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DRESSING ACTIONS ON PROPER DEFINITE AFFINE SPHERES*

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Abstract. We will first clarify the loop group formulations for both hyperbolic and elliptic definite affine spheres in $\mathbb{R}^3$. Then we classify the rational elements with 3 poles or 6 poles in a real twisted loop group, and compute dressing actions of them on such surfaces. Some new examples with pictures will be produced at last.

Key words. Definite affine sphere, Tzitzeica equation, dressing action, Monge-Ampere equation.

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1. Introduction. All intuitive geometrical notions about size and shape developed in school geometry are invariant under rigid motions comprised of translations, rotations and reflections. Following Klein's famous Erlangen Program (1872), affine geometry studies the constructs invariant under the affine group $\mathrm{A}(n) = \mathrm{GL}(n) \ltimes \mathbb{R}^n$ comprised of nondegenerate linear maps and translations. While two fundamental measurements angle and distance in Euclidean geometry are no longer invariant under affine motions, notions for some relative positions such as parallels and midpoints (more generally center of mass) still make sense. As the standard directional derivative operator D is preserved by $\mathrm{A}(n)$, differential geometry of curves and surfaces can also be generalized. Similarly, the symmetry group for equiaffine geometry is $\mathrm{SA}(n) = \mathrm{SL}(n) \ltimes \mathbb{R}^n$ preserving in addition the standard volume form on $\mathbb{R}^n$.

In 1907 Gheorghe Tzitzéica discovered a particular class of hyperbolic surfaces in $\mathbb{R}^3$ whose Gauss curvature at any point p is proportional to the fourth power of the distance from a fixed point to the tangent plane at p. He used the structure equation

$$\omega_{xy} = e^{\omega} - e^{-2\omega} \tag{1.1}$$

to describe the (indefinite) affine spheres. The reader may refer to [20] for a concise survey on affine spheres and to [31] for an extensive survey on their relations to real Monge-Ampère equations, projective structures on manifolds, and Calabi-Yau manifolds. Here we would like to emphasize the Monge-Amprère equation (see Calabi [8]). The graph of a locally strictly convex function f is a mean curvature H affine sphere centered at the origin or infinity if and only if the Legendre transform u of f solves:

$$\det\left(\frac{\partial^2 u}{\partial y_i \partial y_j}\right) = \begin{cases} (Hu)^{-n-2}, & \text{if } H \neq 0, \\ 1, & \text{if } H = 0. \end{cases} \tag{1.2}$$

Cheng & Yau [10] showed that on a bounded convex domain there is for $H < 0$ a unique negative convex solution of (1.2) extending continuously to be 0 on the boundary. This gave a beautiful geometric picture of complete hyperbolic ($H < 0$) affine spheres (first conjectured by Calabi [8]): such a hypersurface is always asymptotic

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to a convex cone with its vertex at the center and the interior of any regular convex cone is a disjoint union of complete hyperbolic affine spheres asymptotic to it with all negative mean curvature values and with centers at the vertex. Even in $\mathbb{R}^3$, it is highly nontrivial to see the above equation is equivalent to the usual surface structure equations (see Simon & Wang [38]):

$$\begin{cases}\psi_{z\bar{z}} + \frac{H}{2}e^{\psi} + |U|^2 e^{-2\psi} = 0,\\ U_{\bar{z}} = 0,\end{cases} \tag{1.3}$$

where $e^{\psi}|\mathrm{d}z|^2$ is the affine metric and $U\mathrm{d}z^3$ is equivalent to the affine cubic form. One may also fix a holomorphic cubic form $\mathbf{U}$ (locally $\mathbf{U} = U\mathrm{d}z^3$) and a conformal metric g on a Riemann surface (such that the affine metric $e^{\psi}|\mathrm{d}z|^2 = e^{\phi}g$). Then a coordinate-free version of (1.3) is:

$$\Delta\phi + 4\|\mathbf{U}\|^2 e^{-2\phi} + 2He^{\phi} - 2\kappa = 0, \tag{1.4}$$

where Δ is the Laplace operator of g, $\|\cdot\|$ is the induced norm on cubic differential, and κ is the Gauss curvature.

The classical Tzitzéica equation (1.1) was rediscovered in many mathematical and physical contexts afterwards.(see, e.g., [13], [18], [21]). In recent years, techniques from soliton theory have been applied to this equation by: Rogers & Schief in the context of gas dynamics ([36]), Kaptsov & Shan'ko on multi-soliton formulas ([26]), Dorfmeister & Eitner on Weierstrass type representation ([14]), Bobenko & Schief on its discretizations ([7], [6]), and Wang ([43]) on dressing actions following Terng & Uhlenbeck ([39]) approach.

