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DEFORMATIONS OF HOMOGENEOUS ASSOCIATIVE SUBMANIFOLDS IN NEARLY PARALLEL G_2-MANIFOLDS*

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Abstract. A nearly parallel G_2-manifold Y is a Riemannian 7-manifold whose cone $C(Y) = \mathbb{R}_{>0} \times Y$ has the holonomy group contained in Spin(7). In other words, it is a spin 7-manifold with a real Killing spinor.

We have a special class of calibrated submanifolds called Cayley submanifolds in $C(Y)$. An associative submanifold in Y is a minimal 3-submanifold whose cone is Cayley. We study its deformations, namely, Cayley cone deformations, explicitly when it is homogeneous in the 7-sphere S^7.

Key words. Associative submanifolds, nearly parallel G_2-manifolds, Cayley cones.

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1. Introduction. For any Riemannian manifold (Y, g), consider its Riemannian cone $(C(Y), \overline{g}) = (\mathbb{R}_{>0} \times Y, dr^2 + r^2 g)$. A Riemannian 7-manifold (Y, g) is called a nearly parallel G_2-manifold if the holonomy group of $\overline{g}$ is contained in Spin(7). The existence of a nearly parallel G_2-structure is equivalent to that of a spin structure with a real Killing spinor ([2]), which is also used in supergravity and superstring theory in physics.

We have a canonical closed 4-form Φ on $C(Y)$, which defines a calibration. A 3-submanifold M in Y is called associative if its cone $C(M)$ is calibrated by Φ. In other words, $C(M)$ is a Cayley submanifold in $C(Y)$. For example, special Legendrian submanifolds in Sasaki-Einstein manifolds are associative (Lemma 2.19), and Lagrangian submanifolds in the sine cones of nearly Kähler 6-manifolds are associative (Lemma 2.30). Here, Lagrangian submanifolds are defined in terms of the vanishing of a non-closed 2-form which characterizes nearly Kähler geometry. These are also called totally real submanifolds.

The deformation of compact calibrated submanifolds was studied by Mclean [19]. Joyce [11, 12, 13, 14, 15] introduced the notion of the stability index of a special Lagrangian cone to study deformations of a special Lagrangian submanifold with a conical singularity. Lotay [17] generalized it to the coassociative case. Associative and Cayley submanifolds behave differently from special Lagrangian and coassociative submanifolds, and hence it is difficult to generalize it directly to the associative or Cayley case. Thus in this paper, we focus on the Cayley case and study the deformations of homogeneous Cayley cones explicitly. It may help to develop the general deformation theory of a Cayley submanifold with a conical singularity. Our approach is based on the representation theory. This is an analogue of Ohnita's approach to special Legendrian submanifolds in [24].

The homogeneous associative submanifolds in S^7 are classified by Lotay [16] into 8 types: A_1, A_2 and A_3 not lying in a totally geodesic nearly Kähler S^6, Lagrangian submanifolds L_1, L_2, L_3, and L_4 in S^6, and the totally geodesic S^3 (Proposition 6.1). Infinitesimal Lagrangian deformations in S^6 are studied in [17], and hence we study the infinitesimal deformations of the others and obtain the following.

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Theorem 1.1. *As an associative submanifold, A_1 is rigid, while A_2 and A_3 are not rigid. The deformation space of A_2 is unobstructed, and all non-trivial associative deformations of A_2 are induced by the* $\mathrm{PGL}(4,\mathbb{C})$*-action on* $\mathbb{C}P^3$ *via the Hopf lift.*

Theorem 1.2. *All the associative and non-Lagrangian deformations of the totally geodesic S^3, L_1, L_2, L_3, and L_4 are trivial. In other words, such deformations are induced from* $\mathrm{Spin}(7) \setminus G_2$.

This paper is organized as follows. In Section 2, we review the fundamental facts of G_2, Spin(7), Sasakian, and nearly Kähler geometry.

In Section 3, we characterize the space of all infinitesimal associative deformations as an eigenspace of a twisted Dirac operator D (Proposition 3.2).

In Section 4 (5), we compute the difference of the dimension between infinitesimal associative and special Legendrian (Lagrangian) deformations. These computations are useful to prove Theorem 1.1 and 1.2 and give the geometrical meanings of some eigenspaces of some differential operators such as the Laplacian.

In Section 6, according to Lotay's classification, we calculate the dimensions of eigenspaces of homogeneous associative submanifolds by the representation theoretical method in Appendix B, and prove Theorem 1.1 and 1.2.

Notation. Let M be a manifold and E be a vector bundle over M. We denote by $C(M,E)$ the space of all continuous sections of $E \to M$, and by $C^\infty(M,E)$ the space of all smooth sections of $E \to M$. Especially, we write $\mathfrak{X}(M) = C^\infty(M,TM)$.

If a Lie group G acts on M, we denote by X^* the vector field generated by $X \in \mathfrak{g} = Lie(G)$.

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2. Preliminaries.

2.1. G_2 and Spin(7) geometry.

Definition 2.1. Define a 3-form φ_0 on $\mathbb{R}^7$ by

$$\varphi_0 = dx_{123} + dx_1(dx_{45} + dx_{67}) + dx_2(dx_{46} - dx_{57}) - dx_3(dx_{47} + dx_{56}),$$

where $(x_1, \cdots, x_7)$ is the standard coordinate system on $\mathbb{R}^7$ and wedge signs are omitted. The Hodge dual of φ_0 is given by

$$*\varphi_0 = dx_{4567} + dx_{23}(dx_{67} + dx_{45}) + dx_{13}(dx_{57} - dx_{46}) - dx_{12}(dx_{56} + dx_{47}).$$

Decompose $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ and denote by x_0 the coordinate on $\mathbb{R}$. Define a self-dual 4-form Φ_0 on $\mathbb{R}^8$ by

$$\Phi_0 = dx_0 \wedge \varphi_0 + *\varphi_0.$$

If we identify $\mathbb{R}^8 \cong \mathbb{C}^4$ via $\mathbb{R}^8 \ni (x_0, \cdots, x_7) \mapsto (x_0+ix_1, x_2+ix_3, x_4+ix_5, x_6+ix_7) =: (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, then Φ_0 is described as

$$\Phi_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}\Omega_0,$$

where $\omega_0 = \frac{i}{2}\sum_{j=1}^4 dz_{j\bar{j}}$ and $\Omega_0 = dz_{1234}$ are the standard Kähler form and the holomorphic volume form on $\mathbb{C}^4$, respectively.

The stabilizers of φ_0 and Φ_0 are the exceptional Lie group G_2 and Spin(7), respectively:

$$G_2 = \{g \in GL(7, \mathbb{R}); g^*\varphi_0 = \varphi_0\}, \qquad \mathrm{Spin}(7) = \{g \in GL(8, \mathbb{R}); g^*\Phi_0 = \Phi_0\}.$$

The Lie group G_2 fixes the standard metric $g_0 = \sum_{i=1}^7 (dx_i)^2$ and the orientation on $\mathbb{R}^7$. They are uniquely determined by φ_0 via

$$6g_0(v_1, v_2)\mathrm{vol}_{g_0} = i(v_1)\varphi_0 \wedge i(v_2)\varphi_0 \wedge \varphi_0, \tag{2.1}$$

where vol_{g_0} is a volume form of g_0, $i(\cdot)$ is the interior product, and $v_i \in T(\mathbb{R}^7)$.

Similarly, Spin(7) fixes the standard metric $h_0 = \sum_{i=0}^7 (dx_i)^2$ and the orientation on $\mathbb{R}^8$. We have the following identities:

$$\Phi_0^2 = 14\mathrm{vol}_{h_0}, \qquad (i(w_2)i(w_1)\Phi_0)^2 \wedge \Phi_0 = 6\|w_1 \wedge w_2\|_{h_0}^2 \mathrm{vol}_{h_0}, \tag{2.2}$$

where vol_{h_0} is a volume form of h_0 and $w_i \in T(\mathbb{R}^8)$.

Definition 2.2. Let Y be an oriented 7-manifold and φ a 3-form on Y. A 3-form φ is called a **G_2-structure** on Y if for each $y \in Y$, there exists an oriented isomorphism between T_yY and $\mathbb{R}^7$ identifying φ_y with φ_0. From (2.1), φ induces the metric g and the volume form on Y. A G_2-structure φ is said to be **nearly parallel** if $d\varphi = 4 * \varphi$. We call a manifold with a nearly parallel G_2-structure a **nearly parallel G_2-manifold** for short. A G_2-structure φ is called **torsion-free** if $d\varphi = d * \varphi = 0$.

Let X be an oriented 8-manifold and Φ a 4-form on X. A 4-form Φ is called a **Spin(7)-structure** on X if for each $x \in X$, there exists an oriented isomorphism between T_xX and $\mathbb{R}^8$ identifying Φ_x with Φ_0. From (2.2), Φ induces the metric h and the volume form on X. A Spin(7)-structure Φ is called **torsion-free** if $d\Phi = 0$.

Lemma 2.3. *[25] A G_2-structure φ is torsion-free if and only if* $\mathrm{Hol}(g) \subset G_2$. *A* Spin(7)*-structure Φ is torsion-free if and only if* $\mathrm{Hol}(h) \subset \mathrm{Spin}(7)$.

Lemma 2.4. *[1] The following are equivalent:*
1. *$d\varphi = 4 * \varphi$ (i.e. The 3-form φ is a nearly parallel G_2-structure.),*
2. *$\nabla\varphi = \frac{1}{4}d\varphi$, where ∇ is the Levi-Civita connection of g,*
3. *$\nabla\varphi = *\varphi$,*
4. *$\nabla_X(*\varphi) = -g(X, \cdot) \wedge \varphi$ for any $X \in TY$,*
5. *$i(X)\nabla_X\varphi = 0$ for any $X \in TY$,*
6. *The Riemannian cone $C(Y) = \mathbb{R}_{>0} \times Y$ admits a torsion-free* Spin(7)*-structure $\Phi = r^3 dr \wedge \varphi + r^4 * \varphi$ with the induced cone metric $\overline{g} = dr^2 + r^2 g$.*

Next, we give a summary of the facts about submanifolds. Let Y be a manifold with a G_2-structure φ and the induced metric g.

Lemma 2.5. *[8] For every oriented k-dimensional subspace $V^k \subset T_pY$ where $p \in Y$ and $k = 3, 4$, we have $\varphi|_{V^3} \leq \mathrm{vol}_{V^3}$, $*\varphi|_{V^4} \leq \mathrm{vol}_{V^4}$. An oriented 3-submanifold $L^3 \subset Y$ is called* **associative** *if $\varphi|_{TL^3} = \mathrm{vol}_{L^3}$. An oriented 4-submanifold L^4 is called* **coassociative** *if $*\varphi|_{TL^4} = \mathrm{vol}_{L^4}$.*

Lemma 2.6. *[8] Define a tangent bundle valued 3-form $\chi \in C^\infty(Y, \wedge^3 T^*Y \otimes TY)$ by*

$$g(v_1, \chi(v_2, v_3, v_4)) = *\varphi(v_1, v_2, v_3, v_4)$$

for $v_i \in TY$. *If* $L^k \subset Y$ *is an oriented* k*-submanifold where* $k = 3, 4$, *then*

$$L^3\text{: associative} \Leftrightarrow \chi|_{TL^3} = 0 \text{ and } \varphi|_{TL^3} > 0,$$
$$L^4\text{: coassociative} \Leftrightarrow \varphi|_{TL^4} = 0 \text{ and } *\varphi|_{TL^4} > 0.$$

Definition 2.7. Define the cross product $\times : TY \times TY \to TY$ by

$$g(u \times v, w) = \varphi(u, v, w)$$

for $u, v, w \in TY$. This satisfies the following relation:

$$\chi(u, v, w) = u \times (v \times w) + g(u, v)w - g(u, w)v. \tag{2.3}$$

Remark 2.8. When L^3 is associative, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ satisfying $e_3 = e_1 \times e_2$ at any point in L^3.

Definition 2.9. Let X be a manifold with a Spin(7)-structure Φ. Then for every oriented 4-dimensional subspace $W \subset T_x X$ where $x \in X$, we have $\Phi|_W \leq \mathrm{vol}_W$. An oriented 4-submanifold $N \subset X$ is called **Cayley** if $\Phi|_{TN} = \mathrm{vol}_N$.

Lemma 2.10. *Let* (Y, φ, g) *be a nearly parallel* G_2*-manifold and* $L \subset Y$ *be an oriented 3-submanifold. By Lemma 2.4,* $C(Y)$ *is a manifold with a torsion-free* Spin(7)*-structure* Φ. *Then* $L \subset Y$ *is associative if and only if* $C(L) \subset C(Y)$ *is Cayley.*

Lemma 2.11. *[16] There are no coassociative submanifolds of a nearly parallel* G_2*-manifold* (Y, φ, g).

Proof. If L is a coassociative submanifold, we have $\varphi|_{TL} = 0$, which implies that $4\mathrm{vol}_L = 4 * \varphi|_{TL} = d\varphi|_{TL} = 0$. This is a contradiction. □

2.2. Sasakian geometry.

Definition 2.12. An odd dimensional Riemannian manifold (S, g) is a **Sasakian manifold** if its Riemannian cone $(C(S), \overline{g}) = (\mathbb{R}_{>0} \times S, dr^2 + r^2 g)$ is a Kähler manifold with respect to some integrable complex structure J over $C(S)$.

Here, r is a standard coordinate of $\mathbb{R}_{>0}$ and we regard r as the function on $C(S)$. We identify S with the submanifold $\{1\} \times S \subset C(S)$.

Lemma 2.13. *Let* (S, g) *be a Sasakian* $(2m + 1)$*-manifold. If* g *is Einstein, the cone* $(C(S), \overline{g})$ *is Ricci-flat. In addition, if there exists a holomorphic volume form* $\Omega \in \Omega^{(m+1,0)}(C(S))$ *such that*

$$\omega^{m+1}/(m + 1)! = (-1)^{m(m+1)/2}(i/2)^{m+1}\Omega \wedge \overline{\Omega}, \tag{2.4}$$

where $\omega = \overline{g}(J\cdot, \cdot)$ *is the associated Kähler form on* $C(S)$, *we call* $(C(S), \overline{g}, J, \omega, \Omega)$ *a Calabi-Yau manifold.*

Lemma 2.14 ([4, Corollary 11.1.8]). *If* S *is a compact simply-connected Sasaki-Einstein manifold,* $C(S)$ *is a Calabi-Yau manifold.*

Remark 2.15. The holomorphic volume form Ω is not unique. For any $\theta \in \mathbb{R}$, $e^{i\theta}\Omega$ also satisfies (2.4).

Let (S, g) be a Sasaki-Einstein 7-manifold with a Calabi-Yau structure on $C(S)$.

LEMMA 2.16. *There exists a 3-form $\varphi \in \Omega^3(S)$ such that (S, φ, g) is a nearly parallel G_2-manifold.*

Proof. Fix a holomorphic volume form Ω. Then a 4-form

$$\Phi = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}\Omega \in \Omega^4(C(S)) \tag{2.5}$$

gives a torsion-free Spin(7)-structure on $C(S)$. A 3-form $\varphi \in \Omega^3(S)$ defined by

$$\Phi_{(r,p)} = r^3 dr \wedge \varphi_p + r^4 * \varphi_p, \qquad \text{where } (r, p) \in \mathbb{R}_{>0} \times S,$$

gives the nearly parallel G_2-structure on S. □

Next, we summarize the facts about submanifolds in Sasakian manifolds.

DEFINITION 2.17. An m-submanifold $L \subset S$ is called **Legendrian** if $C(L) \subset C(S)$ is Lagrangian: $\omega|_{TC(L)} = 0$. Fix a holomorphic volume form Ω on $C(S)$. An m-submanifold $L \subset S$ is called **special Legendrian** if $C(L) \subset C(S)$ is special Lagrangian: $\mathrm{Re}\Omega|_{TC(L)} = \mathrm{vol}_{C(L)} \Leftrightarrow \omega|_{TC(L)} = 0, \mathrm{Im}\Omega|_{TC(L)} = 0$ and $\mathrm{Re}\Omega|_{TC(L)} > 0$.

The following is a well-known fact. For example, see [21, Proposition 4.5].

LEMMA 2.18. *Let $L \subset S$ be a Legendrian submanifold. Then L is minimal if and only if* $\mathrm{Im}(e^{i\theta}\Omega) = 0$ *for some $\theta \in \mathbb{R}$.*

By definition, we obtain the following result.

