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EULER CHARACTERISTIC NUMBERS OF SPACE-LIKE MANIFOLDS*

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Abstract. In this note, we prove that if a compact even dimensional manifold M^n with negative sectional curvature is homotopic to some compact space-like manifold N^n, then the Euler characteristic number of M^n satisfies $(-1)^{\frac{n}{2}}\chi(M^n) > 0$. We also show that the minimal volume conjecture of Gromov is true for all compact even dimensional space-like manifolds.

Key words. Euler characteristic, Hopf conjecture.

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1. Introduction. Let M^n be a compact even dimensional Riemannian manifold with negative sectional curvature. A long-standing conjecture due to H. Hopf [5] in differential geometry asks whether the Euler characteristic number of M^n satisfies $(-1)^{\frac{n}{2}}\chi(M^n) > 0$. When $n = 4$, the proof was given by Chern [2] (who attributed the result to Milnor) by showing that the integrand of Gauss-Bonnet-Chern is positive. However, when $n = 6$, some examples show that the integrand does not have a definite sign in general. On the other hand, Gromov in [4] proved that Hopf conjecture is true when the manifold is Kähler.

In this note, we will consider the Euler characteristic numbers of a class of real Riemannian manifolds. These manifolds N^n are locally embeddable in Lorentz-Minkowski space $R^{n,1}$. In [7], such N^n is called space-like. More precisely, we call a manifold (N^n, g) (see [7]) space-like if there exists a symmetric $(0,2)$ tensor h_{ij} such that the following two equations are fulfilled

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}); \tag{1.1}$$

$$\nabla_i h_{jk} = \nabla_j h_{ik}. \tag{1.2}$$

Here the sign convention for the Riemann curvature tensor R_{ijkl} is made so that R_{ijij} is positive on sphere. Clearly, a space-like n-dimensional submanifold of $R^{n,1}$ satisfies the above two equations (1.1) and (1.2) if we take the tensor h_{ij} to be the second fundamental form induced from $R^{n,1}$. A space-like manifold shares some interesting properties of manifolds with non-positive sectional curvature. For example, it can be shown that the universal cover of a complete space-like manifold is diffeomorphic to the Euclidean space (see Corollary 2.2). One main result of this note is the following theorem:

THEOREM 1.1. *Let M^n be a compact even dimensional Riemannian manifold with negative sectional curvature. Suppose M^n is homotopic to some compact space-like manifold N^n. Then the Euler characteristic number of M^n satisfies*

$$(-1)^{\frac{n}{2}}\chi(M^n) > 0.$$

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Note that in the theorem, we do not assume the curvature of space-like manifold N^n has a sign. The curvature sign is only imposed on the manifold M^n. The motivation for the proof of Theorem 1.1 is from [7], where the second author studied the hyperbolization problem of space-like manifolds by using the intrinsic mean curvature flow. More precisely, it was shown in [7] that if the manifold is compact and $h_{ij} > 0$ in (1.1), then the manifold admits a Riemannian metric of negative constant sectional curvature.

Theorem 1.1 follows from a more general result on space-like manifolds satisfying (1.1) and (1.2):

Theorem 1.2. *Let N^n be an even dimensional compact space-like manifold, then the Euler characteristic number satisfies*

$$(-1)^{\frac{n}{2}}\chi(N^n) \geq 0.$$

The equality holds if and only if the minimal volume of N^n is zero.

According to Gromov [3], the minimal volume $minvol(N^n)$ of a manifold N^n is the infimum of all volumes $vol(N^n, g')$, where g' ranges over all Riemannian metrics with sectional curvatures satisfying $|K_{g'}| \leq 1$. The minimal volume conjecture of Gromov [3] is asking whether there is a number $\varepsilon(n)$ depending only on the dimension n such that $minvol(N^n) < \varepsilon(n)$ implies $minvol(N^n) = 0$. This conjecture was already verified by X. C. Rong [6] in dimension 4. A byproduct of Theorem 1.2 is

Corollary 1.3. *The minimal volume conjecture is true for all compact even dimensional space-like manifolds.*

The proof of above Theorems 1.1 and 1.2 is an elementary application of the mean curvature flow. The detail will be given in the following two sections.