The equation (1.3) for nonzero U is an elliptic version of the classical Tzitzéica equation, or the Bullough-Dodd-Jiber-Shabat equation, or the affine $\mathfrak{a}_2^{(2)}$-Toda field equation, or the first $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ elliptic system (see Terng [40]). Dunajski & Plansangkate [19] have given gauge invariant characterization of (1.3) in terms of $\mathrm{SU}(2,1)$ Hitchin equations or reduction of Anti-Self-Dual-Yang-Mills equations, and linked the local equation around a double pole of U to Painlevé III. Zhou [46] have given Darboux transformations for two dimensional elliptic affine Toda equations. Hildebrand [24] have given analytic formulas for complete hyperbolic affine spheres asymptotic to semi-homogenous convex cones.

In the background of mirror symmetry, Loftin, Yau & Zaslow [30] solved (1.4) globally on Riemann sphere minus 3 points with some prescribed asymptotic behaviours around the singularities. It is still open to construct the corresponding convex affine spheres based on their solutions (also called "trinoids"). This has motivated our extensive studies of such surfaces from integrable system or soliton theory. In this paper we compute the dressing actions and soliton examples. The Permutability Theorem and group structure of dressing actions will be presented in a subsequent paper [44]. Their Weierstrass or DPW representation has been studied in [15], using an Iwasawa decomposition of certain twisted loop group. The equivariant solutions have been constructed in [16]. The conformal parametrizations and the associated families of Hildebrand's complete affine spheres have been presented in [29].

The rest of the paper is organized as follows. In section 2, we will recall the fundamental notions and theorems of equiaffine differential geometry, and the definitions of affine spheres. We specialize to two dimensional proper definite affine spheres in section 3 and give the loop group descriptions of them. We then classify the simple rational elements in certain twisted loop group in section 4 and compute the dressing actions of them on such surfaces in section 5. Some examples will be presented in the last section.

2. Fundamental theorem of affine differential geometry and affine spheres. Classical affine differential geometry studies the properties of an immersed hypersurface $r : M^n \to \mathbb{R}^{n+1}$ invariant under the equiaffine transformations $r \to Ar + b$, where $A \in \mathrm{SL}_{n+1}(\mathbb{R})$ and $b \in \mathbb{R}^{n+1}$. Here $\mathbb{R}^{n+1}$ is viewed as an affine space equipped with standard equiaffine structure: the canonical connection D and the parallel volume form given by the standard determinant. The following form in local coordinates $(u_1, \cdots, u_n)$ is naturally an equiaffine invariant:

$$\Lambda = \sum_{i,j} \det\left(\frac{\partial^2 r}{\partial u_i \partial u_j}, \frac{\partial r}{\partial u_1}, \cdots, \frac{\partial r}{\partial u_n}\right)(\mathrm{d}u_i \mathrm{d}u_j) \otimes (\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n) \tag{2.1}$$

which is independent of choice of local coordinates and thus defines a global quadratic form with values in the line bundle of top forms on M. Denote the determinant coefficients in (2.1) by Λ_{ij} and their determinant by $\det\Lambda$. Assuming $\det\Lambda \neq 0$ everywhere, a pseudo-Riemannian metric g is induced uniquely up to sign (depending on the orientation of M) by the equation

$$\Lambda = g \otimes \mathrm{vol}(g) \qquad (\text{ i.e. volume form of } g\ ),$$

or defined explicitly by

$$g = \pm|\det\Lambda|^{-\frac{1}{n+2}} \sum_{i,j} \Lambda_{ij} \mathrm{d}u_i \mathrm{d}u_j,$$

called the **affine metric**. The hypersurface is said to be **definite** or **indefinite** if this metric g is so. A smooth hypersurface is definite if and only if it is locally strictly convex. Although the Euclidean angle is not invariant under affine transformations, there exists an invariant transversal vector field ξ along $r(M)$ defined by $\triangle^g r\,/\,n$, called the **affine normal**. Here $\triangle^g$ is the Laplace-Beltrami operator of g. Another way to find the affine normal up to sign is by modifying the scale and direction of any transversal vector field (such as the Euclidean normal) to meet two natural characterizing conditions:

(i) $D_X\xi$ or $\mathrm{d}\xi(X)$ is tangent to the hypersurface for any $X \in T_pM$,

(ii) ξ and g induce the same volume measure on M:

$$\det\left(r_*X_1, \cdots, r_*X_n,\ \xi\right)^2 = |\det(g(X_i, X_j))|$$

for any $X_i \in T_pM$.

The formula of Gauss gives the following decomposition into tangential and transverse components:

$$D_X r_* Y = r_*(\nabla_X Y) + g(X, Y)\xi\ , \tag{2.2}$$

which induces a torsion-free affine connection ∇ on M, called **Blaschke connection**. Let $\hat{C}$ denote the difference tensor between the induced Blaschke connection ∇ and g's Levi-Civita connection ∇^g. The affine cubic form measures the difference between the induced Blaschke connection ∇ and g's Levi-Civita connection ∇^g:

$$C(X, Y, Z) := g(\nabla_X Y - \nabla^g_X Y, Z). \tag{2.3}$$

It is actually symmetric in all 3 arguments and is a 3rd order invariant. The **Pick invariant** J is simply $\|C\|_g^2/(n^2-n)$.