LEMMA 2.19. *Let $L \subset S$ be an oriented 3-submanifold. If L is special Legendrian or if the cone $C(L)$ is a complex submanifold in $C(S)$, L is associative.*

Proof. If L is special Legendrian, we have $\frac{1}{2}\omega \wedge \omega|_{TC(L)} = 0$ and $\mathrm{Re}\Omega|_{TC(L)} = \mathrm{vol}_{C(L)}$. If $C(L)$ is a complex submanifold, we have $\frac{1}{2}\omega \wedge \omega|_{TC(L)} = \mathrm{vol}_{C(L)}$ and $\mathrm{Re}\Omega|_{TC(L)} = 0$. By (2.5) and Lemma 2.10, we see that L is associative in both cases. □

2.3. Infinitesimal deformation of special Legendrian submanifolds. Let (S, g) be a Sasaki-Einstein $(2m+1)$-manifold with a Calabi-Yau structure on $C(S)$. Fix a holomorphic volume form Ω and let $L \subset S$ be a special Legendrian submanifold.

LEMMA 2.20 ([24]). *The vector space of all infinitesimal special Legendrian deformations of L is identified with*

$$\{f \in C^\infty(L); \Delta_+ f = (2m+2)f\}, \tag{2.6}$$

where Δ_+ is the Hodge Laplacian for functions on L.

We write the subscript $+$ of Δ_+ since every eigenvalue of this Laplacian is nonnegative if L is compact.

Proof. Let ν be the normal bundle of L in S. Since L is Legendrian, there is a canonical isomorphism $\nu \ni v \mapsto \left(g(v, J(r\frac{\partial}{\partial r})|_{r=1}), -g(Jv, \cdot)\right) \in \mathbb{R} \oplus T^*L$. Via this identification, suppose that $V \in C^\infty(L,\nu)$ corresponds to $(f,\alpha) \in C^\infty(L) \oplus \Omega^1(L)$. Then we have

$$0 = L_V \left(i\left(r\frac{\partial}{\partial r}\right)\omega\right)\bigg|_{TL} = -2\alpha + df, \tag{2.7}$$

$$0 = L_V \left(i\left(r\frac{\partial}{\partial r}\right)\mathrm{Im}\Omega\right)\bigg|_{TL} = d * \alpha + (m+1) f \mathrm{vol}_L, \tag{2.8}$$

which implies the proof. □

The same result is obtained in [6] by using the fact that a cone $C(L)$ of L is special Lagrangian in $C(S)$ and applying the deformation theory of special Lagrangian submanifolds in [19].

2.4. Nearly Kähler geometry.

Definition 2.21. Let (N, k, J, σ) be a real 6-dimensional almost Hermitian manifold with a Hermitian metric k, an almost complex structure J and an associated Kähler form σ. Let $\psi^\pm \in \Omega^3(N)$ be 3-forms on N. A quintuple $(k, J, \sigma, \psi^\pm)$ is called an $SU(3)$**-structure** if we have $\|\psi^\pm\| = 2$ and $\Psi := \psi^+ + \sqrt{-1}\psi^-$ is a $(3,0)$-form with respect to J.

Remark 2.22. The $SU(3)$-structure with a Kähler structure and a holomorphic (3, 0)-form Ψ is a Calabi-Yau structure. In fact, we can prove

$$\sigma \wedge \psi^\pm = 0, \quad \sigma^3/3! = (-1)^{\frac{3(3-1)}{2}} (i/2)^3 \Psi \wedge \bar{\Psi}.$$

Definition 2.23. An $SU(3)$-structure satisfying $d\sigma = 3\psi^+$ and $d\psi^- = -2\sigma^2$ is called **nearly Kähler**.

Lemma 2.24 ([5]). *Let (N, k, J, σ) be a real 6-dimensional almost Hermitian manifold. It admits a nearly Kähler structure if and only if $(\nabla_X J)X = 0$ for every vector field X on N and $\nabla_X J \neq 0$ for every $0 \neq X \in TN$, where ∇ is the Levi-Civita connection of k.*

Lemma 2.25. *Let $(N, k, J, \sigma, \psi^\pm)$ be a nearly Kähler manifold. Then $C(N) = \mathbb{R}_{>0} \times N$ admits a torsion-free G_2-structure $(\varphi, \overline{k})$ with*

$$\begin{aligned} \overline{k} &= dr^2 + r^2 k, \\ \varphi &= r^2 dr \wedge \sigma + r^3 \psi^+ = \tfrac{1}{3} d(r^3 \sigma), \\ *\varphi &= r^3 \psi^- \wedge dr + \tfrac{1}{2} r^4 \sigma^2 = -\tfrac{1}{4} d(r^4 \psi^-). \end{aligned}$$

Lemma 2.26 ([4]). *Let $(N, k, J, \sigma, \psi^\pm)$ be a nearly Kähler manifold. Then $C_s(N) = (0,\pi) \times N$ (a sine cone of N) admits a nearly parallel G_2-structure $(\tilde{\varphi}, \tilde{k})$ with*

$$\begin{aligned} \tilde{k} &= dt^2 + (\sin^2 t) k, \\ \tilde{\varphi} &= (\sin^2 t) dt \wedge \sigma + (\cos t \sin^3 t) \psi^+ - (\sin^4 t) \psi^-, \\ *\tilde{\varphi} &= \tfrac{1}{2} (\sin^4 t) \sigma^2 + (\sin^3 t \cos t) \psi^- \wedge dt - (\sin^4 t) dt \wedge \psi^+. \end{aligned}$$

We canonically identify N with the submanifold $N \times \{\frac{\pi}{2}\} \subset C_s(N)$.

Remark 2.27. Since $C(N)$ admits a torsion-free G_2-structure, $\mathbb{R} \times C(N)$ admits a torsion-free Spin(7)-structure. The nearly parallel G_2-structure on $C_s(N)$ is induced via the identification $C(C_s(N)) = \mathbb{R}_{>0} \times (0, \pi) \times N \ni (r, t, x) \mapsto (r\cos t, r\sin t, x) \in \mathbb{R} \times \mathbb{R}_{>0} \times N = \mathbb{R} \times C(N)$.

Lemma 2.28 ([20]). *Let $(N, k, J, \sigma, \psi^{\pm})$ be a nearly Kähler manifold. Define a map $G : TN \times TN \to TN$ by $k(G(u, v), w) = \psi^+(u, v, w)$ for $u, v, w \in TN$. Then we have*

$$(\nabla_X J)(Y) = G(X, Y), \qquad \nabla_X \psi^+ = -k(X, \cdot) \wedge \sigma,$$

where ∇ is the Levi-Civita connection of k and $X, Y \in \mathfrak{X}(N)$.

Lemma 2.29. *Let $(N, k, J, \sigma, \psi^{\pm})$ be a nearly Kähler manifold. From Lemma 2.25, the cone $C(N) = \mathbb{R}_{>0} \times N$ admits a torsion-free G_2 structure. Let $\Sigma \subset N$ ($L \subset N$) be an oriented 2(3)-submanifold. Then we have*

- *$C(\Sigma) \subset C(N)$ is associative if and only if Σ is a J-holomorphic curve.*
- *$C(L^3) \subset C(N)$ is a coassociative 4-fold if and only if L is Lagrangian: $\sigma|_{TL} = 0$.*

Lemma 2.30. *Let $(N, k, J, \sigma, \psi^{\pm})$ be a nearly Kähler manifold. From Lemma 2.26, the sine cone $C_s(N) = N \times (0, \pi)$ admits a nearly parallel G_2 structure. Let $\Sigma \subset N$ ($L \subset N$) be an oriented 2(3)-submanifold. Then it follows that*

- *$C_s(\Sigma) \subset C_s(N)$ is associative if and only if Σ is a J-holomorphic curve.*
- *$L \times \{\frac{\pi}{2}\} \subset C_s(N)$ is associative if and only if L is Lagrangian: $\sigma|_{TL} = 0$.*

Remark 2.31. On a nearly Kähler manifold, we know that $d\sigma = 3\psi^+$, which implies that a Lagrangian submanifold L satisfies $\psi^+|_{TL} = 0$. Thus Lagrangian submanifolds in a nearly Kähler manifold are regarded as "special Lagrangian"(with phase $-i$).

We know the following as Lemma 2.20.

Lemma 2.32. *The vector space of all infinitesimal Lagrangian deformations of L in a nearly Kähler manifold is identified with*

$$\{v \in \mathfrak{X}(L); \mathrm{rot}(v) = 3v\}, \tag{2.9}$$

where $\mathrm{rot}(v) = \sum_{i=1}^{3} e_i \times \nabla^{\top}_{e_i} v$, $\nabla^{\top}$ is the Levi-Civita connection of the metric k_L on L induced from (N, k) and $\{e_i\}_{i=1,2,3}$ is the local orthonormal frame of TL.

Proof. Since L is Lagrangian, there is a canonical isomorphism between the tangent bundle TL and the normal bundle of L in N via $v \mapsto Jv$. Then a vector field $v \in \mathfrak{X}(L)$ on L corresponds to an infinitesimal Lagrangian deformation of L if and only if

$$0 = L_{Jv}\sigma|_{TL} = 3i(v)\mathrm{vol}_L - d(k_L(v, \cdot)).$$

Note that $\psi^-|_{TL} = -\mathrm{vol}_L$. Then the equations $*(i(v)\mathrm{vol}_L) = k_L(v, \cdot)$ and $*d(k_L(v, \cdot)) = k_L(\mathrm{rot}(v), \cdot)$ imply the proof. ∎

3. Associative deformations in nearly parallel G_2-manifolds. Let (Y, φ, g) be a nearly parallel G_2-manifold, $\iota : M^3 \hookrightarrow Y$ be an associative immersion, and $\{\iota_t : M \hookrightarrow Y\}_{t \in (-\epsilon, \epsilon)}$ be a smooth family of immersions with $\iota_0 = \iota$.

Definition 3.1. A family $\{\iota_t\}$ is called an **associative deformation** of ι if ι_t is an associative immersion for each t. An associative deformation $\{\iota_t\}$ is called **trivial** if $\{\iota_t\}$ is induced by a one-parameter family of automorphisms of (Y, φ, g). If all infinitesimal associative deformations of M come from trivial deformations, M is called **rigid**.

First, we characterize the space of infinitesimal associative deformations of M.

Proposition 3.2. *Let (Y, φ, g) be a nearly parallel G_2-manifold, and $M^3 \subset Y$ be an associative submanifold. Denote by ν the normal bundle of M in Y and by $\nabla^\perp$ the connection on ν induced by the Levi-Civita connection ∇ of (Y, g).*

Taking any local orthonormal frame $\{e_1, e_2, e_3\}$ of TM, define the operator $D : C^\infty(M, \nu) \to C^\infty(M, \nu)$ by

$$D\psi := \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} \psi.$$

Then the vector space of all infinitesimal associative deformations of $M^3 \hookrightarrow Y$ is identified with

$$\{\psi \in C^\infty(M, \nu); D\psi = -\psi\}.$$

Remark 3.3. [19] There exists a rank 4 vector bundle $E \to M$ satisfying $\nu \cong \mathbb{S} \otimes_{\mathbb{H}} E$, where $\mathbb{S} \to M$ is a spinor bundle. Then D is a twisted Dirac operator.

The proof of Proposition 3.2 comes from the following general theory of associative deformations.

Proposition 3.4 ([7, 19]). *Let (Y, φ, g) be a manifold with a G_2-structure and $M^3 \subset Y$ be an associative submanifold. Then the vector space of all infinitesimal associative deformations of $M^3 \hookrightarrow Y$ is identified with* $\ker \tilde{D}$*, where $\tilde{D} : C^\infty(M, \nu) \to C^\infty(M, \nu)$ is defined by*

$$\tilde{D}\psi := -\sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} \psi + \sum_{k=1}^{4} (\nabla_\psi * \varphi)(\eta_k, \omega)\eta_k.$$

Here $\{e_1, e_2, e_3\}$ is an oriented local orthonormal frame of TM, $\omega = e_1 \wedge e_2 \wedge e_3$, and $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ is a local orthonormal frame of ν.

Proof. We give an outline of the proof. Define a map $F : C^\infty(M, \nu) \to C^\infty(M, TY|_M)$ as $F(\sigma) = \exp_\sigma^* \chi(\omega)$, where χ is defined in Lemma 2.6. We know that $\exp_\sigma(M)$ is associative if and only if $F(\sigma)$ vanishes. For any $\psi \in C^\infty(M, \nu)$, we may consider

$$(dF)_0(\psi) = 0.$$

By a direct computation, the left hand side is equal to $-\sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} \psi + \sum_{k=1}^{4} (\nabla_\psi * \varphi)(\eta_k, \omega)\eta_k$, and hence the statement is proved. ∎

By Lemma 2.4, we see the following lemma, which implies Proposition 3.2.

LEMMA 3.5. *If (Y, φ, g) is nearly parallel, then* $\sum_{k=1}^{4}(\nabla_\psi * \varphi)(\eta_k, \omega)\eta_k = -\psi$.

REMARK 3.6. We can prove Proposition 3.2 by using the the fact that a cone $C(M)$ of M is a Cayley submanifold in $C(Y)$ with a torsion-free Spin(7)-structure. Applying the deformation theory of Cayley submanifolds in [19], we consider the Cayley cone deformation of $C(M)$. This is an analogue of the proof of Lemma 2.20 given by [6].

The operator D has the following properties.

LEMMA 3.7. *The operator D is elliptic. There exists a vector field $X \in \mathfrak{X}(M)$ on M satisfying*

$$g(D\psi, \psi') = -\mathrm{div}(X) + g(\psi, D\psi') \tag{3.1}$$

for any $\psi, \psi' \in C^\infty(M, \nu)$. In particular, when M is compact, D is self-adjoint.

Proof. The ellipticity of D is shown in [7]. For any $\psi, \psi' \in C^\infty(M, \nu)$, we compute by Definition 2.7 and Lemma A.1

$$\begin{aligned} g(D\psi, \psi') &= g\left(\sum_{i=1}^{3} e_i \times \nabla_{e_i}\psi, \psi'\right) \\ &= -\sum_{i=1}^{3} g(\nabla_{e_i}\psi, e_i \times \psi') \\ &= \sum_{i=1}^{3} (-e_i(g(\psi, e_i \times \psi')) + g(\psi, \nabla_{e_i}e_i \times \psi')) + g(\psi, D\psi'). \end{aligned}$$

Define the vector field $X \in \mathfrak{X}(M)$ on M by $g(X, v) = g(\psi, v \times \psi')$ for $v \in TM$. Then we obtain (3.1). □

Since D is a twisted Dirac operator, there is a close relation between D^2 and the Laplacian. Choose a local orthonormal frame $\{e_1, e_2, e_3\}$ of TM and define the operators $\nabla^{\perp *}\nabla^\perp, \mathcal{R}, \mathcal{A} : C^\infty(M, \nu) \to C^\infty(M, \nu)$ by

$$\nabla^{\perp *}\nabla^\perp = \sum_{i=1}^{3}(-\nabla^\perp_{e_i}\nabla^\perp_{e_i} + \nabla^\perp_{\nabla^\top_{e_i} e_i}), \qquad \mathcal{R} = \pi_\mathcal{V}(\sum_{i=1}^{3} R(e_i, \cdot)e_i), \mathcal{A} = {}^t\!A \circ A,$$

where $\nabla^\perp$ is the connection on the normal bundle ν induced by the Levi-Civita connection ∇ of (Y, g), $\nabla^\top$ is the orthogonal projection of ∇ to TM, R is the curvature tensor of g, $\pi_\mathcal{V}$ is the orthogonal projection to ν, $A : \nu \ni \psi \mapsto (u \mapsto -\nabla^\top_u \psi) \in SM := \{T : TM \to TM; {}^tT = T\}$ (the second fundamental form), and ${}^tA$ is the transpose of A.

PROPOSITION 3.8. *Let (Y, φ, g) be a nearly parallel G_2-manifold and $M^3 \subset Y$ be an associative submanifold. Then we have*

$$(D - 3id_\nu)(D + id_\nu) = \nabla^{\perp *}\nabla^\perp + \mathcal{R} - \mathcal{A}.$$

The proof is given in the appendix. The right hand side $\mathcal{J} := \nabla^{\perp *}\nabla^\perp + \mathcal{R} - \mathcal{A}$ is called a Jacobi operator, and $\ker \mathcal{J}$ is known to be the space of infinitesimal minimal deformations ([26]). By this formula, $D\psi = -\psi$ implies $\mathcal{J}\psi = 0$, which ensures that associative deformations are minimal deformations.

