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GAUSS-MANIN CONNECTION IN DISGUISE: NOETHER-LEFSCHETZ AND HODGE LOCI*

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Abstract. We give a classification of components of the Hodge locus in any parameter space of smooth projective varieties. This is done using determinantal varieties constructed from the infinitesimal variation of Hodge structures (IVHS) of the underlying family. As a corollary we prove that the minimum codimension for the components of the Hodge locus in the parameter space of m-dimensional hypersurfaces of degree d with $d \geq 2 + \frac{4}{\mathsf{m}}$ and in a Zariski neighborhood of the point representing the Fermat variety, is obtained by the locus of hypersurfaces passing through an $\frac{\mathsf{m}}{2}$-dimensional linear projective space. In the particular case of surfaces in the projective space of dimension three, this is a theorem of Green and Voisin. In this case our classification under a computational hypothesis on IVHS implies a weaker version of the Harris-Voisin conjecture which says that the set of special components of the Noether-Lefschetz locus is not Zariski dense in the parameter space.

Key words. Gauss-Manin connection, Hodge locus, Holomorphic foliations, Hodge filtration, Griffiths transversality, infinitesimal variation of Hodge structures, Kodaira-Spencer map.

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1. Introduction. The Hodge conjecture implies that the components of the Hodge locus in a parameter space of smooth projective varieties are algebraic, and moreover, they are defined over the algebraic closure of the base field. The algebraicity statement has been successfully proved by Cattani, Deligne and Kaplan in [CDK95] using transcendental methods in Hodge theory, and hence, their proof does not give any light into the second part of the above statement. The classification of the components of the Hodge locus according to their codimension is another important challenge in Hodge theory. Here, the Hodge conjecture has not so much to say. A typical evidence to this is the particular case of Noether-Lefschetz locus, where the Hodge conjecture is known as Lefschetz $(1,1)$ theorem. In this case it was observed that there are two classes of components, general and special ones. Ciliberto, Harris and Miranda in [CHM88] proved that general components are dense in the parameter space in both usual and Zariski topology. Harris conjectured that special components must be finite and Voisin found counterexamples to this, see [Voi89, Voi90, Voi91]. She then conjectured that special components are not Zariski dense. We refer to this as the Harris-Voisin conjecture.

In the present article, we first put the Harris-Voisin conjecture in the general framework of Hodge loci and then we give a conjectural description of a proper algebraic subset of the parameter space which contains components of the Hodge locus in a wide range of codimensions. Partial verifications of our conjecture in the case of hypersurfaces give us the precise description of the component which acquires the minimum codimension. Our results are stated using the infinitesimal variation of Hodge structures (IVHS) invented by Carlson, Donagi, Griffiths, Green and Harris in order to avoid the transcendental nature of the variation of Hodge structures, see [CGGH83]. The mentioned authors used IVHS in single points to attack classical problems such as Torelli problem and Noether's Theorem.

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For the proof of our main results we have to go back to the origin of IVHS which is the algebraic Gauss-Manin connection of the corresponding family. From this we construct a holomorphic foliation in a larger parameter space whose leaves and singularities are responsible for the classification of the components of the Hodge locus. Despite the fact that the concept of a foliation/integrable distribution is quit old in differential geometry, their applications to classical problems in algebraic geometry and Hodge theory, in the way we do, must be considered as our main contribution to the literature. The construction of such foliations goes back to the works of the author on differential equations of modular forms and their generalization to Calabi-Yau varieties, see [Mov11, Mov13] and the references therein. For many examples such foliations are given by vector fields which are natural generalizations of Darboux, Halphen and Ramanujan vector fields. By Gauss-Manin connection in disguise we mean such vector fields and foliations. The terminology arose from a private letter of Pierre Deligne to the author [Del09]. For a fast review of the results in Hodge locus the reader is referred to Voisin's expository article [Voi13].

1.1. Main results. Let $Y \to V$ be a family of smooth complex projective varieties and let V be irreducible, smooth and affine. For an even number m, an irreducible component H of the Hodge locus $\mathrm{Ho}_{\mathsf{m}}(Y/V)$ is any irreducible closed subvariety of V with a continuous family of Hodge classes $\delta_t \in H^{\mathsf{m}}(Y_t,\mathbb{Q}) \cap H^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}}$ in varieties $Y_t, t \in H$ such that for points t in a Zariski open subset of H, the monodromy of δ_t to a point in a neighborhood (in the classical topology of V) of t and outside H, is no more a Hodge class. Let

$$H^1(Y_t, T_{Y_t})_\theta \times H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t}) \to H^{\mathsf{m}-k+1}(Y_t, \Omega^{k-1}_{Y_t}) \tag{1}$$

be the k-th infinitesimal variation of Hodge structures at t, IVHS for short, with $k = \frac{\mathsf{m}}{2}+1$ (we will not need the intersection form of IVHS except in §3.10). Here, $H^1(Y_t, T_{Y_t})_\theta \subset H^1(Y_t, T_{Y_t})$ corresponds to projective deformations of Y_t, see §3.1. From (1) we derive

$$H^{\mathsf{m}-k+1}(Y_t, \Omega^{k-1}_{Y_t})^* \to \mathrm{Hom}\left(H^1(Y_t, T_{Y_t})_\theta, H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t})^*\right) \tag{2}$$

where $*$ means dual and we have removed the zero element from both sides of (2). Let $D_{s,t},\ s \in \mathbb{N}_0$ be the determinantal subvariety of the right hand side of (2) consisting of homomorphisms of rank $\leq s$. For simplicity, throughout the text we assume that the Kodaira-Spencer map

$$(T_V)_t \to H_1(Y_t, T_{Y_t})_\theta \tag{3}$$

is surjective, however, all the arguments are valid when we replace $H_1(Y_t, T_{Y_t})_\theta$ with the image of (3).

Theorem 1. *If the image of* (2) *with* $k = \frac{\mathsf{m}}{2}+1$ *does not intersect* $D_{s,t}$ *for some* s *and* $t \in V$ *then there is a Zariski open neighborhood* U *of* t *in* V *such that all the components of the Hodge locus* $\mathrm{Ho}_{\mathsf{m}}(Y/V)$ *intersecting* U *have codimension* $\geq s+1$.

The above theorem is a direct consequence of Voisin's results on the Zariski tangent space of the components of the Hodge locus, see [Voi03] Lamma 5.16, p. 146. For a proof see §3.10. Stated in this format it becomes directly related to a weaker version of Harris-Voisin's conjecture, and it seems to me that its power and importance has been neglected in the literature, see Theorem 2 and Theorem 3 below. In this article we give a new proof of Theorem 1 which is based on the construction of a larger parameter space T, a modular foliation in T and the notion of Hodge locus with constant periods in T. Our proof is purely algebraic, whereas Voisin's proof is based on local analytic study of the Hodge locus. This might open up a new point of view to the Cattani-Deligne-Kaplan theorem discussed at the beginning of the present paper. An advantage of our proof is that it says which part of the Gauss-Manin connection is absent in IVHS and it is needed for a full solution of the Harris-Voisin conjecture. Another advantage is that it gives a precise description of the polynomial equations for the periods of Hodge classes, see for instance the end of §3.4.

Theorem 1 for $s = 0$ is the classical Noether's theorem. It says that if (2) is injective (or equivalently if the map (1) is surjective) then the components of the Hodge locus in V are proper analytic subsets of V. Since we know that the set of such components is enumerable, we conclude that for a generic Y_t, the m-dimensional Hodge classes are complete intersection of Y_t with another variety in the ambient projective space and hence are algebraic, see [CGGH83] page 71 and [Har85] page 56. From now on we assume that (2) is injective and so we can take the induced map after projectivization. In the case of hypersurfaces, IVHS can be computed explicitly and a simple analysis of the hypothesis of Theorem 1 for the Fermat variety gives us:

Theorem 2. *Let V be the parameter space of smooth hypersurfaces of degree d in $\mathbb{P}^{\mathsf{m}+1}$ and let $0 \in V$ be the parameter of the Fermat variety. Assume that $d \geq 2 + \frac{4}{\mathsf{m}}$. There is a Zariski open neighborhood U of $0 \in V$ such that all the components of the Hodge locus $\mathrm{Ho}_{\mathsf{m}}(Y/V)$ intersecting U have codimension $\geq \binom{\frac{\mathsf{m}}{2}+d}{d} - (\frac{\mathsf{m}}{2}+1)^2$. The lower bound is obtained by the locus H of hypersurfaces containing a linear projective space $\mathbb{P}^{\frac{\mathsf{m}}{2}} \subset \mathbb{P}^{\mathsf{m}+1}$.*

