathfrak{S}^{\omega}(E, A)$ the set of surjective maps $\sigma : E \rightarrow A$ such that $\sigma(i_E(e)) = i_A(\sigma(e))$ for all $e \in E$. By Lemma 1, $\{\mathcal{F}_{\sigma} : \sigma \in \mathfrak{S}^{\omega}(E, A)\}$ are exactly the elements of $\mathbf{Fl}_A^{\omega}(V)$ compatible with E (the notation \mathcal{F}_{σ} is introduced in (10)). Let $\sigma_0 \in \mathfrak{S}^{\omega}(E, A)$ be such that $\mathcal{F} = \mathcal{F}_{\sigma_0}$.

The group $\mathbf{W}^{\omega}(E)$ is defined as the group of permutations $w : E \rightarrow E$ such that $w(e) = e$ for all but finitely many $e \in E$ and $w(i_E(e)) = i_E(w(e))$ for all $e \in E$. Note that $\mathbf{W}^{\omega}(E)$ acts on the set $\mathfrak{S}^{\omega}(E, A)$ by the procedure $(w, \sigma) \mapsto \sigma \circ w^{-1}$. Let $\mathbf{W}_{\mathbf{P}}^{\omega}(E)$ be the subgroup of elements $w \in \mathbf{W}^{\omega}(E)$ such that $\sigma_0 \circ w^{-1} = \sigma_0$ and let $\mathbf{W}^{\omega}(E) \cdot \sigma_0 := \{\sigma_0 \circ w^{-1} : w \in \mathbf{W}^{\omega}(E)\}$ be the $\mathbf{W}^{\omega}(E)$ -orbit of σ_0 .

LEMMA 3. *The map $w \mapsto \mathcal{F}_{\sigma_0 \circ w^{-1}}$ induces a bijection between $\mathbf{W}^{\omega}(E)/\mathbf{W}_{\mathbf{P}}^{\omega}(E)$ and the set of E -compatible elements of $\mathbf{Fl}(\mathcal{F}, \omega, E)$.*

The splitting Borel subgroup \mathbf{B} is the subgroup $\mathbf{B} = \mathbf{P}_{\mathcal{F}_0}^{\omega}$ of elements preserving some maximal ω -isotropic generalized flag \mathcal{F}_0 compatible with E . We can write $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ with $F'_{0,e} = \langle e' \in E : e' \prec_{\mathbf{B}} e \rangle$ and $F''_{0,e} = \langle e' \in E : e' \preceq_{\mathbf{B}} e \rangle$, where $\preceq_{\mathbf{B}}$ is a total order on E . Moreover, the fact that \mathcal{F}_0 is ω -isotropic implies that the involution $i_E : E \rightarrow E$ is an anti-automorphism of the ordered set $(E, \preceq_{\mathbf{B}})$.

Number of inversions $n_{\text{inv}}^{\omega}(\sigma)$. Let $\sigma \in \mathfrak{S}^{\omega}(E, A)$. We define an ω -isotropic inversion of σ as a pair $(e, e') \in E \times E$ such that

$$e \prec_{\mathbf{B}} e', \quad e \prec_{\mathbf{B}} i_E(e), \quad e' \neq i_E(e'), \quad \text{and} \quad \sigma(e) \succ_A \sigma(e').$$

Let

$$n_{\text{inv}}^\omega(\sigma) = |\{(e, e') \in E \times E : (e, e') \text{ is an } \omega\text{-isotropic inversion of } \sigma\}|.$$

Partial order \leq_ω on $\mathfrak{S}^\omega(E, A)$. Given $(e, e') \in E \times E$ with $e \neq e'$, $i_E(e) \neq e$, $i_E(e') \neq e'$, we set

$$t_{e, e'}^\omega = t_{e, e'} \circ t_{i_E(e), i_E(e')} \text{ if } e' \neq i_E(e), \quad t_{e, e'}^\omega = t_{e, e'} \text{ if } e' = i_E(e).$$

Thus $t_{e, e'}^\omega \in \mathbf{W}^\omega(E)$. Let $\sigma, \tau \in \mathfrak{S}^\omega(E, A)$. We set $\sigma \hat{<}_\omega \tau$ if $\tau = \sigma \circ t_{e, e'}^\omega$ for a pair (e, e') satisfying $e \prec_{\mathbf{B}} e'$ and $\sigma(e) \prec_A \sigma(e')$. Finally we set $\sigma <_\omega \tau$ if there is a chain $\tau_0 = \sigma \hat{<}_\omega \tau_1 \hat{<}_\omega \tau_2 \hat{<}_\omega \dots \hat{<}_\omega \tau_k = \tau$ of elements of $\mathfrak{S}^\omega(E, A)$.

THEOREM 2. *Let $\mathbf{P}_{\mathcal{F}}^\omega$ be the splitting parabolic subgroup of $\mathbf{G}^\omega(E)$ containing $\mathbf{H}^\omega(E)$, and corresponding to an E -compatible generalized flag $\mathcal{F} = \mathcal{F}_{\sigma_0} \in \mathbf{Fl}_A^\omega(V)$ (with $\sigma_0 \in \mathfrak{S}^\omega(E, A)$). Let \mathbf{B} be any splitting Borel subgroup of $\mathbf{G}^\omega(E)$ containing $\mathbf{H}^\omega(E)$.*

(a) *We have the decomposition*

$$\mathbf{G}^\omega(E)/\mathbf{P}_{\mathcal{F}}^\omega = \mathbf{Fl}(\mathcal{F}, \omega, E) = \bigsqcup_{\sigma \in \mathbf{W}^\omega(E) \cdot \sigma_0} \mathbf{B}\mathcal{F}_\sigma = \bigsqcup_{w \in \mathbf{W}^\omega(E)/\mathbf{W}_{\mathbf{P}}^\omega(E)} \mathbf{B}\mathcal{F}_{\sigma_0 \circ w^{-1}}.$$

(b) *For $\mathcal{G} \in \mathbf{Fl}(\mathcal{F}, \omega, E)$ the map $\sigma_{\mathcal{G}} : E \rightarrow A$ (see (12)) belongs to $\mathbf{W}^\omega(E) \cdot \sigma_0$. Moreover, \mathcal{G} belongs to $\mathbf{B}\mathcal{F}_\sigma$ ($\sigma \in \mathbf{W}^\omega(E) \cdot \sigma_0$) if and only if $\sigma_{\mathcal{G}} = \sigma$.*

(c) *The orbit $\mathbf{B}\mathcal{F}_\sigma$ ($\sigma \in \mathbf{W}^\omega(E) \cdot \sigma_0$) is a locally closed ind-subvariety of $\mathbf{Fl}(\mathcal{F}, \omega, E)$ isomorphic to the affine space $\mathbb{A}^{n_{\text{inv}}^\omega(\sigma)}$ (possibly infinite-dimensional).*

(d) *For $\sigma, \tau \in \mathbf{W}^\omega(E) \cdot \sigma_0$, the inclusion $\mathbf{B}\mathcal{F}_\sigma \subset \overline{\mathbf{B}\mathcal{F}_\tau}$ holds if and only if $\sigma \leq_\omega \tau$.*

4.3. Bruhat decomposition of the ind-group $\mathbf{G} = \mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$. Let $\mathbf{H} = \mathbf{H}(E)$ or $\mathbf{H}^\omega(E)$, and let $\mathbf{W} = \mathbf{W}(E)$ or $\mathbf{W}^\omega(E)$. If $\mathbf{W} = \mathbf{W}(E)$, the linear extension of $w \in \mathbf{W}$ is an element $\hat{w} \in \mathbf{G}(E)$. If $\mathbf{W} = \mathbf{W}^\omega(E)$, we can find scalars $\lambda_e \in \mathbb{K}^*$ ($e \in E$) such that the map $e \mapsto \lambda_e w(e)$ linearly extends to an element $\hat{w} \in \mathbf{G}^\omega(E)$. In both situations it is easy to deduce that \mathbf{W} is isomorphic to the quotient $N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$.

Given a splitting parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ containing \mathbf{H} , we denote by $\mathbf{W}_{\mathbf{P}}$ the corresponding subgroup of \mathbf{W} . The following statement describes the decomposition of the ind-group \mathbf{G} into double cosets. It is a consequence of Theorems 1–2.

COROLLARY 1. *Let \mathbf{G} be one of the ind-groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$, and let \mathbf{P} and \mathbf{B} be respectively a splitting parabolic and a splitting Borel subgroup of \mathbf{G} containing \mathbf{H} . Then we have a decomposition*

$$\mathbf{G} = \bigsqcup_{w \in \mathbf{W}/\mathbf{W}_{\mathbf{P}}} \mathbf{B}\hat{w}\mathbf{P}.$$

REMARK 7. (a) Note that the unique assumption on the splitting parabolic and Borel subgroups \mathbf{P} and \mathbf{B} in Corollary 1 is that they contain a common splitting Cartan subgroup, in particular it is not required that \mathbf{B} be conjugate to a subgroup of \mathbf{P} .

(b) The ind-group \mathbf{G} admits a natural exhaustion $\mathbf{G} = \bigcup_{n \geq 1} G_n$ by finite-dimensional subgroups of the form $G_n = G(E_n)$ or $G_n = G^\omega(E_n)$ (see Section 2.2). Moreover, the

intersections $P_n := \mathbf{P} \cap G_n$ and $B_n := \mathbf{B} \cap G_n$ are respectively a parabolic subgroup and a Borel subgroup of G_n , containing a common Cartan subgroup. Then the decomposition of Corollary 1 can be retrieved by considering usual Bruhat decompositions of the groups G_n into double cosets for P_n and B_n .

4.4. On the existence of cells of finite or infinite dimension. In Theorems 1–2 it appears that the decomposition of an ind-variety of generalized flags into \mathbf{B} -orbits may comprise orbits of infinite dimension. The following result determines precisely the situations in which infinite-dimensional orbits arise.

THEOREM 3. *Let \mathbf{G} be one of the groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$. Let $\mathbf{P}, \mathbf{B} \subset \mathbf{G}$ be splitting parabolic and Borel subgroups containing the splitting Cartan subgroup \mathbf{H} of \mathbf{G} .*

- (a) *The following conditions are equivalent:*
 - (i) \mathbf{B} is conjugate (under \mathbf{G}) to a subgroup of \mathbf{P} ;
 - (ii) At least one \mathbf{B} -orbit of \mathbf{G}/\mathbf{P} is finite dimensional;
 - (iii) One \mathbf{B} -orbit of \mathbf{G}/\mathbf{P} is a single point (and this orbit is necessarily unique).
- (b) *Let $\preceq_{\mathbf{B}}$ be the total order on the basis E induced by \mathbf{B} . Assume that $\mathbf{P} \neq \mathbf{G}$. The following conditions are equivalent:*
 - (i) \mathbf{B} is conjugate (under \mathbf{G}) to a subgroup of \mathbf{P} , and the ordered set $(E, \preceq_{\mathbf{B}})$ is isomorphic (as ordered set) to a subset of (\mathbb{Z}, \leq) ;
 - (ii) Every \mathbf{B} -orbit of \mathbf{G}/\mathbf{P} is finite dimensional.

REMARK 8. (a) Theorem 3 provides in particular a criterion for a given splitting Borel subgroup to be conjugate to a subgroup of a given splitting parabolic subgroup. This criterion is applied in the next section.

(b) Following [4], we call a generalized flag \mathcal{G} a *flag* if the chain (\mathcal{G}, \subset) is isomorphic as ordered set to a subset of (\mathbb{Z}, \leq) . Then the second part of condition (b) (i) in Theorem 3 can be rephrased by saying that the maximal generalized flag \mathcal{F}_0 is a flag. Another characterization of flags is provided by [4, Proposition 7.2] which says that the ind-variety of generalized flags $\mathbf{Fl}(\mathcal{G}, E)$ (resp., $\mathbf{Fl}(\mathcal{G}, \omega, E)$) is *projective* (i.e., isomorphic as ind-variety to a closed ind-subvariety of the infinite-dimensional projective space \mathbb{P}^∞) if and only if \mathcal{G} is a flag.

4.5. Decomposition of ind-grassmannians. A minimal (nontrivial) generalized flag $\mathcal{F} = \{0, F, V\}$ of the space V is determined by the proper nonzero subspace $F \subset V$. If \mathcal{F} is compatible with the basis E , then the surjective map $\sigma_0 : E \rightarrow \{1, 2\}$ such that $F = \langle e \in E : \sigma_0(e) = 1 \rangle$ can be simply viewed as the subset $\sigma_0 \subset E$ such that $F = \langle \sigma_0 \rangle$.

In this case the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ is an *ind-grassmannian* and we denote it by $\mathbf{Gr}(F, E)$.

- If $k := \dim F$ is finite, a subspace $F_1 \subset V$ is E -commensurable with F if and only if $\dim F_1 = k$. Thus the ind-variety $\mathbf{Gr}(F, E)$ only depends on k , and we write $\mathbf{Gr}(k) = \mathbf{Gr}(F, E)$ in this case.
- If $k := \text{codim}_V F$ is finite, the ind-variety $\mathbf{Gr}(F, E)$ depends on E and k (but not on F). It is also isomorphic to $\mathbf{Gr}(k)$. Indeed, the basis $E \subset V$ gives rise to a dual family $E^* \subset V^*$. The linear space $V_* := \langle E^* \rangle$ is then countable dimensional. Let $U^\# := \{\phi \in V_* : \phi(u) = 0 \ \forall u \in U\}$ be the orthogonal subspace in V_* of a subspace $U \subset V$. The map $U \mapsto U^\#$ realizes an isomorphism of ind-varieties between $\mathbf{Gr}(F, E)$ and $\{F' \subset V_* : \dim F' = k\} \cong \mathbf{Gr}(k)$.

- If F is both infinite dimensional and infinite codimensional, the ind-variety $\mathbf{Gr}(F, E)$ depends on (F, E) , although all ind-varieties of this type are isomorphic; their isomorphism class is denoted $\mathbf{Gr}(\infty)$. Moreover, $\mathbf{Gr}(\infty)$ and $\mathbf{Gr}(k)$ are not isomorphic as ind-varieties (see [10]).