Remark 3.9. When M is compact, the space of all infinitesimal minimal and non-associative deformations of M is identified with $\{\psi \in C^\infty(M,\nu); D\psi = 3\psi\}$.

Proof. Since D is elliptic self-adjoint, there is an orthonormal basis $\{\psi_i\}_{i=1}^\infty \subset C^\infty(M,\nu)$ of $L^2(M,\nu)$ consisting of eigensections of D. The set of eigenvalues is discrete and the each eigenspace is finite dimensional. We may assume that $D\psi_i = \lambda_i\psi_i$ for $\lambda_i \in \mathbb{R}$. For any $\psi = \sum_{i=1}^\infty (\psi,\psi_i)_{L^2}\psi_i \in C^\infty(M,\nu)$ where $(\cdot,\cdot)_{L^2}$ is the L^2 inner product, we have

$$\begin{aligned}(D-3id_\nu)(D+id_\nu)\psi &= \sum_{i=1}^\infty ((D-3id_\nu)(D+id_\nu)\psi,\psi_i)_{L_2}\psi_i \\ &= \sum_{i=1}^\infty (\psi,(D-3id_\nu)(D+id_\nu)\psi_i)_{L_2}\psi_i \\ &= \sum_{i=1}^\infty (\lambda_i-3)(\lambda_i+1)(\psi,\psi_i)_{L^2}\psi_i.\end{aligned}$$

Since $\{\psi_i\}_{i=1}^\infty$ is an orthonormal basis, we see that $(D-3id_\nu)(D+id_\nu)\psi = 0$ if and only if $(\lambda_i-3)(\lambda_i+1)(\psi,\psi_i)_{L^2} = 0$ for each i. Thus elements of $\ker(D-3id_\nu)(D+id_\nu)$ are linear combinations of elements of $\ker(D-3id_\nu)$ and $\ker(D+id_\nu)$. ∎

4. Associative deformations of special Legendrian submanifolds in Sasaki-Einstein manifolds. Let (S,g) be a Sasaki-Einstein 7-manifold with a Calabi-Yau structure $(\overline{g},J,\omega,\Omega)$ on $C(S)$. Let $M \subset S$ be a special Legendrian submanifold. By Lemmas 2.16 and 2.19, (S,φ,g) admits a nearly parallel G_2-structure for some $\varphi \in \Omega^3(S)$ and M is associative. We study the infinitesimal associative deformations of M.

4.1. Associative deformations of special Legendrians. Let $\nu \to M$ be the normal bundle of M. First, we rewrite the operator $D: C^\infty(M,\nu) \to C^\infty(M,\nu)$ in Proposition 3.2 in the special Legendrian case. Since M is special Legendrian, there exists canonical isomorphism $TM \oplus \mathbb{R} \ni (v,x) \mapsto Jv + xJ(r\frac{\partial}{\partial r})|_{r=1} \in \nu$. Via this identification, we obtain the following.

Proposition 4.1. *The corresponding operator* $D: \mathfrak{X}(M) \oplus C^\infty(M) \to \mathfrak{X}(M) \oplus C^\infty(M)$ *is described as*

$$D(v,f) = (-\mathrm{grad}(f) + \mathrm{rot}(v) + v, \mathrm{div}(v) + 3f),$$

where $g_M(\mathrm{grad}(f),\cdot) = df$, $\mathrm{div}(v) = \mathrm{tr}(\nabla^\top v)$, *and* $\mathrm{rot}(v) = \sum_{i=1}^3 e_i \times \nabla^\top_{e_i} v$. *Here, we denote by* g_M *the metric on* M *induced from* (Y,g), *by* $\nabla^\top$ *the Levi-Civita connection of* g_M, *by* $\{e_i\}_{i=1,2,3}$ *the local orthonormal frame of* TM, *and* $g_M(v\times w,\cdot) = \varphi(v,w,\cdot) = \mathrm{vol}_M(v,w,\cdot)$ $(v,w \in TM)$.

We first give all the statements in this section and then prove them.

Corollary 4.2. *We have*

$$\begin{aligned}&\dim\{\textit{the infinitesimal associative deformations of } M\} \\ &= \dim\{f \in C^\infty(M); \Delta_+ f = 8f\} + \dim\{v \in \mathfrak{X}(M); \mathrm{rot}(v) = -2v\}.\end{aligned}$$

REMARK 4.3. From Lemma 2.20, $\dim\{v \in \mathfrak{X}(M); \mathrm{rot}(v) = -2v\}$ gives the dimension of infinitesimal associative and non-special Legendrian deformations.

We have the same equations as in the vector analysis.

LEMMA 4.4. *For any $f \in C^\infty(M)$ and $v \in \mathfrak{X}(M)$, we have*

$$\mathrm{rot}(\mathrm{grad}(f)) = 0, \qquad \mathrm{div}(\mathrm{rot}(v)) = 0,$$
$$\mathrm{rot}(\mathrm{rot}(v)) = \nabla^{\top *}\nabla^{\top} v + \mathrm{grad}(\mathrm{div}(v)) + \sum_{i=1}^{3} R(v, e_i)e_i,$$

where $\{e_i\}_{i=1,2,3}$ is the local orthonormal frame of TM, R is the curvature tensor, and $\nabla^{\top *}\nabla^{\top} = \sum_{i=1}^{3}(-\nabla^{\top}_{e_i}\nabla^{\top}_{e_i} + \nabla^{\top}_{\nabla^{\top}_{e_i} e_i})$ is the rough Laplacian.

This lemma implies the following, which corresponds to Proposition 3.8.

COROLLARY 4.5.

$$D^2(v, f) = \left(-4\mathrm{grad}(f) + v + \mathrm{rot}(v) + \nabla^{\top *}\nabla^{\top} v + \sum_{i=1}^{3} R(v, e_i)e_i,\ \Delta_{+} f + 4\mathrm{div}(v) + 9f\right).$$

Now, we give proofs.

Proof of Proposition 4.1. Let $\{e_1, e_2, e_3\} \subset TM$ be a local oriented orthonormal frame. Set $e_4 := r\frac{\partial}{\partial r}|_{r=1}$ and $\eta_j := J(e_j)$ for $1 \le j \le 4$. Then $\{\eta_j\}_{1 \le j \le 4}$ is a local oriented orthonormal frame of ν. Let $\{e^1, \cdots, e^4, \eta^1, \cdots, \eta^4\}$ be the dual coframe, then we have

$$\omega = \sum_{i=1}^{4} e^i \wedge \eta^i, \qquad \Omega = (e^1 + i\eta^1) \wedge \cdots \wedge (e^4 + i\eta^4).$$

Denoting $\nabla^{\top}_{e_i} e_j = \sum_{k=1}^{3} \Gamma^k_{ij} e_k$ and $\nabla^{\perp}_{e_i} \eta_a = \sum_{b=1}^{4} \tilde{\Gamma}^b_{ia} \eta_b$ for $1 \le i, j \le 3$ and $1 \le a \le 4$, we see the following by a direct computation.

LEMMA 4.6. *We have*

$$(e_i \times \eta_a) = \begin{pmatrix} \eta_4 & \eta_3 & -\eta_2 & -\eta_1 \\ -\eta_3 & \eta_4 & \eta_1 & -\eta_2 \\ \eta_2 & -\eta_1 & \eta_4 & -\eta_3 \end{pmatrix},$$
$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}, \qquad \tilde{\Gamma}^4_{ij} = -\delta_{ij}, \qquad \tilde{\Gamma}^k_{i4} = \delta_{ik}, \qquad \tilde{\Gamma}^4_{i4} = 0,$$

for $1 \le i, j, k \le 3, 1 \le a \le 4$.

Then via the identification $\mathfrak{X}(M) \oplus C^\infty(M) \ni (\sum_{j=1}^{3} v_j e_j, f) \mapsto \sum_{j=1}^{3} v_j \eta_j + f\eta_4 \in$

$C^\infty(M,\nu)$ where $v_j, f \in C^\infty(M)$, we have

$$
\begin{aligned}
&D\left(\sum_{j=1}^3 v_j\eta_j + f\eta_4\right)\\
=&\sum_{i=1}^3 e_i \times \nabla^\perp_{e_i}\left(\sum_{j=1}^3 v_j\eta_j + f\eta_4\right)\\
=&\sum_{i,j=1}^3 e_i(v_j)e_i\times\eta_j + \sum_{i,j=1}^3 v_je_i\times\left(\sum_{k=1}^3 \Gamma^k_{ij}\eta_k - \delta_{ij}\eta_4\right) + \sum_{i=1}^3 e_i(f)e_i\times\eta_4 + \sum_{i=1}^3 fe_i\times\eta_i\\
=&\left\{e_2(v_3) - e_3(v_2) + \sum_{j=1}^3 v_j(\Gamma^3_{2j} - \Gamma^2_{3j})\right\}\eta_1 + \left\{e_3(v_1) - e_1(v_3) + \sum_{j=1}^3 v_j(\Gamma^1_{3j} - \Gamma^3_{1j})\right\}\eta_2\\
&+\left\{e_1(v_2) - e_2(v_1) + \sum_{j=1}^3 v_j(\Gamma^2_{1j} - \Gamma^1_{2j})\right\}\eta_3 + \left\{\sum_{i=1}^3 e_i(v_i) + \sum_{i,j=1}^3 v_j\Gamma^i_{ij}\right\}\eta_4\\
&+\sum_{i=1}^3 v_i\eta_i - \sum_{i=1}^3 e_i(f)\eta_i + 3f\eta_4,
\end{aligned}
$$

which gives the proof. □

Proof of Lemma 4.4. The first two equations are easy to prove. We only prove the third equation. By Lemma A.1 and the fact that M is associative, it follows that

$$\mathrm{rot}(\mathrm{rot}(v)) = \sum_{i,j=1}^3 e_i \times (\nabla^\top_{e_i}e_j \times \nabla^\top_{e_j}v + e_j \times \nabla^\top_{e_i}\nabla^\top_{e_j}v).$$

From (2.3) , we have $u\times(v\times w) = -g(u,v)w + g(u,w)v$ for $u,v,w \in TM$ on an associative submanifold M. Hence we have

$$
\begin{aligned}
&\mathrm{rot}(\mathrm{rot}(v))\\
=&\sum_{i,j=1}^3 \left(\Gamma^j_{ii}\nabla^\top_{e_j}v + g(e_i,\nabla^\top_{e_j}v)\nabla^\top_{e_i}e_j - \delta_{ij}\nabla^\top_{e_i}\nabla^\top_{e_j}v + g(e_i,\nabla^\top_{e_i}\nabla^\top_{e_j}v)e_j\right)\\
=&\nabla^{\top *}\nabla^\top v + \sum_{i,j=1}^3 g((\nabla^\top_{e_i}\nabla^\top_{e_j} - \nabla^\top_{\nabla^\top_{e_i}e_j})v, e_i)e_j,
\end{aligned}
$$

where we use the fact that $\Gamma^j_{ii} = -\Gamma^i_{ij}$ since $\{e_i\}_{i=1,2,3}$ is the local orthonormal frame of TM. On the other hand, we have

$$\mathrm{grad}(\mathrm{div}(v)) = \sum_{i,j=1}^3 e_i(g(\nabla^\top_{e_j}v, e_j))e_i = \sum_{i,j=1}^3 g((\nabla^\top_{e_i}\nabla^\top_{e_j} - \nabla^\top_{\nabla^\top_{e_i}e_j})v, e_j)e_i,$$

which implies the proof. □

The proof of Corollary 4.5 is straightforward and we omit it.

Proof of Corollary 4.2. By Proposition 4.1, $D(v,f) = -(v,f)$ is equivalent to $\mathrm{rot}(v) + 2v = \mathrm{grad}(f), \mathrm{div}(v) = -4f$. Considering the divergence of the first equation, we have $\Delta_+ f = -2\mathrm{div}(v)$, which implies that $D(v,f) = -(v,f)$ is equivalent to

$$\begin{cases} \mathrm{rot}(v) + 2v = \mathrm{grad}(f), \\ \Delta_+ f = 8f. \end{cases} \tag{4.1}$$

The second equation is given in (2.6). For any $f \in C^\infty(M)$ satisfying $\Delta_+ f = 8f$, $(v, f) = (\frac{1}{2}\mathrm{grad}(f), f)$ is the solution of (4.1), which corresponds to the infinitesimal special Legendrian deformations of M. (See (2.7)). □

4.2. Associative deformation of homogeneous special Legendrians. The method and the notation in this subsection are summarized in the appendix. We give an explicit description of the operator rot when M is the reductive homogeneous space G/K, where $G \subset \mathrm{Aut}(S, \varphi, g)$ and $K \subset G$ is a closed subgroup. Take an $\mathrm{Ad}(K)$-invariant vector subspace of $\mathfrak{p} \subset \mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

It is well-known that there is an one-to-one correspondence between $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{p}$ and G-invariant metrics on $M = G/K$. Since $G \subset \mathrm{Aut}(S, \varphi, g)$, there exists a G-invariant metric g_M on M induced from (S, g). Denote by $\langle \cdot, \cdot \rangle$ the corresponding $\mathrm{Ad}(K)$-invariant inner product and by $\{e_1, e_2, e_3\} \subset \mathfrak{p}$ an oriented orthonormal basis of $\mathfrak{p}$. Then we have the following.

Lemma 4.7. *The map $G \times_{\mathrm{Ad}} \mathfrak{p} \ni [g, X] \mapsto \frac{d}{dt} g \cdot \exp(tX)K|_{t=0} \in TM$ is an isomorphism. Thus the tangent bundle TM of M is a homogeneous vector bundle.*

Proposition 4.8. *The operator rot : $\mathfrak{X}(M) \to \mathfrak{X}(M)$ is a homogeneous differential operator and induces the map $\widetilde{\mathrm{rot}} : C^\infty(G, \mathfrak{p})^{(K,\mathrm{Ad})} \to C^\infty(G, \mathfrak{p})^{(K,\mathrm{Ad})}$. If we define $\overline{\mathrm{rot}} \in (\mathrm{End}(\mathfrak{p}) \otimes U(\mathfrak{g}))^K$ by*

$$\overline{\mathrm{rot}} = \sum_{i \in \mathbb{Z}/3} e_i^* \otimes (e_{i+1} \wedge e_{i+2}) - \sum_{i \in \mathbb{Z}/3} \langle [e_{i+1}, e_{i+2}]_{\mathfrak{p}}, \cdot \rangle e_i \otimes 1,$$

where $\{e_i^*\}_{i=1,2,3}$ is the dual basis of $\{e_i\}_{i=1,2,3}$, we have

$$\overline{\mathrm{rot}}|_{C^\infty(G,\mathfrak{p})^{(K,\mathrm{Ad})}} = \widetilde{\mathrm{rot}}. \tag{4.2}$$

Remark 4.9. Set $[e_i, e_j]_{\mathfrak{p}} = \sum_{k=1}^3 c_{ij}^k e_k$. Then with respect to $\{e_1, e_2, e_3\}$, $\overline{\mathrm{rot}}$ is described as the following $U(\mathfrak{g})$-valued matrix:

$$\overline{\mathrm{rot}} = \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix} - \begin{pmatrix} c_{23}^1 & c_{23}^2 & c_{23}^3 \\ c_{31}^1 & c_{31}^2 & c_{31}^3 \\ c_{12}^1 & c_{12}^2 & c_{12}^3 \end{pmatrix}.$$

Proof of Proposition 4.8. It is straightforward to show that rot is a homogeneous differential operator. Since $\overline{\mathrm{rot}}$ is independent of the choice of $\{e_i\}_{i=1,2,3}$ and $\mathrm{Ad}(K)$ preserves the orientation and the metric, we see that $\overline{\mathrm{rot}}$ is K-invariant.

From Remark B.8, a homogeneous differential operator is completely determined by its value at a point, and so we only have to compute rot at $eK \in G/K = M$.

For any $\tilde{v} = \sum_{i=1}^3 v_i e_i \in C^\infty(G, \mathfrak{p})^{(K,\mathrm{Ad})}$ where $v_i \in C^\infty(G)$, denote by $v \in \mathfrak{X}(M)$ the induced vector field:

$$v(gK) = \frac{d}{dt} g \cdot \exp\left(t \sum_{i=1}^3 v_i(g) e_i \right) \cdot K \Bigg|_{t=0}.$$

Take local coordinates (y_1, y_2, y_3) around eK defined by $(y_1, y_2, y_3) \mapsto \exp\left(\sum_{i=1}^3 y_i e_i\right)$. Let $\pi : G \to G/K = M$ be the projection and $\tau_g : M \to M$

for $g \in G$ be the left translation. Denoting $\nabla^{\top}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} = \sum_{k=1}^{3} \Gamma_{ij}^k \frac{\partial}{\partial y_k}$, we see the following.