We usually call $0 \in V$ the Fermat point. The above theorem for $\mathsf{m} = 2$ and $U = V$ was conjectured in [CGGH83], I. It was independently proved by Green in [Gre88, Gre89] and Voisin in [Voi91]. In this case it is also proved that H is the only component of codimension $d-3$. Note that for $\mathsf{m} \geq 4$ the Hodge conjecture is not known and Theorem 2 is independent of this. We may analyze IVHS for other single points in the parameter space of hypersurfaces or complete intersections and get further results similar to Theorem 2. One can also take the bundle of IVHS in (2) and try to compute its first order approximation in single points. We do this around the Fermat point and we get the following. Let

$$I_N := \left\{(i_0, i_1, \ldots, i_{\mathsf{m}+1}) \in \mathbb{Z}^{\mathsf{m}+2} \mid 0 \leq i_e \leq d-2, \quad i_0 + i_1 + \cdots + i_{\mathsf{m}+1} = N\right\} \tag{4}$$

for $N = 0, 1, 2, \ldots, (d-2)(\mathsf{m}+2)$ and consider independent variables x_i indexed by $i \in I_{(\frac{\mathsf{m}}{2}+1)d-\mathsf{m}-2}$. For any other i which is not in the set $I_{(\frac{\mathsf{m}}{2}+1)d-\mathsf{m}-2}$, x_i by definition is zero. Let $M := [x_{i+j}]$ be a matrix whose rows and columns are indexed by $i \in I_{\frac{\mathsf{m}}{2}d-\mathsf{m}-2}$ and $j \in I_d$, respectively, and in its (i,j) entry we have x_{i+j}. The matrix M is obtained by IVHS for the Fermat point. For $j, \alpha \in I_d$ and $i \in I_{\frac{\mathsf{m}}{2}d-\mathsf{m}-2}$ such that for a unique $0 \leq \check{e} \leq \mathsf{m}+1$ we have $i_{\check{e}} + j_{\check{e}} \geq d-1$, let us define

$$i +_\alpha j = i + j + \alpha - (0, \cdots, 0, \overbrace{d}^{\check{e}\text{-th place}}, 0, \cdots, 0).$$

For other pairs of (i,j) let $i +_\alpha j = 0 \in \mathbb{Z}^{\mathsf{m}+2}$ (it can be any element outside $I_{(\frac{\mathsf{m}}{2}+1)d-\mathsf{m}-2}$). We define the matrix $\check{M}_\alpha$ in the following way. For (i,j) as above and in the first case, the (i,j) entry of $\check{M}_\alpha$ is $\alpha_{\check{e}} \cdot x_{i+_\alpha j}$, and elsewhere entries are zero. The matrix $N_{j,\alpha}$ is obtained by replacing the j-th column of M with the j-th column of $\check{M}_\alpha$. We define the homogeneous ideal $\mathcal{I}_s^1 \subset \mathbb{C}[x], \quad s = 0, 1, 2, \dots$ to be generated by

$$\text{minor}_{s+1}(M), \tag{5}$$

$$\sum_{j \in I_d} \text{minor}_{s+1}(N_{j,\alpha}), \alpha \in I_d, \tag{6}$$

where "minor" runs through all minors of $(s+1) \times (s+1)$ submatrices of a matrix. Note that once we fix a block of $(s+1) \times (s+1)$ matrix for making a determinant, it is the same for all the matrices $N_{j,\alpha}$ in the sum (6). Let also $\mathcal{I}_s^0 \subset \mathbb{C}[x]$ be the homogeneous ideal generated by (5). This is the ideal of $(s+1) \times (s+1)$ minors of M. We define $s^i_{\max}, \quad i = 0, 1$ to be the maximum s such that $\text{Zero}(\mathcal{I}_s^i) = \{0\}$. The proof of Theorem 1 is reduced to check the equality

$$s^0_{\max} = \binom{\frac{\mathsf{m}}{2} + d}{d} - (\frac{\mathsf{m}}{2} + 1)^2 - 1, \tag{7}$$

that is, if for some x_i's the rank of M is $\leq s^0_{\max}$ then all x_i's are zero, see Proposition 7.

THEOREM 3. *There is a Zariski open subset U of the parameter space V of smooth hypersurfaces in $\mathbb{P}^{\mathsf{m}+1}$ such that all the components of the Hodge locus $\text{Ho}_{\mathsf{m}}(Y/V)$ intersecting U have codimension $\geq s^1_{\max} + 1$.*

We were not able to compute $s^1_{\max}$ neither by hand nor by computer. Some methods using both theoretical and computational aspects of ideals and their Gröbner basis seems to be necessary for computing $s^1_{\max}$.

1.2. Harris-Voisin conjecture. In this section we assume that (2) is injective and hence, Noether's theorem is valid for $Y \to V$. Let

$$\begin{aligned} a &:= \dim H^{\frac{\mathsf{m}}{2}-1}(Y_t, \Omega_{Y_t}^{\frac{\mathsf{m}}{2}+1}) = \# I_{\frac{\mathsf{m}}{2}d - \mathsf{m} - 2}, \\ b &:= \dim H^{\frac{\mathsf{m}}{2}}(Y_t, \Omega_{Y_t}^{\frac{\mathsf{m}}{2}}) = \# I_{(\frac{\mathsf{m}}{2}+1)(d-2)}, \\ r &:= \dim H^1(Y_t, T_{Y_t})_\theta = \# I_d, \\ c &:= \text{ the maximum rank of the image of (2) for generic } t \end{aligned} \tag{8}$$

where I_N is defined in (4). The first equalities/definitions are for a general smooth projective variety Y_t and the second equalities are for smooth hypersurfaces of degree d in $\mathbb{P}^{\mathsf{m}+1}$. By a theorem of Voisin (see Proposition 5.14 [Voi03]) the codimension of the components of $\text{Ho}_{\mathsf{m}}(Y/V)$ is $\leq a$. The main challenge in front of us is to find the maximum value of s for a fixed or a generic $t \in V$ such that the hypothesis of Theorem 1 is true. The vector spaces in (2) are fibers of algebraic bundles over V. Let

$$\mathcal{H}^{b,*} \to \text{Hom}\,(\mathcal{H}^r, \mathcal{H}^{a,*}) \tag{9}$$

be the bundle homomorphism obtained from (2). We denote by $W_{c-s} \subset \mathcal{H}^{b,*}$ the determinantal variety of homomorphisms of rank $\leq s$ of (9) (the reason for this index notation is explained in §2.3). It is the pull-back of the determinantal variety D_s in $\text{Hom}\,(\mathcal{H}^r, \mathcal{H}^{a,*})$ by the map (9). Two important numbers in our study are

$s_{\max} :=$ the maximum s such that the projection $W_{c-s} \to V$ is not dominant,

$$\check{s}_{\max} := a - \left\lceil \sqrt{\left(\frac{r-a}{2}\right)^2 + b} \;-\; \frac{r-a}{2} \right\rceil, \tag{10}$$

where for $x \in \mathbb{R}$ we define $\lceil x \rceil$ the unique integer with $\lceil x \rceil - 1 < x \leq \lceil x \rceil$. We have

$$s^0_{\max} \leq s^1_{\max} \leq s_{\max} \leq \check{s}_{\max}. \tag{11}$$

Conjecture 1. *If the Kodaira-Spencer map* (3) *is surjective then for a generic $t \in V$, the map* (2) *is transversal to the determinantal variety $D_{\check{s}_{\max},t}$ of homomorphisms of rank $\leq \check{s}_{\max}$ (and hence does not intersect it).*

If this conjecture is true then $s_{\max} = \check{s}_{\max}$. In order to explain the content of Conjecture 1 we consider the following case. Let V be the parameter space of smooth complex surfaces of degree d in $\mathbb{P}^3$ and let Y/V be the corresponding family. For $t \in V$ let $f_t = f_t(X_0, X_1, X_2, X_3)$ be the corresponding homogeneous polynomial and $Y_t := \mathbb{P}\{f_t = 0\}$. The map in (1) for $\mathsf{m} = k = 2$ is identified by the multiplication of polynomials:

$$(\mathbb{C}[X]/J)_d \times (\mathbb{C}[X]/J)_{d-4} \to (\mathbb{C}[X]/J)_{2d-4}, \tag{12}$$

where $J := \mathrm{jacob}(f_t)$ is the Jacobian ideal of f_t. In this case the Hodge locus $\mathrm{NL}_d := \mathrm{Ho}_2(Y/V)$ is known as the Noether-Lefschetz locus and we have a good understanding of it, see §3.7 for a review of some results. The components of NL_d have codimension $\leq \binom{d-1}{3}$ and Ciliberto, Harris and Miranda in [CHM88] have constructed infinite number of components of codimension a whose union is dense in V with both usual and Zariski topology. A component of this codimension is called a general component and all others are called special. Joe Harris conjectured that there must be finitely many special components and Voisin gave counterexamples, see [Voi89, Voi90, Voi91]. She then formulated the conjecture below:

Conjecture 2. *(Harris-Voisin) The union of all special components of the Noether-Lefschetz locus is not Zariski dense in V.*

We have:

Corollary 1. *If Conjecture 1 is true for* (12) *with $d \geq 4$ then there is a Zariski open subset U of V such that all components of the Noether-Lefschetz locus intersecting U have codimension bigger than or equal $s_{\max} + 1$, where*

$$s_{\max} = \frac{(d-1)(d-2)(d-3)}{6} - \left\lceil \sqrt{d^4 + \frac{2}{3}d^3 - 16d^2 + \frac{7}{3}d + 48} - (d^2 - 7) \right\rceil. \tag{13}$$

Let y_0 be the quantity inside $\lceil \cdot \rceil$ in (13). For any small real number ϵ we have

$$\frac{1}{3}d - \frac{19}{18} \leq y_0 \leq \frac{1}{3}d - \frac{19}{18} + \epsilon$$

where the left hand side equality is for all $d \geq 4$ and the right hand side equality is for big d depending on ϵ. Therefore, once Conjecture 1 is verified we get a good approximation to the Harris-Voisin conjecture. The reason why IVHS cannot do better than this will be explained during the proof of Theorem 1 and §4.

