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# LOGARITHMIC VERSION OF THE MILNOR FORMULA*

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**Abstract.** In this paper, we propose a logarithmic version of the Milnor formula. It is a formula for the total dimension of vanishing cycles with tamely ramified coefficient sheaves at an isolated log-singular point. We prove this formula in the geometric case. In the geometric case, it implies that the total dimension of vanishing cycles with tamely ramified coefficient sheaves can be computed as an intersection number (in terms of characteristic cycle).

**Key words.** Milnor formula, vanishing cycle, characteristic cycle.

**AMS subject classifications.** 14F20, 14G17.

**Introduction.** Let $S$ be a regular scheme purely of dimension 1 and $s$ a closed point of $S$ with perfect residue field $k$ of characteristic $p$. Let $X$ be a regular scheme, $f : X \to S$ a flat morphism of finite type, and let $x_0$ be an isolated singular point of $f$ such that $f(x_0) = s$. The Milnor formula says that the Milnor number of $f$ at $x_0$ (cf. Definition 1.3) is equal to the total number of vanishing cycles of $f$ at $x_0$. In the geometric case where $S$ is a scheme over $k$, this conjecture was proved by P. Deligne in [5]. In [14], F. Orgogozo showed that the conductor formula of Bloch implies the Milnor formula. In [11], K. Kato and T. Saito showed that the conductor formula is a consequence of an embedded resolution in a strong sense for the reduced closed fiber. Hence, the Milnor formula is true if we assume an embedded resolution. Consequently, the Milnor formula is true if the relative dimension is two. Recently, using Radon transform, T. Saito proved an analogue formula of the Milnor formula with coefficient sheaf even for a normal surface in [17].

In this paper, we propose a logarithmic version of the Milnor formula. We prove this formula in the geometric case. Let $(X, D)$ be a simple normal crossing pair (cf. section 1.1). If $x_0$ is an isolated log-singular point of a morphism $f : X \to S$ in the sense of section 1.1, then similarly we define the logarithmic Milnor number (cf. Definition 1.3). The relation between Milnor number and its logarithmic version will be given in Lemma 1.9. Let $\ell \neq p$ be a prime number. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-vector spaces on $U = X - D$ such that $\mathcal{F}$ is tamely ramified along $D$. Then the logarithmic version of the Milnor formula (Conjecture 1.12) says that the total number of vanishing cycles of $f$ for the sheaf $\mathcal{F}$ at $x_0$ is equal to rank $\mathcal{F}$ times the logarithmic Milnor number of $f$ at $x_0$. In order to prove this formula in the geometric case, we first prove a logarithmic refinement of Elkik's Lemma ([7], Lemme 2). Using this logarithmic refinement of Elkik's Lemma, we can deform a morphism to a curve. Then by a suggestion of Professor A. Abbes, we apply a result of I. Vidal ([20], Corollaire 3.4) by constructing a compactification (cf. Theorem 3.1). Then we can reduce the proof to the case where $\mathcal{F}$ is equal to $\mathbb{F}_\ell$. At last, the logarithmic Milnor formula in the geometric case is derived from Deligne's result [5] (cf. Proposition 1.13).

The content of each section is as follows. In section 1, we propose the logarithmic Milnor formula. In section 2, we formulate and prove a logarithmic version of Elkik's

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Lemma. In section 3, using Elkik's Lemma and pencils, we construct a compactification. In section 4, using Vidal's result and the compactification constructed in section 4, we prove the logarithmic Milnor formula in the geometric case and give an interpretation of this formula in terms of characteristic cycle.

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## 1. Logarithmic Milnor formula.

**1.1. Normal crossing pairs with isolated singularity.** In this section, let $S$ be a regular scheme purely of dimension 1 and $s$ a closed point of $S$ with perfect residue field. A *simple normal crossing pair* stands for a pair $(X, D)$, where $X$ is a regular scheme and $D$ is a simple normal crossing divisor on $X$. Let $(X, D)$ be a simple normal crossing pair and let $f : X \to S$ be a flat morphism of finite type. Let $x_0$ be a closed point of $X$ such that $f(x_0) = s$. We say that $x_0$ is an *isolated log-singular point* of $f$ (with respect to the divisor $D$), if there exists an open neighborhood $U$ of $x_0$ in $X$ such that $f|_{U-\{x_0\}} : U - \{x_0\} \to S$ is smooth and that $D \cap U - \{x_0\}$ is a divisor on $U - \{x_0\}$ with simple normal crossings relatively to $S$ (cf. [9], Exposé XIII, 2.1). The morphism $f$ is locally a logarithmic hypersurface on an open neighborhood of $x_0$ in the following sense:

**1.2. Locally logarithmic hypersurface.** We first recall a definition similar to ([11], Definition 3.1.1).

Definition 1.1. Let $S$ be a scheme. Let $X$ be a scheme locally of finite presentation over $S$. Let $D \subset X$ be a divisor with simple normal crossings. We say that $(X, D)$ is *locally a logarithmic hypersurface* (resp. of virtual relative dimension $n-1$) over $S$ if, for every $x \in X$, there exist an open neighborhood $U$ of $x$ in $X$, a smooth scheme $P$ over $S$ (resp. of relative dimension $n$) , a divisor $E$ on $P$ with simple normal crossings relatively to $S$ and a regular immersion $U \to P$ of codimension 1 over $S$ such that $D \cap U = E \times_P U$.

The following lemma is proved in ([12], Lemma 4.1.1).

Lemma 1.2. *Let $A$ be a complete discrete valuation ring and $S = \mathrm{Spec} A$. Let $X$ be a regular flat scheme of finite type over $S$ and $D$ a divisor on $X$ with simple normal crossings. Assume that the residue field of $A$ is perfect, then $(X, D)$ is locally a logarithmic hypersurface over $S$.*

**1.3. Logarithmic Milnor number.** Let $S$ be a regular scheme purely of dimension 1 and $s$ a closed point of $S$ with perfect residue field. Let $(X, D)$ be a simple normal crossing pair and let $f : X \to S$ be a flat morphism of finite type. Assume that $f$ has a *unique isolated log-singularity* at a closed point $x_0 \in D \cap X_s$ in the sense of section 1.1, i.e., $X - \{x_0\} \to S$ is smooth and that $D - \{x_0\}$ is a divisor on $X - \{x_0\}$ with simple normal crossings relatively to $S$. Consider the following two coherent $\mathcal{O}_X$-modules

$$\begin{aligned}\mathrm{T_{X/S}} &= \mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}, \mathcal{O}_X\right)\\ \mathrm{T^{log}_{(X,D)/S}} &= \mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}(\log D), \mathcal{O}_X\right).\end{aligned}$$

Then the support of $\mathrm{T^{log}_{(X,D)/S}}$ and $\mathrm{T_{X/S}}$ are contained in $\{x_0\}$ and both are of finite length at $x_0$. Using Lemma 1.2, we will give a local expression for $\mathrm{T^{log}_{(X,D)/S}}$ in Lemma 1.7.

Definition 1.3. The *Milnor number* of $X/S$ at $x_0$ is defined to be (see [5])

$$\mu = \mu(X/S, x_0) = \mathrm{length}_{\mathcal{O}_{X,x_0}} \left\{\mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}, \mathcal{O}_X\right)\right\}_{x_0}.$$

We define the *logarithmic Milnor number* to be

$$\mu^{\log} = \mu^{\log}((X,D)/S, x_0) = \mathrm{length}_{\mathcal{O}_{X,x_0}} \left\{\mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}(\log D), \mathcal{O}_X\right)\right\}_{x_0}.$$

For the definition of $\Omega^1_{X/S}(\log D)$, we refer to [10]. If we give $X$ the logarithmic structure defined by $D$ and denote this log scheme by $X^\dagger$, then $\Omega^1_{X/S}(\log D)$ is equal to $\Omega^1_{X^\dagger/S}$, where $S$ is considered as a log scheme with the trivial log structure. More explicitly, the $\mathcal{O}_X$-module $\Omega^1_{X/S}(\log D)$ is canonically isomorphic to (cf. [10], section 1.7)

$$\left(\Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} j_*\mathcal{O}_U^\times)\right) / \left(da - a \otimes a, 1 \otimes b : a \in \mathcal{O}_X \cap j_*\mathcal{O}_U^\times, b \in \mathrm{Im}(f^{-1}\mathcal{O}_S^\times \to \mathcal{O}_X)\right),$$

where $j : U = X - D \to X$ is the open immersion.

Example 1.4. Let $S = \mathrm{Spec}\ A$ be a henselian trait with algebraically closed residue field. Let $G \in A[T_1, \cdots, T_n]$ be a polynomial with coefficients in $A$. Let $X = \mathrm{Spec}\ A[T_1, \cdots, T_n]/(G)$ and $D$ the divisor defined by $T_1 \cdot T_2 \cdots T_r = 0$. Let $x_0 = (0, \cdots, 0) \in X$. Assume that the canonical morphism $f : X \to S$ has a unique isolated log-singularity at $x_0$ with respect to the divisor $D$. Then by Lemma 1.7 below we have

$$\mu^{\log} = \mathrm{length}_A \frac{A[T_1, \cdots, T_n]_{(T_1,\cdots,T_n)}}{(G, T_1\frac{\partial G}{\partial T_1}, \cdots, T_r\frac{\partial G}{\partial T_r}, \frac{\partial G}{\partial T_{r+1}}, \cdots, \frac{\partial G}{\partial T_n})}.$$

**1.4. Localized Chern classes.** In this section, we compute the logarithmic Milnor number using localized Chern classes (cf. [1, 11, 12]). Let $S$ be a henselian trait with closed point $s$ and generic point $\eta$. Assume that the residue field of $S$ at $s$ is perfect. Let $X$ be a regular flat scheme of finite type over $S$. Assume that $X$ is purely of dimension $n+1$. Let $D$ be a divisor on $X$ with simple normal crossings. Let $Z$ be a closed subscheme of $D_s$ such that the following two conditions are satisfied:

(a) $Z$ is proper over $s$.

(b) $X \setminus Z$ is smooth over $S$ and $D \setminus Z$ is a divisor on $X \setminus Z$ with simple normal crossings relatively to $S$.