Lemma 4.10 ([9, 22]). *For a sufficiently small $X \in \mathfrak{p}$, we have*

$$\begin{aligned}
\left(\frac{\partial}{\partial y_i}\right)_{\exp(X)\cdot K} =&((\tau_{\exp(X)})_*)_{eK}(\pi_*)_e \left(\sum_{m=0}^{\infty} \frac{(-\mathrm{ad}(X))^m}{(m+1)!} e_i\right) \\
=&((\tau_{\exp(X)})_*)_{eK} \left(\sum_{m=0}^{\infty} \frac{(-\mathrm{ad}(X))^m}{(m+1)!} \left(\frac{\partial}{\partial y_i}\right)_{eK}\right), \\
g_M\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)_{eK} =&\delta_{ij}, \qquad \Gamma_{ij}^k(eK) = \frac{1}{2}(c_{ki}^j + c_{kj}^i).
\end{aligned}$$

Now we compute $\mathrm{rot}(v)$ at $eK \in G/K = M$. First, we compute $(\nabla^{\top}_{\frac{\partial}{\partial y_i}} v)_{eK}$. Since the metric g_M is G-invariant, we have

$$\langle e_i, e_j \rangle = g_M\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)_{eK} = g_M\left(((\tau_{\exp(X)})_*)_{eK}\left(\frac{\partial}{\partial y_i}\right)_{eK}, ((\tau_{\exp(X)})_*)_{eK}\left(\frac{\partial}{\partial y_j}\right)_{eK}\right)$$

for any $X \in \mathfrak{p}$. Then for sufficiently small $t \in \mathbb{R}$, it follows that

$$\begin{aligned}
g_M\left(v, \frac{\partial}{\partial y_j}\right)_{exp(te_i)\cdot K} &= g_M\left(\sum_{k=1}^{3} v_k(\exp(te_i))(\tau_{\exp(te_i)})_*\left(\frac{\partial}{\partial y_k}\right)_{eK}, \left(\frac{\partial}{\partial y_j}\right)_{\exp(te_i)\cdot K}\right) \\
&= \sum_{k=1}^{3} v_k(\exp(te_i)) \left\langle e_k, \sum_{m=0}^{\infty} \frac{(-t\cdot \mathrm{ad}(e_i))^m}{(m+1)!} e_j \right\rangle, \\
g_M\left(\nabla^{\top}_{\frac{\partial}{\partial y_i}} v, \frac{\partial}{\partial y_j}\right)_{eK} &= \left(\frac{\partial}{\partial y_i}\right)_{eK} g_M\left(v, \frac{\partial}{\partial y_j}\right) - g_M\left(v, \nabla^{\top}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j}\right)_{eK} \\
&= \sum_{k=1}^{3}\left(e_i(v_k)\delta_{kj} + v_k\left\langle e_k, -\frac{1}{2}[e_i, e_j]\right\rangle\right) - g_M\left(v, \sum_{k=1}^{3} \Gamma_{ij}^k(eK)\frac{\partial}{\partial y_k}\right) \\
&= e_i(v_j) - \frac{1}{2}\sum_{k=1}^{3} v_k(c_{ij}^k + c_{ki}^j + c_{kj}^i).
\end{aligned}$$

Hence we obtain

$$(\nabla^{\top}_{\frac{\partial}{\partial y_i}} v)_{eK} = \sum_{j=1}^{3} e_i(v_j)\frac{\partial}{\partial y_j} - \frac{1}{2}\sum_{j,k=1}^{3} v_k(c_{ij}^k + c_{ki}^j + c_{kj}^i)\frac{\partial}{\partial y_j}.$$

Thus we compute

$$\begin{aligned}
(\mathrm{rot}(v))_{eK} =& \left(\frac{\partial}{\partial y_i}\right)_{eK} \times (\nabla^{\top}_{\frac{\partial}{\partial y_i}} v)_{eK} \\
=& \left(-e_3(v_2) + e_2(v_3) - \sum_{j=1}^{3} c_{23}^j\right)\frac{\partial}{\partial y_1} + \left(e_3(v_1) - e_1(v_3) - \sum_{j=1}^{3} c_{31}^j\right)\frac{\partial}{\partial y_2} \\
&+ \left(-e_2(v_1) + e_1(v_2) - \sum_{j=1}^{3} c_{12}^j\right)\frac{\partial}{\partial y_3},
\end{aligned}$$

which implies the proof. □

5. Associative deformations in the sine cone of nearly Kähler manifolds. Let $(N, k, J, \sigma, \psi^{\pm})$ be a nearly Kähler manifold and $L \subset N$ be a Lagrangian submanifold. From Lemma 2.26 and 2.30, the sine cone $C_s(N) = (0, \pi) \times N$ admits nearly parallel G_2-structure $(\tilde{\varphi}, \tilde{k})$ and $\{\frac{\pi}{2}\} \times L \subset C_s(N)$ is associative. We study the infinitesimal associative deformations of $\{\frac{\pi}{2}\} \times L$.

Let $\nu \to \{\frac{\pi}{2}\} \times L$ be the normal bundle of $\{\frac{\pi}{2}\} \times L \subset C_s(N)$. First, we rewrite the operator $D : C^\infty(\{\frac{\pi}{2}\} \times L, \nu) \to C^\infty(\{\frac{\pi}{2}\} \times L, \nu)$ in Proposition 3.2 in this case. Since L is Lagrangian, there exists canonical isomorphism $TL \oplus \mathbb{R} \ni (v, x) \mapsto Jv + x\frac{\partial}{\partial t}|_{t=\frac{\pi}{2}} \in \nu$. Via this identification, we obtain the following.

Proposition 5.1. *The corresponding operator $D : \mathfrak{X}(L) \oplus C^\infty(L) \to \mathfrak{X}(L) \oplus C^\infty(L)$ is described as*

$$D(v, f) = (-\mathrm{grad}(f) - \mathrm{rot}(v) + 2v, \mathrm{div}(v)),$$

where we use the notation in Proposition 4.1.

By Proposition 5.1, we prove Corollary 5.2 as in the case of Corollary 4.2.

Corollary 5.2. *We have*

$$\begin{aligned}&\dim\{\textit{the infinitesimal associative deformations of } \{\tfrac{\pi}{2}\} \times L\} \\ &= \dim\{f \in C^\infty(L); \Delta_+ f = 3f\} + \dim\{v \in \mathfrak{X}(L); \mathrm{rot}(v) = 3v\}.\end{aligned}$$

Remark 5.3. From Lemma 2.32, $\dim\{f \in C^\infty(L); \Delta_+ f = 3f\}$ gives the dimension of infinitesimal associative and non-Lagrangian deformations.

Proof of Proposition 5.1. Let $\{e_1, e_2, e_3\} \subset TL$ be a local oriented orthonormal frame such that $\psi^-(e_1, e_2, e_3) = -1$. Set $\eta_j := J(e_j)$ for $1 \leq j \leq 3$ and $\eta_4 := \frac{\partial}{\partial t}|_{t=\frac{\pi}{2}}$. Then $\{\eta_j\}_{1 \leq j \leq 4}$ is a local oriented orthonormal frame of ν. Let $\{e^1, \cdots, e^3, \eta^1, \cdots, \eta^4\}$ be the dual coframe, then we have

$$\sigma = \sum_{i=1}^{3} e^i \wedge \eta^i, \qquad \Psi = \psi^+ + i\psi^- = -i(e^1 + i\eta^1) \wedge (e^2 + i\eta^2) \wedge (e^3 + i\eta^3).$$

Hence at a point of $L \times \{\frac{\pi}{2}\}$, we have

$$\tilde{\varphi} = \eta^4 \wedge \sum_{i=1}^{3} e^i \wedge \eta^i + e^1(e^{23} - \eta^{23}) - \eta^1(e^2 \wedge \eta^3 + \eta^2 \wedge e^3).$$

As in the Sasakian case, the definition of the Levi-Civita connection gives the following.

Lemma 5.4. *For any $X, Y \in \mathfrak{X}(N \times \{\frac{\pi}{2}\})$, we have*

$$\nabla^{C_s(N)}_X Y|_{N \times \{\frac{\pi}{2}\}} = \nabla^N_X Y, \qquad \nabla^{C_s(N)}_X \frac{\partial}{\partial t}|_{N \times \{\frac{\pi}{2}\}} = 0, \qquad \nabla^{C_s(N)}_{\frac{\partial}{\partial t}} X|_{N \times \{\frac{\pi}{2}\}} = 0,$$

where $\nabla^{C_s(N)}$ and ∇^N are the Levi-Civita connections of $\tilde{k}$ on $C_s(N)$ and k on N, respectively.

Denoting $\nabla^\top_{e_i} e_j = \sum_{k=1}^3 \Gamma^k_{ij} e_k$ for $1 \leq i, j \leq 3$, we see the following from the computations above and Lemma 2.28.

Lemma 5.5.

$$(e_i \times \eta_a) = \begin{pmatrix} \eta_4 & -\eta_3 & \eta_2 & -\eta_1 \\ \eta_3 & \eta_4 & -\eta_1 & -\eta_2 \\ -\eta_2 & \eta_1 & \eta_4 & -\eta_3 \end{pmatrix}, \qquad \nabla^{\perp}_{e_i}\eta_j = \sum_{k=1}^{3}(\epsilon_{ijk} + \Gamma^k_{ij})\eta_k,$$

where ϵ_{ijk} is the permutation symbol and $1 \le i, j \le 3$.

Via the identification $\mathfrak{X}(L) \oplus C^\infty(L) \ni (\sum_{j=1}^3 v_j e_j, f) \mapsto \sum_{j=1}^3 v_j\eta_j + f\eta_4 \in C^\infty(L \times \{\frac{\pi}{2}\}, \nu)$ where $v_j, f \in C^\infty(L)$, we can calculate as in the proof of Proposition 4.1. □

6. Computation in the standard sphere S^7. By Definition 2.1, $\mathbb{C}^4$ admits a torsion-free Spin(7)-structure (Φ_0, h_0), which induces the nearly parallel G_2-structure (φ, g) on S^7 by Lemma 2.16. In this section, we study the deformation spaces of homogeneous associative submanifolds in S^7, and prove Theorem 1.1 and 1.2.

6.1. Classification of homogeneous associative submanifolds in S^7. Mashimo [18] classified homogeneous Lagrangian submanifolds in S^6. Applying this classification, Lotay [16] classified homogeneous associative submanifolds in S^7.

Proposition 6.1 ([16, 18]). *Let A be a connected associative 3-fold in $S^7 \subset \mathbb{C}^4$ which is the orbit of a closed 3-dimensional Lie subgroup of* Spin(7). *If A does not lie in a totally geodesic S^6, then, up to the* Spin(7)*-action, A is either*
1. *$A_1 \cong T^3$ given by Example 6.2,*
2. *$A_2 \cong \mathrm{SU}(2)/\mathbb{Z}_3$, or $A_3 \cong \mathrm{SU}(2)$ given by Example 6.3,*

If A lies in a totally geodesic S^6, then, up to the G_2-action, A is either
1. *the totally geodesic $S^3 \cong \mathrm{SU}(2)$,*
2. *$L_1 \cong \mathrm{SU}(2)$ given by Example 6.5,*
3. *$L_2 \cong \mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$ given by Example 6.6, or*
4. *$L_3 \cong \mathrm{SU}(2)/A_4^*$, or $L_4 \cong \mathrm{SU}(2)/D_3^*$ given by Example 6.7.*

Note that the automorphism group of nearly parallel S^7 is Spin(7) and that of nearly Kähler S^6 is G_2.

Example 6.2. Define the action of $T^3 \cong (\mathbb{R}/2\pi\mathbb{Z})^3$ on $\mathbb{C}^4$ by

$$(\theta_1, \theta_2, \theta_3) \cdot {}^t(z_1, z_2, z_3, z_4) = {}^t(e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3, e^{-i(\theta_1+\theta_2+\theta_3)} z_4),$$

where $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$ and $z_i \in \mathbb{C}$. Then

$$A_1 := T^3 \cdot \frac{1}{2}{}^t(1, 1, 1, i) \cong T^3$$

is special Legendrian given in [8].

Example 6.3. Define the SU(2)-action on $\mathbb{C}^4$ by

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} a^3 & -\sqrt{3}a^2\overline{b} & \sqrt{3}a\overline{b}^2 & -\overline{b}^3 \\ \sqrt{3}a^2 b & a(|a|^2 - 2|b|^2) & -\overline{b}(2|a|^2 - |b|^2) & \sqrt{3}\overline{a}\overline{b}^2 \\ \sqrt{3}ab^2 & b(2|a|^2 - |b|^2) & \overline{a}(|a|^2 - 2|b|^2) & -\sqrt{3}\overline{a}^2\overline{b} \\ b^3 & \sqrt{3}\overline{a}b^2 & \sqrt{3}\overline{a}^2 b & \overline{a}^3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \tag{6.1}$$

where $z_i \in \mathbb{C}$ and $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Set

$$A_2 = \mathrm{SU}(2) \cdot {}^t(1,0,0,0) \cong \mathrm{SU}(2)/\mathbb{Z}_3, \qquad A_3 = \mathrm{SU}(2) \cdot \frac{1}{\sqrt{2}}{}^t(0,1,i,0) \cong \mathrm{SU}(2),$$

where $\mathbb{Z}_3 = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \overline{\zeta} \end{pmatrix} \in \mathrm{SU}(2); \zeta^3 = 1 \right\}$. Then A_2 is the Hopf lift of the Veronese curve in $\mathbb{C}P^3$:

$$\{[x^3 : \sqrt{3}x^2 y : \sqrt{3}xy^2 : y^3] \in \mathbb{C}P^3; [x : y] \in \mathbb{C}P^1\},$$

and hence associative. However, A_3 is an associative submanifold which does not arise from other known geometries.

Remark 6.4. Set $A_2(\theta) = \mathrm{SU}(2) \cdot {}^t(\cos\theta, 0, 0, \sin\theta)$ for $\theta \in [0, \frac{\pi}{4}]$. It is known that all the $A_2(\theta)$ are congruent up to the Spin(7)-action to $A_2 = A_2(0)$, which is U(2)-invariant. In [10], $A_2(\frac{\pi}{4})$ is shown to be special Legendrian.

Next, we give examples of homogeneous Lagrangian submanifolds in S^6.

Example 6.5. Define the SU(2)-action on $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ by

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} (|a|^2 - |b|^2)x_1 - 2\mathrm{Im}(\overline{ab}z_1) \\ 2i\overline{a}bx_1 + \overline{a}^2 z_1 + b^2 \overline{z_1} \\ az_2 - \overline{b}\overline{z_3} \\ \overline{b}\overline{z_2} + az_3 \end{pmatrix}, \tag{6.2}$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Then

$$L_1 := \mathrm{SU}(2) \cdot {}^t(\tfrac{\sqrt{5}}{3}, 0, \tfrac{2}{3}, 0) \cong \mathrm{SU}(2), \tag{6.3}$$

where ${}^t(\frac{\sqrt{5}}{3}, 0, \frac{2}{3}, 0) \in \mathbb{R} \oplus \mathbb{C}^3$, is Lagrangian in S^6 given in [8]. Moreover, L_1 is invariant under a $\mathrm{U}(2) (\subset G_2)$ action.