Organization of the text. The organization of the article and the main ideas behind the proof of our main theorems are as follows. For a proof Theorem 1 using Voisin's results, the reader can go directly to §3.10. Our proof of Theorem 1 is based on the construction of a foliation $\mathcal{F}$ on the variety $\mathsf{T} := V \times \mathbb{A}_{\mathbb{C}}^{\dim(F^{\frac{\mathsf{m}}{2}})}$, where $F^{\frac{\mathsf{m}}{2}}$ is the $\frac{\mathsf{m}}{2}$-th piece of the Hodge filtration of the m-th cohomology bundle of Y/V. Surprisingly, this is not interpreted as the total space of the bundle $F^{\frac{\mathsf{m}}{2}}$ which has been useful in the works [CDK95, Voi13]. In §2 we define the variety T and we construct the foliation $\mathcal{F}$. In T we define the Hodge locus with constant periods and we show that it projects into the classical Hodge locus in V. It turns out that the components of the Hodge locus with constant periods are leaves of $\mathcal{F}$. We discuss the leaves and singularities of $\mathcal{F}$ in §2.4 and §2.5, respectively. In §3 we prove our main results announced in the Introduction. This is based on a precise description of the contribution of IVHS in the algebraic expression of $\mathcal{F}$. The case of Noether-Lefschetz locus is explained in §3.7. Finally, in §4 we discuss some problems and conjectures which might be useful for future works.

Notations. We explain some of our notations. We denote by $\mathsf{h}^{ij},\ i+j=\mathsf{m}$ the Hodge numbers of the m-th cohomology of Y_t. The dimensions of the pieces of the Hodge filtration of Y_t are denoted by

$$\mathsf{h}^i := \mathsf{h}^{\mathsf{m},0} + \mathsf{h}^{\mathsf{m}-1,1} + \cdots + \mathsf{h}^{i,\mathsf{m}-i}. \tag{14}$$

For a $\mathsf{h}^0 \times \mathsf{h}^0$ matrix M we denote by $M^{ij},\ i,j=0,1,2,\ldots,\mathsf{m}$ the $\mathsf{h}^{\mathsf{m}-i,i} \times \mathsf{h}^{\mathsf{m}-j,j}$ submatrix of M corresponding to the decomposition $\mathsf{h}^0 := \mathsf{h}^{\mathsf{m},0} + \mathsf{h}^{\mathsf{m}-1,1} + \cdots + \mathsf{h}^{0,\mathsf{m}}$. We call $M^{ij},\ i,j=0,1,2\ldots,\mathsf{m}$ the (i,j)-th Hodge block of M. In a similar way, for a $\mathsf{h}^0 \times 1$ matrix M we write $M=[M^i]$, where $M^i,\ i=0,1,2,\ldots,\mathsf{m}$ is the $\mathsf{h}^{\mathsf{m}-i,i} \times 1$ submatrix of M corresponding to the decomposition of h^0 into Hodge numbers. For any property "P" of matrices we say that the property "block P" or "Hodge block P" is valid if the property P is valid with respect to the Hodge blocks. For instance, we say that a $\mathsf{h}^0 \times \mathsf{h}^0$ matrix M is block upper triangular if $M^{ij}=0,\ \ i>j$. We always write a basis of a free R-module of rank h^0 or a vector space as a $\mathsf{h}^0 \times 1$ matrix. Note that in (8) we have already used the notation $a=\mathsf{h}^{\frac{\mathsf{m}}{2}+1,\frac{\mathsf{m}}{2}-1}$ and $b=\mathsf{h}^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}}$.

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2. Modular foliations. The theory of modular foliations in the sense that we use it here, was introduced in [Mov11]. In this section we give the necessary definitions in order to handle a particular modular foliation constructed from a projective family.

2.1. Adding new parameters. Around any point of V we can find global sections ω of the m-th relative de Rham cohomology sheaf of Y/V such that ω at each fiber $H^*_{\mathrm{dR}}(Y_t),\ t\in V$ form a basis compatible with the Hodge filtration. If it is necessary we may replace V with a Zariski open subset of V. We take variables

$x_1, x_2, \ldots, x_{\mathsf{h}^{\frac{\mathsf{m}}{2}}}$ and put them in a $\mathsf{h}^0 \times 1$ matrix x as below. The first $\frac{\mathsf{m}}{2}$ Hodge blocks are zero and x_i's are listed in the next blocks:

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^{\frac{\mathsf{m}}{2}} \\ \vdots \\ x^{\mathsf{m}} \end{pmatrix}. \tag{15}$$

We take C a non-zero evaluation of the matrix x by some constants and call it a period vector. For instance, take C a vector with zero entries except for the entry corresponding to x_1, which is one. Let S be any Hodge block lower triangular $\mathsf{h}^0 \times \mathsf{h}^0$ matrix depending on x such that

$$\mathsf{S} \cdot \mathsf{C} = x \tag{16}$$

and define

$$O := \operatorname{Spec}\left(\mathbb{C}\left[x_1, x_2, \ldots, x_{\mathsf{h}^{\frac{\mathsf{m}}{2}}}, \frac{1}{\det(\mathsf{S})}\right]\right).$$

We can take the matrix S the one obtained from the identity matrix by replacing the $\mathsf{h}^{\frac{\mathsf{m}}{2}+1} + 1$ column with x in order to get the equality $\mathsf{SC} = x$. In this way S^{-1} is obtained from S by replacing x_1 with x_1^{-1} and $x_i,\ i \geq 2$ with $-x_i x_1^{-1}$. We consider the family $\mathsf{X} \to \mathsf{T}$, where $\mathsf{X} := Y \times O,\ \mathsf{T} := V \times O$. It is obtained from $Y \to V$ and the identity map $O \to O$. We also define α by

$$\alpha := \mathsf{S}^{-1} \cdot \omega. \tag{17}$$

Let $\nabla : H^{\mathsf{m}}_{\mathrm{dR}}(Y/V) \to \Omega_V \otimes_{\mathcal{O}_V} H^{\mathsf{m}}_{\mathrm{dR}}(Y/V)$ be the algebraic Gauss-Manin connection (see [KO68]). We can write ∇ in the basis ω and define the $\mathsf{h}^0 \times \mathsf{h}^0$ matrix B by the equality:

$$\nabla\omega = \mathsf{B} \otimes \omega.$$

The entries of B are differential 1-forms in V. In a similar way we can compute the Gauss-Manin connection of X/T in the basis α:

$$\nabla\alpha = \mathsf{A} \otimes \alpha,$$

where

$$\mathsf{A} = -\mathsf{S}^{-1} d\mathsf{S} + \mathsf{S}^{-1} \cdot \mathsf{B} \cdot \mathsf{S}. \tag{18}$$

This follows from the construction of the global sections α in (17) and the Leibniz rule. We call B (resp. A) the Gauss-Manin connection matrix of the pair $(Y/V, \omega)$ (resp. $(\mathsf{X}/\mathsf{T}, \alpha)$). From the integrability of the Gauss-Manin connection it follows that

$$d\mathsf{A} + \mathsf{A} \wedge \mathsf{A} = 0. \tag{19}$$

2.2. Modular foliations. Let T be an algebraic variety and A be a $\mathsf{h}^0 \times \mathsf{h}^0$ matrix whose entries are differential 1-forms in T and it satisfies (19). For any $\mathsf{h}^0 \times 1$ matrix C, the entries of AC induce a holomorphic foliation $\mathcal{F}$ in T. The integrability of the distribution given by the kernel of the entries of AC follows from (19):

$$d(\mathsf{A}\cdot\mathsf{C}) = -d\mathsf{A}\cdot\mathsf{C} = \mathsf{A}\wedge(\mathsf{A}\cdot\mathsf{C}).$$

Let $\mathcal{F}$ be a foliation given by a finite collection of differential 1-forms $\alpha_i,\ \ i=1,2,\ldots,a$ in T. Its codimension c is the dimension of the vector space generated by α_i's over the functions field of T. Its singular set is defined to be

$$\mathrm{Sing}(\mathcal{F}) := \{t\in\mathsf{T} \mid \alpha_{i_1}\wedge\alpha_{i_2}\wedge\cdots\wedge\alpha_{i_c}=0,\ \ \forall i_1,i_2,\ldots,i_c=1,2,\ldots,a\}.$$

The singular set $\mathrm{Sing}(\mathcal{F})$ is a proper algebraic subset of T. An analytic irreducible (not necessarily closed) subset L of T is tangent to $\mathcal{F}$ if it is tangent to the kernel of α_i's. It is called a (local) leaf of $\mathcal{F}$ if it is tangent to $\mathcal{F}$ and it is not a proper analytic subset of some $\tilde{L}$ tangent to $\mathcal{F}$. All the leaves of the holomorphic foliation $\mathcal{F}$ in $\mathsf{T}\backslash\mathrm{Sing}(\mathcal{F})$ have the same codimension c and we call them general leaves. We call the others special leaves. In the literature by a leaf one mainly means a general leaf.