Similar to the Euclidean case, the **affine shape operator** S defined by the formula of Weingarten:

$$D_X\xi = -r_*(S(X)),$$

is again self-adjoint with respect to g. The **affine mean curvature** H and the **affine Gauss curvature** K are defined as $H = \mathrm{Tr}S\,/\,n,\ K = \det S$.

Note that the affine normal is also a 3rd order invariant. Hence S is a 4th order invariant, which can actually be computed from g and C:

$$H = R_g - J, \qquad \text{where } R_g = \text{ scalar curvature of } g. \tag{2.4}$$
$$g(S_0(X),Y) = -\frac{2}{n}\mathrm{Tr}\{Z \mapsto (\nabla^g_Z\hat{C})(X,Y)\}, \qquad \text{where } S_0 := S - H\cdot\mathrm{Id}.$$

The readers may refer to several textbooks [5, 28, 34] for more details of the above basic notions.

Theorem 2.1 (Dillen-Nomizu-Vrancken type fundamental theorem [17]). *Given a nondegenerate symmetric 2-form g and a symmetric 3-form C on simply connected M satisfying two compatibility conditions: the apolarity condition $\mathrm{Tr}_g C = 0$, and the conjugate connection $\overline{\nabla} := \nabla^g - \hat{C}$ is projectively flat, there exists a global immersion into $\mathbb{R}^{n+1}$ unique up to equiaffine motions such that (g,C) are the induced affine metric and cubic form respectively.*

Remark 2.2. When we scale the immersion f to ρf for some positive constant ρ, the above invariants changes by:

$$\nabla \to \nabla, \qquad g \to \rho^{\frac{2(n+1)}{n+2}} g, \qquad \xi \to \rho^{\frac{-n}{n+2}}\xi, \qquad C \to \rho^{\frac{2(n+1)}{n+2}} C,$$
$$S \to \rho^{\frac{-2(n+1)}{n+2}} S \qquad H \to \rho^{\frac{-2(n+1)}{n+2}} H, \qquad K \to \rho^{\frac{-2n(n+1)}{n+2}} K. \tag{2.5}$$

Historically the first important class of affine surfaces is the affine spheres studied by Tzitzéica in a series of papers from 1908 to 1910. A **proper affine sphere** is a surface whose affine normal lines all meet in a point (the center), such as all ellipsoids and hyperboloids. An **improper affine sphere** is a surface whose affine normals are all parallel, such as all paraboloids and all ruled surfaces of the form $x_3 = x_1x_2 + f(x_1)$. An equivalent definition is $S = H\cdot$ Id, which implies from the compatibility conditions that the affine mean curvature H must be constant. When $S \equiv 0$, ξ is a constant vector and it is improper. So improper affine spheres are special affine maximal surfaces ($H = 0$) and have been integrated explicitly for dimension 2 in [5]. When $S = H\cdot$ Id $\neq 0$, it is proper and $\xi = -H(x - x_0)$ with some x_0 being the center. For simplicity, we will always make $x_0 = 0$ by translating the surface. In the indefinite case, we can scale the surface and change the sign of ξ if necessary to normalize $H = -1$. However, in the definite case we have a preferred choice of the signs of g and ξ by requiring g to be positive definite, or equivalently by choosing ξ to point to the inside of the convex surface. So we need to distinguish two cases according to the sign of H:

(1) the **elliptic affine spheres** whose center is inside the convex surface ($H > 0$), such as all ellipsoids;

(2) the **hyperbolic affine spheres** whose center is outside the convex surface ($H < 0$), such as all hyperboloids of two sheets and $x_1x_2x_3 = 1$.

Improper definite affine spheres ($H = 0$), are naturally called **parabolic affine spheres**, such as all elliptic paraboloids. While quadrics are the only global examples of elliptic and parabolic affine spheres (by Calabi, Cheng-Yau, Jörgens, Pogorelov), there are many global hyperbolic affine spheres asymptotic to each sharp (or regular) convex cone (by Cheng-Yau). One of the main goals of this paper is to construct families of local examples of both elliptic and hyperbolic affine spheres in $\mathbb{R}^3$ using integrable system techniques.