Example 6.6. Let $L_2 \subset S^6$ be given by

$$L_2 = \left\{(0, z_1, z_2, z_3) \in \mathbb{R} \oplus \mathbb{C}^3; |z_1|^2 + |z_2|^2 + |z_3|^2 = 1, z_1^2 + z_2^2 + z_3^2 = 0\right\}. \tag{6.4}$$

Since L_2 is the link of an complex cone, it is Lagrangian in S^6. Define the SO(3)-action on $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ by the trivial action of $\mathbb{R}$ and the standard (real) action on $\mathbb{C}^3$. Let $\varpi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ be a standard double covering:

$$\varpi : \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mapsto \begin{pmatrix} |a|^2 - |b|^2 & 2\mathrm{Im}(ab) & -2\mathrm{Re}(ab) \\ -2\mathrm{Im}(\overline{a}b) & \mathrm{Re}(a^2 + b^2) & \mathrm{Im}(a^2 + b^2) \\ 2\mathrm{Re}(\overline{a}b) & \mathrm{Im}(-a^2 + b^2) & \mathrm{Re}(a^2 - b^2) \end{pmatrix}, \tag{6.5}$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. By composing these actions, SU(2) acts on $\mathbb{R}^7$, and we have

$$L_2 = \mathrm{SU}(2) \cdot \frac{1}{\sqrt{2}}{}^t(0,0,1,i) \cong \mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{SO}(3).$$

EXAMPLE 6.7. Let $\{\epsilon_1, \cdots, \epsilon_7\}$ be a standard basis for $\mathbb{R}^7$. Identify $\mathbb{R}^7$ with the homogeneous harmonic cubics $\mathcal{H}^3(\mathbb{R}^3)$ on $\mathbb{R}^3$ by:

$$\begin{aligned}
&\epsilon_1 \mapsto \frac{\sqrt{10}}{10}x(2x^2-3y^2-3z^2); && \epsilon_2 \mapsto -\sqrt{6}xyz; \quad \epsilon_3 \mapsto \frac{\sqrt{6}}{2}x(y^2-z^2);\\
&\epsilon_4 \mapsto -\frac{\sqrt{15}}{10}y(4x^2-y^2-z^2); && \epsilon_5 \mapsto -\frac{\sqrt{15}}{10}z(4x^2-y^2-z^2);\\
&\epsilon_6 \mapsto \frac{1}{2}y(y^2-3z^2); && \epsilon_7 \mapsto \frac{1}{2}z(z^2-3y^2).
\end{aligned}$$

Let SU(2) act on $\mathcal{H}^3(\mathbb{R}^3) \cong \mathbb{R}^7$ as $A \cdot f(x,y,z) = f((x,y,z)\varpi(A))$, where $A \in \mathrm{SU}(2)$ and $f \in \mathcal{H}^3(\mathbb{R}^3) \cong \mathbb{R}^7$. Set

$$L_3 := \mathrm{SU}(2)\cdot\epsilon_2, \qquad L_4 := \mathrm{SU}(2)\cdot\epsilon_6.$$

Then $L_3 \cong \mathrm{SU}(2)/A_4^*$ and $L_4 \cong \mathrm{SU}(2)/D_3^*$ are Lagrangian, where A_4^* is a binary tetrahedral group of order 24 generated by

$$k_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad k_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad k_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{\pi i}{4}} & -e^{\frac{-\pi i}{4}} \\ e^{\frac{\pi i}{4}} & e^{\frac{-\pi i}{4}} \end{pmatrix}, \tag{6.6}$$

and D_3^* is a binary dihedral group of order 12 generated by

$$k_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad k_5 = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}. \tag{6.7}$$

6.2. Computations on SU(2)**.** For the convenience of the following computations, we summarize formulas on SU(2). Define the basis of the Lie algebra $\mathfrak{su}(2)$ of SU(2) by

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{6.8}$$

which satisfies the relation $[E_i, E_{i+1}] = 2E_{i+2}$ for $i \in \mathbb{Z}/3$. We see the following by Proposition B.5 and Lemma B.6.

LEMMA 6.8. *Let V_n be a $\mathbb{C}$-vector space of all complex homogeneous polynomials with two variables z_1, z_2 of degree $n (n \geq 0)$ and define the representation $\rho_n : \mathrm{SU}(2) \to \mathrm{GL}(V_n)$ as*

$$\left(\rho_n\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} f\right)(z_1,z_2) = f\left((z_1,z_2)\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}\right).$$

Define the Hermitian inner product $\langle\ ,\ \rangle$ of V_n such that

$$\left\{ v_k^{(n)} = \frac{1}{\sqrt{k!(n-k)!}} z_1^{n-k} z_2^k \right\}_{0 \leq k \leq n}$$

is a unitary basis of V_n. If we denote by $\widehat{\mathrm{SU}(2)}$ the set of all equivalence classes of finite dimensional irreducible representations of SU(2)*, we know that $\widehat{\mathrm{SU}(2)} =$*

$\{(V_n, \rho_n); n \geq 0\}$. *Then every* $\mathbb{C}$*-valued continuous function on* $\mathrm{SU}(2)$ *is uniformly approximated by the* $\mathbb{C}$*-linear combination of the following functions:*

$$\left\{ \langle \rho_n(\cdot) v_i^{(n)}, v_j^{(n)} \rangle; n \geq 0, 0 \leq i, j \leq n \right\},$$

which are mutually orthogonal with respect to the L_2 *inner product.*

By a direct computation, we see the following.

LEMMA 6.9. *Identify* $X \in \mathfrak{su}(2) \subset U(\mathfrak{su}(2))$ *with the left invariant differential operator on* $\mathrm{SU}(2)$. *For* $u = \sum_{l=0}^{n} C_l v_l^{(n)} \in V_n$, *set*

$$u^* = \sum_{l=0}^{n} (-1)^{n-l} \overline{C}_{n-l} v_l^{(n)} \in V_n.$$

Then for any $n \geq 0, 0 \leq k, l \leq n, u, v \in V_n, X \in \mathfrak{su}(2)$, *we have*

$$\begin{aligned}
X \langle \rho_n(\cdot) v, u \rangle &= \langle \rho_n(\cdot) d\rho_n(X) v, u \rangle, \\
(d\rho_n(X)v)(z_1, z_2) &= \left(\frac{\partial v}{\partial z_1}, \frac{\partial v}{\partial z_2} \right) {}^t X \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \\
\overline{\langle \rho_n(\cdot) v_k^{(n)}, u \rangle} &= (-1)^k \langle \rho_n(\cdot) v_{n-k}^{(n)}, u^* \rangle, \\
(-iE_1 + E_2) \langle \rho_n(\cdot) v_k^{(n)}, u \rangle &= \begin{cases} 2i\sqrt{(k+1)(n-k)} \langle \rho_n(\cdot) v_{k+1}^{(n)}, u \rangle, & (k < n) \\ 0, & (k = n) \end{cases} \\
(iE_1 + E_2) \langle \rho_n(\cdot) v_k^{(n)}, u \rangle &= \begin{cases} 2i\sqrt{k(n-k+1)} \langle \rho_n(\cdot) v_{k-1}^{(n)}, u \rangle, & (k > 0) \\ 0, & (k = 0) \end{cases} \\
iE_3 \langle \rho_n(\cdot) v_k^{(n)}, u \rangle &= (-n + 2k) \langle \rho_n(\cdot) v_k^{(n)}, u \rangle.
\end{aligned}$$

The next lemma is useful for the later computations.

LEMMA 6.10. *Suppose that* $\{e_1, e_2, e_3\} = \{pE_1, pE_2, qE_3\}$ *where* $0 \neq p, q \in \mathbb{R}$ *is an oriented orthonormal basis of* $\mathfrak{su}(2)$ *for some metric and orientation. For* $v = \sum_{i=1}^{3} v_i e_i \in C^\infty(\mathrm{SU}(2), \mathfrak{su}(2))$, $\overline{\mathrm{rot}}(v) = \alpha v$ *for* $0 \neq \alpha \in \mathbb{R}$ *is equivalent to*

$$(ie_3 - (2q + \alpha))(v_1 + iv_2) + (-ie_1 + e_2)v_3 = 0, \tag{6.9}$$

$$(ie_1 + e_2)(v_1 + iv_2) + \left(\alpha + \frac{2p^2}{q} + ie_3 \right) v_3 = 0. \tag{6.10}$$

These equations imply that

$$\left\{ \Delta_+ + \left(\frac{4p^2}{q} - 4q \right) ie_3 + \left(-\alpha - \frac{2p^2}{q} + 2q \right)(2q + \alpha) \right\} (v_1 + iv_2) = 0, \tag{6.11}$$

$$\left\{ \Delta_+ - \alpha \left(\alpha + \frac{2p^2}{q} \right) \right\} v_3 = 0, \tag{6.12}$$

where $\Delta_+ = -\sum_{i=1}^{3} e_i^2$ *is a Laplacian. Especially, for any* $n \geq 0, 0 \leq k \leq n, u \in V_n$, *we have*

$$\Delta_+\langle\rho_n(\cdot)v_k^{(n)},u\rangle=\left\{(-p^2+q^2)(n-2k)^2+p^2(n^2+2n)\right\}\langle\rho_n(\cdot)v_k^{(n)},u\rangle, \tag{6.13}$$

$$\begin{aligned}&\left\{\Delta_+ +\left(\frac{4p^2}{q}-4q\right)ie_3+\left(-\alpha-\frac{2p^2}{q}+2q\right)(2q+\alpha)\right\}\langle\rho_n(\cdot)v_k^{(n)},u\rangle\\&=\left\{(-p^2+q^2)(n-2k+2)^2+p^2(n^2+2n)-\alpha\left(\alpha+\frac{2p^2}{q}\right)\right\}\langle\rho_n(\cdot)v_k^{(n)},u\rangle.\end{aligned} \tag{6.14}$$

Remark 6.11. In the case of $\mathrm{SU}(2)/\Gamma$ for some finite subgroup Γ, we have to consider the Γ equivariant solutions of (6.9) and (6.10).

Proof. Note that $[e_1,e_2]=\frac{2p^2}{q}e_3, [e_1,e_3]=-2qe_2, [e_2,e_3]=2qe_1$. Then from Remark 4.9, $\overline{\mathrm{rot}}(v)=\alpha v$ is equivalent to

$$ie_3(v_1+iv_2)+(-ie_1+e_2)(v_3)=(2q+\alpha)(v_1+iv_2), \tag{6.15}$$

$$\mathrm{Re}((ie_1+e_2)(v_1+iv_2))=-\left(\alpha+\frac{2p^2}{q}\right)v_3. \tag{6.16}$$

It is clear that (6.9) and (6.10) imply (6.15) and (6.16). Conversely, suppose that (6.15) and (6.16) hold. Applying (ie_1+e_2) to (6.15), we obtain

$$(ie_3-\alpha)(ie_1+e_2)(v_1+iv_2)+\left(e_1^2+e_2^2+\frac{2p^2}{q}ie_3\right)v_3=0.$$

Considering the real and imaginary parts, we obtain from (6.16)

$$-e_3\mathrm{Im}((ie_1+e_2)(v_1+iv_2))+\alpha\left(\alpha+\frac{2p^2}{q}\right)v_3+(e_1^2+e_2^2)(v_3)=0, \tag{6.17}$$

$$-\alpha e_3(v_3)-\alpha\mathrm{Im}((ie_1+e_2)(v_1+iv_2))=0. \tag{6.18}$$

The equations (6.16) and (6.18) imply (6.10), and hence we obtain the first statement.

Substituting (6.18) into (6.17), we have (6.12). Applying $(-ie_1+e_2)$ to (6.10), we obtain from (6.9)

$$\begin{aligned}&\left(e_1^2+e_2^2-\frac{2p^2}{q}ie_3\right)(v_1+iv_2)\\&=\left(-\alpha-\frac{2p^2}{q}+2q-ie_3\right)(-ie_1+e_2)v_3\\&=\left\{-e_3^2+\left(\frac{4p^2}{q}-4q\right)ie_3+\left(-\alpha-\frac{2p^2}{q}+2q\right)(2q+\alpha)\right\}(v_1+iv_2),\end{aligned}$$

which imply (6.11). Then from Lemma 6.9, we obtain (6.13) and (6.14). □

6.3. The case A_1, A_2, and A_3. First, we study the deformation of homogeneous associative submanifolds which do not lie in a totally geodesic S^6.

6.3.1. The case $A_1\cong T^3$. Define the basis of the Lie algebra $\mathfrak{t}^3$ of T^3 by

$$e_1=(\sqrt{2},0,0),\qquad e_2=(0,\sqrt{2},-\sqrt{2}),\qquad e_3=(-1,1,1)\in\mathbb{R}^3\cong\mathfrak{t}^3,$$

which is an oriented orthonormal basis of $\mathfrak{t}^3$ with respect to the orientation and the metric induced from A_1.

Define the smooth function $f_\gamma \in C^\infty(T^3, \mathbb{C})$ for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3$ on $T^3 \cong (\mathbb{R}/2\pi\mathbb{Z})^3$ by $f_\gamma(\theta_1, \theta_2, \theta_3) = \exp(i \sum_{j=1}^3 \gamma_j \theta_j)$. By a Fourier series expansion, every $\mathbb{C}$-valued continuous function on T^3 is uniformly approximated by the $\mathbb{C}$-linear combination of f_γ's. By a direct computation, we obtain the following.

LEMMA 6.12. *Identifying $e_i \in \mathfrak{t}^3$ with the left invariant differential operator on T^3, we have*

$$e_1(f_\gamma) = \sqrt{2}\gamma_1 i f_\gamma, \qquad e_2(f_\gamma) = \sqrt{2}(\gamma_2 - \gamma_3) i f_\gamma, \qquad e_3(f_\gamma) = (-\gamma_1 + \gamma_2 + \gamma_3) i f_\gamma,$$
$$\Delta_+(f_\gamma) = \{2\gamma_1^2 + 2(\gamma_2 - \gamma_3)^2 + (-\gamma_1 + \gamma_2 + \gamma_3)^2\} f_\gamma.$$

Then we deduce the following.

PROPOSITION 6.13. $\dim_{\mathbb{R}}\{f \in C^\infty(T^3); \Delta_+ f = 8f\} = 12.$

PROPOSITION 6.14. $\dim_{\mathbb{R}}\{v \in C^\infty(T^3, \mathfrak{t}^3); \overline{\mathrm{rot}}(v) = -2v\} = 6.$

By Corollary 4.2, these imply that associative deformations of A_1 are trivial since Spin(7) induces $18 (= \dim_{\mathbb{R}}(\mathrm{Spin}(7)/T^3))$-dimensional associative deformations of A_1. Now, we give proofs.

Proof of Proposition 6.13. For $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3$, we know that

$$\begin{aligned} &2\gamma_1^2 + 2(\gamma_2 - \gamma_3)^2 + (-\gamma_1 + \gamma_2 + \gamma_3)^2 = 8 \\ \Leftrightarrow &(\gamma_1, \gamma_2, \gamma_3) = \pm(2,1,1), \pm(0,1,-1), \pm(1,2,1), \pm(1,1,2), \pm(1,-1,0), \pm(1,0,-1), \end{aligned}$$

which gives the proof by Lemma 6.12. □

Proof of Proposition 6.14. Take any $v = \sum_{i=1}^3 v_i e_i \in C^\infty(T^3, \mathfrak{t}^3)$ where $v_i \in C^\infty(T^3)$. Then from Remark 4.9, $\overline{\mathrm{rot}}(v) = \alpha v$ for $\alpha \in \mathbb{R}$ is equivalent to

$$(ie_3 - \alpha)(v_1 + iv_2) + (-ie_1 + e_2)(v_3) = 0, \tag{6.19}$$
$$\mathrm{Re}((ie_1 + e_2)(v_1 + iv_2)) = -\alpha v_3. \tag{6.20}$$

Eliminating v_3, we have

$$-\alpha(ie_3 - \alpha)(v_1 + iv_2) + (-ie_1 + e_2)\mathrm{Re}((ie_1 + e_2)(v_1 + iv_2)) = 0. \tag{6.21}$$

Set $v_1 + iv_2 = \sum_{\gamma \in \mathbb{Z}^3} C_\gamma f_\gamma$ where $C_\gamma \in \mathbb{C}$. Since $\overline{f}_\gamma = f_{-\gamma}$, (6.21) is equivalent to

$$C_\gamma \left(-\gamma_1^2 - (\gamma_2 - \gamma_3)^2 + \alpha(-\gamma_1 + \gamma_2 + \gamma_3 + \alpha)\right) + \overline{C}_{-\gamma}(\gamma_1 + (\gamma_2 - \gamma_3)i)^2 = 0. \tag{6.22}$$

Take the complex conjugation of (6.22) and replace γ by $-\gamma$, then we obtain

$$C_\gamma(\gamma_1 + (-\gamma_2 + \gamma_3)i)^2 + \overline{C}_{-\gamma}\left(-\gamma_1^2 - (\gamma_2 - \gamma_3)^2 + \alpha(\gamma_1 - \gamma_2 - \gamma_3 + \alpha)\right) = 0. \tag{6.23}$$

Eliminating $\overline{C}_\gamma$ from (6.22) and (6.23), we have

$$\alpha^2 \left\{-2(\gamma_1^2 + (\gamma_2 - \gamma_3)^2) - \alpha^2(-\gamma_1 + \gamma_2 + \gamma_3)^2 + \alpha^2\right\} C_\gamma = 0.$$

Set $\alpha = -2$. Since we know that $-2(\gamma_1^2 + (\gamma_2 - \gamma_3)^2) - (-\gamma_1 + \gamma_2 + \gamma_3)^2 + 4 = 0 \Leftrightarrow (\gamma_1, \gamma_2, \gamma_3) = \pm(1,1,0), \pm(1,0,1), \pm(0,1,1)$, we deduce by (6.22) that

$$\begin{aligned} v_1 + iv_2 = &C_{(1,1,0)} f_{(1,1,0)} - i\overline{C}_{(1,1,0)} f_{(-1,-1,0)} + C_{(1,0,1)} f_{(1,0,1)} + i\overline{C}_{(1,0,1)} f_{(-1,0,-1)} \\ &+ C_{(0,-1,-1)} f_{(0,-1,-1)}. \end{aligned}$$

Thus $v_1 + iv_2$ depends 3 complex parameters $C_{(1,1,0)}, C_{(1,0,1)}, C_{(0,-1,-1)}$, which implies Proposition 6.14. □

6.3.2. The case $A_2 \cong \mathrm{SU}(2)/\mathbb{Z}_3$. By Remark 6.4, $A_2 = A_2(0)$ is congruent to $A_2(\frac{\pi}{4})$, which is special Legendrian. We may compute the dimension of the infinitesimal associative deformations of $A_2(\frac{\pi}{4})$ by Corollary 4.2. The action (6.1) induces an inclusion $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(4)$, where E_1, E_2, E_3 in (6.8) correspond to

$$\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & 2i & 0 \\ 0 & 2i & 0 & \sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}, \begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix},$$

respectively. Set $p_0 = \frac{1}{\sqrt{2}}{}^t(1, 0, 0, 1) \in \mathbb{C}^4$. Then we have

$$(E_1^*)_{p_0} = \sqrt{\frac{3}{2}}{}^t(0, -1, 1, 0), \qquad (E_2^*)_{p_0} = \sqrt{\frac{3}{2}}{}^t(0, i, i, 0), \qquad (E_3^*)_{p_0} = \frac{3i}{\sqrt{2}}{}^t(1, 0, 0, -1),$$

Hence if we set $e_1 := E_1/\sqrt{3}, e_2 := E_2/\sqrt{3}, e_3 := E_3/3$, $\{e_i\}_{1 \le i \le 3}$ is an oriented orthonormal basis of $\mathfrak{su}(2)$ with respect to the orientation and the metric induced from A_2.