Now, consider the Gauss-Manin connection matrix A constructed in §2.1. Let

$$\delta_t\in H^{\mathsf{m}}(\mathsf{X}_t,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C},\ \ t\in(\mathsf{T},t_0)$$

be a continuous family of cycles, that is, δ_t is a flat section of the Gauss-Manin connection: $\nabla\delta_t=0$. Here, (T,t_0) is a small neighborhood of t_0 in T in the usual topology. Let us define

$$L_{\delta_t} := \{\quad t\in(\mathsf{T},t_0)\quad \mid \quad \langle\alpha,\delta_t\rangle=\mathsf{C}\}, \tag{20}$$

where

$$\langle\cdot,\cdot\rangle : H^{\mathsf{m}}_{\mathrm{dR}}(\mathsf{X}_t)\times H^{\mathsf{m}}_{\mathrm{dR}}(\mathsf{X}_t)\to\mathbb{C},\ \ (\beta_1,\beta_2)\mapsto\frac{1}{(2\pi i)^{\mathsf{n}}}\int_{\mathsf{X}_t}\beta_1\cup\beta_2\cup\theta^{\mathsf{n}-\mathsf{m}} \tag{21}$$

and $\theta\in H^2_{\mathrm{dR}}(\mathsf{X}_t)$ is the element obtained by polarization. We have the holomorphic function

$$f:(\mathsf{T},t_0)\to\mathbb{C}^{\mathsf{h}^0},\ \ f(t):=\langle\alpha,\delta_t\rangle-\mathsf{C}$$

which satisfies

$$df=\langle\nabla\alpha,\delta_t\rangle=\mathsf{A}\cdot\langle\alpha,\delta_t\rangle=\mathsf{A}\cdot\mathsf{C}+\mathsf{A}\cdot f. \tag{22}$$

This implies that $\mathsf{A}\cdot\mathsf{C}$ restricted to L_{δ_t}'s is identically zero. More precisely, the local leaves of $\mathcal{F}$ are given by L_{δ_t}'s. Recall the constant period vector C defined in §2.1.

Definition 1. The Hodge locus with constant periods C is defined to be the union of all L_{δ_t} in (20) with $\delta_t\in H^{\mathsf{m}}(\mathsf{X_t},\mathbb{Q})$.

By definition any component of the Hodge locus with constant periods is either inside $\mathrm{Sing}(\mathcal{F})$ or it is a general leaf of $\mathcal{F}$. From the zero blocks of C, it follows that $\delta_t\in H^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}}$ and so δ_t is a Hodge class.

2.3. The algebraic description of modular foliations. We note that the foliation $\mathcal{F}$ in T is given by

$$0 = \mathsf{B}^{\frac{\mathsf{m}}{2}-1,\frac{\mathsf{m}}{2}} x^{\frac{\mathsf{m}}{2}} \tag{23}$$

$$dx^{\frac{\mathsf{m}}{2}} = \mathsf{B}^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}} x^{\frac{\mathsf{m}}{2}} + \mathsf{B}^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}+1} x^{\frac{\mathsf{m}}{2}+1}, \tag{24}$$

$$dx^i = \sum_{j=\frac{\mathsf{m}}{2}}^{\mathsf{m}} \mathsf{B}^{i,j} x^j, \qquad i = \frac{\mathsf{m}}{2}+1,\dots,\mathsf{m}. \tag{25}$$

For this we use (18) and we conclude that $\mathcal{F}$ is given by $(-\mathsf{S}^{-1}d\mathsf{S}+\mathsf{S}^{-1}\cdot\mathsf{B}\cdot\mathsf{S})\mathsf{C}$. Since C is a constant vector and S is an invertible matrix and we have (16), we conclude that $\mathcal{F}$ is given by the entries of $dx-\mathsf{B}x=0$. Opening this equality and using the zero blocks of x in (16) we get (23), (24) and (25). Note that by Griffiths transversality $\mathsf{B}^{i,j}=0$ for $j-i\geq 2$. Let

$$\alpha := \mathsf{B}^{\frac{\mathsf{m}}{2}-1,\frac{\mathsf{m}}{2}} \cdot x^{\frac{\mathsf{m}}{2}}. \tag{26}$$

We will not use more α defined in §2.1. Note that $x^{\frac{\mathsf{m}}{2}}$ is a $b\times 1$ matrix with unknown entries $x_i,\ \ i=1,2,\dots,b$. We consider the entries of α as differential forms in $V\times\mathbb{A}^b_{\mathbb{C}}$. Let c be the dimension of the vector space generated by the entries of α and over the functions field of $V\times\mathbb{A}^b_{\mathbb{C}}$. We define the algebraic set $W_y,\ \ y=0,1,\dots,c$ to be the Zariski closure of

$$\big\{(t,x)\in V\times\mathbb{A}^b_{\mathbb{C}} \mid x\neq 0,\ \ \alpha_{i_1}\wedge\alpha_{i_2}\wedge\cdots\wedge\alpha_{i_{c-y+1}}=0, \\ \forall i_1,i_2,\dots,i_{c-y+1}=1,2,\dots,a\big\}. \tag{27}$$

We have inclusions of algebraic varieties

$$\emptyset = W_{c+1}\subset W_c\subset\cdots\subset W_1\subset W_0 = V\times\mathbb{A}^b_{\mathbb{C}}.$$

The set W_y in (27) does not depend on the variables in $x^i,\ i=\frac{\mathsf{m}}{2}+1,\dots,\mathsf{m}$ and so we define

$$\check{W}_y := W_y\times\mathbb{A}_{\mathbb{C}}^{\mathsf{h}^{\frac{\mathsf{m}}{2}+1}}. \tag{28}$$

The affine variety T is a Zariski open subset of $V\times\mathbb{A}_{\mathbb{C}}^{\mathsf{h}^{\frac{\mathsf{m}}{2}}}$ given by $x_1\neq 0$. From now on we redefine T to be $V\times\mathbb{A}_{\mathbb{C}}^{\mathsf{h}^{\frac{\mathsf{m}}{2}}}$. We have the foliation $\mathcal{F}$ in T given by the differential forms (23),(25) and (24).

2.4. Singularities of modular foliations.

PROPOSITION 1. *The set of singularities of the foliation $\mathcal{F}$ is given by $\check{W}_1\cup\tilde{W}$, where $\tilde{W}\subset\mathsf{T}$ is given by $x^{\frac{\mathsf{m}}{2}}=0$.*

Proof. This follows from the explicit form (23), (24) and (25) and the fact that all the entries of x^i's in (24) and (25) are independent variables. ◻

We would like to understand the geometric meaning of the singular set $\tilde{W}$. Recall the definition of L_{δ_t}'s in (20).

PROPOSITION 2. *There is no component of the Hodge locus with constant periods inside $\tilde{W}$.*

Proof. Using the equalities (17) and (16), we know that a Hodge locus with constant periods is given by $L_{\delta_t}:\langle\omega,\delta_t\rangle = x,\ \delta_t\in H^{\mathsf{m}}(\mathsf{X}_t,\mathbb{Q})$. The first $\frac{\mathsf{m}}{2}$ Hodge blocks of x are already zero and if $L_{\delta_t} \subset \tilde{W}$ then the next $(\frac{\mathsf{m}}{2}+1)$-th Hodge block is also zero. If $\delta_t \in H^{\mathsf{m}}(\mathsf{X}_t, \mathbb{R})$ then we have

$$\langle \omega^i, \delta_t \rangle = \langle \overline{\omega^i}, \delta_t \rangle = 0, \quad i = 0, 1, \ldots, \frac{\mathsf{m}}{2},$$

where $\overline{\omega}$ is the complex conjugation of the differential forms in the entries of ω. Since the entries of $\overline{\omega^i}$ and $\omega^i, i = 0, 1, \ldots, \frac{\mathsf{m}}{2}$ generate the m-th complex de Rham cohomology of each fiber X_t, $t \in \mathsf{T}$, we conclude that $\delta_t = 0$. Therefore, for $L_{\delta_t} \subset \tilde{W}$, the cycle δ_t has not real coefficients. Note that a Hodge locus with constant periods is defined using cohomology classes with rational coefficients. □

2.5. Leaves of modular foliations. Let $\tilde{r} := \dim V$. For simplicity, the reader can take an r-dimensional subvariety of V such that the Kodaira-Spencer map over its points is an isomorphism and so, follow the arguments with $\tilde{r} = r$.