The condition (b) implies that $\Omega^1_{X/S}(\log D)$ is locally free outside $Z$. Hence by ([1], Chapter 3) we can define a localized Chern class

$$c^X_{n+1,Z}\left(\Omega^1_{X/S}(\log D)\right) \in \mathrm{CH}^{n+1}(Z \to X),$$

where $\mathrm{CH}^{n+1}(Z \to X)$ is the bivariant Chow group defined in ([8], Chapter 17), see also ([11], Chapter 2). Then $c^X_{n+1,Z}(\Omega^1_{X/S}(\log D)) \cap [X] \in \mathrm{CH}_0(Z)$ is a zero cycle class in $Z$. Since $Z$ is proper over $s$, we can take the degree map:

Definition 1.5. We define the *localized Euler characteristic* of $(X, D)$ to be the following number (cf. [1], Definition 3.3)

$$c^{loc}_{n+1}(X, D) = \deg\left(c^X_{n+1,Z}\left(\Omega^1_{X/S}(\log D)\right) \cap [X]\right) \in \mathbb{Z}.$$

View $X$ as a logarithmic scheme with logarithmic structure defined by $M_X = \mathcal{O}_X \cap j_*\mathcal{O}^\times_U$. Take a frame $X \to [\mathbb{N}^r]$ defined by $D$ (See [11], Chapter 4), where $r$ is the number of irreducible components of $D$. We also consider $S$ as a logarithmic scheme with trivial logarithmic structure. Consider the log diagonal (See [12], Chapter 1.3)

$$X \to (X \times_S X)^\sim := X \times^{\log}_{S,\mathbb{N}^r} X.$$

By ([11], Corollary 4.2.8), $X \to X \times^{\log}_{S,\mathbb{N}^r} X$ is an exact immersion with conormal sheaf

$$\mathcal{N}_{X/X\times^{\log}_{S,\mathbb{N}^r} X} = \Omega^1_{X/S}(\log D).$$

Lemma 1.6. *Let $(X, D)$ be a simple normal crossing pair. Let $f : X \to S$ be a flat morphism of finite type and purely of relative dimension $n$. Let $x \in D \cap X_s$ be a closed point of $X$. Assume that $f$ has a unique isolated log-singularity at $x$. Put $Z = \{x\}$. Then we have*

$$c^{loc}_{n+1}(X, D) = \mu^{\log}\left((X, D)/S, x\right).$$

*Proof.* Denote by $\chi(X, \cdot)$ the following map:

$$\chi : K_Z(X) \simeq K(Z) \xrightarrow{\deg} \mathbb{Z}.$$

By Lemma 1.7 below,

$$\mu^{\log}((X, D)/S, x) = \chi(X, \Omega^{n+1}_{X/S}(\log D)) = \chi(X, L\Lambda^{n+1}\Omega^1_{X/S}(\log D)).$$

While by ([16], Proposition 4.8(3)),

$$\chi(X, L\Lambda^{n+1}\Omega^1_{X/S}(\log D)) = c^{loc}_{n+1}(X, D).$$

Combined the two formulas above, we obtain Lemma 1.6. ∎

**1.5. Local expression of $\mathrm{T}^{\log}_{(\mathrm{X},\mathrm{D})/\mathrm{S}} = \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_{X/S}(\log D), \mathcal{O}_X)$.** Let $S$ be a henselian trait with perfect residue field. Let $X$ be a regular flat scheme of finite type over $S$. Assume that $X$ is purely of relative dimension $n$ over $S$. Let $D$ be a divisor on $X$ with simple normal crossings. Let $Z$ be the closed subscheme of $X$ defined by the annihilator of $\Omega^{n+1}_{X/S}(\log D)$. Let $e : Z \to X$ denote the canonical embedding. Then $X - Z$ is smooth over $S$ and $D - Z$ is a simple normal crossing divisor relative to $S$ (cf. [11], Lemma 3.1.2 and [12], Lemma 4.2.2). By Lemma 1.2, for any point $x \in X$, there is an open neighborhood $U \subset X$ of $x$, a regular immersion $i : U \to P$ of codimension 1 into a smooth scheme $P$ of relative dimension $n+1$ over $S$ and a divisor $E$ on $P$ with simple normal crossings relative to $S$ such that $D \cap U = E \times_P U$.

Assume that $Z \neq X$, then we have an exact sequence which can be viewed as a locally free resolution of $\Omega^1_{U/S}(\log D)$ (See [12], Lemma 4.2.2)

$$0 \to \mathcal{N}_{U/P} \to \Omega^1_{P/S}(\log E) \otimes_{\mathcal{O}_P} \mathcal{O}_U \to \Omega^1_{U/S}(\log D) \to 0, \tag{1.1}$$

where the injectivity of the second arrow comes from that $\Omega^1_{X/S}(\log D)$ is locally free on the (open dense) subscheme $X - Z$. Applying $\mathcal{H}om(-, \mathcal{O}_U)$ to this exact sequence, we have

$$(i^*\Omega^1_{P/S}(\log E))^\vee \to \mathcal{N}^\vee_{U/P} \to \mathcal{E}xt^1_{\mathcal{O}_U}(\Omega^1_{U/S}(\log D), \mathcal{O}_U) \to 0. \tag{1.2}$$

Lemma 1.7. *There exist isomorphisms*

$$\begin{aligned}\mathcal{E}xt^1_{\mathcal{O}_U}(\Omega^1_{U/S}(\log D), \mathcal{O}_U) &\simeq \Omega^{n+1}_{U/S}(\log D) \otimes_{\mathcal{O}_U} (i^*\Omega^{n+1}_{P/S}(\log E))^\vee \otimes_{\mathcal{O}_U} (\mathcal{N}_{U/P})^\vee \\ &\simeq \mathcal{O}_{Z\cap U} \otimes_{\mathcal{O}_U} (\mathcal{N}_{U/P})^\vee.\end{aligned}$$

*Proof.* By exact sequence (1.1), we have

$$\mathcal{N}_{U/P} \otimes_{\mathcal{O}_U} i^*\Omega^n_{P/S}(\log E) \to i^*\Omega^{n+1}_{P/S}(\log E) \to \Omega^{n+1}_{U/S}(\log D) \to 0.$$

Since $i^*\Omega^{n+1}_{P/S}(\log E)$ is an invertible sheaf, tensoring the above sequence with $(i^*\Omega^{n+1}_{P/S}(\log E))^\vee$, we get

$$\mathcal{N}_{U/P} \otimes_{\mathcal{O}_U} (i^*\Omega^1_{P/S}(\log E))^\vee \to \mathcal{O}_U \to \Omega^{n+1}_{U/S}(\log D) \otimes_{\mathcal{O}_U} (i^*\Omega^{n+1}_{P/S}(\log E))^\vee \to 0. \tag{1.3}$$

Tensor the above sequence with $\mathcal{N}^\vee_{U/P}$ and compared with sequence (1.2), we obtain

$$\mathcal{E}xt^1_{\mathcal{O}_U}(\Omega^1_{U/S}(\log D), \mathcal{O}_U) \simeq \Omega^{n+1}_{U/S}(\log D) \otimes_{\mathcal{O}_U} (i^*\Omega^{n+1}_{P/S}(\log E))^\vee \otimes_{\mathcal{O}_U} (\mathcal{N}_{U/P})^\vee.$$

Since by definition of $Z$, $Z \cap U$ is defined by the ideal sheaf $\mathrm{Im}(\mathcal{N}_{U/P} \otimes_{\mathcal{O}_U} (i^*\Omega^1_{P/S}(\log E))^\vee \to \mathcal{O}_U)$. Hence by the exact sequence (1.3), we obtain an isomorphism

$$\mathcal{O}_{Z\cap U} \simeq \Omega^{n+1}_{U/S}(\log D) \otimes_{\mathcal{O}_U} (i^*\Omega^{n+1}_{P/S}(\log E))^\vee,$$

which finishes the proof. □

Let $\mathcal{L}$ and $\mathcal{E}$ be locally free $\mathcal{O}_X$-modules of rank 1 and $n$ respectively on a scheme $X$. We say that a morphism $\mathcal{L} \to \mathcal{E}$ of sheaves is regular if for every point $x \in X$, after choosing a suitable basis for $\mathcal{L}_x$ and $\mathcal{E}_x$, the $\mathcal{O}_{X,x}$-linear map $\mathcal{L}_x \to \mathcal{E}_x$ is defined by a regular sequence $(a_1, \cdots, a_n)$ of length $n$ in $\mathcal{O}_{X,x}$.

LEMMA 1.8. *Let $\mathcal{L}$ and $\mathcal{E}$ be locally free $\mathcal{O}_X$-modules of rank 1 and $n$ respectively on a scheme $X$. Let $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If the map $\mathcal{L} \to \mathcal{E}$ is regular, then the following Koszul complex*

$$0 \to \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n-1} \otimes \mathcal{E} \to \cdots \to \mathcal{L} \otimes \Lambda^{n-1}\mathcal{E} \to \Lambda^n \mathcal{E} \to \Lambda^n \mathcal{F} \to 0$$

*is exact.*

*Proof.* For the version for modules of this lemma, see ([6], Corollary 17.12). □

LEMMA 1.9. *Let $S$ be a henselian trait with perfect residue field. Let $(X, D)$ be a simple normal crossing pair and let $f : X \to S$ be a flat morphism of finite type. Assume that $f : X \to S$ has a unique isolated log-singular point $x_0 \in D \cap X_s$ with respect to $D$. Let $D_i (i \in I)$ be the irreducible components of $D$. For any subset $J \subset I$, put $D_J = \cap_{j \in J} D_j$ with $D_\emptyset = X$. We have*

(1)

$$\begin{aligned}\mu^{\log}((X, D)/S, x_0) &= \mu(X/S, x_0) + \sum_{q \geq 1} \sum_{J \subset I, |J|=q} \mu(D_J/S, x_0) \\ &= \sum_{J \subset I} \mu(D_J/S, x_0).\end{aligned}$$

(2) *Let $r = |I|$ be the number of irreducible components of $D$, then*

$$\mu(X/S, x_0) \leq \mu^{\log}((X, D)/S, x_0) \leq 2^r \cdot \mu(X/S, x_0).$$

*Proof.* After shrinking $X$, we may assume that there is a regular immersion $X \to P$ of codimension 1 into a smooth scheme $P$ of relative dimension $n+1$ over $S$ and a divisor $E$ on $P$ with simple normal crossings relative to $S$ such that $D = E \times_P X$.