2. Mean curvature flow. In this section, we assume that (N^n, g) is a space-like manifold, i.e., there is a tensor h_{ij} such that equations (1.1) and (1.2) hold. In [7], the second author studied the following flow:

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= -2R_{ij} + 2h_{ik}h_{jl}g^{kl},\\ \frac{\partial h_{ij}}{\partial t} &= \triangle h_{ij} - R_{im}h_{nj}g^{mn} - R_{jm}h_{ni}g^{mn}\\ &\qquad + 2h_{im}h_{jn}h_{kl}g^{mk}g^{ln} - h_{mn}h_{kl}g^{mk}g^{nl}h_{ij}.\end{aligned} \tag{2.1}$$

It was shown in [7], when (N^n, g) is compact, equation (2.1) admits a smooth solution for any initial data (g_0, h_0). Moreover, if equations (1.1) and (1.2) hold at time $t = 0$, then they also continue to hold for time $t > 0$. That is to say, $(N^n, g(t), h(t))$ will remain to be a space-like manifold under the deformation (2.1). In this case, equation (2.1) may be simplified:

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= 2Hh_{ij}\\ \frac{\partial h_{ij}}{\partial t} &= \triangle h_{ij} + 2Hh_{im}h_{jn}g^{mn} - |A|^2h_{ij}\end{aligned} \tag{2.2}$$

where $H = g^{ij}h_{ij}, |A|^2 = g^{ij}g^{kl}h_{ik}h_{jl}$.

We may call equation (2.1) an intrinsic mean curvature flow. Equations (1.1) and (1.2) may be called Gauss and Codazzi equations respectively.

Another approach to solve equations (2.1) or (2.2) is to embed the universal cover $(\tilde{N}, \tilde{g})$ into $R^{n,1}$ as a space-like submanifold Σ in the usual sense, and deform Σ in $R^{n,1}$ by mean curvature flow and prove that the mean curvature flow is invariant under the deck transformation.

Proposition 2.1. *Let N^n be a complete space-like manifold satisfying (1.1) and (1.2). Then its universal cover $\tilde{N}^n$ admits an isometric embedding into $R^{n,1}$ as a space-like submanifold whose second fundamental form is given by the tensor h_{ij} in (1.1) and (1.2).*

Proof. By a monodromy argument, there is a smooth isometric immersion $\varphi :(\tilde{N}^n, \tilde{g}) \to R^{n,1}$ with h_{ij} as the second fundamental form. Let $\pi : R^{n,1} \to R^n$ be the projection to an n-coordinate plane, and $\psi = \pi \circ \varphi : \tilde{N}^n \to R^n$. Let g_0 be the Euclidean metric in R^n, then it is not hard to see $\tilde{g} \leq \psi^* g_0$. This implies ψ is proper, hence a covering map to R^n. From this, we know φ is an embedding. □

Corollary 2.2. *The universal cover $\tilde{N}^n$ of Proposition 2.1 is diffeomorphic to the Euclidean space.*

Now let (N^n, g) be a compact space-like manifold, we deform (g, h) by (2.1) or (2.2).

From (2.2), it is not hard to show

$$\begin{aligned} \frac{\partial H}{\partial t} &= \triangle H - H|A|^2 \\ \frac{\partial |A|^2}{\partial t} &= \triangle |A|^2 - 2|\nabla A|^2 - 2|A|^4 \end{aligned} \tag{2.3}$$

where $|\nabla A|^2 = g^{ij} g^{kl} g^{pq} \nabla_i h_{kp} \nabla_j h_{lq}$.