PROPOSITION 3. *Any component of the analytic set L_{δ_t} which intersects $\mathsf{T} - \check{W}_{y+1}$ has dimension $\leq \tilde{r} - c + y$.*

Proof. Let t be a point in $\mathsf{T} - \check{W}_{y+1}$. We have $t \in \check{W}_k \backslash \check{W}_{k+1}$ for some k in the set $\{0, 1, \ldots, y\}$. By definition of $\check{W}_y$'s, the dimension of the $\mathbb{C}$-vector space A spanned by the differential forms (23) ,(24), (25) is exactly $\mathsf{h}^{\frac{\mathsf{m}}{2}} + c - k$, and so the kernel of such differential forms is of dimension $\tilde{r} - c + k$. Note that $\dim(\mathsf{T}) = \mathsf{h}^{\frac{\mathsf{m}}{2}} + \tilde{r}$. □

Recall the construction of T in §2.1 and let $\mathsf{T} \to V$ be the projection on V.

PROPOSITION 4. *Any component H of the Hodge locus in V is the projection under the map $\mathsf{T} \to V$ of a component L of the Hodge locus with constant periods in T. Moreover,* $\dim(H) = \dim(L)$.

Proof. Let $\delta_{t_0} \in H^{\mathsf{m}}(\mathsf{X}_{t_0}, \mathbb{Q}) \cap H^{\frac{\mathsf{m}}{2}, \frac{\mathsf{m}}{2}}$ be a Hodge class. This is equivalent to say that it is in the set

$$H := \{t \in (\mathsf{T}, t_0) \quad | \quad \langle \omega^i, \delta_t \rangle = 0, \quad i = 0, 1, \cdots, \frac{\mathsf{m}}{2} - 1\}$$

which is the Hodge locus passing through t_0. The Hodge locus L in T with the constant periods C is given by the set of $(t, x) \in \mathsf{T}$ such that

$$\langle \alpha, \delta_t \rangle = \mathsf{S}^{-1} \langle \omega, \delta_t \rangle = \mathsf{C},$$

or equivalently

$$\langle \omega, \delta_t \rangle = x, \tag{29}$$

where we have used (16). The first $\frac{\mathsf{m}}{2}$ Hodge blocks of this equality are just the equalities in the definition of H and hence t must lie in the Hodge locus H. For others, the entries of the left hand side of (29) are independent variables and the entries of the right hand side are holomorphic functions in H (from Deligne-Cattani-Kaplan theorem in [CDK95] it follows that they are actually algebraic functions in H). This implies that these equalities do not produce further constrains on t and so the proposition is proved. □

Note that if H is a component of the Hodge locus in V then Proposition 3 and Proposition 4 imply that the codimension of H in V is less than or equal to $a = \mathsf{h}^{\frac{\mathsf{m}}{2}-1, \frac{\mathsf{m}}{2}+1}$ which is Proposition 5.14 in Voisin's book [Voi03].

3. Proofs. So far we have used the full Gauss-Manin connection in order to construct and study the modular foliation $\mathcal{F}$. In this section we remind which part of the Gauss-Manin connection is IVHS. This will give us the proof of our main theorems. Throughout the present section we will redefine x to be the middle Hodge block $x^{\frac{\mathsf{m}}{2}}$ of the matrix x defined in (15).

3.1. Infinitesimal variation of Hodge structures. For definitions and details of the concepts used below see [CGGH83] and [Voi13]. Let

$$H^{\mathsf{m}}_{\mathrm{dR}}(Y/V) := \cup_{t\in V} H^{\mathsf{m}}_{\mathrm{dR}}(Y_t)$$

be the algebraic de Rham cohomology bundle of Y/V and let $F^k,\ \ k = 0, 2, \dots, \mathsf{m}+1$ be the subbundles of $H^{\mathsf{m}}_{\mathrm{dR}}(Y/V)$ corresponding to Hodge filtration in its fibers. By Griffiths transversality theorem, the Gauss-Manin connection of Y/V induces maps

$$\nabla_k : H^{k,\mathsf{m}-k} \to \Omega^1_V \otimes_{\mathcal{O}_V} H^{k-1,\mathsf{m}-k+1}, \quad k = 1, 2, \dots, \mathsf{m}, \tag{30}$$

where $H^{k,\mathsf{m}-k} := F^k/F^{k+1}$. One usually use the canonical identifications

$$F^k/F^{k+1} \cong H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t}) \tag{31}$$

compose the Gauss-Manin connection with vector fields in V and arrives at

$$(T_V)_t \to \mathrm{Hom}(H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t}),\ H^{\mathsf{m}-k+1}(Y_t, \Omega^{k-1}_{Y_t})).$$

Further, one may use a theorem of Griffiths which says that the above maps are the composition of the Kodaira-Spencer map (3) and

$$\delta_{\mathsf{m},k} = \delta_k : H^1(Y_t, T_{Y_t})_\theta \to \mathrm{Hom}\left(H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t}),\ H^{\mathsf{m}-k+1}(Y_t, \Omega^{k-1}_{Y_t})\right) \tag{32}$$

which is obtained by contraction of differential forms along vector fields. Hopefully, δ_k will not be confused with the topological cycle δ_t used in previous sections. Here θ is the element in $H^1(Y_t, \Omega^1_V)$ induced by the polarization of Y_t and

$$H_1(Y_t, T_{Y_t})_\theta := \{v \in H_1(Y_t, T_{Y_t}) \mid \delta_{2,1}(v)(\theta) = 0\}.$$

The data (32) is known as the infinitesimal variation of Hodge structures at t (IVHS), see [CGGH83]. From this we get

$$\delta_k^* : H^{\mathsf{m}-k+1}(Y_t, \Omega^{k-1}_{Y_t})^* \to \mathrm{Hom}\left(H^1(Y_t, T_{Y_t})_\theta, H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t})^*\right). \tag{33}$$

Let ω be the basis of $H^{\mathsf{m}}_{\mathrm{dR}}(Y/V)$ chosen in §2.1. This induces a basis for both $H^{\mathsf{m}-k+1}(Y_t, \Omega^{k-1}_{Y_t})$ and $H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t})$ which we denote them by ω_* and ϖ_*, respectively.

Around a smooth point of V we choose coordinate system $(t_1, t_2, \dots, t_{\tilde{r}})$ and we denote by the same notation the image of the vector field $\frac{\partial}{\partial t_i},\ \ i = 1, 2, \dots, \tilde{r}$ under the Kodaira-Spencer map (3). Let B be the Gauss-Manin connection matrix of Y/V written in the basis ω and used in §2.1.

Proposition 5. *We have*

$$\mathsf{B}^{k,k-1} = \sum_{j=1}^{\tilde{r}} \mathsf{B}^{k,k-1}_j dt_j \tag{34}$$

where $\mathsf{B}_j^{k,k-1}$ *is the* $a \times b$ *matrix of* $\delta_k(\frac{\partial}{\partial t_j})$ *written in the bases* ω_* *and* ϖ_*.

Proof. This follows from the identifications (31). □

For the proposition below we set $k = \frac{\mathsf{m}}{2} + 1$.

PROPOSITION 6. *For* $y = 0, 1, \ldots, c$ *the determinantal variety of homomorphisms of rank* $\leq c - y$ *of* (33) *is given by* W_y *defined in* (27).

Proof. Let $\mathsf{B} = \mathsf{B}^{k,k-1}$. We have $\mathsf{B} \cdot x = \sum_{j=1}^{\tilde{r}} (\mathsf{B}_j x) dt_j$ and

$$[\mathsf{B}_1 x, \mathsf{B}_2 x, \cdots, \mathsf{B}_{\tilde{r}} x] = \sum_{j=1}^{b} x_j [\mathsf{B}_1^j, \mathsf{B}_2^j, \ldots, \mathsf{B}_{\tilde{r}}^j], \tag{35}$$

where B_j^h is the h-th column of B_j. By definition W_y is the determinantal variety of matrices of rank $\leq c-y$ constructed from the left hand side of (35). The determinantal variety of the right hand side of (35) is the determinantal variety of (33). □

3.2. Proof of Theorem 1 using modular foliations. Let $y + 1 := c - s$. By our assumption the fiber of $\pi_{y+1} : \check{W}_{y+1} \to V$ over t is empty. The variety W_{y+1} is given by homogeneous polynomials in x and with coefficients in $\mathcal{O}_V$. Since projective varieties are complete, the image of π_{y+1} is a closed proper subset of V which does not contain t, see for instance Milne's lecture notes [Mil]. Let $U \subset V$ be the complement of the image of π_{y+1} in V and so $t \in U$. By Proposition 4 we know that a component of the Hodge locus H in U is the projection of a component L of the Hodge locus with constant periods L_{δ_t} in $\mathsf{T} - \check{W}_{y+1}$ and $\dim H = \dim L_{\delta_t}$. Using Proposition 3 we get $\dim(H) \leq \tilde{r} - c + y$. Therefore, the codimension of H in V is $\geq s + 1$.