We have the so-called weight filtration $W_\bullet$ on $\Omega^{n+1}_{X/S}(\log D)$ such that

$$\begin{aligned}W_0(\Omega^{n+1}_{X/S}(\log D)) &= \Omega^{n+1}_{X/S} \\ \mathrm{Gr}^W_q(\Omega^{n+1}_{X/S}(\log D)) &\simeq a^{(q)}_* \Omega^{n+1-q}_{\widetilde{D}_q/S}, \quad \text{for all } 0 \leq q \leq r,\end{aligned}$$

where $\widetilde{D}_q := \coprod_{J \subset I, |J|=q} D_J \xrightarrow{a^{(q)}} X$ is the canonical morphism and $\widetilde{D}_0 := X$. This filtration is constructed in the following way: For any $m \in \mathbb{N}$ and any integer $q \in \mathbb{Z}$, put

$$W_q(\Omega^m_{X/S}(\log D)) := \begin{cases} 0 & \text{if} \quad q < 0 \\ \mathrm{Im}(\Omega^{m-q}_{X/S} \otimes \Omega^q_{X/S}(\log D) \xrightarrow{\wedge} \Omega^m_{X/S}(\log D)) & \text{if} \quad 0 \leq q \leq m \\ \Omega^m_{X/S}(\log D) & \text{if} \quad m < q \end{cases}$$

Now we show that $\mathrm{Gr}^W_q \Omega^t_{X/S}(\log D) \simeq a^{(q)}_* \Omega^{t-q}_{\widetilde{D}_q/S}$ for all $t \in \mathbb{N}$. This is trivial for $t = 0$. The case $t = 1$ follows from the following exact sequence([12], Lemma 4.2.2)

$$0 \to \Omega^1_{X/S} \to \Omega^1_{X/S}(\log D) \to \bigoplus_{i \in I} \mathcal{O}_{D_i} \to 0.$$

By (1.1), we have a locally free resolution of $\Omega^1_{X/S}(\log \mathrm{D})$:

$$0 \to \mathcal{L} := \mathcal{N}_{X/P} \to \mathcal{E} := \Omega^1_{P/S}(\log E) \otimes_{\mathcal{O}_P} \mathcal{O}_X \to \Omega^1_{X/S}(\log D) \to 0.$$

Since $f$ has an isolated log-singularity at $x_0$ and smooth at any other points, the morphism $\mathcal{L} \to \mathcal{E} = \Omega^1_{P/S}(\log E) \otimes_{\mathcal{O}_P} \mathcal{O}_X$ is regular. By Lemma 1.8, we get a long exact sequence

$$0 \to \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m-1} \otimes \mathcal{E} \to \cdots \to \mathcal{L} \otimes \wedge^{m-1}\mathcal{E} \to \wedge^m \mathcal{E} \to \Omega^m_{X/S}(\log D) \to 0. \quad (1.4)$$

Denote by $K^\bullet$ the complex $0 \to \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m-1} \otimes \mathcal{E} \to \cdots \to \mathcal{L} \otimes \wedge^{m-1}\mathcal{E} \to \wedge^m \mathcal{E} \to 0$. Then $K^\bullet$ can be viewed as a resolution of $\Omega^m_{X/S}(\log D)$ by locally free sheaves. The complex $K^\bullet$ has a filtration induced by $W$. Now we compute the graded quotient $\mathrm{Gr}^W_\bullet K^\bullet$. Applying the construction of the exact sequence (1.4) with $X$ replaced by $\widetilde{D}_q/S$, we get a long exact sequence

$$\begin{aligned}0 \to\ & \mathcal{L}^{\otimes m-q} \otimes a^{(q)}_* \Omega^0_{\widetilde{E}_q/S} \otimes_{\mathcal{O}_P} \mathcal{O}_X \to \mathcal{L}^{\otimes m-q-1} \otimes a^{(q)}_* \Omega^1_{\widetilde{E}_q/S} \otimes_{\mathcal{O}_P} \mathcal{O}_X \to \\ & \cdots \to \mathcal{L} \otimes a^{(q)}_* \Omega^{m-q-1}_{\widetilde{E}_q/S} \otimes_{\mathcal{O}_P} \mathcal{O}_X \to a^{(q)}_* \Omega^{m-q}_{\widetilde{E}_q/S} \otimes_{\mathcal{O}_P} \mathcal{O}_X \to a^{(q)}_* \Omega^{m-q}_{\widetilde{D}_q/S} \to 0, \end{aligned} \quad (1.5)$$

where $\widetilde{E}_q := \coprod_{J \subset I, |J|=q} E_J \xrightarrow{a^{(q)}} P$ is the canonical morphism and $\widetilde{E}_0 := P$. Since $P$ is smooth over $S$, by ([2], Proposition 3.6), we have $\mathrm{Gr}^W_q(\Omega^i_{P/S}(\log E)) \simeq a^{(q)}_* \Omega^{i-q}_{\widetilde{E}_q/S}$ for $i \geq q$. We have an exact sequence

$$\begin{aligned}0 \to\ & \mathcal{L}^{\otimes m-q} \otimes \mathrm{Gr}^W_q(\wedge^q \mathcal{E}) \to \mathcal{L}^{\otimes m-q-1} \otimes \mathrm{Gr}^W_q(\wedge^{q+1}\mathcal{E}) \to \\ & \cdots \to \mathcal{L} \otimes \mathrm{Gr}^W_q(\wedge^{m-1}\mathcal{E}) \to \mathrm{Gr}^W_q(\wedge^m \mathcal{E}) \to a^{(q)}_* \Omega^{m-q}_{\widetilde{D}_q/S} \to 0. \end{aligned} \quad (1.6)$$

Thus the complex $\mathrm{Gr}^W_q(K^\bullet)$ is a resolution of $a^{(q)}_* \Omega^{m-q}_{\widetilde{D}_q/S}$. Now consider the spectral sequence

$$E_1^{p,q} = \mathcal{H}^{p+q}(\mathrm{Gr}^W_{-p}(K^\bullet)) \Rightarrow \mathcal{H}^{p+q}(K^\bullet).$$

Replacing $\mathrm{Gr}^W_{-p}(K^\bullet)$ by $a^{(-p)}_* \Omega^{m+p}_{\widetilde{D}_{-p}/S}$ ($p \leq 0$), the above spectral sequence degenerates at $E_1$-terms and we get that $\mathrm{Gr}^W_q \Omega^m_{X/S}(\log D) \simeq a^{(q)}_* \Omega^{m-q}_{\widetilde{D}_q/S}$.

Since the length function is additive on exact sequence([8], Appendix A, Lemma A.1.1), we have

$$\begin{aligned} \mathrm{length}_{\mathcal{O}_{X,x_0}} \Omega^{n+1}_{X/S}(\log D)_{x_0} &= \sum_{q=0}^{r} \mathrm{length}_{\mathcal{O}_{X,x_0}} \left( \mathrm{Gr}^W_q(\Omega^{n+1}_{X/S}(\log D)) \right)_{x_0} \\ &= \sum_{q=0}^{r} \mathrm{length}_{\mathcal{O}_{X,x_0}} \left( \Omega^{n+1-q}_{\widetilde{D}_q/S} \right)_{x_0} \\ &= \sum_{q=0}^{r} \sum_{J \subset I, |J|=q} \mathrm{length}_{\mathcal{O}_{X,x_0}} \left( \Omega^{n+1-q}_{D_J/S} \right)_{x_0}. \end{aligned} \quad (1.7)$$

Since by the local expression of $T^{\log}_{(X,D)/S}$ (Lemma 1.7), we have

$$\begin{aligned}\mu^{\log}((X,D)/S,x_0) &= \mathrm{length}_{\mathcal{O}_{X,x_0}}\Omega^{n+1}_{X/S}(\log D)_{x_0},\\ \mu(X/S,x_0) &= \mathrm{length}_{\mathcal{O}_{X,x_0}}\left(\Omega^{n+1}_{X/S}\right)_{x_0},\\ \mu(D_J/S,x_0) &= \mathrm{length}_{\mathcal{O}_{X,x_0}}(\Omega^{n+1-q}_{D_J/S})_{x_0} = \mathrm{length}_{\mathcal{O}_{D_J,x_0}}\left(\Omega^{n+1-q}_{D_J/S}\right)_{x_0}.\end{aligned}$$

From formula (1.7), we get that

$$\mu^{\log}((X,D)/S,x_0) = \mu(X/S,x_0) + \sum_{q\geq 1}\sum_{J\subset I,|J|=q}\mu(D_J/S,x_0).$$

□

**1.6. Total number of vanishing cycles.** Under the hypothesis of 1.1, denote by $\overline{s}$ the geometric closed point of $S$. Let $S_{(\overline{s})}$ be the spectrum of strict henselization of $S$ at $\overline{s}$ and let $X_{\overline{s}}$ be the inverse image of $X$ over $S_{(\overline{s})}$. Let $\overline{\eta}$ be the spectrum of an algebraic closure of the fraction field $k(\eta)$ of $S_{(\overline{s})}$, let $I = \mathrm{Gal}(k(\overline{\eta})/k(\eta))$ be the inertia group. Let $\ell$ be a prime number different from the residue characteristic of $S$. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-vector spaces on $U = X - D$ which is tamely ramified along $D$, let $j : U \to X$ be the open immersion. For the complex of vanishing cycles $\mathbf{R}\Phi(j_!\mathcal{F})$ on $X_{\overline{s}}$, we have the following result.

Theorem 1.10 ([4], Lemme 2.1.11 and Théorème 2.4.2). *The cohomology groups of the complex of vanishing cycles $\mathbf{R}\Phi(j_!\mathcal{F})$ are sheaves of $\mathbb{F}_\ell$-vector spaces of finite dimension and are concentrated at $\overline{x}_0$. Moreover the complex of vector spaces $\mathbf{R}\Phi_{\overline{x}_0}(j_!\mathcal{F})$ is invariant by base change of traits, i.e., if $S$ is a trait and $S' \to S$ is a morphism of trait. Let $(X',D',j',\mathcal{F}')$ be the object obtained from $(X,D,j,\mathcal{F})$ by base change $S' \to S$. Then $\mathbf{R}\Phi(j'_!\mathcal{F}')$ is also concentrated at $x_0$ and we have an isomorphism $\mathbf{R}\Phi_{\overline{x}_0}(j_!\mathcal{F}) \simeq \mathbf{R}\Phi_{\overline{x}_0}(j'_!\mathcal{F}')$.*

For any vector space $V$ of finite dimension over $\mathbb{F}_\ell$ with a continuous action of the inertia group $I = \mathrm{Gal}(k(\overline{\eta})/k(\eta))$, we can define the Swan conductor $\mathrm{Sw}\ V$ of $V$ and its total dimension

$$\mathrm{dimtot}\ V := \dim_{\mathbb{F}_\ell} V + \mathrm{Sw}\ V,$$

see [5] and [18] for more details.