3. Loop group description of definite affine spheres. From now on we consider only the surface case in $\mathbb{R}^3$ where its affine metric g is positive definite: this means that $r(M)$ is locally strongly convex and oriented so that Λ is positive valued. In particular its Euclidean Gauss curvature is positive and the affine normal ξ is chosen to point to the concave side of the surface. This essentially induces a unique Riemann surface (or complex) structure on M in whose coordinates $g = e^{\psi}|\mathrm{d}z|^2$ and the orientation given by $\mathrm{i}\mathrm{d}z \wedge \mathrm{d}\bar{z}$ is consistent with the orientation induced by ξ. Alternatively we are studying "**affine-conformal**" immersions of any Riemann surface M into $\mathbb{R}^3$:

$$| \, r_z \quad r_{\bar{z}} \quad r_{zz} \, | = 0 \qquad \text{affine-conformal condition.} \tag{3.1}$$

Then the following two determinants completely determine the fundamental invariants:

$$| \, r_z \quad r_{\bar{z}} \quad r_{z\bar{z}} \, | = \frac{\mathrm{i}}{4} e^{2\psi}, \qquad | \, r_z \quad r_{zz} \quad r_{zzz} \, | = \frac{\mathrm{i}}{4} U^2,$$

with $g = e^{\psi}|\mathrm{d}z|^2$ and $C = U\mathrm{d}z^3 + \bar{U}\mathrm{d}\bar{z}^3$. This can also be illustrated by computing the evolution equations for the positively oriented frame $\tilde{F} = (r_z, r_{\bar{z}}, \xi = 2e^{-\psi} r_{z\bar{z}})$:

$$\tilde{F}^{-1}\mathrm{d}\tilde{F} = \begin{pmatrix} \psi_z \mathrm{d}z & \bar{U}e^{-\psi}\mathrm{d}\bar{z} & -H\mathrm{d}z + 2e^{-2\psi}\bar{U}_z \mathrm{d}\bar{z} \\ Ue^{-\psi}\mathrm{d}z & \psi_{\bar{z}}\mathrm{d}\bar{z} & -H\mathrm{d}\bar{z} + 2e^{-2\psi}U_{\bar{z}}\mathrm{d}z \\ \frac{1}{2}e^{\psi}\mathrm{d}\bar{z} & \frac{1}{2}e^{\psi}\mathrm{d}z & 0 \end{pmatrix}. \tag{3.2}$$

The compatibility condition ($\tilde{F}_{z\bar{z}} = \tilde{F}_{\bar{z}z}$) or the flatness of $\tilde{F}^{-1}\mathrm{d}\tilde{F}$ determines every entry in the above matrix one-form satisfying two structure equations (first derived by Radon):

$$H = -2e^{-\psi}\psi_{z\bar{z}} - 2|U|^2 e^{-3\psi} \qquad \text{Gauss equation} \tag{3.3}$$

$$H_{\bar{z}} = 2e^{-3\psi}\bar{U}U_{\bar{z}} - 2e^{-\psi}(e^{-\psi}\bar{U}_z)_z. \qquad \text{Codazzi equation for } S \tag{3.4}$$

Conversely, given a positive definite metric g and fully symmetric cubic form C as above satisfying (3.3) and (3.4), there exits a definite affine surface unique up to equiaffine motions such that (g, C) are the induced affine metric and cubic form respectively. This is *the fundamental theorem for definite affine surface.* Here H is indeed the affine mean curvature function since the affine shape operator S takes the following matrix form in terms of the basis $(\partial_z, \partial_{\bar{z}})$:

$$S = \begin{pmatrix} H & -2e^{-2\psi}\bar{U}_z \\ -2e^{-2\psi}U_{\bar{z}} & H \end{pmatrix}.$$

In summary we obtain the governing equations for definite affine spheres in $\mathbb{R}^3$ (see also Simon-Wang [38]):

$$\begin{cases} \psi_{z\bar{z}} + \frac{H}{2}e^{\psi} + |U|^2 e^{-2\psi} &= 0, \\ U_{\bar{z}} &= 0, \end{cases} \tag{3.5}$$

whose solution determines a unique definite affine sphere (up to equi-affine motions) with constant affine mean curvature H and affine metric $e^{\psi}|\mathrm{d}z|^2$, by integrating the frame equation:

$$\tilde{\alpha} := \tilde{F}^{-1}\mathrm{d}\tilde{F} = \begin{pmatrix} \psi_z \mathrm{d}z & \bar{U}e^{-\psi}\mathrm{d}\bar{z} & -H\mathrm{d}z \\ Ue^{-\psi}\mathrm{d}z & \psi_{\bar{z}}\mathrm{d}\bar{z} & -H\mathrm{d}\bar{z} \\ \frac{1}{2}e^{\psi}\mathrm{d}\bar{z} & \frac{1}{2}e^{\psi}\mathrm{d}z & 0 \end{pmatrix}. \tag{3.6}$$

It is now clear that $U\mathrm{d}z^3$ is a globally defined holomorphic cubic differential (i.e. in $H^0(M, K^3)$ where K is the canonical bundle of M). Recall Pick's Theorem: $C \equiv 0$ if and only if $r(M)$ is part of a quadric surface. So U is nonzero except for the quadrics. Away from its isolated zeroes one could make a holomorphic coordinate change to normalize U to a nonzero constant but we will not do that now. These zeroes of U will be called "planar" points of the affine surface.