From (2.3) and maximum principle, it can be shown that the solution $(g(t), h(t))$ of equation (2.1) (or (2.2)) always exists for all time $0 \leq t < \infty$, and h_{ij} satisfies the estimate

$$0 \leq |A|^2 \leq \frac{1}{2t + 1/|A|^2_{max}(0)}. \tag{2.4}$$

First, we derive two monotonicity formulas for the intrinsic mean curvature flow. We have to mention that all the quantities in the following propositions involving the norm and the volume element dv are computed with respect to the evolving metric $g(t)$.

Proposition 2.3.

$$\frac{d}{dt}\int_{N^n} |H|^n dv = -n(n-1)\int_{N^n} |\nabla H|^2 |H|^{n-2} dv - n\int_{N^n} |H|^n |h_{ij} - \frac{H}{n} g_{ij}|^2 dv \leq 0,$$

and

$$\begin{aligned} \frac{d}{dt}\int_{N^n} |A|^n dv = &-(\frac{n}{2}-1)\frac{n}{2}\int_{N^n} |A|^{(n-4)} |\nabla |A|^2|^2 dv - n\int_{N^n} |A|^{n-2}|\nabla A|^2 dv \\ &- n\int_{N^n} |A|^n |h_{ij} - \frac{H}{n} g_{ij}|^2 dv \leq 0. \end{aligned}$$

Proof. The proof is direct calculations by using equations (2.3). □

PROPOSITION 2.4. *There are constant $C_0 > 0$ and a sequence of times $t_k \to \infty$ such that*

$$0 \leq \int_{N^n} |A|^n dv < C_0, \tag{2.5}$$

for all time $t \geq 0$, and

$$t_k \cdot \int_{N^n} |A|^n |h_{ij} - \frac{H}{n} g_{ij}|^2 dv \, |_{t=t_k} \to 0 \quad as \quad k \to \infty. \tag{2.6}$$

Proof. (2.5) follows from the second formula of Proposition 2.3. Integrating the formula, we have

$$\int_0^\infty \int_{N^n} |A|^n |h_{ij} - \frac{H}{n} g_{ij}|^2 dv \leq C_0. \tag{2.7}$$

If (2.6) does not hold, there are constants $C > 0$ and $\delta > 0$ such that for all $t > C$, we have

$$\int_{N^n} |A|^n |h_{ij} - \frac{H}{n} g_{ij}|^2 dv > \frac{\delta}{t},$$

which is a contradiction with (2.7). □

Since

$$\frac{d}{dt} vol(N^n, t) = \int_{N^n} H^2 dv \leq \Big(\int_{N^n} |H|^n dv \Big)^{\frac{2}{n}} (vol(N^n, t))^{1-\frac{2}{n}},$$

we have

$$\frac{d}{dt} vol(N^n, t)^{\frac{2}{n}} \leq \frac{2}{n} (\int_{N^n} |H|^n dv)^{\frac{2}{n}}. \tag{2.8}$$

PROPOSITION 2.5. *There is a constant $C_1 > 0$ such that for all $t > 0$, we have*

$$vol(N^n, t) \leq C_1 (t+1)^{\frac{n}{2}}, \tag{2.9}$$

$$\frac{1}{1+t} \int_{N^n} |A|^{n-2} dv \leq C_1. \tag{2.10}$$

Moreover

$$\limsup_{t\to\infty} \frac{vol(N^n, t)}{(1+t)^{\frac{n}{2}}} \leq (\frac{2}{n})^{\frac{n}{2}} \lim_{t\to\infty} \int_{N^n} |H|^n dv. \tag{2.11}$$

Proof. From (2.8) we have

$$\frac{vol(N^n, t)^{\frac{2}{n}} - vol(N^n, 0)^{\frac{2}{n}}}{t+1} \leq \frac{2}{n} \frac{\int_0^t (\int_{N^n} |H|^n dv)^{\frac{2}{n}} dt}{t+1}.$$