Definition 1.11. We define the *total number of vanishing cycles* of $(X,D)/S$ at $\overline{x}_0$ to be the integer

$$w(j_!\mathcal{F},\overline{x}_0) := (-1)^n\mathrm{dimtot}\ \mathbf{R}\Phi_{\overline{x}_0}(j_!\mathcal{F}) := (-1)^n\sum_{i\geq 0}(-1)^i\mathrm{dimtot}\ \mathbf{R}^i\Phi_{\overline{x}_0}(j_!\mathcal{F}),$$

where $n$ is the relative dimension of $X$ over $S$.

Conjecture 1.12. [Logarithmic Milnor Formula] Let $S$ be a regular scheme purely of dimension 1 and $s$ a closed point of $S$ with perfect residue field. Let $(X,D)$ be a simple normal crossing pair and let $f : X \to S$ be a flat morphism of finite type. Let $x_0 \in D \cap X_s$ be a closed point of $X$. Assume that $x_0$ is the unique isolated log-singular point of $f$. Let $\ell$ be a prime number different from the residue characteristic of $S$ at $s$. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-vector spaces on $U = X - D$. Assume that $\mathcal{F}$ is tamely ramified along $D$. Let $j : U \to X$ be the open immersion. Then $w(j_!\mathcal{F}, \overline{x}_0)$ is equal to rank$(\mathcal{F})$ times $\mu^{\log}$, i.e.,

$$w(j_!\mathcal{F}, \overline{x}_0) = \text{rank } \mathcal{F} \cdot \mu^{\log}\left((X, D)/S, x_0\right). \tag{1.8}$$

The classical Milnor formula says that

$$w(\mathbb{F}_\ell, \overline{x}_0) = \mu(X/S, x_0).$$

Notice that, since we assume that the residue field of $S$ at $s$ is perfect, the Swan conductor is defined. The classical Milnor formula is true in the following cases (see [5, 14]):

- $n = 0, 1, 2$
- $X \to S$ has an ordinary quadratic singularity at $x$.
- $S$ is of equal-characteristic.

In [14], the classical Milnor formula is shown to follow from the conjecture of Bloch conductor formula.

Proposition 1.13. *Let $X$ be a regular scheme and $S$ be a regular scheme purely of dimension 1. Let $f : X \to S$ be a flat morphism of finite type such that $f$ has a unique isolated log-singular point $x_0 \in D \cap X_s$ with respect to a simple normal crossing divisor $D$ on $X$. For any subset $J \subset I$, put $D_J = \cap_{j \in J} D_j$. When $J$ is empty, we put $D_J = X$. Assume that $w(\mathbb{F}_{\ell, D_J}, \overline{x}_0) = \mu(D_J/S, x_0)$ for any subset $J \subset I$. Then we have*

$$w(j_!\mathbb{F}_\ell, \overline{x}_0) = \mu^{\log}\left((X, D)/S, x_0\right).$$

*Proof.* Let $f : X \to S$ be a flat morphism of finite type with a unique isolated log-singular point $x_0 \in D \cap X_s$. Let $D_i (i \in I)$ be the irreducible components of $D$. Then we have an exact sequence,

$$0 \to j_!\mathbb{F}_\ell \to \mathbb{F}_{\ell, X} \to \bigoplus_{i \in I} \mathbb{F}_{\ell, D_i} \to \cdots \to \bigoplus_{J \subset I, \text{Card}(J) = q} \mathbb{F}_{\ell, D_J} \to \cdots.$$

By this exact sequence, we have

$$w(j_!\mathbb{F}_\ell, \overline{x}_0) = w(\mathbb{F}_\ell, \overline{x}_0) + \sum_{r \geq 1} \sum_{J \subset I, |J| = r} w(\mathbb{F}_{\ell, D_J}, \overline{x}_0). \tag{1.9}$$

Since by assumption,

$$\mu(X/S, x_0) + \sum_{r \geq 1} \sum_{J \subset I, |J| = r} \mu(D_J/S, x_0) = w(\mathbb{F}_\ell, \overline{x}_0) + \sum_{r \geq 1} \sum_{J \subset I, |J| = r} w(\mathbb{F}_{\ell, D_J}, \overline{x}_0).$$

By Lemma 1.9,

$$\mu^{\log}((X, D)/S, x_0) = \mu(X/S, x_0) + \sum_{r \geq 1} \sum_{J \subset I, |J| = r} \mu(D_J/S, x_0).$$

Combined with equation (1.9), we obtain $w(j_!\mathbb{F}_\ell, \overline{x}_0) = \mu^{\log}((X, D)/S, x_0)$, which finishes the proof. □

**2. Logarithmic version of Elkik's Lemma.** In this section, we formulate and prove a logarithmic version of Elkik's Lemma [7].

LEMMA 2.1. *Let $(A, I)$ be a henselian pair (cf. [15], Chapters XI), $B = A[X_1, \cdots, X_N]/J$ be an $A$-algebra of finite presentation such that $J = (f_1, \cdots, f_q)$ with $f_i \in A[X_1, \cdots, X_N]$. Let $M$ be a non-negative integer such that $M \leq N$. Let $\Delta$ be the ideal of $A[X_1, \cdots, X_N]$ generated by order-q-minors of the logarithmic Jacobian matrix*

$$\mathrm{Jac}^{\log} = \begin{pmatrix} X_1\frac{\partial f_1}{\partial X_1} & \cdots & X_M\frac{\partial f_1}{\partial X_M} & \frac{\partial f_1}{\partial X_{M+1}} & \cdots & \frac{\partial f_1}{\partial X_N} \\ X_1\frac{\partial f_2}{\partial X_1} & \cdots & X_M\frac{\partial f_2}{\partial X_M} & \frac{\partial f_2}{\partial X_{M+1}} & \cdots & \frac{\partial f_2}{\partial X_N} \\ \vdots & & \vdots & \vdots & & \vdots \\ X_1\frac{\partial f_q}{\partial X_1} & \cdots & X_M\frac{\partial f_q}{\partial X_M} & \frac{\partial f_q}{\partial X_{M+1}} & \cdots & \frac{\partial f_q}{\partial X_N} \end{pmatrix}.$$

*For any $\mathbf{x} = (x_1, \cdots, x_N) \in A^N$, we define a homomorphism $\psi_{\mathbf{x}} \colon A[X_1, \cdots, X_N] \to A$ by mapping $X_i$ to $x_i$. For any ideal $\mathcal{E}$ of $A[X_1, \cdots, X_N]$, the value $\mathcal{E}(\mathbf{x})$ of $\mathcal{E}$ at $\mathbf{x}$ is defined to be the image $\psi_{\mathbf{x}}(\mathcal{E})$. It is easy to see that $\mathcal{E}(\mathbf{x})$ is an ideal of $A$.*

*Let $n$ and $h$ be two integers such that $n > 2h$ and $\mathbf{a} = (a_1, \cdots, a_N) \in A^N$ such that*

$$J(\mathbf{a}) \subset I^n \quad \textit{and} \quad I^h \subset \Delta(\mathbf{a}),$$

*where the ideal $J(\mathbf{a}) = (f_1(\mathbf{a}), \cdots, f_q(\mathbf{a})) \subset I$ (resp. $\Delta(\mathbf{a})$) is the value of $J$ (resp. $\Delta$) at $\mathbf{a}$. Then there exists an element $\mathbf{b} = (b_1, \cdots, b_N) \in A^N$ such that the ideal $J(\mathbf{b})$ (value of $J$ at $\mathbf{b}$) is zero and*

$$b_v = a_v(1 + \epsilon_v), \qquad \epsilon_v \equiv 0 \mod I^{n-h}, \ \textit{for } v = 1, 2, \cdots, M \tag{2.1}$$
$$b_s \equiv a_s \mod I^{n-h}, \ \textit{for } s = M+1, M+2, \cdots, N. \tag{2.2}$$

*Proof.* Let $(t_1, \cdots, t_r)$ be a system of generators of $I^h$. Since $t_j \in I^h \subset \Delta(\mathbf{a})$, we can find a $N \times q$-matrix $E_j$ such that

$$\mathrm{Jac}^{\log}(\mathbf{a})E_j = t_j \cdot Id_{q\times q} \quad \text{for all } j = 1, \cdots, r.$$

By the assumption $J(\mathbf{a}) \subset I^n = I^{n-2h} \cdot I^h \cdot I^h$, we can write

$$\begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} = \sum_{i,j} t_i t_j \begin{pmatrix} e_{ij,1} \\ \vdots \\ e_{ij,q} \end{pmatrix} = \sum_i t_i \cdot \mathrm{Jac}^{\log}(\mathbf{a}) \sum_j E_j \begin{pmatrix} e_{ij,1} \\ \vdots \\ e_{ij,q} \end{pmatrix}$$

with $e_{ij,k} \in I^{n-2h}$. For any $i = 1, \cdots, r$, we only need to find $\mathbf{u}^i = (u_1^i, \cdots, u_N^i) \in A^N$ verifying $u_j^i \in I^{n-2h}$ for all $j = 1, \cdots, N$ such that

$$f_k\left(\mathbf{a} + \sum_{i=1}^r t_i \widetilde{\mathbf{u}}^i\right) = 0, \text{ where } \widetilde{\mathbf{u}}^i = (a_1 u_1^i, \cdots, a_M u_M^i, u_{M+1}^i, \cdots, u_N^i), \text{ for all } k = 1, \cdots, q.$$