Proposition 6.15. $\dim_{\mathbb{R}}\{f \in C^\infty(A_2); \Delta_+ f = 8f\} = 19$.

Proposition 6.16. $\dim_{\mathbb{R}}\{v \in \mathfrak{X}(A_2); \mathrm{rot}(v) = -2v\} = 11$.

On the other hand, Spin(7) induces $17 (= \dim_{\mathbb{R}}(\mathrm{Spin}(7)/\mathrm{U}(2)))$-dimensional associative deformations of A_2. By Corollary 4.2 and Remark 6.4, we have a 30-dimensional infinitesimal associative deformation space of A_2, and hence A_2 can have non-trivial associative deformations. In fact, we obtain the following.

Proposition 6.17. *All non-trivial associative deformations of A_2 are induced by the* $\mathrm{PGL}(4, \mathbb{C})$*-action on* $\mathbb{C}P^3$ *via the Hopf lift.*

Remark 6.18. ([24, 3]) As a special Legendrian submanifold, $A_2(\frac{\pi}{4})$ is not rigid, either. By a non-standard projection $p_2 : S^7 \to \mathbb{C}P^3$, $p_2(A_2(\frac{\pi}{4}))$ is a horizontal holomorphic curve in $\mathbb{C}P^3$, and for any horizontal holomorphic curve Σ, $p_2^{-1}(\Sigma) \subset S^7$ is a special Legendrian submanifold.

Since the group of biholomorphic maps which preserve the horizontal distribution is $\mathrm{PSp}(2, \mathbb{C})$, all non-trivial special Legendrian deformations of $A_2(\frac{\pi}{4})$ are given by the induced action of $\mathrm{PSp}(2, \mathbb{C})$ on $\mathbb{C}P^3$.

Now, we give proofs. First, we prove the following lemma.

Lemma 6.19. *Let $\{(v_l^{(n)})^* = \langle \cdot, v_l^{(n)} \rangle\}$ be the dual basis of $\{v_l^{(n)}\}$. Then we have*

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{Z}_3}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}) \\ &= \{L \in \mathrm{Hom}_{\mathbb{C}}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}); L(\rho_n(k)v) = \mathrm{Ad}(k)L(v) \textit{ for any } k \in \mathbb{Z}_3, v \in V_n\} \\ &= \mathrm{span}_{\mathbb{C}} \left\{ (v_i^{(n)})^* \otimes X; X = \left\{ \begin{array}{ll} e_3 & (n - 2i \in 3\mathbb{Z}) \\ e_1 + ie_2 & (n - 2i \in 3\mathbb{Z} + 1) \\ e_1 - ie_2 & (n - 2i \in 3\mathbb{Z} + 2) \end{array} \right. \right\}, \\ &\mathrm{Hom}_{\mathbb{Z}_3}(V_n, \mathbb{C}) = \mathrm{span}_{\mathbb{C}} \left\{ (v_i^{(n)})^*; n - 2i \in 3\mathbb{Z} \right\}. \end{aligned}$$

Proof. Take any $v \in V_n$ and $k = \begin{pmatrix} \zeta & 0 \\ 0 & \overline{\zeta} \end{pmatrix} \in \mathbb{Z}_3$ where $\zeta^3 = 1$. By definition, we see that

$$\begin{aligned} \mathrm{Ad}(k)e_1 &= \mathrm{Re}(\zeta)e_1 - \mathrm{Im}(\zeta)e_2, \\ \mathrm{Ad}(k)e_2 &= \mathrm{Im}(\zeta)e_1 + \mathrm{Re}(\zeta)e_2, \\ \mathrm{Ad}(k)e_3 &= e_3, \\ \rho_n(k)v_l^{(n)} &= \zeta^{n-2l}v_l^{(n)}. \end{aligned}$$

Setting $L = \sum_{l=0}^n \sum_{i=1}^3 C_{li}(v_l^{(n)})^* \otimes e_i \in \mathrm{Hom}_{\mathbb{C}}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C})$ where $C_{ki} \in \mathbb{C}$, we know that $L \in \mathrm{Hom}_{\mathbb{Z}_3}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C})$ if and only if

$$\zeta^{n-2l} \sum_{i=1}^3 C_{li}e_i = C_{l1}(\mathrm{Re}(\zeta)e_1 - \mathrm{Im}(\zeta)e_2) + C_{l2}(\mathrm{Im}(\zeta)e_1 + \mathrm{Re}(\zeta)e_2) + C_{l3}e_3$$

for any $0 \leq l \leq n$ and $\zeta^3 = 1$. This is equivalent to

$$\begin{aligned} (\zeta^{n-2l} - 1)C_{l3} &= 0, \\ (\zeta^{n-2l+1} - 1)(C_{l2} - iC_{l1}) &= 0, \\ (\zeta^{n-2l+2} - 1)(C_{l2} + iC_{l1}) &= 0, \end{aligned}$$

which implies the first statement. The second is proven in the same way. □

Proof of Proposition 6.15. From (6.13), the solution f of $\Delta_+ f = 8f$ is contained in $\mathrm{span}_{\mathbb{C}} \left\{ \langle \rho_n(\cdot)v_k^{(n)}, v_l^{(n)} \rangle ; (n,k) = (6,0), (6,6), (4,2), 0 \leq l \leq n \right\}$, which are $\mathbb{Z}_3$ invariant. Hence we obtain Proposition 6.15. □

Proof of Proposition 6.16. First, we consider $\dim_{\mathbb{R}}\{v \in \mathfrak{X}(A_2); \overline{\mathrm{rot}}(v) = -2v\}$. Set $(p,q,\alpha) = (\frac{1}{\sqrt{3}}, \frac{1}{3}, -2)$ in Lemma 6.10. Since we know that

$$\begin{aligned} -\frac{2}{9}(n - 2k + 2)^2 + \frac{1}{3}(n^2 + 2n) = 0 &\Leftrightarrow (n,k) = (4,0), \\ -\frac{2}{9}(n - 2k)^2 + \frac{1}{3}(n^2 + 2n) = 0 &\Leftrightarrow (n,k) = (0,0), \end{aligned}$$

we have $v_1 + iv_2 = \langle \rho_4(\cdot)v_0^{(4)}, u \rangle$ for $u \in V_4$ and v_3 is constant. We see that $v = \sum_{i=1}^3 v_i e_i$ satisfies (6.9), (6.10), and is $\mathbb{Z}_3$ equivariant. Hence we obtain $\dim_{\mathbb{R}}\{v \in \mathfrak{X}(A_2); \mathrm{rot}(v) = -2v\} = 11$. □

Proof of Proposition 6.17. We find $13(= 30 - 17)$-dimensional family of non-trivial associative deformations.

Let $p_1 : S^7 \to \mathbb{C}P^3$ be the Hopf fibration. By Lemma 2.19, for any holomorphic curve $\Sigma \subset \mathbb{C}P^3$, the Hopf lift $p_1^{-1}(\Sigma) \subset S^7$ of Σ is an associative submanifold. Since $p_1(A_2)$ is a holomorphic curve in $\mathbb{C}P^3$, the group of biholomorphic map of $\mathbb{C}P^3$, which is known to be $\mathrm{PGL}(4,\mathbb{C})$, induces the associative deformations of A_2 via the Hopf lift.

The $\mathrm{PGL}(4,\mathbb{C})$-action included in the $\mathrm{Spin}(7)$-action is the standard $\mathrm{SU}(4)$-action on S^7. Thus the dimension of non-trivial associative deformations of A_2 induced by

PGL$(4,\mathbb{C})$ is given by

$$\begin{aligned}&\dim_{\mathbb{R}} \mathrm{PGL}(4,\mathbb{C}) - \dim_{\mathbb{R}}\{g \in \mathrm{PGL}(4,\mathbb{C}); g\cdot p_1(A_2) \subset p_1(A_2)\}\\ &- (\dim_{\mathbb{R}} \mathrm{SU}(4) - \dim_{\mathbb{R}}\{h \in \mathrm{SU}(4); h\cdot A_2 \subset A_2\})\\ =&\dim_{\mathbb{R}} \mathrm{PGL}(4,\mathbb{C}) - \dim_{\mathbb{R}} \mathrm{PGL}(2,\mathbb{C}) - \dim_{\mathbb{R}} \mathrm{SU}(4) + \dim_{\mathbb{R}} \mathrm{U}(2)\\ =&30-6-15+4=13,\end{aligned}$$

which gives the proof. □

6.3.3. The case $A_3 \cong \mathrm{SU}(2)$. Since A_3 is not special Legendrian, we cannot apply Corollary 4.2 to this case. First, we describe the operator D explicitly. Define $E_i \in \mathfrak{su}(2)$ as (6.8). We denote by e_1, e_2, e_3 the left invariant vector fields on $\mathrm{SU}(2) \cong A_3$ induced by $\frac{1}{\sqrt{7}}E_1, \frac{1}{\sqrt{7}}E_2, E_3$, respectively. If we define the vectors η_k for $1 \le k \le 4$ as

$$\eta_1 = \sqrt{\frac{7}{3}}\left(Je_1 + \frac{2}{\sqrt{7}}e_4\right), \qquad \eta_2 = \sqrt{\frac{7}{3}}\left(Je_2 + \frac{2}{\sqrt{7}}e_3\right),$$
$$\eta_3 = \sqrt{\frac{7}{3}}\left(Je_3 - \frac{2}{\sqrt{7}}e_2\right), \qquad \eta_4 = \sqrt{\frac{7}{3}}\left(Je_4 - \frac{2}{\sqrt{7}}e_1\right),$$

where J is the standard complex structure on $\mathbb{C}^4$ and e_4 is the position vector, then $\{e_1, \cdots, e_3\}$ is the orthonormal frame of TA_3 and $\{\eta_1, \cdots, \eta_4\}$ is the orthonormal frame of ν. At $p_0 = \frac{1}{\sqrt{2}}{}^t(0,1,i,0)$, we have

$$e_1 = \frac{1}{\sqrt{14}}\begin{pmatrix}\sqrt{3}\\ 2i\\ -2\\ -\sqrt{3}i\end{pmatrix}, e_2 = \frac{1}{\sqrt{14}}\begin{pmatrix}\sqrt{3}i\\ -2\\ 2i\\ -\sqrt{3}\end{pmatrix}, e_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ i\\ 1\\ 0\end{pmatrix}, e_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ i\\ 0\end{pmatrix},$$

$$\eta_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\ 0\\ 0\\ 1\end{pmatrix}, \eta_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 0\\ 0\\ -i\end{pmatrix}, \eta_3 = \frac{1}{\sqrt{42}}\begin{pmatrix}-2\sqrt{3}i\\ -3\\ 3i\\ 2\sqrt{3}\end{pmatrix}, \eta_4 = \frac{1}{\sqrt{42}}\begin{pmatrix}-2\sqrt{3}\\ 3i\\ -3\\ 2\sqrt{3}i\end{pmatrix}.$$

LEMMA 6.20. *We have*

$$\nabla^{\top}_{e_i} e_i = 0 \textit{ for } i = 1,2,3, \qquad [e_1, e_2] = \frac{2}{7}e_3, \qquad [e_1, e_3] = -2e_2, \qquad [e_2, e_3] = 2e_1,$$

$$(\nabla^{\perp}_{e_i}\eta_j) = \frac{3}{7}\begin{pmatrix}-\eta_4 & -\eta_3 & \eta_2 & \eta_1\\ \eta_3 & -\eta_4 & -\eta_1 & \eta_2\\ 7\eta_2 & -7\eta_1 & -5\eta_4 & 5\eta_3\end{pmatrix}, (e_i \times \eta_j) = \begin{pmatrix}\eta_4 & \eta_3 & -\eta_2 & -\eta_1\\ -\eta_3 & \eta_4 & \eta_1 & -\eta_2\\ \eta_2 & -\eta_1 & \eta_4 & -\eta_3\end{pmatrix}.$$

Proof. Since the SU(2)-action preserves the G_2-structure on S^7, we only have to consider at p_0. The equations of $\nabla^{\top}_{e_i} e_i$ and $[e_i, e_j]$ is shown easily. By a direct computation, we have

$$(\nabla^{\mathbb{C}^4}_{e_i}\eta_j) = \frac{3}{7}\begin{pmatrix}-\eta_4 & -\eta_3 & \eta_2 & \eta_1\\ \eta_3 & -\eta_4 & -\eta_1 & \eta_2\\ 7\eta_2 & -7\eta_1 & -5\eta_4 & 5\eta_3\end{pmatrix} + \frac{2\sqrt{3}}{7}\begin{pmatrix}-e_1 & -e_2 & 2e_3 & 0\\ e_2 & -e_1 & 0 & -2e_3\\ 0 & 0 & 2e_1 & -2e_2\end{pmatrix},$$

and hence we obtain $\nabla^{\perp}_{e_i}\eta_j$. To prove the equations of $e_i \times \eta_j$, let h_0 be the standard metric on $\mathbb{C}^4$, ω_0 be the standard Kähler form on $\mathbb{C}^4$, and Ω_0 be the standard holomorphic volume form on $\mathbb{C}^4$. Define $e^i = h_0(e_i, \cdot), \eta^j = h_0(\eta_j, \cdot)$. Then $\{e^1, \cdots, e^4, \eta^1, \cdots, \eta^4\}$ is the dual coframe of $\{e_1, \cdots, e_4, \eta_1, \cdots, \eta_4\}$. We compute

$$\begin{pmatrix} e^1(J\cdot) \\ e^2(J\cdot) \\ e^3(J\cdot) \\ e^4(J\cdot) \end{pmatrix} = \frac{2}{\sqrt{7}} \begin{pmatrix} e^4 \\ e^3 \\ -e^2 \\ -e^1 \end{pmatrix} - \sqrt{\frac{3}{7}} \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \\ \eta^4 \end{pmatrix}, \quad \begin{pmatrix} \eta^1(J\cdot) \\ \eta^2(J\cdot) \\ \eta^3(J\cdot) \\ \eta^4(J\cdot) \end{pmatrix} = \sqrt{\frac{3}{7}} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} + \frac{2}{\sqrt{7}} \begin{pmatrix} -\eta^4 \\ -\eta^3 \\ \eta^2 \\ \eta^1 \end{pmatrix}.$$

Since we know $h_0 = \sum_{i=1}^4 ((e^i)^2 + (\eta^i)^2)$, we obtain

$$\omega_0 = h_0(J\cdot, \cdot) = \sqrt{\frac{3}{7}} \sum_{i=1}^4 e^i \wedge \eta^i + \frac{2}{\sqrt{7}}(-e^{14} - e^{23} + \eta^{14} + \eta^{23}).$$