We would like to emphasize that the immersion is analytic for any definite affine sphere, since the defining equation is elliptic (cf. [5] §76).

The following observation is crucial for the integrability of definite affine spheres: the system (3.5) is invariant under $U \to e^{i\theta}U$ with any constant $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Thus an S^1-family of definite affine spheres r^{θ} can be associated to any given one, with the same affine metric but different affine cubic forms $C = e^{i\theta}U\mathrm{d}z^3 + e^{-i\theta}\bar{U}\mathrm{d}\bar{z}^3$.

To further reveal the hidden symmetry, we will scale the surface to normalize $H = \pm 2$ using (2.5) for elliptic or hyperbolic case, also replace U by $e^{i\theta}U$ in the frame equation (3.6), and then use $\mathrm{diag}(2\sqrt{\mp 1}\lambda^{-1}e^{-\psi/2}, 2\sqrt{\mp 1}\lambda e^{-\psi/2}, 1)$ to gauge the Maurer-Cartan form $\tilde{\alpha}$ to:

$$\alpha_\lambda = \begin{pmatrix} \frac{1}{2}(\psi_z\mathrm{d}z - \psi_{\bar{z}}\mathrm{d}\bar{z}) & \lambda^{-1}\bar{U}e^{-\psi}\mathrm{d}\bar{z} & \sqrt{\mp 1}\lambda\, e^{\psi/2}\mathrm{d}z \\ \lambda U e^{-\psi}\mathrm{d}z & \frac{1}{2}(\psi_{\bar{z}}\mathrm{d}\bar{z} - \psi_z\mathrm{d}z) & \sqrt{\mp 1}\lambda^{-1}e^{\psi/2}\mathrm{d}\bar{z} \\ \sqrt{\mp 1}\lambda^{-1}e^{\psi/2}\mathrm{d}\bar{z} & \sqrt{\mp 1}\lambda e^{\psi/2}\mathrm{d}z & 0 \end{pmatrix} \tag{3.7}$$

Although α_λ has real geometric meaning only for $|\lambda| = 1$, it is actually flat for all $\lambda \in \mathbb{C} \setminus \{0\}$. Furthermore it takes value in certain twisted loop algebra of $\mathrm{sl}(3, \mathbb{C})$, i.e. satisfying two reality conditions:

$$\tau(\alpha_{1/\bar{\lambda}}) = \alpha_\lambda, \qquad \sigma(\alpha_{e^{-2\pi \mathrm{i}/6}\lambda}) = \alpha_\lambda, \tag{3.8}$$

where $\tau(X) = T\bar{X}T^{-1}$ and $\sigma(X) = -PX^tP^{-1}$ for

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -H/2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \epsilon^4 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \epsilon = e^{\frac{\pi i}{3}}. \tag{3.9}$$

We denote τ_1 for τ-reality condition in hyperbolic case, τ_2 in elliptic case, and T_1 for T in hyperbolic case, T_{-1} in elliptic case, i.e.

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note that τ is a conjugate linear involution of $\mathrm{sl}(3,\mathbb{C})$ whose fixed point set is isomorphic to $\mathrm{sl}(3,\mathbb{R})$; and σ is an order 6 automorphism of $\mathrm{sl}(3,\mathbb{C})$ giving the following eigenspace decomposition or $\mathbb{Z}_6$-gradation:

$$sl(3,\mathbb{C}) = \oplus_{j=0}^{5}\mathcal{G}_j, \quad [\mathcal{G}_j,\mathcal{G}_k] \subset \mathcal{G}_{j+k}.$$

with

$$\mathcal{G}_0 = \left\{\begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}, \quad \mathcal{G}_1 = \left\{\begin{pmatrix} 0 & 0 & s_1 \\ s_2 & 0 & 0 \\ 0 & s_1 & 0 \end{pmatrix}\right\},$$
$$\mathcal{G}_2 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ -s & 0 & 0 \end{pmatrix}\right\}, \quad \mathcal{G}_3 = \left\{\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -2s \end{pmatrix}\right\},$$
$$\mathcal{G}_4 = \left\{\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & -s & 0 \end{pmatrix}\right\}, \quad \mathcal{G}_5 = \left\{\begin{pmatrix} 0 & s_1 & 0 \\ 0 & 0 & s_2 \\ s_2 & 0 & 0 \end{pmatrix}\right\}.$$

We verify that $\sigma\tau = \tau\sigma$ and they define a 6-symmetric space "$\mathrm{SL}(3,\mathbb{R})$"$/\,\mathrm{SO}(2,\mathbb{R})$. The induced automorphisms on $\mathrm{SL}(3,\mathbb{C})$ (still denoted by τ and σ) are:

$$\tau(g) = T\bar{g}T^{-1}, \qquad \sigma(g) = P\,(g^t)^{-1}P^{-1}. \tag{3.10}$$

If we solve $F_\lambda^{-1}\mathrm{d}F_\lambda = \alpha_\lambda$ uniquely with certain initial condition $F_\lambda(p_0)$ at any base point p_0 of M, it is easy to show that $F_\lambda(p_0)^{-1}F_\lambda$ satisfies the reality conditions (3.8) and therefore lies in the corresponding twisted loop group.