Then by Taylor expansion, we have

$$\begin{aligned}
\begin{pmatrix} f_1(\mathbf{a}+\sum_{i=1}^r t_i\widetilde{\mathbf{u}}^i) \\ \vdots \\ f_q(\mathbf{a}+\sum_{i=1}^r t_i\widetilde{\mathbf{u}}^i) \end{pmatrix} &= \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} + \sum_{i=1}^r t_i \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_N} \\ \vdots & \ldots & \vdots \\ \frac{\partial f_q}{\partial X_1} & \cdots & \frac{\partial f_q}{\partial X_N} \end{pmatrix} \widetilde{\mathbf{u}}^i + \sum_{i,j} t_i t_j Q_{i,j} \\
&= \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} + \sum_{i=1}^r t_i \cdot \mathrm{Jac}^{\log}(\mathbf{a})\mathbf{u}^i + \sum_{i,j} t_i t_j Q_{i,j} \\
&= \sum_i t_i \cdot \mathrm{Jac}^{\log}(\mathbf{a}) \left\{ \sum_j E_j \begin{pmatrix} e_{ij,1} \\ \vdots \\ e_{ij,q} \end{pmatrix} + \mathbf{u}^i + \sum_j E_j Q_{i,j} \right\}
\end{aligned}$$

where $Q_{i,j}$ is a column-vector whose components are polynomial in $\mathbf{a}$ and $\mathbf{u}^i$ such that the minimal degree in $\mathbf{u}^i$ is at least 2. Therefore we reduce the problem to the following equation (in variables $u^i_j$)

$$\begin{pmatrix} u^i_1 \\ \vdots \\ u^i_N \end{pmatrix} + \sum_j E_j \left( Q_{i,j} + \begin{pmatrix} e_{ij,1} \\ \vdots \\ e_{ij,q} \end{pmatrix} \right) = 0.$$

After modulo $I^{n-2h}$, 0 is a solution of this system of equations, and the jacobian of this system is congruent to the identity matrix. Since $(A, I)$ is a henselian pair, we can lift this solution (mod $I^{n-2h}$) to a real solution which finished the proof. □

LEMMA 2.2. *Let $S = \mathrm{Spec}\ R$ be an affine noetherian scheme. Let $f : X = \mathrm{Spec}\ Q \to S$ be a morphism of finite type. Assume $X$ is regular and $D \subset X$ is a divisor on $X$ with simple normal crossings. Let $Z \subset X$ be a closed subscheme of $X$ defined by an ideal $I \subset Q$. Assume that $Z = \{x\}$ contains only one closed point $x$ of $X$, $X - Z$ is smooth over $S$ and $D - Z \cap D = D \times_X (X - Z)$ is a divisor on $X - Z$ with simple normal crossings relatively to $S$. Let $\widetilde{X}$ be the henselization of $X$ along $Z$ (cf. [15], Chapitres XI).*

*Then there exist integers $h \geq 1$ and $n > 2h$ such that for any morphism $g : X \to S$ satisfying $g \equiv f \mod I^n$, there exists an isomorphism $p : \widetilde{X} \to \widetilde{X}$ inducing an isomorphism on $D \times_X \widetilde{X}$ such that the following diagram is cartesian*

$$\begin{array}{ccc}
D \times_X \widetilde{X} & \xrightarrow{p} & D \times_X \widetilde{X} \\
\downarrow & & \downarrow \\
\widetilde{X} & \xrightarrow{p} & \widetilde{X} \\
{\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\
S & = & S
\end{array} \tag{2.3}$$

*Moreover, $p$ is congruent to the identity modulo $I^{n-h}\mathcal{O}_{\widetilde{X}}$.*

*Proof.* For a scheme $T$ over $X$, let $T_f$ denote $T$ regarded as a scheme over $S$ with respect to the composition with $f : X \to S$ and similarly for $T_g$ for any morphism $g : X \to S$. For an integer $r \geq 1$ and for a scheme $T$ over $X$, let $T_r \subset T$ denote the closed subscheme $T \times_X \mathrm{Spec}\ Q/I^r$.

After replacing $X_f$ by an étale neighborhood of $Z = \{x\}$ , we can choose a presentation of $Q$

$$Q \simeq R[X_1, \cdots, X_N]/J, \quad J = (f_1, \cdots, f_q),$$

such that $D \subset X_f$ is the pullback of the divisor $D' = V(X_1 \cdots X_M) \subset X' = \text{Spec } R[X_1, \cdots, X_N]$ by the map $X \to X'$. Let $\Delta'_f$ be the ideal of $R[X_1, \cdots, X_N]$ defined by the logarithmic Jacobian matrix as in Lemma 2.1 for the canonical morphism $f' : X' \to S$ and the divisor $D'$. Let $\Delta_f$ be the ideal of $Q$ defined by the image of $\Delta'_f$. For any morphism $q : Y \to S$, let $\widetilde{f}$ be the base change $X \times_S Y \to Y$ of $f$ by $q : Y \to S$. Then the ideal $\Delta_{\widetilde{f}}$ defined by $\widetilde{f}$ and the divisor $D \times_S Y$ is equal to the pull back of $\Delta_f$. Since, by assumption, $X_f - Z$ is smooth over $S$ and $D - Z \cap D = D \times_X (X - Z)$ is a divisor on $X_f - Z$ with simple normal crossings relatively to $S$, there exists an integer $h \geq 1$ such that $I^h \subset \Delta_f$. Let $n$ be any integer such that $n > 2h$.

Now, let $g : X \to S$ be a morphism satisfying $g \equiv f \mod I^n$. Let $\widetilde{X} = \text{Spec } A$ and $\widetilde{D} = \widetilde{X} \times_X D$. Let Spec $B$ be the fiber product $\widetilde{X}_g \times_S X_f$ and $D_B$ the fiber product $(\text{Spec } B) \times_{X_f} D$. Since $g \equiv f \mod I^n$, $\widetilde{X}_{r,g} = \widetilde{X}_{r,f} (=: \widetilde{X}_r)$ and $\widetilde{D}_{r,g} = \widetilde{D}_{r,f} (=: \widetilde{D}_r)$ for all $r \leq n$ and we have a cartesian diagram

$$\begin{array}{ccc} \widetilde{D}_n & \longrightarrow & \widetilde{X}_n \\ \downarrow & & \downarrow{\scriptstyle \text{diag}} \\ D_B & \longrightarrow & \text{Spec } B \\ \downarrow & & \downarrow{\scriptstyle \text{pr}_2} \\ D_f & \longrightarrow & X_f, \end{array} \tag{2.4}$$

where $\widetilde{X}_n \xrightarrow{\text{diag}} \text{Spec } B = \widetilde{X}_g \times_S X_f$ is defined by the closed immersion $\widetilde{X}_n \hookrightarrow \widetilde{X}_g$ and the composition $\widetilde{X}_n \hookrightarrow \widetilde{X}_f \to X_f$.

We apply Lemma 2.1 to the henselian pair $(A, IA)$ and $B$. The morphism $\widetilde{X}_n \to \text{Spec } B$ corresponds to an $A$-homomorphism $B \simeq A[X_1, \cdots, X_N]/J \to A/I^n A$. The latter map is defined by an element $\mathbf{a} = (a_1, \cdots, a_N) \in A^N$ such that the condition $J(\mathbf{a}) \subset I^n A$ in Lemma 2.1 is satisfied. Here we still use $J$ to denote the image of $J \in R[X_1, \cdots, X_N]$ in $A[X_1, \cdots, X_N]$. Let $\Delta$ be the ideal of $B$ defined by the morphism $\text{pr}_1 : \text{Spec } B \to \widetilde{X}_g$ and the divisor $D_B$. Notice that since $\text{pr}_1 : \text{Spec } B \to \widetilde{X}_g$ is the base change of $f : X \to S$ by the morphism $\widetilde{X}_g \to S$, $\Delta$ is equal to the pull back of $\Delta_f$. Now the condition $I^h A \subset \Delta(\mathbf{a})$ in Lemma 2.1 is also satisfied by our choice of $h$. By Lemma 2.1, we can find an element $\mathbf{b} \in A^N$ satisfying the congruences (2.1) in Lemma 2.1 such that $J(\mathbf{b}) = \mathbf{0}$, i.e., we have a morphism $\widetilde{X}_g = \text{Spec } A \to \text{Spec } B$ such that the following diagram is commutative

$$\begin{array}{ccccc} \widetilde{X}_{n-h} & \hookrightarrow & \widetilde{X}_g & \hookleftarrow & \widetilde{D}_g \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{X}_n & \xrightarrow{\text{diag}} & \text{Spec } B & \hookleftarrow & D_B \end{array} \tag{2.5}$$

The right square in (2.5) is cartesian by the congruences (2.1) and $1+\epsilon_r \in 1+IA \subset A^\times$ for all $r$. Let $p'$ be the composition $\widetilde{X}_g \to \text{Spec} B \to X_f$. Since $(A, IA)$ is a henselian pair, the morphism $p' : \widetilde{X}_g \to X_f$ factors through a morphism $p : \widetilde{X}_g \to \widetilde{X}_f$ such that the following diagram commutes

$$\begin{array}{ccccc} \widetilde{D}_g & \hookrightarrow & \widetilde{X}_g & & \\ \downarrow & & \downarrow{\scriptstyle p} & \searrow^{p'} & \\ \widetilde{D}_f & \hookrightarrow & \widetilde{X}_f & \longrightarrow & X_f. \end{array} \tag{2.6}$$

Now we show that $p : \widetilde{X}_g \to \widetilde{X}_f$ is an isomorphism. Let $W$ be an etale neighborhood of $\widetilde{X}_1 \to X_g$ such that the morphism $p'$ is induced by $W \to X_f$. Since $p : \widetilde{X}_g \to \widetilde{X}_f$ induces the identity on $\widetilde{X}_2 \subset \widetilde{X}_{n-h}$, the endomorphisms induced by $W \to X_f$ on the completions of the local rings of $\widetilde{X}_g$ and $\widetilde{X}_f$ at all points of $\widetilde{X}_1$ are surjections and hence automorphisms. Thus the endomorphisms induced by $W \to X_f$ on the henselizations of the local rings of $\widetilde{X}_g$ and $\widetilde{X}_f$ at all points of $\widetilde{X}_1$ are also automorphisms and the morphism $W \to X_f$ is etale on a neighborhood $W' \subset \widetilde{X}_g$ of $\widetilde{X}_1$. Hence, by replacing $W$ by $W'$, we may assume that $W \to X_f$ itself is etale. Since $(\widetilde{X}, \widetilde{X}_1)$ is a henselian pair, the morphism $p : \widetilde{X}_g \to \widetilde{X}_f$ is an isomorphism. □