Let $\mathfrak{S}(E)$ be the set of subsets $\sigma \subset E$. The group $\mathbf{W}(E)$ acts on $\mathfrak{S}(E)$ in a natural way. The $\mathbf{W}(E)$ -orbit of σ_0 is the subset $\mathbf{W}(E) \cdot \sigma_0 = \{\sigma \in \mathfrak{S}(E) : |\sigma_0 \setminus \sigma| = |\sigma \setminus \sigma_0| < +\infty\}$. We write $F_\sigma = \langle \sigma \rangle$ (for $\sigma \in \mathfrak{S}(E)$).

The following statement describes the decomposition of the ind-grassmannian $\mathbf{Gr}(F, E)$ into \mathbf{B} -orbits. It is a direct consequence of Theorem 1.

PROPOSITION 6. *Let $\mathbf{B} \subset \mathbf{G}(E)$ be a splitting Borel subgroup containing $\mathbf{H}(E)$.*

(a) *We have the decomposition*

$$\mathbf{Gr}(F, E) = \bigsqcup_{\sigma \in \mathbf{W}(E) \cdot \sigma_0} \mathbf{B}F_\sigma.$$

(b) *For $F' \in \mathbf{Gr}(F, E)$, we have $F' \in \mathbf{B}F_\sigma$ if and only if*

$$\sigma = \sigma_{F'} := \{e \in E : F' \cap \langle e' \in E : e' \prec_{\mathbf{B}} e \rangle \neq F' \cap \langle e' \in E : e' \preceq_{\mathbf{B}} e \rangle\}.$$

(c) *For $\sigma \in \mathbf{W}(E) \cdot \sigma_0$, the orbit $\mathbf{B}F_\sigma$ is a locally closed ind-subvariety of $\mathbf{Gr}(F, E)$ isomorphic to an affine space \mathbb{A}^{d_σ} of (possibly infinite) dimension*

$$d_\sigma = n_{\text{inv}}(\sigma) := |\{(e, e') \in E \times E : e \prec_{\mathbf{B}} e', e \notin \sigma, e' \in \sigma\}|.$$

(d) *For $\sigma, \tau \in \mathbf{W}(E) \cdot \sigma_0$, the inclusion $\mathbf{B}F_\sigma \subset \overline{\mathbf{B}F_\tau}$ holds if and only if $\sigma \leq \tau$, where the relation $\sigma \leq \tau$ means that, if $e_1 \prec_{\mathbf{B}} e_2 \prec_{\mathbf{B}} \dots \prec_{\mathbf{B}} e_\ell$ are the elements of $\sigma \setminus \tau$ and $f_1 \prec_{\mathbf{B}} f_2 \prec_{\mathbf{B}} \dots \prec_{\mathbf{B}} f_\ell$ are the elements of $\tau \setminus \sigma$, then $e_i \prec_{\mathbf{B}} f_i$ for all $i \in \{1, \dots, \ell\}$.*

EXAMPLE 2 (Case of the ind-grassmannian $\mathbf{Gr}(k)$). Let $\mathfrak{S}_k(E)$ be the set of subsets $\sigma \subset E$ of cardinality k . Given $\sigma_0 \in \mathfrak{S}_k(E)$, set $F = \langle \sigma_0 \rangle$, and consider the splitting parabolic subgroup $\mathbf{P}_F = \{g \in \mathbf{G}(E) : g(F) = F\}$ and the ind-grassmannian $\mathbf{Gr}(k) = \mathbf{Gr}(F, E) = \mathbf{G}(E)/\mathbf{P}_F$. By Proposition 6 (a), we have the decomposition

$$\mathbf{Gr}(k) = \bigsqcup_{\sigma \in \mathfrak{S}_k(E)} \mathbf{B}F_\sigma.$$

By Proposition 6 (c), the cell $\mathbf{B}F_\sigma$ is finite dimensional if and only if σ is contained in a finite ideal of the ordered set $(E, \preceq_{\mathbf{B}})$, i.e., there is a finite subset $\bar{\sigma} \subset E$ satisfying $(e \in \bar{\sigma} \text{ and } e' \preceq_{\mathbf{B}} e \Rightarrow e' \in \bar{\sigma})$ and containing σ . It easily follows that there are finite-dimensional \mathbf{B} -orbits in $\mathbf{Gr}(k)$ if and only if the maximal generalized flag \mathcal{F}_0 corresponding to \mathbf{B} contains a subspace M of dimension k . By Theorem 3, \mathbf{B} is conjugate to a subgroup of the splitting parabolic subgroup \mathbf{P}_F exactly in this case. By Theorem 3 (or directly), we note that all cells $\mathbf{B}F_\sigma \subset \mathbf{Gr}(k)$ are finite dimensional if and only if $(E, \preceq_{\mathbf{B}})$ is isomorphic to (\mathbb{N}, \leq) as an ordered set, in other words \mathcal{F}_0 is a flag of the form

$$\mathcal{F}_0 = (F_{0,0} \subset F_{0,1} \subset F_{0,2} \subset \dots) \quad \text{with} \quad \dim F_{0,i} = i \quad \text{for all } i \geq 0. \tag{13}$$

By Proposition 6 (d), given $\sigma = \{e_1 \prec_{\mathbf{B}} e_2 \prec_{\mathbf{B}} \dots \prec_{\mathbf{B}} e_k\}$ and $\tau = \{f_1 \prec_{\mathbf{B}} f_2 \prec_{\mathbf{B}} \dots \prec_{\mathbf{B}} f_k\}$, we have $\mathbf{B}F_\sigma \subset \overline{\mathbf{B}F_\tau}$ if and only if $e_i \preceq_{\mathbf{B}} f_i$ for all $i \in \{1, \dots, k\}$.

Now let $\tau_0 \subset E$ be an infinite subset whose complement $E \setminus \tau_0$ is finite of cardinality k . Let $M = \langle \tau_0 \rangle$ be the corresponding subspace of V of codimension k and

let $\mathbf{P}_M \subset \mathbf{G}(E)$ be the corresponding splitting parabolic subgroup. We consider the ind-grassmannian $\mathbf{Gr}(M, E) = \mathbf{G}(E)/\mathbf{P}_M$ which is isomorphic to $\mathbf{Gr}(k) = \mathbf{G}(E)/\mathbf{P}_F$ as mentioned at the beginning of Section 4.5. If \mathcal{F}_0 is as in (13) and \mathbf{B} is the corresponding splitting Borel subgroup, then it follows from Proposition 6 (c) that every \mathbf{B} -orbit of $\mathbf{Gr}(M, E)$ is infinite dimensional. By Theorem 3, this shows in particular that the splitting parabolic subgroups \mathbf{P}_F and \mathbf{P}_M are not conjugate under $\mathbf{G}(E)$.

EXAMPLE 3 (Case of the infinite-dimensional projective space). Assume that $k = \dim F = 1$. In this case $\mathbf{Gr}(k)$ is the infinite-dimensional projective space \mathbb{P}^∞ (see Example 1). The decomposition becomes

$$\mathbb{P}^\infty = \bigsqcup_{e \in E} \mathbf{C}_e$$

where $\mathbf{C}_e = \mathbf{B}(e) = \{L \text{ line} : L \subset \langle e' \in E : e' \preceq_{\mathbf{B}} e \rangle, L \not\subset \langle e' \in E : e' \prec_{\mathbf{B}} e \rangle\}$ for all $e \in E$. The cell \mathbf{C}_e is isomorphic to an affine space of dimension $\dim \mathbf{C}_e = |\{e' \in E : e' \prec_{\mathbf{B}} e\}|$. Moreover, $\mathbf{C}_e \subset \overline{\mathbf{C}}_f$ if and only if $e \preceq_{\mathbf{B}} f$.

In this case the maximal generalized flag $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ corresponding to \mathbf{B} can be retrieved from the cell decomposition:

$$F''_{0,e} = \sum_{L \in \overline{\mathbf{C}}_e} L \quad \text{and} \quad F'_{0,e} = \sum_{L \in \overline{\mathbf{C}}_e \setminus \mathbf{C}_e} L \quad \text{for all } e \in E.$$

More generally, let (A, \preceq) be a totally ordered set and let $\mathbb{P}^\infty = \bigsqcup_{\alpha \in A} \mathbf{C}_\alpha$ be a linear cell decomposition such that $\mathbf{C}_\alpha \subset \overline{\mathbf{C}}_\beta$ whenever $\alpha \preceq \beta$. By “linear” we mean that each $\overline{\mathbf{C}}_\alpha$ is a projective subspace of \mathbb{P}^∞ , i.e., we can find a subspace $F''_\alpha \subset V$ such that $\overline{\mathbf{C}}_\alpha = \mathbb{P}(F''_\alpha)$. Setting $F'_\alpha = \sum_{\beta \prec \alpha} F''_\beta$, we get a generalized flag $\mathcal{F}_0 := \{F'_\alpha, F''_\alpha : \alpha \in A\}$ such that $\mathbb{P}(F''_\alpha) \setminus \mathbb{P}(F'_\alpha)$ is a (possibly infinite-dimensional) affine space for all α . The last property ensures that $\dim F''_\alpha / F'_\alpha = 1$, i.e., \mathcal{F}_0 is a maximal generalized flag. In this way we obtain a correspondence between maximal generalized flags (not necessarily compatible with a given basis) and linear cell decompositions of the infinite-dimensional projective space \mathbb{P}^∞ .

EXAMPLE 4 (Case of the ind-grassmannian $\mathbf{Gr}(\infty)$). Assume that the basis E is parametrized by \mathbb{Z} , in other words let $E = \{e_i\}_{i \in \mathbb{Z}}$. We consider the splitting Borel subgroup \mathbf{B} corresponding to the natural order \leq on \mathbb{Z} .

Let $F = \langle e_i : i \leq 0 \rangle$. Then the ind-variety $\mathbf{Gr}(F, E)$ is isomorphic to $\mathbf{Gr}(\infty)$. We have $\mathbf{B} \subset \mathbf{P}_F$. It follows from Theorem 3 that every \mathbf{B} -orbit of $\mathbf{Gr}(F, E)$ is finite dimensional.

Let $F' = \langle e_i : i \in 2\mathbb{Z} \rangle$. Again the ind-variety $\mathbf{Gr}(F', E)$ is isomorphic to $\mathbf{Gr}(\infty)$. However in this case we see from Proposition 6 (c) that every \mathbf{B} -orbit of $\mathbf{Gr}(F', E)$ is infinite dimensional.

We now suppose that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω and the basis E is ω -isotropic with corresponding involution $i_E : E \rightarrow E$. Then a minimal ω -isotropic generalized flag is of the form $\mathcal{F} = (0 \subset F \subset F^\perp \subset V)$ with $F \subset V$ proper and nontrivial, possibly $F = F^\perp$. Assuming that F is compatible with the basis E , there is a subset $\sigma_0 \subset E$ such that $F = \langle \sigma_0 \rangle$ and $i_E(\sigma_0) \cap \sigma_0 = \emptyset$ as the generalized flag is ω -isotropic. The ind-variety $\mathbf{Fl}(\mathcal{F}, \omega, E)$ is also denoted $\mathbf{Gr}(F, \omega, E)$ and called *isotropic ind-grassmannian*.

- If $\dim F = k$ is finite, the ind-variety $\mathbf{Gr}(F, \omega, E)$ is the set of all k -dimensional subspaces $M \subset V$ such that $M \subset M^\perp$. This ind-variety does not depend on (F, E) and we denote it also by $\mathbf{Gr}^\omega(k)$.
- If $\dim F$ is infinite, the isomorphism class of the ind-variety $\mathbf{Gr}(F, \omega, E)$ also depends on the dimension of the quotient F^\perp/F . A special situation is when $\dim F^\perp/F \in \{0, 1\}$, in which case $\mathbf{Gr}(F, \omega, E)$ is formed by maximal isotropic subspaces.

We denote by $\mathfrak{S}^\omega(E)$ the set of subsets $\sigma \subset E$ such that $i_E(\sigma) \cap \sigma = \emptyset$. The group $\mathbf{W}^\omega(E)$ acts on $\mathfrak{S}^\omega(E)$ in a natural way. The orbit $\mathbf{W}^\omega(E) \cdot \sigma_0$ is the set of subsets $\sigma \in \mathfrak{S}^\omega(E)$ such that $|\sigma \setminus \sigma_0| = |\sigma_0 \setminus \sigma| < +\infty$. From Theorem 2 we obtain the following description of the \mathbf{B} -orbits of $\mathbf{Gr}(F, \omega, E)$.

PROPOSITION 7. *Let \mathbf{B} be a splitting Borel subgroup of $\mathbf{G}^\omega(E)$ corresponding to a total order $\preceq_{\mathbf{B}}$ on E . Recall that i_E is an anti-automorphism of the ordered set $(E, \preceq_{\mathbf{B}})$.*

(a) *We have the decomposition*

$$\mathbf{Gr}(F, \omega, E) = \bigsqcup_{\sigma \in \mathbf{W}^\omega(E) \cdot \sigma_0} \mathbf{B}F_\sigma$$

where as before $F_\sigma = \langle \sigma \rangle$.