3.3. IVHS for hypersurfaces. Let V be the parameter space of smooth hypersurfaces of degree d in $\mathbb{P}^{\mathsf{m}+1}$. For $t \in V$ let $f_t(X_0, X_1, \ldots, X_{\mathsf{m}+1})$ be the corresponding homogeneous polynomial, $Y_t := \mathbb{P}\{f_t = 0\}$. We have the identifications

$$H_1(Y_t, T_{Y_t})_\theta \cong (\mathbb{C}[X]/J)_d \tag{36}$$

$$H^{\mathsf{m}-k,k} \cong (\mathbb{C}[X]/J)_{(k+1)d-\mathsf{m}-2}, \quad k = 0, 1, \ldots, \mathsf{m} \tag{37}$$

where $J := \mathrm{jacob}(f_t)$ is the Jacobian ideal of f_t (for $k = \frac{\mathsf{m}}{2}$ one must use the primitive part of $H^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}}$). Note that we have changed the role of k in §3.1 with $\mathsf{m} - k$. For g in the right hand side of (36) the corresponding deformation of f_t is given by $f_t + \epsilon g$ and for g in the right hand side of (37) we have

$$\omega_g := \mathrm{Residue}\left(\frac{g \cdot \sum_{i=0}^{\mathsf{m}+1} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{\mathsf{m}+1}}{f_t^{k+1}}\right) \in H^{\mathsf{m}-k,k}, \tag{38}$$

where we have used the residue map $H_{\mathrm{dR}}^{\mathsf{m}+1}(\mathbb{P}^{\mathsf{m}+1} \backslash Y_t) \to H_{\mathrm{dR}}^{\mathsf{m}}(Y_t)$, see for instance [Gri69]. After these identifications, the map δ_k in (32) turns out to be obtained by multiplication of polynomials, that is, we get

$$(\mathbb{C}[X]/J)_d \times (\mathbb{C}[X]/J)_{(k+1)d-\mathsf{m}-2} \to (\mathbb{C}[X]/J)_{(k+2)d-\mathsf{m}-2}, \quad (F, G) \mapsto FG, \tag{39}$$

see [CGGH83, Har85]. Due to Hodge classes we are mainly interested in m even and $k = \frac{\mathsf{m}}{2} - 1$. Recall the set I_N in (4) and let I be the union of all I_N's. We are going to work in a neighborhood of the Fermat point $0 \in V$ and so we take

$$f_t := X_0^d + X_1^d + \cdots + X_{\mathsf{m}+1}^d - d \cdot \sum_{j \in I_d} t_j X^j, \tag{40}$$

and work with $V = \mathrm{Spec}\left(\mathbb{C}[t, \frac{1}{\Delta}]\right)$, where $t = (t_j)_{j\in I_d}$ and $\Delta(t) = 0$ is the locus of parameters $t \in V$ such that the monomials

$$X^j := X_0^{j_0} X_1^{j_1} \cdots X_{\mathsf{m}+1}^{j_{\mathsf{m}+1}}, \quad 0 \le j_e \le d-2 \tag{41}$$

do not form a basis of $\mathbb{C}[X]/J$. The matrices $\mathsf{B}_j^{k,k-1}$ in (34) have entries in $\mathbb{C}[t, \frac{1}{\Delta}]$. The Kodaira-Spencer map (3) in this case is an isomorphism of fiber bundles in V. In general, it is always useful to choose a coordinate system $(t_1, t_2, \dots, t_r)$ such that the image of the vector field $\frac{\partial}{\partial t_j}$, $j = 1, 2, \dots, r$ under $(T_V)_t \to H^1(Y_t, T_{Y_t})_\theta$ form a basis of $H^1(Y_t, T_{Y_t})_\theta$.

3.4. IVHS for the Fermat variety. Let us consider the case $t = 0$, that is, we are going to deal with IVHS of the Fermat variety. In this case, the map (12) can be computed easily. A monomial $X^i \in \mathbb{C}[X]/J$ is zero if and only if for some e, $i_e \ge d-1$. Therefore, for $(X^j, X^i) \in (\mathbb{C}[X]/J)_d \times (\mathbb{C}[X]/J)_{(k+1)d-\mathsf{m}-2}$, the product X^{i+j} is a member of the canonical basis of $(\mathbb{C}[X]/J)_{(k+2)d-\mathsf{m}-2}$ if $i_e + j_e \le d-2$ for all e and it is zero otherwise. From now on we use I_N as an index set. Take variables

$$x_i, \quad i \in I_{(k+2)d-\mathsf{m}-2}.$$

Assume that for any other i which is not in $I_{(k+2)d-\mathsf{m}-2}$, x_i by definition is zero. The matrix $\mathsf{B}_j^{k,k-1}$, $j \in I_d$ is therefore a $a \times b$-matrix with entries 0 everywhere except at $(i, i+j)$ entries which is one. Therefore, We have

$$\mathsf{B}_j^{k,k-1} x = [x_{i+j}], \quad j \in I_d, \quad i \in I_{(k+1)d-\mathsf{m}-2} \tag{42}$$

and we can compute the $a \times r$ matrix:

$$M^{k,k-1} := [\mathsf{B}_1^{k,k-1}x, \mathsf{B}_2^{k,k-1}x, \dots, \mathsf{B}_r^{k,k-1}x] = [x_{i+j}], \quad j \in I_d, \quad i \in I_{(k+1)d-\mathsf{m}-2} \tag{43}$$

where i counts the rows and j counts the columns. The differential forms α_j defined in (26) with $k = \frac{\mathsf{m}}{2} - 1$ and evaluated at the Fermat point $t = 0$ are given by

$$\alpha_i = \sum_{j \in I_d} x_{i+j} dt_j, \quad i \in I_{(k+1)d-\mathsf{m}-2}.$$

3.5. Proof of Theorem 2. Throughout the present section set $k := \frac{\mathsf{m}}{2} - 1$ for the content of §3.4. We will reuse the letter k for an element in $I_{(\frac{\mathsf{m}}{2}+1)d-\mathsf{m}-2}$. We get the following Olympiad problem:

PROPOSITION 7. *If*

$$\mathrm{rank}([x_{i+j}]) < \binom{\frac{\mathsf{m}}{2}+d}{d} - (\frac{\mathsf{m}}{2}+1)^2 \tag{44}$$

then all $x_i, i \in I_{(\frac{\mathsf{m}}{2}+1)d-(\mathsf{m}+2)}$ *are zero.*

Proof. Take any additive ordering for $\mathbb{N}_0^{\mathsf{m}+2}$, that is, $i < j$ if and only if $i + k < j + k$ for all $i, j, k \in \mathbb{N}_0^{\mathsf{m}+2}$. For instance take the lexicographical ordering. We use decreasing induction on $k \in \mathbb{Z}^{\mathsf{m}+2}$, $|k| = (\frac{\mathsf{m}}{2}+1)d - (\mathsf{m}+2)$. For very big k, we have $k \notin I_{(\frac{\mathsf{m}}{2}+1)d-(\mathsf{m}+2)}$ and so we have automatically $x_k = 0$. Let us assume that $x_{\tilde{k}} = 0$ for all $\tilde{k} > k$. We collect all

$$k = i_e + j_e, \; i_e \in I_{\frac{\mathsf{m}}{2}d-\mathsf{m}-2}, \; j_e \in I_d, \quad e = 1, 2, \cdots, f$$

and order them according to the decreasing order of i_e's, that is, $i_{e_1} > i_{e_2} > \cdots$. We find a $f \times f$-submatrix of $[x_{i+j}]$ which is lower triangular and in its diagonal we have only x_k. Therefore, its determinant is x_k^f. Now our proposition follows from an even more elementary problem. □

Proposition 8. *For any $k \in I_{(\frac{\mathsf{m}}{2}+1)d-(\mathsf{m}+2)}$ we have*

$$\#\left\{(i,j) \in I_{\frac{\mathsf{m}}{2}d-(\mathsf{m}+2)} \times I_d \mid k = i+j\right\} \quad \geq \quad \binom{\frac{\mathsf{m}}{2}+d}{d} - (\frac{\mathsf{m}}{2}+1)^2. \tag{45}$$

Proof. Let A_k be the set in the left hand side of (45) for $k = (k_0, k_1, \ldots, k_{\mathsf{m}+1})$. It is easy to see that the lower bound in (45) is obtained by elements k such that $\frac{\mathsf{m}}{2}+1$ number of k_e's are zero and the rest (exactly the next half) is $d-2$. Let us assume that k is not of the mentioned format. For simplicity we can assume that $0 < k_0 < k_1 < d-2$. We prove that

$$\#A_{(k_0,k_1,\cdots)} \geq \#A_{(k_0-1,k_1+1,\cdots)} \tag{46}$$

and so repeating the same argument for $(k_0-1, k_1+1, \cdots)$ we get an element with only $d-2$ and 0 as its entries. In order to prove (46) we define a map

$$A_{(k_0-1,k_1+1,\cdots)} \to A_{(k_0,k_1,\cdots)}$$

and prove that it is injective. It sends the pair $((i_0, i_1, \ldots), (j_0, j_1, \ldots))$ to the pair $((i_0+1, i_1-1, \ldots), (j_0, j_1, \ldots))$ if $i_1 \neq 0$ and to $((0, i_0, \ldots), (k_0, k_1 - i_0, \ldots))$ if $i_1 = 0$. It is easy to see that this map is injective and so (46) is valid. □

In Proposition 7 the number in the right hand side of (44) is the biggest one with such a property. It is enough to find complex numbers x_k such that $\mathrm{rank}([x_{i+j}])$ is the number in the right hand side of (44). Such numbers are the periods of the projective space $\mathbb{P}^{\frac{\mathsf{m}}{2}}$ inside the Fermat variety V_0 given by $x_0 - \zeta x_1 = x_2 - \zeta x_3 = \cdots = x_{\mathsf{m}} - \zeta x_{\mathsf{m}+1} = 0$, where $\zeta^d + 1 = 0$. More precisely

$$x_k := \int_{\mathbb{P}^{\frac{\mathsf{m}}{2}}} \omega_{g_k},$$

where $g_k := X_0^{k_0} X_1^{k_1} \cdots X_{\mathsf{m}+1}^{k_{\mathsf{m}+1}}$ and ω_{g_k} is defined in (38).