## 3. Compactification and Pencils of desired Jet.

THEOREM 3.1. *Let $S$ be a smooth curve over an algebraic closed field $k$ of characteristic $p > 0$. Let $\ell$ be a prime number distinct from $p$. Let $(X, D)$ be a simple normal crossing pair. Let $f : X \to S$ be a flat morphism of finite type. Assume that $f$ is purely of relative dimension $n$ and has a unique isolated log-singular point $x \in D$. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-vector spaces on $X \setminus D$ such that $\mathcal{F}$ is tamely ramified along $D$. Then there exist a projective variety $Y$ over $k$, purely of dimension $n+1$, a morphism $g : Y \to \mathbb{P}^1_k$ , a divisor $D_Y \subset Y$ and a closed point $y \in D_Y$ of $Y$ such that*

1. *$g(y) = 0 \in \mathbb{P}^1_k$ is the origin.*
2. *there exists an open neighborhood $U \subset \mathbb{P}^1_k$ of the origin $0 \in \mathbb{P}^1_k$ such that the base change of $g$ by $U$ gives a morphism with a unique isolated log-singular point, i.e., the morphism $g : Y \times_{\mathbb{P}^1_k} U \to U$ has a unique isolated log-singular point $y$ with respect to the divisor $D_Y \times_{\mathbb{P}^1_k} U$.*
3. *If $h_1 : \mathrm{Spec} k[[t]] \to \mathbb{P}^1_k$ (resp $h_2 : \mathrm{Spec} k[[t]] \to S$) is the canonical isomorphism between $\mathrm{Spec} k[[t]]$ and the completion of the strict henselization of $\mathbb{P}^1_k$ (resp. $S$) at $0$ (resp. $s = f(x)$), there exists an $\mathrm{Spec} k[[t]]$-isomorphism $h$ between the strict henselization of $h_2^* X$ at $x$ and the strict henselization of $h_1^* Y$ at $y$ such that $D$ is the pullback of $D_Y$ under this isomorphism.*
4. *there exists a locally constant and constructible sheaf $\mathcal{G}$ of $\mathbb{F}_\ell$-vector spaces on $Y \setminus D_Y$ such that $\mathcal{G}$ is tamely ramified along $D_Y$ and $h$ induces an isomorphism between the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to the strict henselizations.*

In order to prove this theorem, we briefly recall the definition of dual varieties and pencils. For more details, see [13]. Denote by $\mathbb{P}^N$ the projective space of dimension $N$ over $k$, let $\check{\mathbb{P}}^N$ be the dual projective space of $\mathbb{P}^N$. The points in $\check{\mathbb{P}}^N$ are considered as hyperplanes in $\mathbb{P}^N$. Consider the Grassmann variety $\mathrm{Gr}(1, \check{\mathbb{P}}^N)$ whose points are considered as lines in $\check{\mathbb{P}}^N$. A point $L \in \mathrm{Gr}(1, \check{\mathbb{P}}^N)$ is called a hyperplane pencil in $\mathbb{P}^N$. Considering $L$ as a line in $\check{\mathbb{P}}^N$, we write $L = \{H_t\}_{t \in L}$ where $H_t$ is the hyperplane of $\mathbb{P}^N$ defined by $t \in L$.

Let $P$ be a proper smooth and irreducible closed subscheme of $\mathbb{P}^N$, let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_{\mathbb{P}^N}$ which defines $P$ in $\mathbb{P}^N$. Then the conormal sheaf $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$ is locally free on $P$. Consider the projective bundle $\mathbb{P}(\mathcal{N})$ over $P$ formed by lines in the dual $\check{\mathcal{N}}$. The bundle $\mathbb{P}(\mathcal{N})$ can be identified with the sub-variety of $P \times \check{\mathbb{P}}^N$ formed by pairs $(x, H)$ such that the hyperplane $H$ is tangent to $P$ at $x$. Let $\phi : \mathbb{P}(\mathcal{N}) \to \check{\mathbb{P}}^N$ be the composition $\mathbb{P}(\mathcal{N}) \hookrightarrow P \times \check{\mathbb{P}}^N \xrightarrow{\mathrm{pr}_2} \check{\mathbb{P}}^N$. Then the dual variety $\check{P}$ of $P$ is defined to be the image of $\phi$. The dual variety $\check{P}$ is the set of hyperplanes which do not meet $P$ transversally.

Let $Q \subset \mathbb{P}^N$ be a smooth quasi-projective variety over $k$. Recall that we say another irreducible smooth subscheme $Z \subset \mathbb{P}^N$ meets $Q$ transversally, if the scheme-theoretic intersection $Q \cdot Z = Q \times_{\mathbb{P}^N} Z$ is smooth and of pure codimension $\mathrm{codim}_{\mathbb{P}^N} Z$ in $Q$. Then the Bertini theorem says that for infinite field $k$, there exists an $k$-rational hyperplane $H \subset \mathbb{P}^N$ which meets $Q$ transversally. As pointed out by a referee, the Bertini theorem is an immediate consequence of the fact that the dual variety of $Q \subset \mathbb{P}^N$ is not dense in $\check{\mathbb{P}}^N$.

*Proof of Theorem 3.1.* Put $m = n + 1$. By taking an etale morphism $S \to \mathbb{P}^1_k$ on a neighborhood of $s = f(x)$, we may assume that $S = \mathbb{P}^1_k$ and $s = f(x) = (1 : 0)$ is the origin of $S$. Since $X$ is smooth over a perfect field $k$ and $D$ is a divisor with simple normal crossings, by shrinking $X$, we may assume there is an etale map $e : X \to X' := \mathbb{A}^m_k = \mathrm{Spec} k[T_1, \cdots, T_m]$ such that $D$ is the pull-back of the divisor $D'$ defined by $T_1 \cdots T_r$. Moreover, we may assume $x' := e(x) = (0)$ is the origin of $X' = \mathbb{A}^m_k$. By ([9], Expose XIII, Corollaire 5.3), we have an isomorphism between tame fundamental groups $\pi_1^t(X_{(x)} \times_X (X \setminus D)) \simeq \prod_{\eta \neq p} \mathbb{Z}_\eta^r \simeq \pi_1^t(X' \setminus D')$ which is compatible with numbering of the irreducible components of $D$ and $D'$. By the isomorphism, we find a locally constant constructible sheaf $\mathcal{F}'$ of $\mathbb{F}_\ell$-vector spaces on $X' \setminus D'$ such that $\mathcal{F}'$ is tamely ramified along $D'$ and $\mathcal{F}|_{X_{(x)}} \simeq \mathcal{F}'|_{X_{(x)}}$.

Let $P = \mathbb{P}^m$ be the projective space of dimension $m$ with coordinates $(T_0 : T_1 : \cdots : T_m)$, let $X' \hookrightarrow P$ be the open immersion given by $(x_1, \cdots, x_m) \mapsto (1 : x_1 : \cdots : x_m)$. Let $\overline{D}$ be the closure of $D'$ in $P$ with irreducible components $\overline{D}_i (i = 1, \cdots, r)$. For any subset $I \subset \{1, \cdots, r\}$, put $\overline{D}_I = \cap_{i \in I} \overline{D}_i$ and $\overline{D}_\emptyset = P$. Let $y = (1 : 0 : \cdots : 0)$ be the origin of $P$. Consider a Segre imbedding of degree $d$ with $d$ large enough:

$$P \hookrightarrow E := \mathbb{P}^N,$$

where $N = \binom{m+d}{d} - 1$. In the following, $d$ will be taken such that the map $J_z$ below (cf. (3.1)) is surjective for all $z \in P \setminus \{y\}$. We need to find a hyperplane pencil $L = (H_t)_{t \in L} \in \mathrm{Gr}(1, \check{E})$ such that

(a) The axis of $L$ does not contain $y$ and meets $P$ transversally. Let $t_0 \in L$ be the image of $y$, then $H_{t_0}$ meets $\overline{D}_I \setminus \{y\}$ transversally for all subset $I$ of $\{1, \cdots, r\}$.

(b) There exists an open neighborhood $U \subset L$ of $t_0$ such that for all $t \in U \setminus \{t_0\}$, the hyperplane $H_t$ meets $\overline{D}_I$ transversally for all subset $I$ of $\{1, \cdots, r\}$.

(c) The line $L$ in $\check{E}$ is defined by two sections $F, G \in \Gamma(P, \mathcal{O}_P(d))$ (viewed as linear forms in $\Gamma(E, \mathcal{O}_E(1))$) such that $F(y) \neq 0$ and $e^*(\frac{G}{F}) \equiv f \mod \mathfrak{m}^\lambda$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,x}$ and $\lambda$ is an integer given in the following way: By Lemma 2.2, there exists an integer $N'$ such that if $\lambda \geq N'$, the condition $(c)$ implies the condition (3). Now we choose any integer $\lambda$ such that $\lambda \geq N'$.

Let $\mathbb{H} = \{(x, H) | x \in H\} \subset E \times \check{E}$ be the universal hyperplane. If such $L$ exists, let $Y' = P \times_E \mathbb{H} \subset P \times \check{E}$ be the incidence subscheme and put $Y = Y' \times_{\check{E}} L$ and let $D_Y \subset Y$ be the inverse image of $\overline{D}$ by the first projection $\mathrm{pr}_1 : Y = Y' \times_{\check{E}} L \to P$. Choose an isomorphism $L \simeq \mathbb{P}^1$ such that $t_0$ maps to the origin point of $\mathbb{P}^1$. Put $g := \mathrm{pr}_2 : Y \to L \simeq \mathbb{P}^1$. By the choice of $\lambda$, the condition $(c)$ implies the condition (3) above. Let $\mathcal{G}$ be the pullback of $\mathcal{F}'$ to $Y \setminus D_Y$, then $\mathcal{G}$ satisfies the condition (4). Since $(b)$ implies the condition (2), the morphism $g : Y \to \mathbb{P}^1$ will satisfy our requirement.

Now we first choose a linear form $F \in \Gamma(E, \mathcal{O}_E(1))$ such that the hyperplane $H_F \in \check{E}$ defined by $F$ does not meet the origin $y$ and this hyperplane meets $\overline{D}_I$ transversally for all subset $I$ of $\{1, \cdots, r\}$. We can apply Bertini theorem to find such $F$. If one can choose a linear form $G$ satisfying conditions $(a)$ and $(c)$ above, we let $H_G \in \check{E}$ be the hyperplane of $E$ define by $G$. Then the line $L \subset \check{E}$ through the points $H_F$ and $H_G$ will satisfying our requirement. Indeed by assumption on $G$, the hyperplane $H_G$ meets $\overline{D}_I \setminus \{y\}$ transversally for all subset $I$ of $\{1, \cdots, r\}$. Hence the condition $(a)$ is satisfied if we put $t_0 \in L$ corresponds to $H_G$. Because $H_F$ meets $\overline{D}_I$ transversally for all subset $I$ of $\{1, \cdots, r\}$, the line $L$ is not contained in the union of dual varieties $\cup_I \check{\overline{D}}_I$. Since $\cup_I \check{\overline{D}}_I$ is of codimension 1 in $\check{E}$, there exists an open neighborhood $U \subset L$ of $t_0$ in $L$ such that the condition $(b)$ is satisfied.