(b) *For $F' \in \mathbf{Gr}(F, \omega, E)$ we have $\sigma_{F'} \in \mathbf{W}^\omega(E) \cdot \sigma_0$ (see Proposition 6(b)), moreover $F' \in \mathbf{B}F_\sigma$ if and only if $\sigma = \sigma_{F'}$.*

(c) *For $\sigma \in \mathbf{W}^\omega(E) \cdot \sigma_0$, the orbit $\mathbf{B}F_\sigma$ is a locally closed ind-subvariety of $\mathbf{Gr}(F, \omega, E)$ isomorphic to an affine space of (possibly infinite) dimension*

$$n_{\text{inv}}^\omega(\sigma) := |\{(e, e') \in E \times E : e \prec_{\mathbf{B}} e' \neq i_E(e'), e \prec_{\mathbf{B}} i_E(e), ((e \notin \sigma, e' \in \sigma) \text{ or } (i_E(e) \in \sigma, i_E(e') \notin \sigma))\}|.$$

(d) *For $\sigma, \tau \in \mathbf{W}^\omega(E) \cdot \sigma_0$, the inclusion $\mathbf{B}F_\sigma \subset \overline{\mathbf{B}F_\tau}$ holds if and only if $\sigma \leq \tau$, where the relation $\sigma \leq \tau$ is defined as in Proposition 6(d).*

EXAMPLE 5 (Case of the isotropic ind-grassmannian $\mathbf{Gr}^\omega(k)$). In this case the cells $\mathbf{B}F_\sigma$ are parametrized by the set $\mathfrak{S}_k^\omega(E)$ of finite subsets $\sigma \subset E$ of cardinality k such that $i_E(\sigma) \cap \sigma = \emptyset$. The cell $\mathbf{B}F_\sigma$ is finite dimensional if and only if σ is contained in a finite ideal $\bar{\sigma}$ of the ordered set $(E, \preceq_{\mathbf{B}})$. Thereby the ind-variety $\mathbf{Gr}^\omega(k)$ has finite-dimensional \mathbf{B} -orbits if and only if the ordered set $(E, \preceq_{\mathbf{B}})$ has a finite ideal with k elements. Equivalently, the maximal generalized flag \mathcal{F}_0 corresponding to \mathbf{B} has a subspace $M \in \mathcal{F}_0$ of dimension k . Since \mathcal{F}_0 is maximal and ω -isotropic, it is of the form

$$\mathcal{F}_0 = \{0 = F_{0,0} \subset F_{0,1} \subset \dots \subset F_{0,k} \subset (\dots) \subset F_{0,k}^\perp \subset \dots \subset F_{0,1}^\perp \subset F_{0,0}^\perp = V\}$$

with infinitely many terms between $F_{0,k}$ and $F_{0,k}^\perp$. Hence the ordered set (\mathcal{F}_0, \subset) is not isomorphic to a subset of (\mathbb{Z}, \leq) . By Theorem 3, this implies that $\mathbf{Gr}^\omega(k)$ admits infinite-dimensional \mathbf{B} -orbits. Therefore, contrary to the case of the ind-grassmannian $\mathbf{Gr}(k)$ (see Example 2), there is no splitting Borel subgroup $\mathbf{B} \subset \mathbf{G}^\omega(E)$ for which all \mathbf{B} -orbits of the isotropic ind-grassmannian $\mathbf{Gr}^\omega(k)$ are finite dimensional.

Assume that ω is skew symmetric and $k = 1$. Then $\mathbf{Gr}^\omega(k)$ coincides with the entire infinite-dimensional projective space \mathbb{P}^∞ . The above discussion shows that, for every splitting Borel subgroup \mathbf{B} of $\mathbf{G}^\omega(E)$, there are infinite-dimensional \mathbf{B} -orbits in

the projective space \mathbb{P}^∞ . We know however from Examples 2–3 that, for a well-chosen splitting Borel subgroup of $\mathbf{G}(E)$, the infinite-dimensional projective space \mathbb{P}^∞ admits a decomposition into finite-dimensional orbits. Therefore the realizations of \mathbb{P}^∞ as $\mathbf{Gr}(1)$ and $\mathbf{Gr}^\omega(1)$ yield different sets of cell decompositions on \mathbb{P}^∞ .

EXAMPLE 6 (An isotropic ind-grassmannian with decomposition into finite-dimensional cells). Let $E = \{e_i : i \in 2\mathbb{Z} + 1\}$ be an ω -isotropic basis of V such that $\omega(e_i, e_j) = 0$ unless $i + j = 0$. For $k \geq 1$, we let $F = \langle e_i : i \leq -k \rangle$ and consider the ind-grassmannian $\mathbf{Gr}(F, \omega, E)$. Let \mathbf{B} be the splitting Borel subgroup of $\mathbf{G}^\omega(E)$ corresponding to the natural total order \leq on $2\mathbb{Z} + 1$. We then have $\mathbf{B} \subset \mathbf{P}_F^\omega := \{g \in \mathbf{G}^\omega(E) : g(F) = F\}$, hence by Theorem 3 (b) all the \mathbf{B} -orbits of the ind-grassmannian $\mathbf{Gr}(F, \omega, E)$ are finite dimensional.

5. Proof of the results stated in Section 4. Throughout this section let $\mathbf{G} = \mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$, and \mathbf{W} is the corresponding group $\mathbf{W}(E)$ or $\mathbf{W}^\omega(E)$ (see Sections 4.1–4.2). The proofs of the results stated in Section 4 are given in Sections 5.3–5.5. They rely on preliminary facts presented in Section 5.1 (which is concerned with the combinatorics of the group \mathbf{W}) and Section 5.2 (where we review some standard facts on Schubert decomposition of finite-dimensional flag varieties).

5.1. Combinatorial properties of the group \mathbf{W} . We first recall certain features of the group \mathbf{W} :

- $\mathbf{W} \cong N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ where $\mathbf{H} \subset \mathbf{G}$ is the splitting Cartan subgroup of elements diagonal in the basis E ; specifically, to an element $w \in \mathbf{W}$, we can associate an explicit representative $\hat{w} \in N_{\mathbf{G}}(\mathbf{H})$ (see Section 4.3).
- We have a natural exhaustion

$$\mathbf{W} = \bigcup_{n \geq 1} W_n$$

where $W_n = W(E_n)$ (resp. $W_n = W^\omega(E_n)$) is the Weyl group of $G_n = G(E_n)$ (resp. $G_n = G^\omega(E_n)$).

- Let $E' = E$ if $\mathbf{G} = \mathbf{G}(E)$ and $E' = \{e \in E : i_E(e) \neq e\}$ if $\mathbf{G} = \mathbf{G}^\omega(E)$, and let

$$\hat{E} = \{(e, e') \in E' \times E' : e \neq e'\}.$$

For $(e, e') \in \hat{E}$, set $s_{e,e'} = t_{e,e'}$ if $\mathbf{G} = \mathbf{G}(E)$ and $s_{e,e'} = t_{e,e'}^\omega$ if $\mathbf{G} = \mathbf{G}^\omega(E)$ (see Sections 4.1–4.2). In both cases for each pair $(e, e') \in \hat{E}$, we get an element $s_{e,e'} \in \mathbf{W}$. Clearly $\{s_{e,e'} : (e, e') \in \hat{E}\}$ is a system of generators of \mathbf{W} .

5.1.1. Analogue of Bruhat length. As seen in Sections 4.1–4.2, fixing a splitting Borel subgroup \mathbf{B} of \mathbf{G} with $\mathbf{B} \supset \mathbf{H}$ is equivalent to fixing a total order $\preceq_{\mathbf{B}}$ on the basis E (resp., such that the involution $i_E : E \rightarrow E$ becomes an anti-automorphism of ordered set, in the case where $\mathbf{G} = \mathbf{G}^\omega(E)$). This total order allows us to define a system of simple transpositions for \mathbf{W} by letting

$$S_{\mathbf{B}} = \{s_{e,e'} : e, e' \text{ are consecutive elements of } (E', \preceq_{\mathbf{B}})\}.$$

Note however that in general $S_{\mathbf{B}}$ does not generate the group \mathbf{W} . For instance, in the extreme case where $(E, \preceq_{\mathbf{B}}) = (\mathbb{Q}, \leq)$ we have $S_{\mathbf{B}} = \emptyset$.

For $w \in \mathbf{W}$, we define

$$\ell_{\mathbf{B}}(w) = \min\{m \geq 0 : w = s_1 \cdots s_m \text{ for some } s_1, \dots, s_m \in S_{\mathbf{B}}\}$$

if the set on the right-hand side is nonempty, and

$$\ell_{\mathbf{B}}(w) = +\infty$$

otherwise.

For every $n \geq 1$, the order $\preceq_{\mathbf{B}}$ induces a total order on the finite subset $E_n \subset E$, and thus a system of simple reflections $S_{\mathbf{B},n} := \{s_{e,e'} : e, e' \text{ are consecutive elements of } (E_n \cap E', \preceq_{\mathbf{B}})\}$ of the Weyl group W_n . Let $\ell_{\mathbf{B},n}(w)$ be the usual Bruhat length of $w \in W_n$ with respect to $S_{\mathbf{B},n}$.

PROPOSITION 8. *Let $w \in \mathbf{W}$. Then*

- (a) $\ell_{\mathbf{B}}(w) = \lim_{n \rightarrow \infty} \ell_{\mathbf{B},n}(w)$;
- (b) $\ell_{\mathbf{B}}(w) = \begin{cases} |\{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e' \text{ and } w(e) \succ_{\mathbf{B}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}(E), \\ |\{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e', e \prec_{\mathbf{B}} i_E(e) \text{ and } w(e) \succ_{\mathbf{B}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}^\omega(E); \end{cases}$
- (c) $\ell_{\mathbf{B}}(w) = +\infty$ if and only if there is $e \in E$ such that the set $\{e' \in E : e \prec_{\mathbf{B}} e' \prec_{\mathbf{B}} w(e)\}$ is infinite.

Proof. Denote by $m_{\mathbf{B}}(w)$ the quantity in the right-hand side of (b). Then

$$\ell_{\mathbf{B}}(w) \geq \lim_{n \rightarrow \infty} \ell_{\mathbf{B},n}(w) = m_{\mathbf{B}}(w) \tag{14}$$

(the inequality is a consequence of the definitions of $\ell_{\mathbf{B}}(w)$ and $\ell_{\mathbf{B},n}(w)$ while the equality follows from properties of (finite) Weyl groups).

Let $I_e(w) = \{e' \in E : e \prec_{\mathbf{B}} e' \prec_{\mathbf{B}} w(e)\}$. We claim that

$$m_{\mathbf{B}}(w) = +\infty \Leftrightarrow \exists e \in E \text{ such that } |I_e(w)| = +\infty. \tag{15}$$

We first check the implication \Rightarrow in (15). The assumption yields an infinite sequence $\{(e_i, e'_i)\}_{i \in \mathbb{N}}$ such that $e_i \prec_{\mathbf{B}} e'_i$ and $w(e_i) \succ_{\mathbf{B}} w(e'_i)$. Since w fixes all but finitely many elements of E , one of the sequences $\{e_i\}_{i \in \mathbb{N}}$ and $\{e'_i\}_{i \in \mathbb{N}}$ has a stationary subsequence, and thus along a relabeled subsequence $\{(e_i, e'_i)\}_{i \in \mathbb{N}}$ we have $e_i = e$ for all $i \in \mathbb{N}$ and some $e \in E$, or $e'_i = e'$ for all $i \in \mathbb{N}$ and some $e' \in E$. In the former case, the set $\{e'_i : w(e'_i) = e'_i\}$ is infinite and contained in $I_e(w)$. In the latter case, we similarly obtain that the set $\{f \in E : w(e') \prec_{\mathbf{B}} f \prec_{\mathbf{B}} e'\}$ is infinite, and since w has finite order, this implies that $I_{w^r(e')}(w)$ is infinite for some $r \geq 1$.

Next we check the implication \Leftarrow in (15). We assume that $|I_e(w)| = +\infty$ for some $e \in E$. (Then, necessarily, $w(e) \neq e$, hence $e \neq i_E(e)$ in the case where $\mathbf{G} = \mathbf{G}^\omega(E)$.) Since w fixes all but finitely many elements of E , the set $\{e' \in I_e(w) : w(e') = e'\}$ is infinite. Therefore, there are infinitely many couples $(e, e') \in \hat{E}$ such that $e \prec_{\mathbf{B}} e'$ and $w(e) \succ_{\mathbf{B}} w(e')$. Moreover, in the case where $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e, e') by $(i_E(e'), i_E(e))$, we get infinitely many such couples satisfying $e \prec_{\mathbf{B}} i_E(e)$. This implies $m_{\mathbf{B}}(w) = +\infty$, and (15) is proved.

In view of (14) and (15), to complete the proof of the proposition, it remains to show the relation $\ell_{\mathbf{B}}(w) \leq m_{\mathbf{B}}(w)$. We argue by induction on $m_{\mathbf{B}}(w)$.

If $m_{\mathbf{B}}(w) = 0$, we get $w = \text{id}$, and thus $\ell_{\mathbf{B}}(w) = 0$. Now let $w \in \mathbf{W}$ such that $0 < m_{\mathbf{B}}(w) < +\infty$ and assume that $\ell_{\mathbf{B}}(w') \leq m_{\mathbf{B}}(w')$ holds for all $w' \in \mathbf{W}$ such that $m_{\mathbf{B}}(w') < m_{\mathbf{B}}(w)$. Let $e \in E'$ be minimal such that there is $e' \in E'$ with $e \prec_{\mathbf{B}} e'$ and $w(e) \succ_{\mathbf{B}} w(e')$. Choose e' maximal for this property. We claim that

$$\text{the set } \{i \in E : w(e) \succ_{\mathbf{B}} i \succ_{\mathbf{B}} w(e')\} \text{ is finite.} \tag{16}$$

Assume the contrary. Since w fixes all but finitely many elements of E , there are infinitely many $i \in E$ such that $w(e) \succ_{\mathbf{B}} i = w(i) \succ_{\mathbf{B}} w(e')$. Note that we have $e \prec_{\mathbf{B}} i$ by the minimality of e . Thus there are infinitely many elements in the set $I_e(w)$. In view of (15), this is impossible, and (16) is established.