Since $\int_{N^n} |H|^n dv$ is monotonically decreasing from Proposition 2.3, we know

$$\limsup_{t\to\infty} \frac{vol(N^n, t)^{\frac{2}{n}}}{1+t} \leq \frac{2}{n} \lim_{t\to\infty} (\int_{N^n} |H|^n dv)^{\frac{2}{n}},$$

which gives (2.11) and (2.9). Now (2.10) follows from (2.5)(2.9),

$$\int_{N^n} |A|^{n-2} dv \leq (\int_{N^n} |A|^n dv)^{1-\frac{2}{n}} vol(N^n, t)^{\frac{2}{n}} \leq C(1+t).$$

□

3. Proof of the main theorem. To prove Theorem 1.2, we may assume N^n is orientable. Since the dimension n is even, it is well-known that by Gauss-Bonnet-Chern theorem, the Euler Characteristic number $\chi(N^n)$ may be expressed as a curvature integral (see Chern [1]):

$$\int_{N^n} \Omega = \chi(N^n) \tag{3.1}$$

where

$$\Omega = \frac{1}{2^n \pi^{\frac{n}{2}} (\frac{n}{2})!} \sum \varepsilon^{i_1, i_2, \cdots, i_n} \Omega_{i_1 i_2} \wedge \cdots \Omega_{i_{n-1} i_n},$$

and $\Omega_{i_1 i_2}$ is the curvature form. From equation (1.1) and direct calculations, it follows

$$\Omega = (-1)^{\frac{n}{2}} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{\det(h)}{\det(g)} dv = (-1)^{\frac{n}{2}} \frac{2}{vol(S^n)} \frac{\det(h)}{\det(g)} dv, \tag{3.2}$$

and

$$\chi(N^n) = (-1)^{\frac{n}{2}} \frac{2}{vol(S^n)} \int_{N^n} \frac{\det(h)}{\det(g)} dv. \tag{3.3}$$

We remark that equation (3.3) holds for any $t > 0$, since equation (1.1) holds for any time $t > 0$.

For any fixed $p \in N^n$, choose an orthonormal frame $e_i, i = 1, 2, \cdots, n$, such that h_{ij} is diagonalied in this frame, i.e., $h_{ij} = \lambda_i \delta_{ij}$. Then we have

$$|\frac{\det(h)}{\det(g)} - (\frac{H}{n})^n| = |\lambda_1 \cdots \lambda_n - (\frac{H}{n})^n| \le n|A|^{n-1} |h_{ij} - \frac{H}{n} g_{ij}|.$$

Hence

$$\begin{aligned} &\int_{N^n} |\frac{\det(h)}{\det(g)} - (\frac{H}{n})^n| dv \\ &\le n \int_{N^n} |A|^{n-1} |h_{ij} - \frac{H}{n} g_{ij}| dv \\ &\le n(\int_{N^n} |A|^n |h_{ij} - \frac{H}{n} g_{ij}|^2 dv)^{\frac{1}{2}} \cdot (\int_{N^n} |A|^{n-2} dv)^{\frac{1}{2}}. \end{aligned}$$

Let t_k be the time sequence chosen in Proposition 2.4, it follows from (2.6) and (2.10) that at $t = t_k$:

$$\int_{N^n} |A|^n |h_{ij} - \frac{H}{n} g_{ij}|^2 dv \cdot \int_{N^n} |A|^{n-2} dv \,|_{t=t_k} \to 0 \tag{3.4}$$

as $k \to \infty$. This implies

$$\lim_{k \to \infty} \int_{N^n} |\frac{\det(h)}{\det(g)} - (\frac{H}{n})^n| dv \,|_{t=t_k} = 0. \tag{3.5}$$

Combining (3.3) and (3.5), we have

$$(-1)^{\frac{n}{2}} \chi(N^n) = \lim_{k \to \infty} \frac{2}{vol(S^n)} \int_{N^n} (\frac{H}{n})^n dv \,|_{t=t_k} . \tag{3.6}$$

Because n is even, we know $(-1)^{\frac{n}{2}}\chi(N^n) \geq 0$. This finishes the main part of Theorem 1.2.