We show that such $G$ exists when $d$ is large enough. Let $\Gamma := \Gamma(P, \mathcal{O}_P(d))$ and $\mathcal{L} = \mathcal{O}_{\overline{D}_I}(d)$ where $I$ is a subset of $\{1, \cdots, r\}$. When the set $I$ is empty, we put $\overline{D}_I = P$. For any point $z \in \overline{D}_I \setminus \{y\}$, consider the following map

$$\begin{aligned} J_z : \Gamma &\to \mathcal{O}_{P,y}/\mathfrak{m}_y^\lambda \times \mathcal{L}_z/\mathfrak{m}_z^2\mathcal{L}_z. \\ G &\mapsto (G/F \mod \mathfrak{m}_y^\lambda, \;\; G \mod \mathfrak{m}_z^2\mathcal{L}_z) \end{aligned} \tag{3.1}$$

By ([17], Lemma 3.3(2)), there exists an integer $M \geq 0$ such that for any $d \geq M$ and any integer $\lambda \geq N'$, the canonical map

$$\begin{aligned} \Gamma &\to \mathcal{O}_{P,y}(d)/\mathfrak{m}_y^\lambda \times \mathcal{O}_{P,x}(d)/\mathfrak{m}_z^2 \\ G &\mapsto (G \mod \mathfrak{m}_y^\lambda, \;\; G \mod \mathfrak{m}_z^2) \end{aligned}$$

is surjective for all $z \in P \setminus \{y\}$. Now we choose an integer $d \geq M$ and an integer $\lambda \geq N'$. This implies that the map $J_z$ is also surjective for all $z \in P \setminus \{y\}$. Put $\Gamma' = \{G \in \Gamma : G/F \equiv f \mod \mathfrak{m}_y^\lambda\}$ and for any $z \in \overline{D}_I \setminus \{y\}$ put

$$\Gamma'_I(z) = \{G \in \Gamma' : G \equiv 0 \mod \mathfrak{m}_z^2\mathcal{L}_z \otimes \mathcal{O}_{\overline{D}_I, z}\}.$$

Then for any $G \in \Gamma'_I(z)$, the hyperplane $H_G$ defined by $G$ satisfies that $z$ is a singular point of $H_G \cap \overline{D}_I$. Since $J_z$ is surjective, $\Gamma'_I(z)$ is of codimension $\dim_k \mathcal{L}_z/\mathfrak{m}_z^2\mathcal{L}_z = 1 + \dim \overline{D}_I$ in $\Gamma'$, hence the union $\Gamma'_I := \bigcup_{z \in \overline{D}_I \setminus \{y\}} \Gamma'_I(z)$ is of codimension $(1 + \dim \overline{D}_I) - \dim \overline{D}_I = 1$ in $\Gamma'$. Moreover the union $\bigcup_I \Gamma'_I$ is of codimension 1 in $\Gamma'$. For any $G_1 \in \Gamma' \setminus \bigcup_I \Gamma'_I$, $H_{G_1} \cap \overline{D}_I$ is smooth outside $y$ for all subset $I$.

For any $z \in P \cap H_F$ ($y \notin P \cap H_F$), put

$$\Gamma'_F(z) = \left\{G \in \Gamma' : G \equiv 0 \mod \mathfrak{m}_z^2\mathcal{L}_z \otimes \mathcal{O}_{P \cap H_F, z}\right\}.$$

Then the union $\Gamma'_F := \bigcup_{z \in P \cap H_F} \Gamma'_F(z)$ is of codimension 1 in $\Gamma'$. For any $G_2 \in \Gamma' \setminus \Gamma'_F$, let $L \subset \check{E}$ be the line through the points $H_F$ and $H_{G_2}$, then the axis $A_L$ of the line $L$ meets $P$ transversally.

Now we choose any $G \in \Gamma' \setminus \left\{\bigcup_I \Gamma'_I \bigcup \Gamma'_F\right\}$. Let $L \subset \check{E}$ be the line through the points $H_F$ and $H_G$. Then $H_G \cap \overline{D}_I$ is smooth outside $y$ for all subset $I$ and the axis $A_L$ of $L$ meets $P$ transversally. Hence the pencil $L$ will satisfy our conditions $(a)$, $(b)$ and $(c)$. We finished the proof. □

## 4. Main theorem: geometric case.

THEOREM 4.1. *Let $S$ be a henselian trait of equal characteristic $p > 0$ and perfect residue field , let $s$ be the closed point of $S$. Let $(X, D)$ be a simple normal crossing pair. Let $f : X \to S$ be a flat morphism of finite type and purely of relative dimension $n$. Assume that a closed point $x \in D$ is the unique isolated log-singular point of $f$ and $f(x) = s$. Let $\ell \neq p$ be a prime number. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-vector spaces on $U = X \setminus D$ such that $\mathcal{F}$ is tamely ramified along $D$, let $j : U \to X$ be the open immersion. Then $w(j_!\mathcal{F})$ is equal to* rank $\mathcal{F}$ *times $\mu^{\log}$,i.e.,*

$$w(j_!\mathcal{F}, \overline{x}) = \text{rank } \mathcal{F} \cdot \mu^{\log}((X, D)/S, x).$$

*Proof.* Let $S'$ be the strict hensenlian of $S$, and put $X' = X \times_S S'$ and $D', \mathcal{F}'$ be the pullback of $D$ and $\mathcal{F}$ respectively. By Theorem 1.10, the assertion 4.1 for $(X, D) \to S$ and $\mathcal{F}$ is equivalent to 4.1 for $(X', D') \to S'$ and $\mathcal{F}'$. Hence we may assume the residue field of $S$ is separably closed. But the residue field is perfect by assumption, hence the residue field of $S$ is algebraically closed.

For such $S$, we let $S'$ be the completion of $S$. By the same reason as above, we may assume $S$ is also complete, hence $S$ is isomorphic to Spec $k[[t]]$ with $k$ algebraically closed.

By Theorem 3.1, there exists a projective and flat $S$-scheme $Y$ with a simple normal crossing divisor $E$ such that

1. $g : Y \to S$ is a morphism purely of relative dimension $n$ such that $y \in E \cap Y_s$ is the unique isolated log-singular point of $g$ with respect to the divisor $E$.
2. There exists an $S$-isomorphism between strict henselizations $X_{(x)}$ and $Y_{(y)}$ such that the following diagram

$$\begin{array}{ccc} D \times_X X_{(x)} & \xrightarrow{\cong} & E \times_Y Y_{(y)} \\ \downarrow & & \downarrow \\ X_{(x)} & \xrightarrow{\cong} & Y_{(y)} \end{array}$$

   is cartesian.
3. There exists a locally constant and constructible sheaf $\mathcal{F}'$ of $\mathbb{F}_\ell$-vector spaces on $Y \setminus E$ such that $\mathcal{F}'$ is tamely ramified along $E$ and the pullback of $\mathcal{F}'$ by the isomorphism $X_{(x)} \cong Y_{(y)}$ is equal to $\mathcal{F}|_{X_{(x)}}$.

Let $j' : Y \setminus E \to Y$ be the open immersion. Then we have

$$\text{dimtot } \mathbf{R}\Phi_{\overline{x}}(j_!\mathcal{F}, f) = \text{dimtot } \mathbf{R}\Phi_{\overline{y}}(j'_!\mathcal{F}', g).$$

Let $\overline{\eta}$ be an algebraic closure of the fraction field of $S$. Now $y \in Y$ is the unique isolated non smooth point of $(Y, j'_!\mathcal{F}')$. Hence by ([4], 2.4.6.3), we have the following exact sequence

$$\cdots \to (\mathbf{R}^q g_* j'_!\mathcal{F}')_s \to (\mathbf{R}^q g_* j'_!\mathcal{F}')_{\overline{\eta}} \to \mathbf{R}^q\Phi_{\overline{y}}(j'_!\mathcal{F}', g) \to \cdots.$$

Denote the restriction $Y \setminus E \to S$ also by $g$, then by definition, $Rg_!\mathcal{F}' = Rg_*j'_!\mathcal{F}'$. The exact sequence above shows that

$$\text{dimtot } \mathbf{R}\Phi_{\overline{y}}(j'_!\mathcal{F}', g) = \text{dimtot } (\mathbf{R}g_!\mathcal{F}')_{\overline{\eta}}.$$

Since $\mathcal{F}'$ and $\mathbb{F}_\ell^{\mathrm{rank}\mathcal{F}}$ have the same rank and both are tamely ramified along $E$, by Vidal's result ([20], Corollaire 3.4), we have

$$\mathrm{dimtot}\ (\mathbf{R}g_!\mathcal{F}')_{\overline{\eta}} = \mathrm{rank}\ \mathcal{F} \cdot \mathrm{dimtot}\ (\mathbf{R}g_!\mathbb{F}_\ell)_{\overline{\eta}}.$$

Hence we have

$$\begin{aligned}\mathrm{dimtot}\ \mathbf{R}\Phi_{\overline{x}}(j_!\mathcal{F}, f) &= \mathrm{dimtot}\ (\mathbf{R}g_!\mathcal{F}')_{\overline{\eta}} \\ &= \mathrm{rank}\ \mathcal{F} \cdot \mathrm{dimtot}\ (\mathbf{R}g_!\mathbb{F}_\ell)_{\overline{\eta}} \\ &= \mathrm{rank}\ \mathcal{F} \cdot \mathrm{dimtot}\ \mathbf{R}\Phi(j'_!\mathbb{F}_\ell, g)_{\overline{y}} \\ &= \mathrm{rank}\ \mathcal{F} \cdot \mathrm{dimtot}\ \mathbf{R}\Phi_{\overline{x}}(j_!\mathbb{F}_\ell, f).\end{aligned}$$