3.6. Proof of Theorem 3. We compute the first approximation of IVHS around the Fermat point. More precisely, we compute the image of the Zariski tangent space of W_{c-s} at each point $(0,x)$ and under the derivation of the projection map $\pi : W_{c-s} \to V$. We prove that for any $s \leq s^1_{\max}$ and $(0,x) \in \pi^{-1}(0),\ x \neq 0$, the derivation of π, which maps the Zariski tangent space of W_{c-s} at $(0,x)$ to the Zariski tangent space of V at 0, is not surjective. This implies that π is not dominant. Note that the fibers of π are given by homogeneous polynomials and so one may take their projectivization.

Let us consider the IVHS (39) for the polynomial f_t in (40) with fixed parameters $t = (t_\alpha)_{\alpha \in I_d}$ and arbitrary k. We need to emphasize that the matrix $\mathsf{B}^{k,k-1}$ depends on t and so we write $\mathsf{B}^{k,k-1}(t) := \mathsf{B}^{k,k-1}$. In this way, for fixed $j \in I_d$, $\mathsf{B}_j^{k,k-1}(t)$ is the $a \times b$ matrix of the IVHS (39) written in the basis (41). We write the Taylor series of the matrix $\mathsf{B}_j^{k,k-1}(t)$ in the variables $t_\alpha,\ \alpha \in I_d$:

$$\mathsf{B}_j^{k,k-1}(t) = \mathsf{B}_j^{k,k-1}(0) + \sum_{\alpha \in I_d} \mathsf{B}_{j,\alpha}^{k,k-1}(0) \cdot t_\alpha + \cdots \tag{47}$$

where $\cdots$ means sum of homogeneous polynomials of degree ≥ 2 in t_α's and with coefficients depending on x. Let K be the ideal of $\mathbb{C}[t_\alpha,\ \alpha \in S_d]$ generated by $t_\alpha t_\beta,\ \ \alpha, \beta \in S_d$. Therefore, $\cdots$ in the above equality means it belong to K. As we mentioned in §3.4, $\mathsf{B}_j^{k,k-1}(0)$ is a $a \times b$ matrix with 1 in its $(i, i+j)$ entries and 0 for entries elsewhere. In order to compute $\mathsf{B}_{j,\alpha}^{k,k-1}(0)$ we notice that

$$X^i X^j = \begin{cases} X^{i+j} & \forall e \ \ i_e + j_e \leq d-2, \\ \sum_{\alpha \in I_d} t_\alpha \cdot \alpha_{\check{e}} \cdot \frac{X^{i+j+\alpha}}{X_{\check{e}}^d} & \forall e \ \ i_e + j_e \leq d-2, \text{ except for exactly one } e = \check{e}, \\ 0 & \text{otherwise.} \end{cases} \tag{48}$$

The last two equalities are written modulo both ideals jacob(f_t) and K. From this we derive the fact that the matrix $\mathsf{B}_{j,\alpha}^{k,k-1}(0)$ in its $(i, i +_\alpha j)$ entry has $\alpha_{\check{e}}$ for those (i, j) in the second equality in (48), and elsewhere entries are zero. We define an $a \times r$ matrix $\check{M}_\alpha^{k,k-1}$ in the following way. For (i, j) in the second equality of (48), the (i, j) entry of $\check{M}_\alpha^{k.k-1}$ is $\alpha_{\check{e}} \cdot x_{i+_\alpha j}$, and elsewhere entries are zero. In this way, $\mathsf{B}_{j,\alpha}^{k,k-1}(0) \cdot x$ is the j-th column of $M_\alpha^{k,k-1}$. Let $N_{j,\alpha}^{k,k-1}$ be the $a \times r$ matrix obtained by replacing the j-th column of $M^{k,k-1}$ with the j-th column of $\check{M}_\alpha^{k,k-1}$.

From now on, we set $k = \frac{\mathsf{m}}{2} - 1$, we do not write the $k, k-1$ upper index of our matrices and we have the same notation as in the Introduction. The variety W_{c-s} is given by $(s+1) \times (s+1)$ minors of the $a \times r$ matrix

$$\left[\mathsf{B}_1^{k,k-1}(t) \cdot x \ , \ \mathsf{B}_2^{k,k-1}(t) \cdot x \ , \ \ldots \ , \ \mathsf{B}_r^{k,k-1}(t) \cdot x \right]$$

and so the Zariski tangent space of W_{c-s} under the derivation of π maps to

$$\left\{ (v_\alpha)_{\alpha \in S_d} \quad | \quad \sum_{\alpha \in S_d} \left(\sum_{j \in I_d} \text{minor}_{s+1}(N_{j,\alpha}^{k,k-1}) \right) v_\alpha = 0 \right\}.$$

Now, we use the definition of $s^1_{\max}$ and Theorem 3 is proved.

3.7. Noether-Lefschetz locus. Max Noether's theorem asserts that every curve on a general hypersurface X of degree $d \geq 4$ in $\mathbb{P}^3$ is a complete intersection with another surface. In other words, the Picard group of X is free of rank one. Noether's argument was just the plausibility of the statement and the first rigorous proof of this theorem was given by S. Lefschetz in [Lef50] by using a monodromy argument. A new Hodge-theoretic proof is given by Carlson, Green, Griffiths and Harris in [CGGH83, GH85]. Let $V \subset \mathbb{P}^N$ be the parameter space of smooth hypersurface X of degree $d \geq 4$ in $\mathbb{P}^3$. Noether-Lefschetz locus NL_d in V is the locus of smooth surfaces with Picard group different from $\mathbb{Z}$. By Lefschetz $(1,1)$ theorem any two dimensional Hodge class in X is algebraic and so NL_d is a particular example of a Hodge locus. It is a countable union of proper algebraic subset of V and a surface with parameters outside NL_d is called a general surfaces. Let H be an irreducible component of NL_d and $\mathrm{codim}_V(H)$ be its codimension in V. We have

$$d - 3 \leq \mathrm{codim}_V(H) \leq \binom{d-1}{3}$$

where the upper bound is the $(2,0)$ Hodge number h^{20} of X. Ciliberto, Harris and Miranda in [CHM88] proved that for $d \geq 4$, NL_d contains infinitely many general components, that is those components H such that $\mathrm{codim}_V(H) = \mathsf{h}^{20}$, and the union of these components is Zariski dense in V. Components of codimensions $< \mathsf{h}^{20}$ are called special (or exceptional). Green and Voisin in a series of paper showed that $d-3$ is the minimum codimension for the components of NL_d and for $d \geq 5$ the only component of codimension $d-3$ is the family of surfaces containing a line, see [Gre88, Gre89, Voi88]. Voisin in [Voi89] showed that for $d \geq 5$ the second biggest component of NL_d is of codimension $2d-7$, which consists of those surfaces containing a conic. This implies a conjecture of J. Harris is true in the case $d=5$: there should be only finitely many special components of NL_d. More evidences for Harris' conjecture came from the work [Cox90] of Cox in the case of elliptic surfaces, Debarre and Laszlo's work [DL90] for abelian varieties and Voisin's work [Voi90] for hyepersurfaces of degree $d=6,7$. For $d=4s$ arbitrarily large, Voisin in [Voi91] constructed an infinity of special components for NL_d and hence gave counterexamples to Harris' conjecture. All such components are contained in a proper algebraic subvariety of the set of algebraic surfaces of degree d and so she formulated Conjecture 2. In this case the numbers (8) are given by

$$
\begin{aligned}
c = a &= \binom{d-1}{3} \qquad (49)\\
b &= (d-1)^3 - (d-1)^2 + (d-1) - 2\binom{d-1}{3}\\
r &= \binom{d+3}{3} - 16.
\end{aligned}
$$

General components are of codimension a and so using Proposition 4 the modular foliation $\mathcal{F}$ in $\mathsf{T} := V \times \mathbb{A}_{\mathbb{C}}^{a+b}$ is of the maximal codimension a. The differential forms (23) are linearly independent over the function field of T and so $c=a$ and W_1 is the Zariski closure of

$$
\left\{(t,x) \in V \times \mathbb{A}_{\mathbb{C}}^{b} \mid x \neq 0, \quad \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_a = 0 \right\}. \qquad (50)
$$

3.8. K3 surfaces. Let us consider the case of hypersurfaces X of degree 4 in $\mathbb{P}^3$, that is, $\mathsf{m} = d-2 = 2$. In this case X is called a K3 surface. We have canonical identifications $H^{2,0} \cong \mathbb{C}$, $H_1(Y_t, T_{Y_t})_\theta \cong H^{11}_{\mathrm{prim}} \cong (\mathbb{C}[X]/J)_d$. We choose a basis of $(\mathbb{C}[X]/J)_d$ and hence we obtain a basis of both $H_1(Y_t, T_{Y_t})_\theta$ and H^{11}_{prim}. Using (39) we get

$$
\mathsf{B}^{0,1} = [dt_1, dt_2, \ldots, dt_b], \quad \alpha_1 := x_1 \cdot dt_1 + x_2 \cdot dt_2 + \cdots + x_b \cdot dt_b.
$$

We conclude that the variety W_1 is empty. The singular set of the modular foliation $\mathcal{F}$ in T is given by $\tilde{W}$. An expert in holomorphic foliations might be interested to know whether the components of the Noether-Lefschetz locus with constant periods, are the only algebraic leaves of $\mathcal{F}$. A similar discussion is also valid for the case of four dimensional cubic hypersurfaces, that is, $d=3, \mathsf{m}=4$. In this case $a=1, b=r=20$. Note that in this case the Hodge conjecture is well-known, see [Zuc77].