By (16) we can find $i \in E'$ such that $w(e') \prec_{\mathbf{B}} i$ and $w(e'), i$ are consecutive in E' . Choose $e'' \in E'$ such that $i = w(e'')$. By the maximality of e' , we have $e'' \prec_{\mathbf{B}} e'$. In the case where $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e'', e') by $(i_E(e''), i_E(e'))$ if necessary, we may assume that $e'' \prec_{\mathbf{B}} i_E(e'')$. Hence we have found $e'', e' \in E'$ with the following properties:

$$\begin{aligned} e'' \prec_{\mathbf{B}} e'; \quad w(e') \prec_{\mathbf{B}} w(e'') \text{ are consecutive in } E'; \\ e'' \prec_{\mathbf{B}} i_E(e'') \text{ (in the case where } \mathbf{G} = \mathbf{G}^\omega(E)). \end{aligned}$$

It is straightforward to deduce that $m_{\mathbf{B}}(s_{w(e'), w(e'')}w) = m_{\mathbf{B}}(w) - 1$. Using the induction hypothesis, we derive: $\ell_{\mathbf{B}}(w) \leq \ell_{\mathbf{B}}(s_{w(e'), w(e'')}w) + 1 \leq m_{\mathbf{B}}(s_{w(e'), w(e'')}w) + 1 = m_{\mathbf{B}}(w)$. The proof is now complete. \square

COROLLARY 2. *The following conditions are equivalent.*

- (i) $S_{\mathbf{B}}$ generates \mathbf{W} ;
- (ii) $\ell_{\mathbf{B}}(w) < +\infty$ for all $w \in \mathbf{W}$;
- (iii) $(E, \preceq_{\mathbf{B}})$ is isomorphic as an ordered set to a subset of (\mathbb{Z}, \leq) .

Proof. The equivalence (i) \Leftrightarrow (ii) is immediate. Note that condition (iii) is equivalent to requiring that, for all $e, e' \in E$, the interval $\{e'' \in E : e \prec_{\mathbf{B}} e'' \prec_{\mathbf{B}} e'\}$ is finite. Thus the implication (iii) \Rightarrow (ii) is guaranteed by Proposition 8 (c). Conversely, if (ii) holds true, then we get $\ell_{\mathbf{B}}(s_{e, e'}) < +\infty$ for all $(e, e') \in \hat{E}$, whence (by Proposition 8 (c)) the set $\{e'' \in E : e \prec_{\mathbf{B}} e'' \prec_{\mathbf{B}} e'\}$ is finite. This implies (iii). \square

5.1.2. Relation with parabolic subgroups. In addition to the splitting Borel subgroup \mathbf{B} , we consider a splitting parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ containing \mathbf{H} . Recall that the subgroup \mathbf{P} gives rise (in fact, is equivalent) to each of the following data:

- an E -compatible generalized flag \mathcal{F} (which is ω -isotropic in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$) such that $\mathbf{P} = \{g \in \mathbf{G} : g(\mathcal{F}) = \mathcal{F}\}$;
- a totally ordered set (A, \preceq_A) and a surjective map $\sigma_0 : E \rightarrow A$ such that $\mathcal{F} = \mathcal{F}_{\sigma_0}$ (which is equipped with an anti-automorphism $i_A : A \rightarrow A$ satisfying $\sigma_0 \circ i_E = i_A \circ \sigma_0$ in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$);
- a partial order $\preceq_{\mathbf{P}}$ on E satisfying property (11), such that $e \prec_{\mathbf{P}} e'$ if and only if $\sigma_0(e) \prec_A \sigma_0(e')$.

Moreover, \mathbf{P} gives rise to a subgroup of \mathbf{W} :

$$\mathbf{W}_{\mathbf{P}} = \{w \in \mathbf{W} : \sigma_0 \circ w^{-1} = \sigma_0\} = \{w \in \mathbf{W} : e \not\prec_{\mathbf{P}} w(e) \text{ and } w(e) \not\prec_{\mathbf{P}} e, \forall e \in E\}.$$

Note that we do not assume that \mathbf{B} is contained in \mathbf{P} .

LEMMA 4. *The following conditions are equivalent:*

- (i) $\mathbf{B} \subset \mathbf{P}$;
- (ii) for all $e, e' \in E$, $e \prec_{\mathbf{P}} e' \Rightarrow e \prec_{\mathbf{B}} e'$, i.e., the total order $\preceq_{\mathbf{B}}$ refines the partial order $\preceq_{\mathbf{P}}$;
- (iii) for all $e, e' \in E$, $e \preceq_{\mathbf{B}} e' \Rightarrow \sigma_0(e) \preceq_A \sigma_0(e')$, i.e., the map σ_0 is nondecreasing.

Proof. By the definition of the generalized flag \mathcal{F}_{σ_0} , conditions (i) and (iii) are equivalent. Since the relation $e \prec_{\mathbf{P}} e'$ is equivalent to $\sigma_0(e') \not\preceq_A \sigma_0(e)$, we obtain that (ii) and (iii) are equivalent. \square

For all $w \in \mathbf{W}$, we let

$$m_{\mathbf{B}}^{\mathbf{P}}(w) = \begin{cases} |\{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e', w(e) \succ_{\mathbf{P}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}(E) \\ |\{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e', e \prec_{\mathbf{B}} i_E(e), w(e) \succ_{\mathbf{P}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}^\omega(E). \end{cases}$$

Note that

$$m_{\mathbf{B}}^{\mathbf{P}}(w) = \begin{cases} n_{\text{inv}}(\sigma_0 \circ w) & \text{if } \mathbf{G} = \mathbf{G}(E) \\ n_{\text{inv}}^\omega(\sigma_0 \circ w) & \text{if } \mathbf{G} = \mathbf{G}^\omega(E) \end{cases} \quad (17)$$

(see Sections 4.1–4.2). We also know that $m_{\mathbf{B}}^{\mathbf{B}}(w) = \ell_{\mathbf{B}}(w)$ (see Proposition 8 (b)). In the following proposition, we characterize the property that \mathbf{B} is conjugate to a subgroup of \mathbf{P} in terms of $m_{\mathbf{B}}^{\mathbf{P}}(w)$.

PROPOSITION 9. *For $w \in \mathbf{W}$, recall that $\hat{w} \in \mathbf{G}$ is a representative of w in $N_{\mathbf{G}}(\mathbf{H})$.*

- (a) *We have $\mathbf{B} \subset \hat{w}\mathbf{P}\hat{w}^{-1}$ if and only if $m_{\mathbf{B}}^{\mathbf{P}}(w^{-1}) = 0$.*
- (b) *The following conditions are equivalent:*
 - (i) *there is $w \in \mathbf{W}$ such that $\mathbf{B} \subset \hat{w}\mathbf{P}\hat{w}^{-1}$;*
 - (ii) *there is $w \in \mathbf{W}$ such that $m_{\mathbf{B}}^{\mathbf{P}}(w) < +\infty$.*

Proof. Note that $\hat{w}\mathbf{P}\hat{w}^{-1} \subset \mathbf{G}$ is the isotropy subgroup of the generalized flag $\mathcal{F}_{\sigma_0 \circ w^{-1}}$. Thus part (a) follows from Lemma 4 and the definition of $m_{\mathbf{B}}^{\mathbf{P}}(w^{-1})$.

(b) The implication (i) \Rightarrow (ii) follows from part (a). Now assume that (ii) holds. Choose $w \in \mathbf{W}$ such that $m_{\mathbf{B}}^{\mathbf{P}}(w)$ is minimal. By (a), it suffices to show that $m_{\mathbf{B}}^{\mathbf{P}}(w) = 0$. Assume, to the contrary, that $m_{\mathbf{B}}^{\mathbf{P}}(w) > 0$. Hence there is a couple $(e, e') \in \hat{E}$ satisfying $e \prec_{\mathbf{B}} e'$, $w(e) \succ_{\mathbf{P}} w(e')$. We can assume that e is minimal such that there is e' with this property, and that e' is maximal possible. We claim that

$$\text{the set } \{i \in E' : w(e) \succ_{\mathbf{P}} i \succ_{\mathbf{P}} w(e')\} \text{ is finite.} \quad (18)$$

Otherwise, there are infinitely many $i \in E$ for which $w(e) \succ_{\mathbf{P}} i = w(i) \succ_{\mathbf{P}} w(e')$. By the minimality of e , we have $e \prec_{\mathbf{B}} i$. Whence there are infinitely many couples $(e, i) \in \hat{E}$ with $e \prec_{\mathbf{B}} i$ and $w(e) \succ_{\mathbf{P}} w(i)$ (in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e, i) by $(i_E(i), i_E(e))$, we may also assume that $e \prec_{\mathbf{B}} i_E(e)$). Consequently, $m_{\mathbf{B}}^{\mathbf{P}}(w) = +\infty$, a contradiction. This establishes (18).

By (18) we can find $i \in E'$ minimal (with respect to the order $\preceq_{\mathbf{P}}$) such that $w(e) \succeq_{\mathbf{P}} i \succ_{\mathbf{P}} w(e')$. Let $e'' \in E$ with $w(e'') = i$. The maximality of e' forces $e'' \prec_{\mathbf{B}} e'$. Altogether, we have found a couple $(e'', e') \in \hat{E}$ such that $e'' \prec_{\mathbf{B}} e'$, $w(e'') \succ_{\mathbf{P}} w(e')$, and $w(e'')$ is minimal (with respect to the order $\preceq_{\mathbf{P}}$). For $f \in E$, let $C_{\mathbf{P}}(f)$ denote the class of f for the equivalence relation defined in (11). We may assume that e'' and e' are respectively a minimal element of $w^{-1}(C_{\mathbf{P}}(w(e'')))$ and a maximal element of $w^{-1}(C_{\mathbf{P}}(w(e')))$ (with respect to the order $\preceq_{\mathbf{B}}$). Moreover, in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e'', e') by $(i_E(e''), i_E(e'))$, we may assume that $e'' \prec_{\mathbf{B}} i_E(e'')$. Then it is straightforward to check that

$$\begin{aligned} & \{(f, f') \in \hat{E} : f \prec_{\mathbf{B}} f', s_{w(e''), w(e'')}w(f) \succ_{\mathbf{P}} s_{w(e'), w(e'')}w(f')\} \\ & \subset \{(f, f') \in \hat{E} : f \prec_{\mathbf{B}} f', w(f) \succ_{\mathbf{P}} w(f')\} \setminus \{(e'', e')\}. \end{aligned}$$

Whence $m_{\mathbf{B}}^{\mathbf{P}}(s_{w(e''), w(e'')}w) < m_{\mathbf{B}}^{\mathbf{P}}(w)$, which contradicts the minimality of $m_{\mathbf{B}}^{\mathbf{P}}(w)$. \square

Finally, the following proposition points out the relation between $m_{\mathbf{B}}^{\mathbf{P}}(w)$ and $\ell_{\mathbf{B}}(w)$.

PROPOSITION 10. Assume that there is $w_0 \in \mathbf{W}$ such that $m_{\mathbf{B}}^{\mathbf{P}}(w_0^{-1}) = 0$. Then, for all $w \in \mathbf{W}$, we have

$$m_{\mathbf{B}}^{\mathbf{P}}(w) = \inf\{\ell_{\mathbf{B}}(w_0 w' w) : w' \in \mathbf{W}_{\mathbf{P}}\}.$$

Proof. Note that, for all $e, e' \in E'$, we have $e \prec_{\mathbf{P}} e'$ if and only if $w_0(e) \prec_{\hat{w}_0 \mathbf{P} \hat{w}_0^{-1}} w_0(e')$. This yields $m_{\mathbf{B}}^{\mathbf{P}}(w) = m_{\hat{w}_0 \mathbf{P} \hat{w}_0^{-1}}^{\hat{w}_0 \mathbf{P} \hat{w}_0^{-1}}(w_0 w)$ and $w_0 \mathbf{W}_{\mathbf{P}} w_0^{-1} = \mathbf{W}_{\hat{w}_0 \mathbf{P} \hat{w}_0^{-1}}$. Thus, invoking also Proposition 9 (a), up to replacing \mathbf{P} by $\hat{w}_0 \mathbf{P} \hat{w}_0^{-1}$, we may suppose that $\mathbf{B} \subset \mathbf{P}$ and $w_0 = \text{id}$.

By the definition of $\mathbf{W}_{\mathbf{P}}$, Lemma 4, and Proposition 8 (b), for every $w' \in \mathbf{W}_{\mathbf{P}}$ we obtain

$$\begin{aligned} m_{\mathbf{B}}^{\mathbf{P}}(w) &= |\{(e, e') \in \hat{E}_{\mathbf{B}} : \sigma_0(w(e)) \succ_A \sigma_0(w(e'))\}| \\ &= |\{(e, e') \in \hat{E}_{\mathbf{B}} : \sigma_0(w' w(e)) \succ_A \sigma_0(w' w(e'))\}| \\ &\leq |\{(e, e') \in \hat{E}_{\mathbf{B}} : w' w(e) \succ_{\mathbf{B}} w' w(e')\}| = \ell_{\mathbf{B}}(w' w), \end{aligned}$$

where $\hat{E}_{\mathbf{B}} = \{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e'\}$ if $\mathbf{G} = \mathbf{G}(E)$, and $\hat{E}_{\mathbf{B}} = \{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e', e \prec_{\mathbf{B}} i_E(e)\}$ if $\mathbf{G} = \mathbf{G}^\omega(E)$. If $m_{\mathbf{B}}^{\mathbf{P}}(w) = +\infty$, the result is established. So we assume next that $m_{\mathbf{B}}^{\mathbf{P}}(w) < +\infty$.

Claim 1. There is $w' \in \mathbf{W}_{\mathbf{P}}$ such that the set $\mathcal{I}(w' w) := \{e \in E : \sigma_0(e) = \sigma_0(w' w(e)) \text{ and } w' w(e) \neq e\}$ is empty.

For any $w' \in \mathbf{W}_{\mathbf{P}}$, the set $\mathcal{I}(w' w)$ is finite. Let $w' \in \mathbf{W}_{\mathbf{P}}$ such that $|\mathcal{I}(w' w)|$ is minimal. We claim that $\mathcal{I}(w' w) = \emptyset$. For otherwise, assume that there is $e \in \mathcal{I}(w' w)$. Thus $\sigma_0(w' w(e)) = e$. Either $\sigma_0((w' w)^\ell(e)) = \sigma_0(e)$ for all $\ell \in \mathbb{Z}$, or there is $\ell \in \mathbb{Z}$ such that $\sigma_0((w' w)^{\ell-1}(e)) \neq \sigma_0((w' w)^\ell(e)) = \sigma_0((w' w)^{\ell+1}(e))$. In the former case we set $w'' = s_{(w' w)^{m-2}(e), (w' w)^{m-1}(e)} \cdots s_{(w' w)(e), (w' w)^2(e)} s_{e, (w' w)(e)}$, where $m \geq 2$ is minimal such that $(w' w)^m(e) = e$. In the latter case we set $w'' = s_{(w' w)^\ell(e), (w' w)^{\ell+1}(e)}$. In both cases one has $w'' \in \mathbf{W}_{\mathbf{P}}$, and it easy to check that $\mathcal{I}(w'' w' w) \subsetneq \mathcal{I}(w' w)$, a contradiction. Hence Claim 1 holds.