That is to say $w(j_!\mathcal{F}, \overline{x}) = \mathrm{rank}\mathcal{F} \cdot w(j_!\mathbb{F}_\ell, \overline{x})$. Since the geometric case of Milnor formula is true [5], by Proposition 1.13, we have

$$w(j_!\mathbb{F}_\ell, \overline{x}) = \mu^{\log}((X, D)/S, x).$$

Finally we get that

$$w(j_!\mathcal{F}, \overline{x}) = \mathrm{rank}\ \mathcal{F} \cdot \mu^{\log}((X, D)/S, x).$$

□

Now we give an interpretation of the main result in terms of characteristic cycle. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $S$ be a smooth curve over $k$ and $(X, D)$ a simple normal crossing pair over $k$. Let $f : X \to S$ be a flat morphism of finite type such that the closed point $x_0 \in D$ is the unique isolated log-singular point of $f$. Let $T^*S$ (resp. $T^*X$) be the cotangent bundle of $S$ (resp. $X$). We have an induced morphism $T^*S \times_S X \to T^*X$ on vector bundles. We choose a local coordinate $t$ of $S$ on an open neighborhood $S'$ of $s = f(x_0)$. When replacing $S$ (resp. $X$) by $S'$ (resp. $X' = X \times_S S'$), the values on both sides of (1.8) do not change. We may therefore assume that the local coordinate $t$ is defined on $S$. Let $S \to T^*S$ be the section of $T^*S \to S$ defined by $dt$. By base change, we obtain a section $dt\colon X \to T^*S \times_S X$. Let $df\colon X \to T^*S \times_S X \to T^*X$ be the composition of the section $dt\colon X \to T^*S \times_S X$ with $T^*S \times_S X \to T^*X$.

For a regular immersion $X \to P$ of schemes, the conormal bundle $T^*_X P$ is the vector bundle over $X$ defined by the symmetric algebra $S^\bullet(\mathcal{N}_{X/P})^\vee$ where $\mathcal{N}_{X/P} = \mathcal{I}_X/\mathcal{I}_X^2$ is the conformal sheaf and $\mathcal{I}_X \subset \mathcal{O}_P$ is the ideal sheaf of $X$ in $P$. Let $D_1, \cdots, D_r$ be the irreducible components of $D$. For any subset $I \subset \{1, \cdots, r\}$, let $D_I = \cap_{i \in I} D_i$ with $D_\emptyset = X$. We define $T^*_{D_I}X \subset T^*X$ to be the conormal bundle associated to the regular immersion $D_I \hookrightarrow X$.

Let $\ell$ be a prime number distinct from $p$. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-modules on $U = X \setminus D$. Assume that $\mathcal{F}$ is tamely ramified along $D$. Denote by $j : U \to X$ the open immersion. The characteristic cycle $\mathrm{Char}(j_!\mathcal{F})$ is defined by (see [17])

$$\mathrm{Char}(j_!\mathcal{F}) = (-1)^m \cdot \sum_{I \subset \{1, \cdots, r\}} \mathrm{rank}\mathcal{F} \cdot [T^*_{D_I}X],$$

where $m = \dim(X)$.

Corollary 4.2. *Under the conditions above, we have*

$$-\mathrm{dimtot}\ \mathbf{R}\Phi_{\overline{x}_0}(j_!\mathcal{F}) = (\mathrm{Char}(j_!\mathcal{F}), [df(X)])_{T^*X, x_0}.$$

*Proof.* By Theorem 4.1, we only need to show that

$$\mu^{\log}((X,D)/S,x_0)=\left(\sum_{I\subset\{1,\cdots,r\}}[T^*_{D_I}X],\quad[df(X)]\right)_{T^*X,x_0}.$$

However, by Lemma 1.9, we have

$$\mu^{\log}((X,D)/S,x_0)=\sum_{J\subset\{1,\cdots,r\}}\mu(D_J/S,x_0).$$

Hence we only need to show that for any subset $I\subset\{1,\cdots,r\}$

$$\mu(D_I/S,x_0)=\big([T^*_{D_I}X],[df(X)]\big)_{T^*X,x_0}.$$

If $I$ is empty, we show that $\mu(X/S,x_0)=([T^*_XX],[df(X)])_{T^*X,x_0}$. Let $(x_i)_{1\le i\le m}$ be a system of local coordinates of $X$ at $x_0$. Then $\Omega^1_{X/k}$ has a basis $\{dx_i\}$. We also choose a local coordinate $t:S\to\mathbb{A}^1_k$ at $s=f(x_0)$. Denote still by $f$ the composition $t\circ f$. Let $\mathcal{I}=\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_m}\right)\subset\mathcal{O}_{T^*X}=\mathcal{O}_X\left[\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_m}\right]$ be the ideal sheaf defining the zero section $X=T^*_XX\to T^*X$, let $\mathcal{J}=\left(\frac{\partial f}{\partial x_1}-\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1},\cdots,\frac{\partial f}{\partial x_m}-\frac{\partial f}{\partial x_m}\frac{\partial}{\partial x_m}\right)\subset\mathcal{O}_{T^*X}$ be the ideal sheaf defining the closed immersion $df(X)\to T^*X$. Since $\dim T^*_XX\cap df(X)=\dim\mathcal{O}_X/(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_m})=0$, we have $\dim T^*_XX+\dim df(X)=\dim T^*X+\dim T^*_XX\cap df(X)$. Hence, the closed subschemes $T^*_XX$ and $df(X)$ intersect properly and we can use Serre's tor-formula to calculate $([T^*_XX],[df(X)])_{T^*X,x_0}$. Since $X=T^*_XX\to T^*X$ is a regular immersion and $\mathcal{O}_{T^*X}/\mathcal{J}$ is Cohen-Macaulay at every point of $T^*_XX\cap df(X)$, by ([19], Chapter V, Theorem 4), we have $\mathcal{T}or_i^{\mathcal{O}_{T^*X}}(\mathcal{O}_{T^*X}/\mathcal{I},\mathcal{O}_{T^*X}/\mathcal{J})=0$ for all $i>0$. Thus we have

$$\begin{aligned}\sum_{i\ge0}(-1)^i\left[\mathcal{T}or_i^{\mathcal{O}_{T^*X}}(\mathcal{O}_{T^*X}/\mathcal{I},\mathcal{O}_{T^*X}/\mathcal{J})\right]&=[\mathcal{O}_{T^*X}/\mathcal{I}\otimes_{\mathcal{O}_{T^*X}}\mathcal{O}_{T^*X}/\mathcal{J}]\\&=\left[\mathcal{O}_X\Big/\left(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_m}\right)\right].\end{aligned}$$

Hence $([T^*_XX],[df(X)])_{T^*X,x_0}=\mathrm{length}\left(\mathcal{O}_{X,x_0}\Big/\left(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_m}\right)\right)=\mu(X/S,x_0)$.

When the subset $I\subset\{1,\cdots,r\}$ is not empty. By the following two cartesian diagram,

$$\begin{array}{ccccc} & & D_I & \longrightarrow & X\\ & & \downarrow & & \downarrow{\scriptstyle df}\\ T^*_{D_I}X & \longrightarrow & T^*X\times_X D_I & \longrightarrow & T^*X\\ \downarrow & & \downarrow & & \\ D_I & \longrightarrow & T^*D_I & & \end{array}$$

using projection formula, we have

$$\big([T^*_{D_I}X],[df(X)]\big)_{T^*X,x_0}=\big([T^*_{D_I}D_I],[df(D_I)]\big)_{T^*D_I,x_0}.$$

By the result we proved in the case $I = \emptyset$, we get that

$$\big([T^*_{D_I}X], [df(X)]\big)_{T^*X,x_0} = \mu(D_I/S, x_0).$$

$\square$

## REFERENCES


[1] A. Abbes, *Cycles on arithmetic surfaces*, Compositio Math., 122 (2000), pp. 23–111.

[2] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lectures Notes in mathematics, 163, Springer-Verlag, 1970.

[3] P. Deligne, *Théorie de Hodge II*, Publications Mathématiques de IHES, 40 (1971), pp. 5–58.

[4] P. Deligne, *Le formalisme des cycles évanescents*, Groupes de Monodromie en Géométrie Algébrique, Lecture Notes in Mathematics, (1973), pp. 82–115.

[5] P. Deligne, *La formule de Milnor*, Groupes de Monodromie en Géométrie Algébrique, Lecture Notes in Mathematics, (1973), pp. 197–211.

[6] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Grad. Texts Math. 150, Springer, New York-Heidelberg-Berlin, 1994.

[7] R. Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, Annales Scientifiques de l'École Normale Supérieure, 6:4 (1973), pp. 553–603.

[8] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, second edition, 1998.

[9] A. Grothendieck and M. Raynaud, *Revêtements Étales et Groupe Fondamental*, Lecture Notes in Mathematics, 224, Springer-Verlag (1971).

[10] K. Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory, (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD (1989), pp. 191–224.

[11] K. Kato and T. Saito, *On the conductor formula of Bloch*, Publications Mathématiques de l'IHÉS, 100 (2004), pp. 5–151.

[12] K. Kato and T. Saito, *Ramification theory for varieties over a local field*, Publications Mathématiques de l'IHÉS, 117 (2010), pp. 1–178.

[13] N. Katz, *Pinceaux de Lefschetz: théorème d'existence*, Groupes de Monodromie en Géométrie Algébrique, Lecture Notes in Mathematics, (1973), pp. 212–253.

[14] F. Orgogozo, *Conjecture de Bloch et nombres de Milnor*, Annales de l'Institut Fourier 53, 6 (2003), pp. 1739–1754.

[15] M. Raynaud, *Anneaux Locaux Henséliens*, Lecture Notes in mathematics, 169, Springer-Verlag, (1970).

[16] T. Saito, *Parity in Bloch's conductor formula in even dimension*, Journal de théorie des nombres de Bordeaux, (2004), pp. 403–421.

[17] T. Saito, *Characteristic cycle and the Euler number of a constructible sheaf on a surface*, Kodaira Centennial issue of the Journal of Mathematical Sciences, the University of Tokyo, 22 (2015), pp. 387-442.

[18] J.-P. Serre, *Corps Locaux*, Hermann, Paris, Deuxieme edition, 1968.

[19] J.-P. Serre, *Local Algebra*, Springer Monographs in Mathematics, 2000.

[20] I. Vidal, *Théorie de Brauer et conducteur de Swan*, J. Algebraic Geom., 13 (2004), pp. 349–391.