3.9. Conjecture 1 and the proof of Corollary 1. We know that the codimension of the determinantal variety $D_{s,t}$ of homomorphisms of rank $\leq s$ in the right hand side of (2) is $(a-s)(r-s)$. Therefore, we may hope that for generic t the map (2) is transversal to $D_{s,t}$ (it meets $D_{s,t}$ properly in the terminology of [Eis88]). If this happens then the codimension of the pull-back $W_{c-s,t}$ of $D_{s,t}$ in the left hand side of

(2) is the same as the codimension of $D_{t,s}$ in the right hand side of (2). Here, the codimension of the empty set is defined to be any number bigger than the dimension of the ambient space. Therefore, the projectivization of the map (2) does not intersect $D_{s,t}$ if

$$b \leq (a-s)(r-s) = s^2 - (r+a)s + ar. \tag{51}$$

The biggest s which satisfies this property is $\check{s}_{\max}$ defined in (10). Corollary 1 follows from the computation of $\check{s}_{\max}$ from the data in (49).

3.10. Second proof of Theorem 1 . Since determinantal varieties are homogeneous and projective varieties are complete, there is a Zariski open neighborhood U of $t \in V$ such that the hypothesis of Theorem 1 is valid for all points in U and so it is enough to prove that a component H of the Hodge locus passing through t has codimension $\geq s+1$.

First, note that the algebraic cup product in de Rham cohomology (21) after canonical identifications (31) gives us isomorphisms

$$H^{\mathsf{m}-k}(Y_t, \Omega^k_{Y_t})^* \cong H^k(Y_t, \Omega^{\mathsf{m}-k}_{Y_t}), \quad k = 0, 1, \ldots, \mathsf{m} \tag{52}$$

and under these isomorphisms, the map (2) constructed from the $(\mathsf{m}-k)$-th IVHS is identified with the k-th IVHS:

$$H^{k-1}(Y_t, \Omega^{\mathsf{m}-k+1}_{Y_t}) \to \mathrm{Hom}\left(H^1(Y_t, T_{Y_t})_\theta, H^k(Y_t, \Omega^{\mathsf{m}-k}_{Y_t})\right). \tag{53}$$

Composing the right hand side of (53) with the Kodaira-Spencer map (3) one arrives at Voisin's ${}^t\bar{\nabla}$ map:

$${}^t\bar{\nabla}: H^{k-1}(Y_t, \Omega^{\mathsf{m}-k+1}_{Y_t}) \to \mathrm{Hom}\left((T_V)_t, H^k(Y_t, \Omega^{\mathsf{m}-k}_{Y_t})\right). \tag{54}$$

Now consider the case $k = \frac{\mathsf{m}}{2}+1$. Let $\delta_t \in H^{\mathsf{m}}_{\mathrm{dR}}(Y_t)$ be the Hodge class whose locus is the component H. It induces an element $\delta_t^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}} \in H^{k-1}(Y_t, \Omega^{\mathsf{m}-k+1}_{Y_t})$ and Voisin in [Voi03] 5.3.3 has shown that $\ker({}^t\bar{\nabla}\delta_t^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}})$ is the Zariski tangent space of H at t. Therefore

$$\mathrm{codim}_V H \geq \dim V - \dim\left(\ker({}^t\bar{\nabla}\delta_t^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}})\right) \geq \mathrm{rank}\left({}^t\bar{\nabla}\delta_t^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}}\right) \geq s+1.$$

4. Final remarks. We can use (17) and we can interpret the variety T defined in §2.1 in the following way: Let $\tilde{\mathsf{T}}$ be the total space of the $F^{\frac{\mathsf{m}}{2}}$ bundle mines the total space of $F^{\frac{\mathsf{m}}{2}+1}$ bundle. In $\tilde{\mathsf{T}}$ we have a canonical line bundle L obtained by the choice of an element in $F^{\frac{\mathsf{m}}{2}}$. Let us denote by F^i the pull-back of the bundles with the same name by the projection $\tilde{\mathsf{T}} \to V$. For $i \leq \frac{\mathsf{m}}{2}$ we have a canonical embedding $L \subset F^i$ and we define $\tilde{F}^i = F^i/L$. For other cases we define $\tilde{F}^i = F^i$. The variety T is the total space of choices of bases for F^i's and L with a fixed variation in all $\tilde{F}^i$'s. These kind of total spaces in a refined format of moduli spaces were extensively used by the author to give geometric interpretation of quasi-modular forms and to construct analytic objects which transcend the classical automorphic forms, see [Mov13] and the references therein. The notion of modular foliations based on the historical examples of Darboux, Halphen and Ramanujan has been introduced by the author in [Mov11].

A systematic solution to the Harris-Voisin conjecture and its generalizations involves the study of the Hodge block $\mathsf{B}^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}}$ of the Gauss-Manin connection used in the algebraic expression of the foliation $\mathcal{F}$ given in (23), (24), (25). Note that this is not covered in the IVHS. This together with two other matrices computed from IVHS, namely $\mathsf{B}^{\frac{\mathsf{m}}{2}-1,\frac{\mathsf{m}}{2}}, \mathsf{B}^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}+1}$, govern the codimensions of the leaves of $\mathcal{F}$.

We know that the number c is less than or equal the minimum of r and a. In the case of hypersurfaces of dimension two we saw that $c = a < r$. For hypersurfaces of dimension ≥ 4 and degree $d \geq 2\frac{\mathsf{m}+2}{\mathsf{m}-2}$ we have $r \leq a$ and one may conjecture that $c = r$. For the r-dimensional parameter space in (40), this implies that the foliation $\mathcal{F}$ is zero dimensional and so a generic hypersurface has isolated Hodge classes with constant periods. However, this does not imply the following stronger statement:

CONJECTURE 3. *There is a Zariski open subset U of the parameter space V of smooth hypersurfaces of degree $d \geq 2\frac{\mathsf{m}+2}{\mathsf{m}-2}$ in $\mathbb{P}^{\mathsf{m}+1}$, $\mathsf{m} \geq 4$ such that Hodge classes of Y_t, $t \in U$ are isolated, that is, all the components of the Hodge locus in U are the orbits of* $\mathrm{PGL}(\mathsf{m}+2, \mathbb{C})$ *acting on V.*

Note that IVHS, and hence Theorem 1, is not enough for verifying this conjecture as we need to study the Hodge block $\mathsf{B}^{\frac{\mathsf{m}}{2},\frac{\mathsf{m}}{2}}$ of the Gauss-Manin connection. Even if Conjecture 1 is true, the number $\check{s}_{\max} + 1$ defined in (10) cannot be $\geq r$.

We were not able to verify the hypothesis of Theorem 1 for some examples of hypersurfaces and with the help of a computer code which uses Gröbner basis for ideals. Even in the case of the Fermat variety, where we were able to compute IVHS and prove Proposition 7 all without any computer assistance, writing a simple minded code to prove Proposition 7 fails. Such computational difficulties deserve to be treated in a separate work. There are few examples which one might be able to treat them by hand. One of them is the famous Dwork family of Calabi-Yau varieties:

$$X_0^{\mathsf{m}+2} + X_1^{\mathsf{m}+2} + \cdots + X_{\mathsf{m}+1}^{\mathsf{m}+2} - t \cdot X_0 X_1 \cdots X_{\mathsf{m}+1} = 0. \tag{55}$$

It is left to the reader to analyze the hypothesis of Theorem 1 in this case and get results similar to Theorem 2.

The degree of the components of the Noether-Lefschetz locus in the case of K3 surfaces is related to Gromov-Witten theory, see for instance [MP12]. It would be interesting to know whether such relations can be generalized beyond K3 surfaces. The degree of the component of the Hodge locus in Theorem 2 for $\mathsf{m} = 4$ can be computed explicitly, see for instance Vainsencher's article [Vai14].

For the entire collection of problems, computations and forthcoming articles related to Gauss-Manin connection in disguise the reader is referred to the author's web page. The case of Calabi-Yau varieties and its application in Topological String Theory can be found in [AMSY16].

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