The holomorphic volume form Ω_0 is of the form $C \cdot (e^1 + ig(e_1, \cdot)) \wedge \cdots (e^4 + ig(e_4, \cdot)) = C \cdot (e^1 - ie^1(J\cdot)) \wedge \cdots (e^4 - ie^4(J\cdot))$ for $C > 0$, and from the relation $\omega_0^4/4! = (i/2)^4 \Omega_0 \wedge \overline{\Omega_0}$, we have $C = 7/3$. Hence the G_2-structure $\varphi \in \Omega^3(S^7)$ on S^7 is described as

$$\begin{aligned} \varphi &= i(e_4)\left(\frac{1}{2}\omega_0^2 + \mathrm{Re}\Omega_0\right) \\ &= -e^{123} + e^1 \wedge (\eta^{14} + \eta^{23}) + e^2 \wedge (-\eta^{13} + \eta^{24}) + e^3 \wedge (\eta^{12} + \eta^{34}), \end{aligned}$$

which implies the lemma. □

Proposition 6.21. *By the trivialization of ν via $\{\eta_1, \cdots, \eta_4\}$, $D : C^\infty(\mathrm{SU}(2), \mathbb{R}^4) \cong C^\infty(A_3, \nu) \to C^\infty(A_3, \nu) \cong C^\infty(\mathrm{SU}(2), \mathbb{R}^4)$ is described as follows:*

$$D \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & -e_3 & e_2 & -e_1 \\ e_3 & 0 & -e_1 & -e_2 \\ -e_2 & e_1 & 0 & -e_3 \\ e_1 & e_2 & e_3 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{15}{7} & & & \\ & -\frac{15}{7} & & \\ & & 3 & \\ & & & 3 \end{pmatrix} \right\} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

Setting $\Psi_1 = \psi_1 + i\psi_2$, $\Psi_2 = \psi_3 - i\psi_4$, we have

$$D \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \left\{ \begin{pmatrix} ie_3 & -ie_1 + e_2 \\ -(ie_1 + e_2) & -ie_3 \end{pmatrix} + \begin{pmatrix} -\frac{15}{7} & \\ & 3 \end{pmatrix} \right\} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Proof. Take $\psi = \sum_{a=1}^4 \psi_a \eta_a \in C^\infty(A_3, \nu)$ for $\psi_a \in C^\infty(A_3)$. By the lemma above, we see

$$\begin{aligned} D\psi &= \sum_{i,a} (e_i(\psi_a) e_i \times \eta_a + \psi_a e_i \times \nabla^{\perp}_{e_i} \eta_a) \\ &= (-e_3(\psi_2) + e_2(\psi_3) - e_1(\psi_4) - \tfrac{15}{7}\psi_1)\eta_1 + (e_3(\psi_1) - e_1(\psi_3) - e_2(\psi_4) - \tfrac{15}{7}\psi_2)\eta_2 \\ &+ (-e_2(\psi_1) + e_1(\psi_2) - e_3(\psi_4) + 3\psi_3)\eta_3 + (e_1(\psi_1) + e_2(\psi_2) + e_3(\psi_3) + 3\psi_4)\eta_4, \end{aligned}$$

which gives the proof. □

From these descriptions, we compute the following.

Proposition 6.22. $\dim_{\mathbb{R}}\{\psi \in C^\infty(\mathrm{SU}(2), \mathbb{R}^4); D\psi = -\psi\} = 34.$

On the other hand, Spin(7) induces 18($=\dim_{\mathbb{R}} \mathrm{Spin}(7)/\mathrm{SU}(2)$)-dimensional associative deformations of A_3, and hence A_3 could potentially have 16-dimensional nontrivial associative deformations. However, we do not know whether there exists actual 16-dimensional nontrivial deformations.

Proof of Proposition 6.22. By Proposition 6.21, $D\psi = \alpha\psi$ for $\alpha \in \mathbb{R}$ is equivalent to

$$\left(ie_3 - \left(\frac{15}{7} + \alpha\right)\right)\Psi_1 + (-ie_1 + e_2)\Psi_2 = 0, \tag{6.24}$$

$$-(ie_1 + e_2)\Psi_1 + (-ie_3 + (3-\alpha))\Psi_2 = 0. \tag{6.25}$$

Applying $(ie_1 + e_2)$ to (6.24), we obtain

$$\left(ie_3 - \left(\frac{1}{7} + \alpha\right)\right)(ie_1 + e_2)\Psi_1 + \left(e_1^2 + e_2^2 + \frac{2}{7}ie_3\right)\Psi_2 = 0. \tag{6.26}$$

Substituting (6.25) into (6.26), we have $(-7\Delta_+ + 24ie_3 + (7\alpha+1)(\alpha-3))\,\Psi_2 = 0$. By using the notation in Lemma 6.8 and Lemma 6.9, we obtain

$$\begin{aligned}&(-7\Delta_+ + 24ie_3 + (7\alpha+1)(\alpha-3))\,\langle \rho_n(\cdot)v_k^{(n)}, u\rangle \\ &= \left\{-6(n-2k+2)^2 - n^2 - 2n + 24 + (7\alpha+1)(\alpha-3)\right\}\langle \rho_n(\cdot)v_k^{(n)}, u\rangle,\end{aligned}$$

for $n \geq 0, 0 \leq k \leq n, u \in V_n$.

Set $\alpha = -1$. Since we know that $-6(n-2k+2)^2 - n^2 - 2n + 48 = 0 \Leftrightarrow (n,k) = (6,4), (4,2), (4,4)$, we deduce that

$$\Psi_2 = \langle \rho_6(\cdot)v_4^{(6)}, u_1\rangle + \langle \rho_4(\cdot)v_2^{(4)}, u_2\rangle + \langle \rho_4(\cdot)v_4^{(4)}, u_3\rangle,$$

for $u_1 \in V_6, u_2, u_3 \in V_4$. From (6.24), we see that

$$\Psi_1 = -i\sqrt{\frac{7}{10}}\langle \rho_6(\cdot)v_5^{(6)}, u_1\rangle - 2i\sqrt{\frac{7}{6}}\langle \rho_4(\cdot)v_3^{(4)}, u_2\rangle.$$

Hence we obtain $\dim_{\mathbb{R}}\{\psi \in C^\infty(\mathrm{SU}(2), \mathbb{R}^4); D\psi = -\psi\} = 14 + 2\cdot 10 = 34$. □

6.4. The case S^3, L_1, L_2, L_3 and L_4. Next, we study the deformations of homogeneous associative submanifolds which lie in a totally geodesic S^6. These Lagrangian deformation spaces are studied in [17]. Hence we only consider associative and non-Lagrangian deformations by Remark 5.3.

6.4.1. The totally geodesic $S^3 \cong \mathrm{SU}(2)$. In this case, $\{e_1, e_2, e_3\} = \{E_1, E_2, E_3\}$ gives an orthonormal basis of $\mathfrak{su}(2)$ with respect to the induced metric from the totally geodesic S^3. We easily see the following by (6.13).

PROPOSITION 6.23. $\dim_{\mathbb{R}}\{f \in C^\infty(S^3); \Delta_+ f = 3f\} = 4$.

This implies that associative and non-Lagrangian deformations of the totally geodesic S^3 are trivial since G_2 induces 8($=\dim_{\mathbb{R}} G_2/\mathrm{SO}(4)$)-dimensional Lagrangian deformations of S^3 and Spin(7) induces 12-dimensional associative deformations of S^3, whose space is known to be $\mathrm{Spin}(7)/K$, where $K \cong \mathrm{SU}(2)^3/\mathbb{Z}_2$ is a Lie subgroup of Spin(7) ([8, Theorem.IV.1.38]).

6.4.2. The case $L_1 \cong \mathrm{SU}(2)$. Set $p_0 = \frac{\sqrt{5}}{3}\epsilon_1 + \frac{2}{3}\epsilon_4 = {}^t(\frac{\sqrt{5}}{3}, 0, \frac{2}{3}, 0) \in \mathbb{R} \oplus \mathbb{C}^3$. Then we have

$$(E_1^*)_{p_0} = -\frac{2\sqrt{5}}{3}\epsilon_3 - \frac{2}{3}\epsilon_6, \qquad (E_2^*)_{p_0} = -\frac{2\sqrt{5}}{3}\epsilon_2 - \frac{2}{3}\epsilon_7, \qquad (E_3^*)_{p_0} = \frac{2}{3}\epsilon_5.$$

Thus $\{e_1, e_2, e_3\} = \{\frac{\sqrt{6}}{4}E_1, \frac{\sqrt{6}}{4}E_2, \frac{3}{2}E_3\}$ gives an orthonormal basis of $\mathfrak{su}(2)$. We easily see the following by (6.13).

Proposition 6.24. $\dim_{\mathbb{R}}\{f \in C^\infty(S^3); \Delta_+ f = 3f\} = 7$.

This implies that associative and non-Lagrangian deformations of L_1 are trivial since $\mathrm{Spin}(7) \setminus G_2$ induces 7-dimensional associative deformations of L_1.

6.4.3. The case $L_2 \cong \mathrm{SU}(2)/\mathbb{Z}_2$. Set $p_0 = \frac{1}{\sqrt{2}}(\epsilon_4 + \epsilon_7) = \frac{1}{\sqrt{2}}{}^t(0, 0, 1, i) \in \mathbb{R} \oplus \mathbb{C}^3$. Then we have

$$(E_1^*)_{p_0} = \sqrt{2}\epsilon_3, \qquad (E_2^*)_{p_0} = \sqrt{2}\epsilon_2, \qquad (E_3^*)_{p_0} = \sqrt{2}(\epsilon_5 - \epsilon_6).$$

Thus $\{e_1, e_2, e_3\} = \{\frac{1}{\sqrt{2}}E_1, \frac{1}{\sqrt{2}}E_2, \frac{1}{2}E_3\}$ gives an orthonormal basis of $\mathfrak{su}(2)$.

Lemma 6.25. *If n is even, we have*

$$\mathrm{Hom}_{\mathbb{Z}_2}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_n, \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}), \qquad \mathrm{Hom}_{\mathbb{Z}_2}(V_n, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_n, \mathbb{C}).$$

If n is odd, both spaces are $\{0\}$.

From this Lemma, we see the following by (6.13).

Proposition 6.26. $\dim_{\mathbb{R}}\{f \in C^\infty(L_2); \Delta_+ f = 3f\} = 6$.

This implies that associative and non-Lagrangian deformations of L_2 are trivial since $\mathrm{Spin}(7) \setminus G_2$ induces 6-dimensional associative deformations of L_2. Note that L_2 is invariant under the action of $\{\mathrm{diag}(e^{-3it}, e^{it}, e^{it}, e^{it}); t \in \mathbb{R}\} \subset \mathrm{Spin}(7) \setminus G_2$.

6.4.4. The case $L_3 \cong \mathrm{SU}(2)/A_4^*$. We have

$$(E_1^*)_{\epsilon_2} = \sqrt{10}\epsilon_4 - \sqrt{6}\epsilon_6, \qquad (E_2^*)_{\epsilon_2} = \sqrt{10}\epsilon_5 - \sqrt{6}\epsilon_7, \qquad (E_3^*)_{\epsilon_2} = -4\epsilon_3.$$

Thus $\{e_1, e_2, e_3\} = \{E_1/4, E_2/4, E_3/4\}$ gives an orthonormal basis of $\mathfrak{su}(2)$.

Lemma 6.27.

$$\mathrm{Hom}_{A_4^*}(V_6, \mathbb{C}) = \mathbb{C}\left((v_1^{(6)})^* - (v_5^{(6)})^*\right).$$

Proof. Recall that A_4^* is generated by k_1, k_2, k_3 in (6.6). Take $L = \sum_{l=0}^{10} C_l(v_l^{(10)})^* \in \mathrm{Hom}_{\mathbb{C}}(V_{10}, \mathbb{C})$ where $C_{li} \in \mathbb{C}$ and consider the condition

$$L(\rho_{10}(k)v) = L(v), \tag{6.27}$$

for $k \in A_4^*$ and $v \in V_{10}$. As for $k = k_1, k_2$, (6.27) is equivalent to

$$(-1)^l i^6 C_l = C_l, \qquad (-1)^l C_{6-l} = C_l.$$

Thus L is of the form $C\left((v_1^{(6)})^* - (v_5^{(6)})^*\right)$ for $C \in \mathbb{C}$, and we see that $(v_1^{(6)})^* - (v_5^{(6)})^*$ is invariant by k_3. □

Proposition 6.28. $\dim_{\mathbb{R}}\{f \in C^\infty(L_3); \Delta_+ f = 3f\} = 7$.

This implies that associative and non-Lagrangian deformations of L_3 are trivial since $\mathrm{Spin}(7) \setminus G_2$ induces 7-dimensional associative deformations of L_3.

Proof. The solution f of $\Delta_+ f = 3f$ is contained in $\mathrm{span}_{\mathbb{C}}\left\{\langle \rho_6(\cdot)v_a^{(6)}, v_b^{(6)}\rangle; 0 \leq a, b \leq 6\right\}$ from (6.13). From Lemma 6.27, A_4^* invariant solutions of $\Delta_+ f = 3f$ are of the form $f = \langle \rho_6(\cdot)(v_1^{(6)} - v_5^{(6)}), u\rangle$ for $u \in V_6$. Imposing that f is $\mathbb{R}$-valued, we have $\dim_{\mathbb{R}}\{f \in C^\infty(L_3); \Delta_+ f = 3f\} = 7$. □

6.4.5. The case $L_4 \cong \mathrm{SU}(2)/D_3^*$. We have

$$(E_1^*)_{\epsilon_6} = \sqrt{6}\epsilon_2, \qquad (E_2^*)_{\epsilon_6} = \sqrt{6}\epsilon_3, \qquad (E_3^*)_{\epsilon_6} = 6\epsilon_7.$$

Thus $\{e_1, e_2, e_3\} = \{E_1/\sqrt{6}, E_2/\sqrt{6}, E_3/6\}$ gives an orthonormal basis of $\mathfrak{su}(2)$.

Lemma 6.29. *The space* $\mathrm{Hom}_{D_3^*}(V_n, \mathbb{C})$ *is spanned by the following functions:*

1. *In case $n = 6m$ where $m \in \mathbb{Z}_{\geq 0}$,*

$$(v_{3j}^{(n)})^* + (-1)^j (v_{n-3j}^{(n)})^* \qquad \text{for } 0 \leq j \leq m.$$

2. *In case $n = 6m + 2$,*

$$(v_{3j+1}^{(n)})^* + (-1)^{j+1} (v_{n-(3j+1)}^{(n)})^* \qquad \text{for } 0 \leq j \leq m.$$

3. *In case $n = 6m + 4$,*

$$(v_{3j+2}^{(n)})^* + (-1)^j (v_{n-(3j+2)}^{(n)})^* \qquad \text{for } 0 \leq j \leq m.$$

In case $n \in 2\mathbb{Z} + 1$, we have $\mathrm{Hom}_{D_3^*}(V_n, \mathbb{C}) = \{0\}$.

Proof. Recall that D_3^* is generated by k_4, k_5 in (6.7). Take $L = \sum_{l=0}^n C_l (v_l^{(n)})^* \otimes e_i \in \mathrm{Hom}_{\mathbb{C}}(V_n, \mathbb{C})$ where $C_{li} \in \mathbb{C}$. Consider the condition (6.27) for $k = k_4, k_5$, it is equivalent to

$$(-1)^{n-l} C_{n-l} = C_l, \qquad (e^{\frac{\pi i}{3}})^{n-2l} C_l = C_l.$$

Then we easily see Lemma 6.29. □

Proposition 6.30. $\dim_{\mathbb{R}}\{f \in C^\infty(L_4); \Delta_+ f = 3f\} = 7$.

This implies that associative and non-Lagrangian deformations of L_4 are trivial since $\mathrm{Spin}(7) \setminus G_2$ induces 7-dimensional associative deformations of L_4.

Proof. From (6.13), the solution f of $\Delta_+ f = 3f$ is contained in the space spanned by $\langle \rho_6(\cdot)v_j^{(6)}, v_a^{(6)}\rangle$ where $j = 0, 6$ and $0 \leq a \leq 6$. From Lemma 6.29, D_3^* invariant solutions of $\Delta_+ f = 3f$ are of the form $f = \langle \rho_6(\cdot)(v_0^{(6)} + v_6^{(6)}), u\rangle$ for $u \in V_6$. Imposing that f is $\mathbb{R}$-valued, we have $\dim_{\mathbb{R}}\{f \in C^\infty(L_3); \Delta_+ f = 3f\} = 7$. □

Appendix A. Proof of Proposition 3.8. We follow the proof of [7]. First, we show the following lemma.