Remark 3.1. *Hereafter we will always choose the initial loop* $F_\lambda(p_0) = \mathrm{I}$. *Then we may conjugate the complex frame to a real* $\mathrm{SL}(3,\mathbb{C})$*-frame:*

$$F^{\mathbb{R}} := \mathrm{Ad}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{-\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & \sqrt{H/2} \end{pmatrix} \cdot F_\lambda.$$

In fact $F^{\mathbb{R}} = (e_1, e_2, \xi)$ *with* $\{e_1, e_2\}$ *being simply an orthonormal tangent frame w.r.t. the affine metric. So we obtain an affine sphere immersion* $r = -H^{-1}\xi = \mp\frac{\xi}{2}$. *It is clear now that we may also simply take the real part of the last column of* F_λ *to get an equivalent affine sphere modulo affine motions.*

As in [14] and [43], we write

$$P = \begin{pmatrix} 0 & \epsilon^4 & \\ \epsilon^2 & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \epsilon^4 & & \\ & \epsilon^2 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = QP_{12} = P_{12}Q^{-1},$$

where

$$Q = \begin{pmatrix} \epsilon^4 & & \\ & \epsilon^2 & \\ & & 1 \end{pmatrix}, \quad P_{12} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}. \tag{3.11}$$

It follows that

$$\sigma(g) = P(g^t)^{-1}P = QP_{12}(g^t)^{-1}P_{12}Q^{-1} = \sigma_2(\sigma_1(g)),$$
$$\sigma_1(g(\lambda)) = g(-\lambda), \quad \sigma_2(g(\lambda)) = g(\epsilon^{-2}\lambda). \tag{3.12}$$

where

$$\sigma_1(g) = P_{12}(g^t)^{-1}P_{12}, \quad \sigma_2(g) = QgQ^{-1}, \tag{3.13}$$

and $\sigma = \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$.

Let $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. We adopt the following notations for loop groups:
$\Lambda G = \{$ holomorphic maps from $\mathbb{C}^\times \cap (I_r \cup I_{\frac{1}{r}})$ to $G\}$,
$\Lambda_{\mathbb{C}^\times} G = \{$ holomorphic maps from $C^\times$ to $G\}$,
$\Lambda_I G = \{$ holomorphic maps from I to G $\}$,
$\Lambda^{\sigma,\tau} G = \{ g \in \Lambda G$: $\tau(g(\bar{\lambda}^{-1})) = g(\lambda)$, $\sigma(g(\lambda)) = g(\epsilon\lambda)\}$,
$\Lambda^{\sigma,\tau}_{\mathbb{C}^\times} G = \{ g \in \Lambda_{\mathbb{C}^\times} G$: $\tau(g(\bar{\lambda}^{-1})) = g(\lambda)$, $\sigma(g(\lambda)) = g(\epsilon\lambda)\}$,
$\Lambda^{\sigma,\tau}_I G = \{ g \in \Lambda_I G$: $\tau(g(\bar{\lambda}^{-1})) = g(\lambda)$, $\sigma(g(\lambda)) = g(\epsilon\lambda)\}$,
where $0 < r < 1$ is sufficiently small and

$$I_r = \{\lambda \in \mathbb{C} : |\lambda| < r\}, \quad I_{\frac{1}{r}} = \{r \in \mathbb{C} \cup \{\infty\} : |\lambda| > \frac{1}{r}\}, \quad I = I_r \cup I_{\frac{1}{r}}.$$

Similar notations can apply to their Lie algebras. Then α_λ in (3.7) is a $\Lambda^{\sigma,\tau}_{\mathbb{C}^\times}\,\mathrm{sl}(3,\mathbb{C})$-valued flat connection, and the corresponding frame F_λ lies in $\Lambda^{\sigma,\tau}_{\mathbb{C}^\times}\,\mathrm{SL}(3,\mathbb{C})$ for any $0 < r < 1$. It is remarkable that a simple "algebraic" condition characterize such extended frames of proper definite affine spheres:

Theorem 3.2 (Loop group formulation for proper definite affine spheres). *Let $F(z,\bar{z})$ be any smooth map from a domain in $\mathbb{C}$ to the twisted loop group $\Lambda^{\sigma,\tau}_{\mathbb{C}^\times}\,\mathrm{SL}_3(\mathbb{C})$. If $F^{-1}F_z$ is linear in the loop parameter λ, i.e. of the form $A\lambda+B$, and A_{13} is nowhere zero, then $F^{-1}\mathrm{d}F$ is gauge equivalent to the Maurer-Cartan form* (3.7) *of proper definite affine spheres. In other words, F is gauge equivalent to the extended frame of a proper definite affine sphere if and only if $[F]$ defines a primitive harmonic map into the 6-symmetric space "$\mathrm{SL}_3(\mathbb{R})$"$/\,\mathrm{SO}_2(\mathbb{R})$ with some nondegeneracy condition.*

Proof. For simplicity, we only show the hyperbolic ($H = -1$) case in the flat connection (3.7). The positive case is completely parallel.

The reality conditions (3.8) guarantee that $F^{-1}F_z$ must be linear in λ. So we have

$$F^{-1}F_z = A\lambda + B, \qquad F^{-1}F_{\bar{z}} = C\lambda^{-1} + D, \tag{3.14}$$

with $A \in \mathfrak{g}_{-1}$, $B \in \mathfrak{g}_0$, $C = \tau(A)$, and $D = \tau(B)$. The fixed points of both σ and τ are of the form $\mathrm{diag}(e^{i\beta}, e^{-i\beta}, 1)$. Gauging by them respect the reality conditions. Let $e^{i\beta} = \pm\frac{A_{13}}{|A_{13}|}$. Use it to gauge A_{13} to a real positive function which is then set to $e^{\psi/2}$. The rest follows from the zero curvature equations for $F^{-1}\mathrm{d}F$. □

4. Simple elements in $\Lambda^{\sigma,\tau}_I GL(3,\mathbb{C})$. A rational element in any loop group is usually called **a simple element** if it has the least number of simple poles. To construct the simple elements in the twisted loop group $\Lambda^{\sigma,\tau}_I GL(3,\mathbb{C})$, we will handle order 3 σ_2-twisting first:

LEMMA 4.1. *([7]) A simple element $g \in \Lambda_I^{\sigma_2} GL(3,\mathbb{C})$ has three simple poles and always take the following form:*

$$g(\lambda) = A\,(I + \frac{R}{\lambda-\alpha} + \frac{\epsilon^2 Q^{-1}RQ}{\lambda-\epsilon^2\alpha} + \frac{\epsilon^4 QRQ^{-1}}{\lambda-\epsilon^4\alpha}), \tag{4.1}$$

where A is diagonal and

$$Q = diag(\epsilon^4, \epsilon^2, 1), \quad \epsilon = e^{\frac{\pi i}{3}}, \quad r < |\alpha| < 1/r.$$

To construct simple element in $\Lambda_I^{\sigma} GL(3,\mathbb{C})$, plug (4.1) into σ_1-reality condition in (3.12), we obtain that $g \in \Lambda_I^{\sigma} GL(3,\mathbb{C})$ iff

$$g(-\lambda)\; P_{12}\; g^t(\lambda) = P_{12},$$

where P_{12} is given in (3.11), i.e.

$$g(-\lambda)\; P_{12}\; \left(I + \frac{R^t}{\lambda-\alpha} + \frac{\epsilon^2 QR^tQ^{-1}}{\lambda-\epsilon^2\alpha} + \frac{\epsilon^4 Q^{-1}R^tQ}{\lambda-\epsilon^4\alpha}\right) A^t = P_{12}. \tag{4.2}$$

Compute the LHS of (4.2) at $\lambda = \infty$ to get

$$A\; P_{12}\; A^t = P_{12}.$$

Write $A = \text{diag}(d_1, d_2, d_3)$, and we have

$$d_1 d_2 = 1, \quad d_3^2 = 1.$$

So

$$A = \text{diag}(d, d^{-1}, \pm 1), \tag{4.3}$$

where $d \in \mathbb{C}^\times$.

Compute the residue of the LHS of (4.2) at $\lambda = \alpha$ to get

$$g(-\alpha)\; P_{12}\; R^t A^t = 0, \tag{4.4}$$

i.e.

$$A(I + \frac{1}{\alpha}(-\frac{R}{2} + \epsilon^{-2}Q^{-1}RQ + \epsilon^2 QRQ^{-1}))\; P_{12}\; R^t A^t = 0.$$

Write $R = (b_{ij})_{1\le i,j\le 3}$. Then direct computation implies

$$\begin{pmatrix} d & & \\ & d^{-1} & \\ & & \pm 1 \end{pmatrix} \left\{ I + \frac{1}{\alpha}\left(-\frac{3}{2}R + 3\begin{pmatrix} 0 & 0 & b_{13} \\ b_{21} & 0 & 0 \\ 0 & b_{32} & 0 \end{pmatrix}\right)\right\} P_{12}\; R^t A^t = 0. \tag{4.5}$$