Clearly, if the minimal volume of N^n is zero, then the Euler characteristic number $\chi(N^n)$ is zero. This follows directly from the Gauss-Bonnet-Chern formula (3.1). To see the converse, let $\chi(N^n) = 0$, from (3.6) we have

$$\lim_{k\to\infty}\int_{N^n} |H|^n dv \,|_{t=t_k}= 0. \tag{3.7}$$

Combining (2.11), it implies

$$\limsup_{k\to\infty} \frac{vol(N^n, t_k)}{(1+t_k)^{\frac{n}{2}}} = 0. \tag{3.8}$$

Note that (2.4) and (1.1) implies $|Rm|(g_{t_k}) \leq Ct_k^{-1}$. So $\{Ct_k^{-1}g_{t_k}\}$ is a sequence of Riemannian metrics with sectional curvatures satisfying $|K| \leq 1$ but their volumes converge to zero as $k \to \infty$. This shows the minimal volume of N^n is zero. The proof of Theorem 1.2 is completed.

To prove Theorem 1.1, we recall a result of Gromov [3]: the simplicial volume of a compact manifold X^n with negative sectional curvature is positive. In our case, we have the simplicial volume of M^n is positive, so is N^n by the homotopic invariance of simplicial volume. In paper [3], Gromov proved that the minimal volume is always bounded from below by the simplicial volume multiplied by a constant depending only on the dimension. Theorem 1.1 follows from this result and Theorem 1.2. Finally, we mention one corollary:

Corollary 3.1. *Let (N^n, g, h) be an even-dimensional compact space-like manifold. Then*

$$\frac{1}{vol(S^n)}\int_{N^n} |H|^n dv \geq (-1)^{\frac{n}{2}}\frac{n^n}{2}\chi(N^n). \tag{3.9}$$

Equality holds if and only if either (N^n, g, h) is hyperbolic or flat.

Proof. By (3.6) and the first formula of Proposition 2.3, we know (3.9) holds. Suppose the equality in (3.9) holds, we know

$$\int_{N^n} |\nabla H|^2|H|^{n-2}dv = 0$$

and

$$\int_{N^n} |H|^n|h_{ij} - \frac{H}{n}g_{ij}|^2 dv = 0$$

hold for all $t \geq 0$. The first equation implies $H = const.$. If $H \neq 0$, the second equality implies that (N^n, g, h) is hyperbolic. Suppose now $H \equiv 0$ for all t, from (2.2), we know $\frac{\partial g_{ij}}{\partial t} = 0$, and $R = |A|^2$. Since $\frac{\partial R}{\partial t} = 0$ and $|A|^2 \leq Ct^{-1}$, we know $R = |A|^2 \equiv 0$, which implies that (N^n, g, h) is flat. □

Remark 3.2. It is desirable to generalize Theorem 1.2 to higher codimensional case. Namely, we may consider the manifold (N^n, g) which is locally embeddable as a space-like higher codimensional submanifold of $R^{n,m}$. In this case, a formula similar to the first one in Proposition 2.3 can still hold:

$$\frac{d}{dt}\int_{N^n}|H|_+^n dv \leq 0, \tag{3.10}$$

where H is the (time-like) mean curvature vector and $|H|_+ = \sqrt{-\langle H, H\rangle}$. This in particular implies

$$\limsup_{t\to\infty}\frac{vol(N^n,t)^{\frac{2}{n}}}{1+t} \leq \frac{2}{n}\lim_{t\to\infty}(\int_{N^n}|H|_+^n dv)^{\frac{2}{n}} < \infty. \tag{3.11}$$

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