Note that $m_{\mathbf{B}}^{\mathbf{P}}(w' w) = m_{\mathbf{B}}^{\mathbf{P}}(w)$. Up to dealing with $w' w$ instead of w , we may assume that $\mathcal{I}(w) = \emptyset$. For $\alpha \in A$, let $I_\alpha(w) = \{e \in \sigma_0^{-1}(\alpha) : w(e) \neq e\}$. Since $\mathcal{I}(w) = \emptyset$, one has $I_\alpha(w) = I_\alpha^+(w) \sqcup I_\alpha^-(w)$ with

$$I_\alpha^+(w) = \{e \in \sigma_0^{-1}(\alpha) : \sigma_0(w^{-1}(e)) \succ_A \alpha\} \quad \text{and} \quad I_\alpha^-(w) = \{e \in \sigma_0^{-1}(\alpha) : \sigma_0(w^{-1}(e)) \prec_A \alpha\}.$$

Claim 2. There is $w' \in \mathbf{W}_{\mathbf{P}}$ with $w'(e) = e$ whenever $w(e) = e$, and satisfying the following property: for every $\alpha \in A$, the set $\{e' \in \sigma_0^{-1}(\alpha) : w'(e) \prec_{\mathbf{B}} e'\}$ is finite whenever $e \in I_\alpha^+(w)$, and the set $\{e' \in \sigma_0^{-1}(\alpha) : w'(e) \succ_{\mathbf{B}} e'\}$ is finite whenever $e \in I_\alpha^-(w)$.

Let $e \in I_\alpha^+(w)$. There is $\ell(e) \geq 2$ minimal such that $\sigma_0(w^{-\ell(e)}(e)) \preceq_A \alpha$. Since $m_{\mathbf{B}}^{\mathbf{P}}(w) < +\infty$, the set $\{e' \in \sigma_0^{-1}(\alpha) : w^{-\ell(e)}(e) \prec_{\mathbf{B}} e'\}$ is finite. Set $w'(e) = w^{-\ell(e)}(e)$. Similarly, given $e \in I_\alpha^-(w)$, there is $m(e) \geq 2$ minimal such that $\sigma_0(w^{-m(e)}(e)) \succeq_A \alpha$, and the set $\{e' \in \sigma_0^{-1}(\alpha) : w^{-m(e)}(e) \succ_{\mathbf{B}} e'\}$ is finite; we set $w'(e) = w^{-m(e)}(e)$ in this case. If $e \in \sigma_0^{-1}(\alpha) \setminus I_\alpha(w)$, we set $w'(e) = e$. It is readily seen that the so-obtained map $w' : \sigma_0^{-1}(\alpha) \rightarrow \sigma_0^{-1}(\alpha)$ is bijective. Collecting these maps for all $\alpha \in A$, we obtain an element $w' \in \mathbf{W}_{\mathbf{P}}$ satisfying the desired properties. This shows Claim 2.

Set $\hat{w} = w' w$ with $w' \in \mathbf{W}_{\mathbf{P}}$ as in Claim 2. For every $\alpha \in A$, the set

$$\begin{aligned} J_\alpha(\hat{w}) &= \{e \in \sigma_0^{-1}(\alpha) : (\exists e' \in \sigma_0^{-1}(\alpha) \text{ with } e' \preceq_{\mathbf{B}} e \text{ and } \sigma_0(\hat{w}^{-1}(e')) \succ_A \alpha) \\ &\quad \text{or } (\exists e' \in \sigma_0^{-1}(\alpha) \text{ with } e' \succeq_{\mathbf{B}} e \text{ and } \sigma_0(\hat{w}^{-1}(e')) \prec_A \alpha)\} \end{aligned}$$

is finite (by Claim 2). We write $J_\alpha(\hat{w}) = \{e_i^\alpha\}_{i=1}^{k_\alpha}$ so that $\hat{w}^{-1}(e_1^\alpha) \prec_{\mathbf{B}} \dots \prec_{\mathbf{B}} \hat{w}^{-1}(e_{k_\alpha}^\alpha)$. There is $w'' \in \mathbf{W}_{\mathbf{P}}$ with $w''(e) = e$ whenever $e \notin \bigcup_{\alpha \in A} J_\alpha(\hat{w})$ and such that

$$w''(J_\alpha(\hat{w})) = J_\alpha(\hat{w}) \text{ and } w''(e_1^\alpha) \prec_{\mathbf{B}} \dots \prec_{\mathbf{B}} w''(e_{k_\alpha}^\alpha) \text{ for all } \alpha \in A.$$

Taking the construction of w'' into account, one can check that there is no couple $(e, e') \in \hat{E}$ with $e \prec_{\mathbf{B}} e'$, $w''\hat{w}(e) \succ_{\mathbf{B}} w''\hat{w}(e')$, and $\sigma_0(w''\hat{w}(e)) = \sigma_0(w''\hat{w}(e'))$. Therefore, $m_{\mathbf{B}}^{\mathbf{P}}(w) = \ell_{\mathbf{B}}(w''\hat{w}) = \ell_{\mathbf{B}}((w''w')w)$ with $w''w' \in \mathbf{W}_{\mathbf{P}}$. The proof is complete. \square

5.2. Review of (finite-dimensional) flag varieties. We consider an E -compatible generalized flag $\mathcal{F} = \mathcal{F}_{\sigma_0}$ corresponding to a surjection $\sigma_0 : E \rightarrow A$. Let $I \subset E$ be a finite subset (resp., i_E -stable, if the form ω is considered). In this section we recall standard properties of the Schubert decomposition of the flag varieties $\text{Fl}(\mathcal{F}, I)$ and $\text{Fl}(\mathcal{F}, \omega, I)$ (see Section 3.3). We refer to [1, §3.2–3.5], [2, §1.2], [9, §10.8] for more details.

PROPOSITION 11. *Let $\mathbf{G} = \mathbf{G}(E)$. Let \mathbf{B} be a splitting Borel subgroup of \mathbf{G} containing \mathbf{H} and let $B(I) := G(I) \cap \mathbf{B}$ be the corresponding Borel subgroup of the group $G(I)$. Let $H(I) = G(I) \cap \mathbf{H}$. Let $W(I) \subset \mathbf{W}$ be the Weyl group of $G(I)$.*

(a) *We have the decomposition*

$$\text{Fl}(\mathcal{F}, I) = \bigcup_{w \in W(I)} B(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}.$$

Moreover, $\mathcal{F}_{\sigma_0 \circ w^{-1}}$ is the unique element of $B(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}$ fixed by the maximal torus $H(I)$.

(b) *Each subset $B(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}$, for $w \in W(I)$, is a locally closed subvariety isomorphic to an affine space of dimension $|\{(e, e') \in I \times I : e \prec_{\mathbf{B}} e', \sigma_0 \circ w^{-1}(e') \prec_A \sigma_0 \circ w^{-1}(e)\}|$.*

(c) *Given $w, w' \in W(I)$, the inclusion $B(I)\mathcal{F}_{\sigma_0 \circ w^{-1}} \subset B(I)\mathcal{F}_{\sigma_0 \circ w'^{-1}}$ holds if and only if $\sigma_0 \circ w^{-1} \leq \sigma_0 \circ w'^{-1}$ for the order \leq defined in Section 4.1.*

(d) *Let $J \subset E$ be another finite subset such that $I \subset J$. Let $\iota_{I,J} : \text{Fl}(\mathcal{F}, I) \hookrightarrow \text{Fl}(\mathcal{F}, J)$ be the embedding constructed in Section 3.3. Then, for all $w \in W(I)$, the image of the Schubert cell $B(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}$ by the map $\iota_{I,J}$ is an affine subspace of $B(J)\mathcal{F}_{\sigma_0 \circ w^{-1}}$.*

PROPOSITION 12. *Let $\mathbf{G} = \mathbf{G}^\omega(E)$. Let \mathbf{B} be a splitting Borel subgroup of \mathbf{G} containing \mathbf{H} and let $B^\omega(I) := G^\omega(I) \cap \mathbf{B}$ be the corresponding Borel subgroup of the group $G^\omega(I)$. Let $H^\omega(I) = G^\omega(I) \cap \mathbf{H}$. Let $W^\omega(I) \subset \mathbf{W}$ be the Weyl group of $G^\omega(I)$.*

(a) *We have the decomposition*

$$\text{Fl}(\mathcal{F}, \omega, I) = \bigcup_{w \in W^\omega(I)} B^\omega(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}.$$

Moreover, $\mathcal{F}_{\sigma_0 \circ w^{-1}}$ is the unique element of $B^\omega(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}$ fixed by the maximal torus $H^\omega(I)$.

(b) *Each subset $B^\omega(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}$, for $w \in W^\omega(I)$, is a locally closed subvariety isomorphic to an affine space of dimension $|\{(e, e') \in I \times I : e \prec_{\mathbf{B}} e', e \prec_{\mathbf{B}} i_E(e), e' \neq i_E(e'), \sigma_0 \circ w^{-1}(e') \prec_A \sigma_0 \circ w^{-1}(e)\}|$.*

(c) *Given $w, w' \in W^\omega(I)$, the inclusion $B^\omega(I)\mathcal{F}_{\sigma_0 \circ w^{-1}} \subset \overline{B^\omega(I)\mathcal{F}_{\sigma_0 \circ w'^{-1}}}$ holds if and only if $\sigma_0 \circ w^{-1} \leq_\omega \sigma_0 \circ w'^{-1}$, for the order \leq_ω defined in Section 4.2.*

(d) *Let $J \subset E$ be another i_E -stable finite subset such that $I \subset J$. Let $\iota_{I,J}^\omega : \text{Fl}(\mathcal{F}, \omega, I) \hookrightarrow \text{Fl}(\mathcal{F}, \omega, J)$*

$\text{Fl}(\mathcal{F}, \omega, I) \hookrightarrow \text{Fl}(\mathcal{F}, \omega, J)$ be the embedding constructed in Section 3.3. Then, for all $w \in W^\omega(I)$, the image of the Schubert cell $B^\omega(I)\mathcal{F}_{\sigma_0 \circ w^{-1}}$ by the map $\iota_{I,J}^\omega$ is an affine subspace of $B^\omega(J)\mathcal{F}_{\sigma_0 \circ w^{-1}}$.

5.3. Proof of Lemmas 2 and 3. We consider the map

$$\phi : \mathbf{W}(E) \rightarrow \mathbf{Fl}_A(V), \quad w \mapsto \mathcal{F}_{\sigma_0 \circ w^{-1}}$$

and, in the proof of Lemma 3, we also consider its restriction $\phi^\omega : \mathbf{W}^\omega(E) \rightarrow \mathbf{Fl}_A^\omega(V)$.

Proof of Lemma 2. Let $\mathbf{Fl}'(\mathcal{F}, E) \subset \mathbf{Fl}(\mathcal{F}, E)$ denote the subset of E -compatible generalized flags. By definition the generalized flag $\phi(w)$ is E -compatible for all $w \in \mathbf{W}(E)$. Moreover, it is easily seen that $\phi(w) = \hat{w}(\mathcal{F}_{\sigma_0})$ where $\hat{w} \in \mathbf{G}(E)$ is the element for which $\hat{w}(e) = w(e)$ for all $e \in E$. Thus $\phi(w)$ is E -commensurable with $\mathcal{F} = \mathcal{F}_{\sigma_0}$ (see Proposition 2). Consequently, $\phi(w) \in \mathbf{Fl}'(\mathcal{F}, E)$ for all $w \in \mathbf{W}(E)$.

Conversely, let $\mathcal{G} \in \mathbf{Fl}'(\mathcal{F}, E)$. Choosing n such that $\mathcal{G} \in \text{Fl}(\mathcal{F}, E_n)$, we have that \mathcal{G} is fixed by the maximal torus $H(E_n) \subset G(E_n)$. Using Proposition 11 (a), we find $w \in W(E_n) \subset \mathbf{W}(E)$ such that $\mathcal{G} = \mathcal{F}_{\sigma_0 \circ w^{-1}} = \phi(w)$.

Finally, for $w, w' \in \mathbf{W}(E)$, we have $\phi(w) = \phi(w')$ if and only if $\sigma_0 \circ w^{-1} = \sigma_0 \circ w'^{-1}$, and the latter condition reads as $w'^{-1}w \in \mathbf{W}_P(E)$. Therefore, ϕ induces a bijection $\mathbf{W}(E)/\mathbf{W}_P(E) \rightarrow \mathbf{Fl}'(\mathcal{F}, E)$. \square

Proof of Lemma 3. Let $\mathbf{Fl}'(\mathcal{F}, \omega, E) = \mathbf{Fl}'(\mathcal{F}, E) \cap \mathbf{Fl}(\mathcal{F}, \omega, E)$. From Lemma 2 we know that $\phi^\omega(w)$ is E -compatible and E -commensurable with $\mathcal{F} = \mathcal{F}_{\sigma_0}$, whence $\phi^\omega(w) \in \mathbf{Fl}'(\mathcal{F}, \omega, E)$ for all $w \in \mathbf{W}^\omega(E)$.

Let $\mathcal{G} \in \mathbf{Fl}'(\mathcal{F}, \omega, E)$. Choosing n such that $\mathcal{G} \in \text{Fl}(\mathcal{F}, \omega, E_n)$, we have that \mathcal{G} is a fixed point of the maximal torus $H^\omega(E_n) \subset G^\omega(E_n)$, hence we can find $w \in W^\omega(E_n)$ such that $\mathcal{G} = \mathcal{F}_{\sigma_0 \circ w^{-1}} = \phi^\omega(w)$.