Lemma A.1. *For any vector fields $u, v, w, z, X \in \mathfrak{X}(Y)$, we have*

$$\begin{aligned}
\nabla_X(u \times v) =& (\nabla_X u) \times v + u \times (\nabla_X v) - \chi(X, u, v), \\
R(w, z)(u \times v) =& (R(w, z)u) \times v + u \times (R(w, z)v) + \varphi(z, u, v)w - \varphi(w, u, v)z \\
& - g(w, u)v \times z - g(w, v)z \times u + g(z, u)v \times w + g(z, v)w \times u.
\end{aligned}$$

When $M^3 \subset Y$ is associative, we have $TM \times TM \subset TM$, $TM \times \nu \subset \nu$, and $\nu \times \nu \subset TM$. Thus for any $X, u, v \in C^\infty(M, TM), \eta \in C^\infty(M, \nu)$, we have

$$\begin{aligned}\nabla^\top_X(u \times v) =& (\nabla^\top_X u) \times v + u \times (\nabla^\top_X v) - (\chi(X, u, v))^\top,\\ \nabla^\perp_X(u \times \eta) =& (\nabla^\top_X u) \times \eta + u \times (\nabla^\perp_X \eta) - (\chi(X, u, \eta))^\perp.\end{aligned}$$

Proof. Let $\{f_k\}_{k=1,\cdots,7}$ be any local orthonormal frame of TY. Then

$$\begin{aligned}\nabla_X(u \times v) &= \sum_{i=1}^{7}\{(\nabla_X\varphi)(u, v, f_i)f_i + \varphi(\nabla_X u, v, f_i)f_i + \varphi(u, \nabla_X v, f_i)f_i\}\\ &= -\chi(X, u, v) + (\nabla_X u) \times v + u \times (\nabla_X v)\end{aligned}$$

since $\nabla g = 0$ and $\nabla\varphi = *\varphi$. For $R(w, z) = \nabla_w\nabla_z - \nabla_z\nabla_w - \nabla_{[w,z]}$, we see the following by a direct computation.

$$R(w, z)(u \times v) = (R(w, z)u) \times v + u \times (R(w, z)v) - (\nabla_w\chi)(z, u, v) + (\nabla_z\chi)(w, u, v).$$

Then, the equation $\nabla_w\chi = \sum_k i(f_k)(\nabla_w * \varphi) \otimes f_k = -\sum_k i(f_k)(g(w, \cdot) \wedge \varphi) \otimes f_k = -\varphi \otimes w + \sum_k(g(w, \cdot) \wedge i(f_k)\varphi) \otimes f_k$ proves the lemma. ∎

Next, we compute D^2. Let $\{e_i\}_{i=1,\cdots,3}$ be any local orthonormal frame satisfying $e_3 = e_1 \times e_2$ and $\{\eta_k\}_{k=1,\cdots,4}$ be any local orthonormal frame of ν. Then by Lemma A.1, it follows that

$$D^2\psi = \sum_{i,j=1}^{3} e_i \times \nabla^\perp_{e_i}(e_j \times \nabla^\perp_{e_j}\psi) = I_1 + I_2,$$

where

$$\begin{aligned}I_1 &= \sum_{i,j=1}^{3} e_i \times (\nabla^\top_{e_i} e_j \times \nabla^\perp_{e_j}\psi + e_j \times \nabla^\perp_{e_i}\nabla^\perp_{e_j}\psi),\\ I_2 &= -\sum_{i,j=1}^{3} e_i \times (\chi(e_i, e_j, \nabla^\perp_{e_j}\psi))^\perp.\end{aligned}$$

From (2.3), the following holds:

$$\begin{aligned}I_2 &= \sum_{i,j} e_i \times ((e_i \times e_j) \times \nabla^\perp_{e_j}\psi)\\ &= -\sum_{i,j}(e_i \times (e_i \times e_j)) \times \nabla^\perp_{e_j}\psi = 2\sum_j e_j \times \nabla^\perp_{e_j}\psi = 2D\psi.\end{aligned}$$

By the computation in [7], we have $I_1 = \nabla^{\perp *}\nabla^\perp\psi + \pi_\mathcal{V}(I_3) + I_4$, where

$$I_3 = -\frac{1}{2}\sum_{i,j}(e_i \times e_j) \times R(e_i, e_j)\psi, \qquad I_4 = \sum_{i,j,k} g(A_{(e_i\times e_j)\times\eta_k} e_i, A_\psi e_j)\eta_k.$$

From the next lemma, we obtain Proposition 3.8.

LEMMA A.2.

$$I_3 = \sum_{i=1}^{3} R(e_i, \psi)e_i + 3\psi, \qquad I_4 = -\mathcal{A}\psi.$$

Proof. By using the relation $e_i \times e_{i+1} = e_{i+2}$ for $i \in \mathbb{Z}/3$ and the Bianchi identity, we have

$$\begin{aligned} I_3 &= -\sum_{i\in\mathbb{Z}/3} e_i \times R(e_{i+1}, e_{i+2})\psi \\ &= \sum_{i\in\mathbb{Z}/3} e_i \times \left(R(\psi, e_{i+1})e_{i+2} + R(e_{i+2}, \psi)e_{i+1}\right), \end{aligned}$$

$$\begin{aligned} e_{i+2} \times R(e_{i+1}, \psi)e_i =& R(e_{i+1}, \psi)e_{i+1} - (R(e_{i+1}, \psi)e_{i+2}) \times e_i \\ & - \varphi(\psi, e_{i+2}, e_i)e_{i+1} + \varphi(e_{i+1}, e_{i+2}, e_i)\psi \\ =& R(e_{i+1}, \psi)e_{i+1} + e_i \times (R(e_{i+1}, \psi)e_{i+2}) + \psi, \end{aligned}$$

since $e_i \times e_{i+1} = e_{i+2}$ for $i \in \mathbb{Z}/3$, $g(e_i, \psi) = 0$, and $\varphi(e_i, e_{i+1}, e_{i+2}) = 1$. Hence we obtain $I_3 = \sum_{i=1}^{3} R(e_i, \psi)e_i + 3\psi$. For I_4, we have by Lemma A.1

$$\begin{aligned} A_{(e_i\times e_j)\times\eta_k} e_i =& -\nabla^{\top}_{e_i}((e_i \times e_j) \times \eta_k) \\ =& -\nabla^{\perp}_{e_i}(e_i \times e_j) \times \eta_k - (e_i \times e_j) \times (\nabla^{\top}_{e_i}\eta_k) + \chi(e_i, e_i \times e_j, \eta_k)^{\top} \\ =& -\{(\nabla^{\perp}_{e_i}e_i) \times e_j + e_i \times (\nabla^{\top}_{e_i}e_j)\} \times \eta_k + (e_i \times e_j) \times A_{\eta_k}e_i \\ & + \chi(e_i, e_i \times e_j, \eta_k)^{\top}. \end{aligned}$$

Since an associative submanifold is minimal, it follows that $\sum_i \nabla^{\perp}_{e_i}e_i = 0$. Moreover, we see $\sum_i e_i \times \nabla^{\perp}_{e_i}e_j = 0$ for $j = 1, 2, 3$ by the relation $e_3 = e_1 \times e_2$. Hence we obtain $I_4 = I_5 + I_6$, where

$$I_5 = \sum_{i,j,k} g((e_i \times e_j) \times A_{\eta_k}e_i, A_\psi e_j)\eta_k, \qquad I_6 = \sum_{i,j,k} g(\chi(e_i, e_i \times e_j, \eta_k)^{\top}, A_\psi e_j)\eta_k.$$

It is shown that $I_5 = -\mathcal{A}\psi$ in [7]. As for I_6, we compute $\chi(e_i, e_i \times e_j, \eta_k) = \eta_k \times (e_i \times (e_i \times e_j)) = \eta_k \times (-e_j + \delta_{ij}e_i)$, and obtain $\sum_i \eta_k \times (-e_j + \delta_{ij}e_i) = -2\eta_k \times e_j \in C^\infty(M, \nu)$, which implies that $I_6 = 0$. □

Appendix B. Harmonic analysis on a homogeneous vector bundle. We give a summary of harmonic analysis on a homogeneous vector bundle from [27].

B.1. Homogeneous vector bundles.

DEFINITION B.1. Let G be a Lie group and let K be a closed subgroup of G. Set $M := G/K$. A vector bundle $E \to M$ is called a **homogeneous vector bundle** if G acts on E on the left and the G-action satisfies:

1. $g \cdot E_x = E_{g\cdot x}$ for $g \in G, x \in M$,
2. $g\cdot : E_x \to E_{g\cdot x}$ is linear for $g \in G, x \in M$,

where E_x is the fiber of E at $x \in M$.

LEMMA B.2. *Let (τ, E_0) be a finite dimensional representation of K. Then the associated vector bundle $E := G \times_\tau E_0 = G \times E_0/\sim$, where $(g, v) \sim (g \cdot k, \tau(k)^{-1}v)$, is a homogeneous vector bundle over M.*

All homogeneous vector bundles are described as above by the following lemma.

LEMMA B.3. *Let $E \to M$ be a homogeneous vector bundle. Let $E_0 = E_{eK}$ and $\tau : K \to \mathrm{End}(E_0)$ be the induced action from 2 of Definition B.1. Then we have $E \cong G \times_\tau E_0$.*

B.2. Fourier series expansion. Let G be a compact Lie group, K be a closed subgroup of G, (τ, E_0) be a finite dimensional unitary representation of K, and $E \to M$ be the homogeneous vector bundle associated with (τ, E_0). Assume that $M = G/K$ is orientable. Setting

$$C(G, E_0)^{(K,\tau)} := \{f \in C(G, E_0); f(g \cdot k) = \tau(k)^{-1} f(g) \text{ for any } g \in G, k \in K)\},$$

we have the following.

LEMMA B.4. *For $f \in C(M, E)$, define $\tilde{f} \in C(G, E_0)^{(K,\tau)}$ by $\tilde{f}(g) = g^{-1} f(gK) \in E_{eK} \cong E_0$. Then the map $f \mapsto \tilde{f}$ gives an isomorphism $C(M, E) \cong C(G, E_0)^{(K,\tau)}$. The map $f \mapsto \tilde{f}$ extends to the isomorphism $A : L^2(M, E) \xrightarrow{\cong} L^2(G, E_0)^{(K,\tau)}$.*

Let $\hat{G}$ be the set of all equivalence classes of finite dimensional irreducible unitary representations of G. For each $\gamma = [(\pi_\gamma, V_\gamma)] \in \hat{G}$, we assign a map $A_\gamma : V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, E_0) \to C(G, E_0)^{(K,\tau)}$, where $\mathrm{Hom}_K(V_\gamma, E_0) = \{L \in \mathrm{Hom}(V_\gamma, E_0); L(k \cdot v) = \tau(k)L(v) \text{ for any } k \in K, v \in V_\gamma)\}$, by $A_\gamma(v \otimes L)(g) = L(g^{-1} \cdot v)$.

PROPOSITION B.5 (Fourier expansion). *The algebraic direct sum*

$$\sum_{\gamma \in \hat{G}} A_\gamma(V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, E_0))$$

is uniformly dense in $C(G, E_0)^{(K,\tau)}$ relative to the uniform topology.

LEMMA B.6 (Schur orthogonality relations). *Let (π, V) and (π', V') be irreducible unitary representations of a compact group G. Let $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ be inner products on V and V', respectively. Then for $u, v \in V$ and $u', v' \in V'$, we have*

$$\int_G (\pi(g)u, v)\overline{(\pi'(g)u', v')'} dg = \begin{cases} 0 & (\pi \not\simeq \pi') \\ (u, u')\overline{(v, v')}/\dim V & (\pi \simeq \pi'). \end{cases}$$

B.3. Homogeneous differential operators.

DEFINITION B.7. Let G be a Lie group and let K be a closed subgroup of G. Set $M = G/K$. Let $E \to M$ and $F \to M$ be homogeneous vector bundles, and (τ, E_0) and (σ, F_0) be the representations of K associated with E and F, respectively.

A differential operator $D : C^\infty(M, E) \to C^\infty(M, F)$ is called a **homogeneous differential operator** if $g \cdot Df = D(g \cdot f)$ for $g \in G, f \in C^\infty(M, E)$. Here, $(g \cdot f)(x) = gf(g^{-1}x)$ for $x \in M, g \in G, f \in C^\infty(M, E)$ or $C^\infty(M, F)$.

REMARK B.8. The map D is completely determined by its value at a point, i.e., given $(Df)_{eK}$ for any $f \in C^\infty(M, E)$, we can determine $(Df)_{gK}$ for each $g \in G, f \in C^\infty(M, E)$.

We give an explicit description of the homogeneous differential operators.

Let $U(\mathfrak{g}) = \otimes^* \mathfrak{g}/I(\mathfrak{g})$, where $I(\mathfrak{g})$ is the two-sided ideal in $\otimes^* \mathfrak{g}$ generated by $\{X \otimes Y - Y \otimes X - [X, Y]; X, Y \in \mathfrak{g}\}$. (In other words, $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.) Let $\xi : \otimes^* \mathfrak{g} \to U(\mathfrak{g})$ be the canonical projection and $U^i(\mathfrak{g}) := \xi(\sum_{k \leq i} \otimes^k \mathfrak{g})$.

Set $D(G)$ be the space of all left invariant differential operators on G. For any $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, define $Xf \in C^\infty(G)$ by $Xf(g) = (d/dt)f(g \cdot \exp(tX))|_{t=0}$. The map $X \mapsto (f \mapsto Xf)$ gives the inclusion $\mathfrak{g} \hookrightarrow D(G)$, from which an isomorphism $U(\mathfrak{g}) \xrightarrow{\cong} D(G)$ is induced.

Lemma B.9. *The algebra $U(\mathfrak{g})$ is isomorphic to $D(G)$. If $\{X_1, \cdots, X_n\}$ is a basis of $\mathfrak{g}$, then $\{X_1^{m_1} \cdots X_n^{m_n}; m_j \geq 0\}$ forms a basis of $U(\mathfrak{g})$.*

Similarly, for $L \otimes X \in \mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g})$ and $f \in C^\infty(G, E_0)$, set $(L \otimes X)f = L \cdot Xf$. Thus the element of $\mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g})$ is considered as a differential operator $C^\infty(G, E_0) \to C^\infty(G, F_0)$.

Let K act on $\mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g})$ as $\mu(k)(L \otimes X) = \sigma(k) L \tau(k)^{-1} \otimes \mathrm{Ad}(k)X$ for $L \in \mathrm{Hom}(E_0, F_0)$ and $X \in U(\mathfrak{g})$. Then $(\mu, \mathrm{Hom}(E_0, F_0) \otimes U^j(\mathfrak{g}))$ is a representation of K for each j. Setting

$$(\mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g}))^K = \{D \in \mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g}); \mu(k)D = D \text{ for any } k \in K\},$$

we have the following.

Lemma B.10. *For any $D \in (\mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g}))^K$, we have $DC^\infty(G, E_0)^{(K,\tau)} \subset C^\infty(G, F_0)^{(K,\sigma)}$. Conversely, if $D \in \mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g})$ satisfies $DC^\infty(G, E_0)^{(K,\tau)} \subset C^\infty(G, F_0)^{(K,\sigma)}$, then $\mu(k)D|_{C^\infty(G,E_0)^{(K,\tau)}} = D|_{C^\infty(G,E_0)^{(K,\tau)}}$.*

Definition B.11. Let $\mathfrak{k}$ be the Lie algebra of K. A homogeneous space G/K is called **reductive** if there exists an $\mathrm{Ad}(K)$-invariant vector subspace $\mathfrak{p} \subset \mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Proposition B.12. *Suppose that $M = G/K$ is a reductive homogeneous space. Let $D : C^\infty(M, E) \to C^\infty(M, F)$ be a homogeneous differential operator of order j and let $\tilde{D}$ be the corresponding map from $C^\infty(G, E_0)^{(K,\tau)}$ to $C^\infty(G, F_0)^{(K,\sigma)}$.*

Then there exists $\overline{D} \in \mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g})$ so that $\overline{D}|_{C^\infty(G,E_0)^{(K,\tau)}} = \tilde{D}$. If K is compact, $\overline{D}$ may be taken to be in $(\mathrm{Hom}(E_0, F_0) \otimes U(\mathfrak{g}))^K$.

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