As in the proof of Lemma 2 it is easy to conclude that ϕ^ω induces a bijection $\mathbf{W}^\omega(E)/\mathbf{W}_P^\omega(E) \rightarrow \mathbf{Fl}'(\mathcal{F}, \omega, E)$. \square

5.4. Proof of Theorems 1 and 2.

Proof of Theorem 1. Recall the exhaustions (3) and (8) of the ind-group $\mathbf{G}(E)$ and the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$. For all $n \geq 1$, the subgroups $H(E_n) := G(E_n) \cap \mathbf{H}(E)$, $B_n := G(E_n) \cap \mathbf{B}$, and $P_n := G(E_n) \cap \mathbf{P}$ are respectively a maximal torus, a Borel subgroup, and a parabolic subgroup of $G(E_n)$.

(a) Let $\mathcal{G} \in \mathbf{Fl}(\mathcal{F}, E)$. By Proposition 11 (a), for any $n \geq 1$ large enough so that $\mathcal{G} \in \text{Fl}(\mathcal{F}, E_n)$, the B_n -orbit of \mathcal{G} contains a unique element of the form $\mathcal{F}_{\sigma_0 \circ w^{-1}}$ with $w \in W(E_n)$. Therefore, every element $\mathcal{G} \in \mathbf{Fl}(\mathcal{F}, E)$ lies in the \mathbf{B} -orbit of \mathcal{F}_σ for a unique $\sigma \in \mathbf{W}(E) \cdot \sigma_0$.

(b) Let $\mathcal{G} = \{G'_\alpha, G''_\alpha : \alpha \in A\} \in \mathbf{Fl}(\mathcal{F}, E)$. According to part (a) of the proof, there is a unique $\sigma \in \mathbf{W}(E) \cdot \sigma_0$ such that $\mathcal{G} \in \mathbf{B}\mathcal{F}_\sigma$, say $\mathcal{G} = b(\mathcal{F}_\sigma)$, where $b \in \mathbf{B}$. Thus

$$G''_\alpha \cap F'_{0,e} = b(F''_{\sigma,\alpha} \cap F'_{0,e}) \quad \text{and} \quad G''_\alpha \cap F''_{0,e} = b(F''_{\sigma,\alpha} \cap F''_{0,e})$$

(because $F'_{0,e}, F''_{0,e}$ are b -stable). This clearly implies that $\sigma_{\mathcal{G}} = \sigma_{\mathcal{F}_\sigma}$. Moreover, from the definition of \mathcal{F}_σ we see that $F''_{\sigma,\alpha} \cap F''_{0,e} \neq F''_{\sigma,\alpha} \cap F'_{0,e}$ if and only if $\sigma(e) \leq_A \alpha$. Whence $\sigma(e) = \min\{\alpha \in A : F''_{\sigma,\alpha} \cap F''_{0,e} \neq F''_{\sigma,\alpha} \cap F'_{0,e}\} = \sigma_{\mathcal{F}_\sigma}(e)$ for all $e \in E$. Thus $\sigma_{\mathcal{G}} = \sigma$. Note that the last equality guarantees in particular that $\sigma_{\mathcal{G}} \in \mathfrak{S}(E, A)$.

(c) follows from Proposition 11 (b) and (d).

(d) We consider $\sigma, \tau \in \mathbf{W}(E) \cdot \sigma_0$ and let $n \geq 1$ be such that $\mathcal{F}_\sigma, \mathcal{F}_\tau \in \mathbf{Fl}(\mathcal{F}, E_n)$. Assume that $\sigma \hat{<} \tau$, i.e., $\tau = \sigma \circ t_{e,e'}$ for a pair $(e, e') \in E \times E$ with $e \prec_{\mathbf{B}} e'$ and $\sigma(e) \preceq_A \sigma(e')$. Up to choosing n larger if necessary, we may assume that $e, e' \in E_n$. Then, by Proposition 11 (c), we get $B_n \mathcal{F}_\sigma \subset \overline{B_n \mathcal{F}_\tau}$. Whence $\mathbf{B} \mathcal{F}_\sigma \subset \overline{\mathbf{B} \mathcal{F}_\tau}$. This argument also shows that the latter inclusion holds whenever $\sigma \leq \tau$. Conversely, assume that $\mathcal{F}_\sigma \in \overline{\mathbf{B} \mathcal{F}_\tau}$. Hence $\mathcal{F}_\sigma \in \overline{B_n \mathcal{F}_\tau}$ for $n \geq 1$ large enough. Once again, by Proposition 11 (c), this yields $\sigma \leq \tau$. The proof of Theorem 1 is complete. \square

Proof of Theorem 2. The proof of Theorem 2 follows exactly the same scheme as the proof of Theorem 1, relying this time on Proposition 12 instead of Proposition 11. We skip the details. \square

5.5. Proof of Theorem 3.

Proof of Theorem 3. (a) Condition (i) means that there is $g \in \mathbf{G}$ such that $\mathbf{B} \subset g \mathbf{P} g^{-1}$. This equivalently means that the element $g \mathbf{P} \in \mathbf{G}/\mathbf{P}$ is fixed by \mathbf{B} , i.e., that \mathbf{G}/\mathbf{P} comprises a \mathbf{B} -orbit reduced to a single point. We have shown the equivalence (i) \Leftrightarrow (iii). The implication (iii) \Rightarrow (ii) is immediate, while the implication (ii) \Rightarrow (i) follows from Proposition 9, relation (17), and Theorems 1(c)–2(c).

(b) The implication (i) \Rightarrow (ii) is a consequence of part (a), Corollary 2, Proposition 10, relation (17), and Theorems 1(c)–2(c). Assume that (ii) holds. From part (a), there is $g \in \mathbf{G}$ such that $\mathbf{B} \subset g \mathbf{P} g^{-1}$. Up to dealing with $g \mathbf{P} g^{-1}$ instead of \mathbf{P} , we may assume that $\mathbf{B} \subset \mathbf{P}$. Arguing by contradiction, say that $(E, \preceq_{\mathbf{B}})$ is not isomorphic to a subset of (\mathbb{Z}, \leq) . Thus there are $e, e' \in E$ such that the set $\{e'' \in E : e \prec_{\mathbf{B}} e'' \prec_{\mathbf{B}} e'\}$ is infinite. Since the surjective map $\sigma_0 : E \rightarrow A$, corresponding to \mathbf{P} , is nondecreasing (by Lemma 4) and nonconstant (because $\mathbf{P} \neq \mathbf{G}$), we find \hat{e}, \hat{e}' with $\hat{e} \preceq_{\mathbf{B}} e \prec_{\mathbf{B}} e' \preceq_{\mathbf{B}} \hat{e}'$ such that $\sigma_0(\hat{e}) \prec_A \sigma_0(\hat{e}')$. Then, $\dim \mathbf{B} \mathcal{F}_{\sigma_0 \circ s_{\hat{e}, \hat{e}'}} = +\infty$ (by Theorems 1(c)–2(c)), a contradiction. \square

6. Smoothness of Schubert ind-varieties. In this section \mathbf{G} is one of the ind-groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$ and \mathbf{B} is a splitting Borel subgroup of \mathbf{G} which contains the splitting Cartan subgroup $\mathbf{H} = \mathbf{H}(E)$ or $\mathbf{H}^\omega(E)$. We consider the *Schubert ind-varieties* defined as the closures of the Schubert cells $\mathbf{B} \mathcal{F}_\sigma$ in the ind-varieties of generalized flags $\mathbf{Fl}(\mathcal{F}, E)$ or $\mathbf{Fl}(\mathcal{F}, \omega, E)$. Specifically, we study the smoothness of Schubert ind-varieties. The general principle (Theorem 4) is straightforward: the ind-variety $\overline{\mathbf{B} \mathcal{F}_\sigma}$ is smooth if and only if its intersections with suitable finite-dimensional flag subvarieties of $\mathbf{Fl}(\mathcal{F}, E)$ or $\mathbf{Fl}(\mathcal{F}, \omega, E)$ are smooth. Note however that this fact is not immediate: see Remark 9 below. As an example, in Section 6.3 we give a combinatorial interpretation of this result in the case of ind-varieties of maximal generalized flags and in the case of ind-grassmannians.

6.1. General facts on the smoothness of ind-varieties. The notion of smooth point of an ind-variety is defined in Section 2.1. We refer to [8, Chapter 4] or [11] for more details. In this section, for later use, we present some general facts regarding the smoothness of ind-varieties.

We start with the following simple smoothness criterion (see [8]).

LEMMA 5. *Let \mathbf{X} be an ind-variety with an exhaustion $\mathbf{X} = \bigcup_{n \geq 1} X_n$. Let $x \in \mathbf{X}$. Suppose that there is a subsequence $\{X_{n_k}\}_{k \geq 1}$ such that x is a smooth point of X_{n_k} for all $k \geq 1$. Then x is a smooth point of \mathbf{X} . In particular, if \mathbf{X} admits an exhaustion by smooth varieties, then \mathbf{X} is smooth.*

EXAMPLE 7. It easily follows from Lemma 5 that the infinite-dimensional affine

space \mathbb{A}^∞ and the infinite-dimensional projective space \mathbb{P}^∞ are smooth. More generally, it follows from Propositions 4–5 and Lemma 5 that the ind-varieties of the form $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$ are smooth.

REMARK 9. The converse of Lemma 5 is clearly false. Consider for instance $\mathbf{X} = \mathbb{A}^\infty = \bigcup_{n \geq 1} \mathbb{A}^n$ and let $x \in \mathbb{A}^1$. For each $n \geq 1$, let $X'_n \subset \mathbb{A}^{n+1}$ be an n -dimensional affine subspace containing x and distinct of \mathbb{A}^n , and set $X_n = \mathbb{A}^n \cup X'_n$. The subvarieties X_n exhaust \mathbb{A}^∞ . Clearly x is a singular point of every X_n . However x is a smooth point of \mathbb{A}^∞ (which is a smooth ind-variety).

The following partial converse of Lemma 5 is used in Section 6.2 for studying the smoothness of Schubert ind-varieties.

LEMMA 6. *Let \mathbf{X} be an ind-variety and let $\mathbf{X} = \bigcup_{n \geq 1} X_n$ be an exhaustion by algebraic varieties. Assume that each inclusion $X_n \subset X_{n+1}$ has a left inverse $r_n : X_{n+1} \rightarrow X_n$ in the category of algebraic varieties. Then, if $x \in \mathbf{X}$ is a singular point of X_{n_0} for some $n_0 \geq 1$, x is a singular point of \mathbf{X} .*

Proof. We start with a preliminary fact. Let Y be an algebraic variety and $Z \subset Y$ be a subvariety such that there is a retraction $r : Y \rightarrow Z$, i.e., a left inverse of the inclusion map $i : Z \hookrightarrow Y$. Let $x \in Z$. We consider the local rings $\mathcal{O}_{Z,x}$, $\mathcal{O}_{Y,x}$ and their maximal ideals $\mathfrak{m}_{Z,x}$, $\mathfrak{m}_{Y,x}$. The map r induces a ring homomorphism $r^\sharp : \mathcal{O}_{Z,x} \rightarrow \mathcal{O}_{Y,x}$ such that $r^\sharp(\mathfrak{m}_{Z,x}^k) \subset \mathfrak{m}_{Y,x}^k$ for all $k \geq 1$. Thus r^\sharp induces maps

$$r_k^\sharp : S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2) \rightarrow S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) \quad \text{and} \quad \tilde{r}_k^\sharp : \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1} \rightarrow \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1},$$

which are respective right inverses of the maps $i_k^\sharp : S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) \rightarrow S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2)$ and $\tilde{i}_k^\sharp : \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1} \rightarrow \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1}$ induced by the inclusion $i : Z \hookrightarrow Y$. Moreover the diagrams

$$\begin{array}{ccc} S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2) & \xrightarrow{\alpha_{Z,k}} & \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1} & \text{and} & S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) & \xrightarrow{\alpha_{Y,k}} & \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1} \\ r_k^\sharp \downarrow & & \downarrow \tilde{r}_k^\sharp & & i_k^\sharp \downarrow & & \downarrow \tilde{i}_k^\sharp \\ S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) & \xrightarrow{\alpha_{Y,k}} & \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1} & & S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2) & \xrightarrow{\alpha_{Z,k}} & \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1} \end{array}$$

are commutative, where $\alpha_{Z,k}$ and $\alpha_{Y,k}$ are defined in a natural way.

In the setting of the lemma, for every $n \geq 1$, we denote $\mathfrak{m}_{n,x} := \mathfrak{m}_{X_n,x}$. The retraction $r_n : X_{n+1} \rightarrow X_n$ induces maps $(r_n)_k^\sharp : S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) \rightarrow S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}_{n+1,x}^2)$ and $(\tilde{r}_n)_k^\sharp : \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1} \rightarrow \mathfrak{m}_{n+1,x}^k/\mathfrak{m}_{n+1,x}^{k+1}$, which are respective right inverses of the maps $(i_n)_k^\sharp : S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}_{n+1,x}^2) \rightarrow S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2)$ and $(\tilde{i}_n)_k^\sharp : \mathfrak{m}_{n+1,x}^k/\mathfrak{m}_{n+1,x}^{k+1} \rightarrow \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1}$ induced by the inclusion $X_n \subset X_{n+1}$. Moreover the diagrams

$$\begin{array}{ccc} S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) & \xrightarrow{\alpha_{n,k}} & \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1} & \text{and} \\ (r_n)_k^\sharp \downarrow & & \downarrow (\tilde{r}_n)_k^\sharp & \\ S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}_{n+1,x}^2) & \xrightarrow{\alpha_{n+1,k}} & \mathfrak{m}_{n+1,x}^k/\mathfrak{m}_{n+1,x}^{k+1} & \\ \\ S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}_{n+1,x}^2) & \xrightarrow{\alpha_{n+1,k}} & \mathfrak{m}_{n+1,x}^k/\mathfrak{m}_{n+1,x}^{k+1} & \\ (i_n)_k^\sharp \downarrow & & \downarrow (\tilde{i}_n)_k^\sharp & \\ S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) & \xrightarrow{\alpha_{n,k}} & \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1} & \end{array}$$

commute, where $\alpha_{n,k} = \alpha_{X_n,k}$ (see also (2)).

Since $x \in X_{n_0}$ is singular, there is $k \geq 2$ such that the map $\alpha_{n_0,k} : S^k(\mathfrak{m}_{n_0,x}/\mathfrak{m}_{n_0,x}^2) \rightarrow \mathfrak{m}_{n_0,x}^k/\mathfrak{m}_{n_0,x}^{k+1}$ is not injective, i.e., there is $a_{n_0} \in \ker \alpha_{n_0,k} \setminus \{0\}$. We define the sequence $\{a_n\}$ by letting

$$a_n = \begin{cases} (i_n)_k^\sharp \circ \cdots \circ (i_{n_0-2})_k^\sharp \circ (i_{n_0-1})_k^\sharp(a_{n_0}) & \text{if } 1 \leq n \leq n_0 \\ (r_{n-1})_k^\sharp \circ \cdots \circ (r_{n_0+1})_k^\sharp \circ (r_{n_0})_k^\sharp(a_{n_0}) & \text{if } n \geq n_0. \end{cases}$$

Then $a_n \in S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2)$ and $(i_n)_k^\sharp(a_{n+1}) = a_n$ for all $n \geq 1$. Thus the sequence $a := \{a_n\}$ is an element of the inverse limit $\lim_{\leftarrow} S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2)$. Moreover, we have $a \in \ker \hat{\alpha}_k \setminus \{0\}$, where $\hat{\alpha}_k := \lim_{\leftarrow} \alpha_{n,k}$. Therefore $\hat{\alpha}_k$ is not injective, and so x is a singular point of \mathbf{X} . \square

6.2. Smoothness criterion for Schubert ind-varieties. Let $\mathbf{G} = \mathbf{G}(E)$ (resp., $\mathbf{G} = \mathbf{G}^\omega(E)$).

Let (A, \leq_A) be a totally ordered set (resp., equipped with an anti-automorphism i_A). A surjective map $\sigma : E \rightarrow A$ (resp., such that $i_A \circ \sigma = \sigma \circ i_E$) gives rise to an E -compatible generalized flag $\mathcal{F}_\sigma = \{F'_{\sigma,\alpha}, F''_{\sigma,\alpha}\}_{\alpha \in A}$ (see (10)) and to the corresponding ind-variety $\mathbf{X} = \mathbf{Fl}(\mathcal{F}_\sigma, E)$ (resp., $\mathbf{X} = \mathbf{Fl}(\mathcal{F}_\sigma, \omega, E)$) (see Section 3). We consider the Schubert cell $\mathbf{BF}_\sigma \subset \mathbf{X}$. We denote its closure in \mathbf{X} by \mathbf{X}_σ (resp., \mathbf{X}_σ^ω) and call it *Schubert ind-variety*. Note that \mathbf{X}_σ and \mathbf{X}_σ^ω depend on the choice of the splitting Borel subgroup $\mathbf{B} \subset \mathbf{G}$.

By Theorems 1 (c), (d) and 2 (c), (d), the Schubert ind-variety \mathbf{X}_σ (resp., \mathbf{X}_σ^ω) admits a cell decomposition into Schubert cells \mathbf{BF}_τ for $\tau \leq \sigma$ (resp., $\tau \leq_\omega \sigma$).

If $I \subset E$ is a finite subset, then the (finite-dimensional) flag variety $\mathbf{Fl}(\mathcal{F}_\sigma, I)$ (defined in Section 3.3) embeds in a natural way in the ind-variety $\mathbf{Fl}(\mathcal{F}_\sigma, E)$. The intersection $X_{\sigma,I} := \mathbf{X}_\sigma \cap \mathbf{Fl}(\mathcal{F}_\sigma, I)$ is a Schubert variety in the usual sense. In the case of $\mathbf{G} = \mathbf{G}^\omega(E)$, if the subset $I \subset E$ is i_E -stable, the flag variety $\mathbf{Fl}(\mathcal{F}_\sigma, \omega, I)$ embeds in the ind-variety $\mathbf{Fl}(\mathcal{F}_\sigma, \omega, E)$. Again, the intersection $X_{\sigma,I}^\omega := \mathbf{X}_\sigma^\omega \cap \mathbf{Fl}(\mathcal{F}_\sigma, \omega, I)$ is a Schubert variety in the usual sense.

Note that the Schubert ind-variety \mathbf{X}_σ depends on the generalized flag \mathcal{F}_σ and on the splitting Borel subgroup \mathbf{B} . Recall that \mathbf{B} is the stabilizer of a maximal generalized flag \mathcal{F}_0 (see Propositions 1, 3). Our singularity criterion (Theorem 4 below) requires a technical assumption on \mathbf{B} and \mathcal{F}_σ :

- (H) At least one of the following conditions holds:
 - (i) \mathcal{F}_0 is a flag (i.e., (\mathcal{F}_0, \subset) is isomorphic as ordered set to a subset of (\mathbb{Z}, \leq));
 - (ii) \mathcal{F}_σ is a flag, and $\dim F''_{\sigma,\alpha}/F'_{\sigma,\alpha}$ is finite whenever $0 \neq F'_{\sigma,\alpha} \subset F''_{\sigma,\alpha} \neq V$.

By $\text{Sing}(X)$ we denote the set of singular points of a variety or an ind-variety X .

THEOREM 4. *Let $\mathbf{G} = \mathbf{G}(E)$ (resp., $\mathbf{G} = \mathbf{G}^\omega(E)$). Let $\sigma, \mathbf{X}_\sigma, \mathbf{X}_\sigma^\omega, X_{\sigma,I}, X_{\sigma,I}^\omega$ be as above. Assume that hypothesis (H) holds. The following alternative holds: either*

- (i) *the variety $X_{\sigma,I}$ (resp., $X_{\sigma,I}^\omega$) is smooth for all (resp., i_E -stable) finite subsets $I \subset E$; then the ind-variety \mathbf{X}_σ (resp., \mathbf{X}_σ^ω) is smooth;*

or

- (ii) *there is a finite subset $I_0 \subset E$ such that, for every (resp., i_E -stable) finite subset $I \subset E$ with $I \supset I_0$, the variety $X_{\sigma,I}$ (resp., $X_{\sigma,I}^\omega$) is singular; then \mathbf{X}_σ (resp., \mathbf{X}_σ^ω) is singular and*

$$\text{Sing}(\mathbf{X}_\sigma) = \bigcup_{I \supset I_0} \text{Sing}(X_{\sigma,I}) \quad (\text{resp., } \text{Sing}(\mathbf{X}_\sigma^\omega) = \bigcup_{I \supset I_0, i_E\text{-stable}} \text{Sing}(X_{\sigma,I}^\omega)).$$

Proof. We provide the proof only for the case $\mathbf{G} = \mathbf{G}(E)$ (the proof in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$ follows the same scheme).

We need preliminary constructions and notation. For a finite subset $I \subset E$ and an element $\tau \in W(I) \cdot \sigma$, we define closed subgroups of $G(I)$ and $B(I)$ by letting

$$\begin{aligned} G_\tau(I) &:= \{g \in G(I) : g(e) - e \in \langle e' \in I : \tau(e') \succ_A \tau(e) \rangle \quad \forall e \in E\} \text{ and} \\ B_\tau(I) &:= \{g \in G(I) : g(e) - e \in \langle e' \in I : e' \prec_{\mathbf{B}} e, \tau(e') \succ_A \tau(e) \rangle \quad \forall e \in E\} \\ &= B(I) \cap G_\tau(I). \end{aligned}$$

It is well known that the set

$$U_\tau(I) := \{g\mathcal{F}_\tau : g \in G_\tau(I)\}$$

is an open subvariety of $\text{Fl}(\mathcal{F}_\sigma, I)$, and the maps

$$\Phi_\tau : G_\tau(I) \rightarrow U_\tau(I), \quad g \mapsto g\mathcal{F}_\tau \quad \text{and} \quad \Phi'_\tau = \Phi_\tau|_{B_\tau(I)} : B_\tau(I) \rightarrow B(I)\mathcal{F}_\tau$$

are isomorphisms of algebraic varieties. Thus, for every $\tau \in \mathbf{W}(E) \cdot \sigma$, we obtain an open ind-subvariety of $\mathbf{Fl}(\mathcal{F}_\sigma, E)$ by letting

$$\mathbf{U}_\tau := \bigcup_I U_\tau(I),$$

where the union is taken over finite subsets $I \subset E$ such that $\tau \in W(I) \cdot \sigma$. Clearly $\mathbf{B}\mathcal{F}_\tau \subset \mathbf{U}_\tau$, hence by Theorem 1 (a) the open subsets \mathbf{U}_τ (for $\tau \in \mathbf{W}(E) \cdot \sigma$) cover the ind-variety $\mathbf{Fl}(\mathcal{F}_\sigma, E)$.

Let $I, J \subset E$ be finite subsets such that $I \subset J$. Let $\text{Fl}(\mathcal{F}_\sigma, I)$, $\text{Fl}(\mathcal{F}_\sigma, J)$ be corresponding finite-dimensional flag varieties, and let $\iota_{I,J} : \text{Fl}(\mathcal{F}_\sigma, I) \rightarrow \text{Fl}(\mathcal{F}_\sigma, J)$ be the embedding defined in Section 3.3. As noted in Proposition 11, we have $\iota_{I,J}(B(I)\mathcal{F}_\sigma) \subset B(J)\mathcal{F}_\sigma$, hence $\iota_{I,J}(X_{\sigma,I}) \subset X_{\sigma,J}$.

Let $\tau \in W(I) \cdot \sigma$. The inclusion $G_\tau(I) \subset G_\tau(J)$ holds. Moreover, using that $g(e) = e$ for all $g \in G_\tau(I)$, all $e \in J \setminus I$, in view of the definition of the map $\iota_{I,J}$, we have $\iota_{I,J}(g\mathcal{F}_\tau) = g\mathcal{F}_\tau \in U_\tau(J)$ for all $g \in G_\tau(I)$. Hence the map $\iota_{I,J}$ restricts to an embedding $\iota'_{I,J} : U_\tau(I) \cap X_{\sigma,I} \rightarrow U_\tau(J) \cap X_{\sigma,J}$.

Claim 1. Let $I, J \subset E$ be finite subsets such that $I \subset J$ and let $\tau \in W(I) \cdot \sigma$. Then, $\iota'_{I,J}$ restricts to an embedding $U_\tau(I) \cap \text{Sing}(X_{\sigma,I}) \rightarrow U_\tau(J) \cap \text{Sing}(X_{\sigma,J})$.

Let $H \subset G(J)$ be the torus formed by the elements $h \in G(J)$ such that $h(e) = e$ for all $e \in I$ and $h(e) \in \mathbb{K}^*e$ for all $e \in J \setminus I$. The torus H acts on $X_{\sigma,J}$. From [7], it follows that $\text{Sing}((X_{\sigma,J})^H) \subset \text{Sing}(X_{\sigma,J})$, where $(X_{\sigma,J})^H \subset X_{\sigma,J}$ stands for the subset of H -fixed points. On the other hand, it is easy to see that the equality $\iota'_{I,J}(U_\tau(I) \cap X_{\sigma,I}) = U_\tau(J) \cap (X_{\sigma,J})^H$ holds. Thereby,

$$\iota'_{I,J}(U_\tau(I) \cap \text{Sing}(X_{\sigma,I})) = U_\tau(J) \cap \text{Sing}((X_{\sigma,J})^H) \subset \text{Sing}(X_{\sigma,J}).$$

This shows Claim 1.

Claim 2. Let $I, J \subset E$ be finite subsets such that $J = I \cup \{e_J\}$ and let $\tau \in W(I) \cdot \sigma$. Assume that at least one of the following conditions holds:

- (i) $e_J \prec_{\mathbf{B}} e$ for all $e \in I$;
- (ii) $e_J \succ_{\mathbf{B}} e$ for all $e \in I$;
- (iii) $\tau(e_J) \preceq_A \tau(e)$ for all $e \in I$;

(iv) $\tau(e_J) \succeq_A \tau(e)$ for all $e \in I$.

Then the map $\iota'_{I,J} : U_\tau(I) \cap X_{\sigma,I} \rightarrow U_\tau(J) \cap X_{\sigma,J}$ admits a left inverse $r'_{I,J} : U_\tau(J) \cap X_{\sigma,J} \rightarrow U_\tau(I) \cap X_{\sigma,I}$.

We write an element $g \in \mathbf{G}(E)$ as a matrix $(g_{e',e})_{e',e \in E}$ such that $g(e) = \sum_{e' \in E} g_{e',e} e'$. Let $G_\tau(J) \rightarrow G_\tau(I)$, $g \mapsto g'$ and $R_{I,J} : G_\tau(J) \rightarrow G_\tau(I)$, $g \mapsto \tilde{g}$ be the maps defined by

$$g'_{e',e} = \begin{cases} 0 & \text{if } e \neq e' = e_J \\ g_{e',e} & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{g}_{e',e} = \begin{cases} 0 & \text{if } e \neq e' \text{ and } e_J \in \{e, e'\} \\ g_{e',e} & \text{otherwise.} \end{cases} \tag{19}$$

The map $R_{I,J}$ induces a morphism of algebraic varieties $r_{I,J} : U_\tau(J) \rightarrow U_\tau(I)$, $g\mathcal{F}_\tau \mapsto \tilde{g}\mathcal{F}_\tau$. It is clear that $\tilde{g} = g$ whenever $g \in G_\tau(I)$, hence $r_{I,J}(\iota_{I,J}(\mathcal{G})) = \mathcal{G}$ whenever $\mathcal{G} \in U_\tau(I)$.

We claim that

$$\mathcal{G} \in U_\tau(J) \cap \mathbf{X}_\sigma \Rightarrow r_{I,J}(\mathcal{G}) \in \mathbf{X}_\sigma. \tag{20}$$

Let $\mathcal{G} = g\mathcal{F}_\tau$ with $g \in G_\tau(J)$. Assume that $\mathcal{G} \in \mathbf{X}_\sigma$. We first check that

$$\mathcal{G}' := g'\mathcal{F}_\tau \in \mathbf{X}_\sigma \tag{21}$$

with g' as in (19). We distinguish four cases depending on the conditions (i)–(iv) of Claim 2.

- Assume that condition (i) holds. Let $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ be the maximal generalized flag corresponding to \mathbf{B} , i.e., $F'_{0,e} = \langle e' : e' \prec_{\mathbf{B}} e \rangle$ and $F''_{0,e} = \langle e' : e' \preceq_{\mathbf{B}} e \rangle$ (see Section 4.1). In view of condition (i) and the definition of the map $g \mapsto g'$, for any $F \in \mathcal{F}_0$ and any linear combination $\sum_{e \in J} \lambda_e e \in \langle J \rangle$, we have

$$\sum_{e \in J} \lambda_e g(e) \in F \Rightarrow \sum_{e \in J} \lambda_e g'(e) \in F.$$

This implication yields $\dim g'(M) \cap \langle J \rangle \cap F \geq \dim g(M) \cap \langle J \rangle \cap F$ for all $M \in \mathcal{F}_\tau$, all $F \in \mathcal{F}_0$. It is well known that this property implies $g'\mathcal{F}_\tau \in \overline{B(J)g\mathcal{F}_\tau} \subset \mathbf{X}_\sigma$ (see, e.g., [1, §3.2.9]).

- Assume that condition (ii) holds. Then every $\mathcal{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A} \in B(J)\mathcal{F}_\sigma$ satisfies $F''_\alpha \subset \langle E \setminus \{e_J\} \rangle$ whenever $\alpha \prec_A \sigma(e_J)$. The same property holds whenever $\mathcal{F} \in \overline{B(J)\mathcal{F}_\sigma} = \text{Fl}(\mathcal{F}_\sigma, J) \cap \mathbf{X}_\sigma$. Applying this observation to $\mathcal{F} = g\mathcal{F}_\tau$ (and noting that $\tau(e_J) = \sigma(e_J)$ because $\tau \in W(I) \cdot \sigma$), we deduce that $g_{e_J,e} = 0$ for all $e \neq e_J$, whence $g' = g$. This clearly yields (21) in this case.
- Assume that condition (iii) holds. Then the definition of $G_\tau(J)$ yields $g_{e_J,e} = 0$ for all $e \in I$, whence $g' = g$. This implies (21).
- Finally, assume that condition (iv) holds. Then the definition of $G_\tau(J)$ implies that $g(e_J) = e_J$. For $t \in \mathbb{K}^*$, let $\tilde{h}_t \in \mathbf{H}(E)$ be defined by

$$\tilde{h}_t(e) = \begin{cases} e & \text{if } e \neq e_J \\ te_J & \text{if } e = e_J \end{cases} \quad \text{for all } e \in E. \tag{22}$$

We have $g'\mathcal{F}_\tau = \lim_{t \rightarrow 0} \tilde{h}_t g\mathcal{F}_\tau$. Since $\tilde{h}_t g\mathcal{F}_\tau \in \mathbf{X}_\sigma$ for all $t \in \mathbb{K}^*$, we get $g'\mathcal{F}_\tau \in \mathbf{X}_\sigma$, whence (21).

Therefore (21) holds true in all the cases. Moreover, we have

$$\tilde{g}\mathcal{F}_\tau = \lim_{t \rightarrow \infty} \tilde{h}_t g' \mathcal{F}_\tau$$

with \tilde{h}_t as in (22). Since $g' \mathcal{F}_\tau \in \mathbf{X}_\sigma$ (by (21)) and \tilde{h}_t stabilizes \mathbf{X}_σ , we conclude that $r_{I,J}(\mathcal{G}) = \tilde{g}\mathcal{F}_\tau \in \mathbf{X}_\sigma$. Whence (20).

By (20), the map $r'_{I,J} : U_\tau(J) \cap X_{\sigma,J} \rightarrow U_\tau(I) \cap X_{\sigma,I}$ obtained by restriction of $r_{I,J}$ is well defined and fulfills the conditions of Claim 2.

Relying on Claims 1 and 2, the proof of the theorem is carried out as follows. If $X_{\sigma,I}$ is smooth for all finite subsets $I \subset E$, then Lemma 5 guarantees that \mathbf{X}_σ is a smooth ind-variety. We now assume that there is a finite subset $I_0 \subset E$ such that X_{σ,I_0} is singular. In this case Lemma 5 yields an inclusion

$$\text{Sing}(\mathbf{X}_\sigma) \subset \bigcup_{I \supset I_0} \text{Sing}(X_{\sigma,I})$$

where the union is taken over all finite subsets $I \subset E$ such that $I \supset I_0$. For completing the proof it is sufficient to prove that

$$\text{Sing}(X_{\sigma,I}) \subset \text{Sing}(\mathbf{X}_\sigma) \tag{23}$$

for each finite subset $I \subset E$ with $I \supset I_0$. To show this, let $\mathcal{G} \in \text{Sing}(X_{\sigma,I})$. There is $\tau \in W(I) \cdot \sigma$ such that $\mathcal{G} \in U_\tau(I)$. We consider the two cases involved in assumption (H).

- If (H) (i) holds, then let $e_0 = \min I$ and $e_1 = \max I$ (for the order $\preceq_{\mathbf{B}}$), and set $I' = \{e \in E : e_0 \preceq_{\mathbf{B}} e \preceq_{\mathbf{B}} e_1\}$. The set I' is finite (by (H) (i)). Moreover, again relying on (H) (i), we can find a filtration $E = \bigcup_{n \geq 1} E_n$ with $E_1 = I'$ and $E_n = E_{n-1} \cup \{e_n\}$ for all $n \geq 2$, where e_n is either the minimum or the maximum of $(E_n, \preceq_{\mathbf{B}})$.
- If (H) (ii) holds, then let $\alpha_0 = \min\{\tau(e) : e \in I\}$ and $\alpha_1 = \max\{\tau(e) : e \in I\}$ (for the order \preceq_A), and set $I' = I \cup \{e \in E : \alpha_0 \prec_A \tau(e) \prec_A \alpha_1\}$. The first part of (H) (ii) ensures that there are at most finitely many $\alpha \in A$ such that $\alpha_0 \prec_A \alpha \prec_A \alpha_1$, while the second part of (H) (ii) (together with the fact that $\tau \in \mathbf{W}(E) \cdot \sigma$) implies that $\tau^{-1}(\alpha)$ is finite for each such α , hence the set I' is finite. Again relying on (H) (ii), we can construct a filtration $E = \bigcup_{n \geq 1} E_n$ with $E_1 = I'$ and $E_n = E_{n-1} \cup \{e_n\}$ for all $n \geq 2$, where e_n satisfies either $\tau(e_n) \preceq_A \tau(e)$ for all $e \in E_{n-1}$ or $\tau(e_n) \succeq_A \tau(e)$ for all $e \in E_{n-1}$.

In both cases, we get a filtration $\{E_n\}_{n \geq 1}$ of E by finite subsets such that $I \subset E_1$ and, for every $n \geq 2$, the pair (E_{n-1}, E_n) satisfies one of the conditions (i)–(iv) of Claim 2. We obtain an exhaustion of the open subset $U_\tau \cap \mathbf{X}_\sigma$ of \mathbf{X}_σ given by the chain

$$U_{\sigma,\tau,1} \xrightarrow{\iota_1} U_{\sigma,\tau,2} \xrightarrow{\iota_2} U_{\sigma,\tau,3} \hookrightarrow \dots \hookrightarrow U_{\sigma,\tau,n} \xrightarrow{\iota_n} \dots$$

where $U_{\sigma,\tau,n} = U_\tau(E_n) \cap X_{\sigma,E_n}$ and $\iota_n = \iota'_{E_n, E_{n+1}}$. Claim 1 implies that \mathcal{G} is a singular point of $U_{\sigma,\tau,1}$. By Claim 2, we can apply Lemma 6 which implies that \mathcal{G} is a singular point of $U_\tau \cap \mathbf{X}_\sigma$, hence of \mathbf{X}_σ . Therefore the inclusion (23) holds. The proof is complete. \square

REMARK 10. (a) Note that hypothesis (H) is valid in the case where $\mathbf{Fl}(\mathcal{F}_\sigma, E)$ is an ind-grassmannian.

(b) Hypothesis (H) is needed in the proof of Theorem 4 for showing Claim 2 which is necessary for applying Lemma 6. We have no indication whatsoever that Theorem 4 is not valid in general (without hypothesis (H)).

REMARK 11. The Schubert ind-varieties \mathbf{X}_σ considered in this paper form a narrower class than the ones considered by H. Salmasian [12]. Indeed, a closed ind-subvariety $\mathbf{X} \subset \mathbf{Fl}(\mathcal{F}, E)$ such that $\mathbf{X} \cap \mathbf{Fl}(\mathcal{F}, I)$ is a Schubert variety for all finite subsets $I \subset E$ is a Schubert ind-variety in the sense of [12], and it may happen that \mathbf{X} has no open \mathbf{B} -orbit and admits no smooth point in this case (see [12, Section 2]). On the other hand, the ind-variety \mathbf{X}_σ defined in Section 6.2 always contains the open \mathbf{B} -orbit \mathbf{BF}_σ , and the points of \mathbf{BF}_σ are smooth in \mathbf{X}_σ .

6.3. Examples. A consequence of Theorem 4 is that the smoothness criteria for Schubert varieties of (finite-dimensional) flag varieties that are expressed in terms of pattern avoidance, may pass to the limit at infinity.

For example, let us apply Theorem 4 to the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ for an E -compatible maximal generalized flag \mathcal{F} . In this case we have two total orders on the basis E : the first one $\preceq_{\mathbf{B}}$ corresponds to the splitting Borel subgroup \mathbf{B} , and the second order $\preceq_{\mathcal{F}}$ corresponds to the maximal generalized flag \mathcal{F} , i.e., $\mathcal{F} = \{F'_e, F''_e : e \in E\}$ is given by

$$F'_e = \langle e' \in E : e' \prec_{\mathcal{F}} e \rangle, \quad F''_e = \langle e' \in E : e' \preceq_{\mathcal{F}} e \rangle.$$

By Theorem 1, the Schubert ind-varieties \mathbf{X}_σ of $\mathbf{Fl}(\mathcal{F}, E)$ are parametrized by the permutations $\sigma \in \mathbf{W}(E)$, and we have

$$\dim \mathbf{X}_\sigma = n_{\text{inv}}(\sigma) = |\{(e, e') \in E \times E : e \prec_{\mathbf{B}} e', \sigma(e') \prec_{\mathcal{F}} \sigma(e)\}|.$$

From Theorem 4 and the known characterization of smooth Schubert varieties of full flag varieties in terms of pattern avoidance (see [1, §8]) we obtain the following criterion.

COROLLARY 3. *Assume that \mathcal{F} or \mathcal{F}_0 is a flag, so that hypothesis (H) is satisfied. Let $\sigma \in \mathbf{W}(E)$. Then the Schubert ind-variety \mathbf{X}_σ is singular if and only if there exist $e_1, e_2, e_3, e_4 \in E$ such that $e_1 \prec_{\mathbf{B}} e_2 \prec_{\mathbf{B}} e_3 \prec_{\mathbf{B}} e_4$ and $(\sigma(e_3) \prec_{\mathcal{F}} \sigma(e_4) \prec_{\mathcal{F}} \sigma(e_1) \prec_{\mathcal{F}} \sigma(e_2) \text{ or } \sigma(e_4) \prec_{\mathcal{F}} \sigma(e_2) \prec_{\mathcal{F}} \sigma(e_3) \prec_{\mathcal{F}} \sigma(e_1))$.*

REMARK 12. (a) Corollary 3 shows in particular that, if the basis E comprises infinitely many pairwise disjoint quadruples (e_1, e_2, e_3, e_4) such that $e_1 \prec_{\mathbf{B}} e_2 \prec_{\mathbf{B}} e_3 \prec_{\mathbf{B}} e_4$ and, say, $e_3 \prec_{\mathcal{F}} e_4 \prec_{\mathcal{F}} e_1 \prec_{\mathcal{F}} e_2$, then for every permutation $\sigma \in \mathbf{W}(E)$, the Schubert ind-variety \mathbf{X}_σ is singular. Thus, there exist pairs $(\mathbf{B}, \mathcal{F})$ such that all Schubert ind-varieties of the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ are singular.

(b) In the case where the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ has finite-dimensional Schubert cells, it has one cell equal to a single point (see Theorem 3), hence has at least one smooth Schubert ind-variety. Note that $\mathbf{Fl}(\mathcal{F}, E)$ may have smooth Schubert ind-varieties although all its Schubert cells are infinite dimensional. Take for instance $E = \{e_i\}_{i \in \mathbb{Z}}$, let the order $\preceq_{\mathbf{B}}$ be the natural order on \mathbb{Z} , and let the order $\preceq_{\mathcal{F}}$ be the inverse order, i.e., $i \preceq_{\mathcal{F}} j$ if and only if $i \geq j$. Then every Schubert cell of $\mathbf{Fl}(\mathcal{F}, E)$ is infinite dimensional, but the permutation $\sigma = \text{id}_E \in \mathbf{W}(E)$ avoids the two forbidden patterns of Corollary 3, hence \mathbf{X}_σ is smooth.

As a second example, we apply Theorem 4 to the case of the ind-grassmannian $\mathbf{Gr}(2)$. In this case, for a splitting Borel subgroup \mathbf{B} , the Schubert ind-varieties \mathbf{X}_σ are parametrized by the surjective maps $E \rightarrow \{1, 2\}$ such that $|\sigma^{-1}(1)| = 2$, or equivalently by the pairs of elements $\sigma = \{\sigma_1, \sigma_2\} \subset E$. From Theorem 4 and [1, §9.3.3] we have:

COROLLARY 4. *Let $\sigma = \{\sigma_1, \sigma_2\} \subset E$ with $\sigma_1 \prec_{\mathbf{B}} \sigma_2$. The Schubert ind-variety \mathbf{X}_σ is smooth if and only if σ_1 is the smallest element of the ordered set $(E, \preceq_{\mathbf{B}})$ or σ_1, σ_2 are two consecutive elements of $(E, \preceq_{\mathbf{B}})$.*

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