From (4.5) we deduce that $rank(R) = 3$ is impossible. If $rank(R) = 0$, we get $g(\lambda) = A$ is trivial. If $rank(R) = 1$, we assume that

$$R = \begin{pmatrix} b_2 \\ c_2 \\ e_2 \end{pmatrix} \begin{pmatrix} b_1, & c_1, & e_1 \end{pmatrix}. \tag{4.6}$$

Substituting (4.6) into (4.5) and computing directly, we have

$$R = \frac{2\alpha}{3}\begin{pmatrix} \frac{c_1}{2b_1c_1-1} \\ b_1 \\ 1 \end{pmatrix}\begin{pmatrix} b_1, & c_1, & 1 \end{pmatrix}, \quad 2b_1c_1 - 1 \neq 0. \tag{4.7}$$

Residue of LHS of (4.2) at $\lambda = \epsilon^2\alpha$ gives

$$g(-\epsilon^2\alpha)\ P_{12}\ QR^tQ^{-1}A^t = 0,$$

which is equivalent to

$$Qg(-\epsilon^2\alpha)Q^{-1}\ P_{12}\ R^tA^t = 0,$$

i.e.

$$\sigma_2(g(-\epsilon^2\alpha))\ P_{12}\ R^tA^t = 0.$$

This is equivalent to (4.4). Residue at $\lambda = \epsilon^4\alpha$ is also equivalent to (4.4). So the loop group element of **rank 1 type** is as follows:

$$g(\lambda) = \begin{pmatrix} d & & \\ & d^{-1} & \\ & & \pm 1 \end{pmatrix}\left[I + \frac{2}{\lambda^3 - \alpha^3}\begin{pmatrix} \frac{\alpha^3 b_1c_1}{2b_1c_1-1} & \frac{\alpha\lambda^2 {c_1}^2}{2b_1c_1-1} & \frac{\alpha^2\lambda c_1}{2b_1c_1-1} \\ \alpha^2\lambda b_1^2 & \alpha^3 b_1c_1 & \alpha\lambda^2 b_1 \\ \alpha\lambda^2 b_1 & \alpha^2\lambda c_1 & \alpha^3 \end{pmatrix}\right]; \tag{4.8}$$

By explicit computation, if $rank(R) = 2$, we have

$$R = \frac{2\alpha}{3}\begin{pmatrix} \frac{b_1c_1-1}{2b_1c_1-1} & \frac{-{c_1}^2}{2b_1c_1-1} & \frac{c_1}{2b_1c_1-1} \\ b_1^2 & (1-b_1c_1) & -b_1 \\ -b_1 & c_1 & 0 \end{pmatrix}, \quad 2b_1c_1 - 1 \neq 0,$$

with the corresponding **rank 2 type** loop group element

$$g(\lambda) = \begin{pmatrix} d & & \\ & d^{-1} & \\ & & \pm 1 \end{pmatrix}\left[I + \frac{2}{\lambda^3 - \alpha^3}\begin{pmatrix} \frac{\alpha^3(b_1c_1-1)}{2b_1c_1-1} & \frac{-\alpha\lambda^2 {c_1}^2}{2b_1c_1-1} & \frac{\alpha^2\lambda c_1}{2b_1c_1-1} \\ \alpha^2\lambda b_1^2 & \alpha^3(1-b_1c_1) & -\alpha\lambda^2 b_1 \\ -\alpha\lambda^2 b_1 & \alpha^2\lambda c_1 & 0 \end{pmatrix}\right]. \tag{4.9}$$

We also compute that the determinant of $g(\lambda)$ for the both cases are $\pm[(\lambda^3+\alpha^3)/(\lambda^3-\alpha^3)]^{rank(R)}$, i.e. only depending on the poles and the rank of the residues.

Let $l := (b, c, 1)$, ℓ be the line $\mathbb{C}\cdot l$, and introduce the following 'cone':

$$\Delta := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid 2z_1z_2 = z_3^2, \text{or } z_3 = 0\}.$$

Then $2bc \neq 1$ is equivalent to $\ell \nsubseteq \Delta$. We observe that Image($\mathrm{Res}_\alpha\ g^t$) for rank 1 type and Kernel($\mathrm{Res}_\alpha\ g\ P_{12}$) for rank 2 type are both $\ell^t := \mathbb{C}\cdot l^t$.

Notation. Since a line ℓ not in Δ, a complex number α and a diagonal matrix A (4.3) determine $g(\lambda)$ uniquely in both types, we always use $Ag_{\alpha,\ell}(\lambda)$ to denote the rank 1 type element (4.8) and use $Am_{\alpha,\ell}(\lambda)$ to denote the rank 2 type element (4.9).

We summarize the above computations and some basic but useful facts into the following proposition:

PROPOSITION 4.2. *Any simple element in* $\Lambda_I^\sigma GL(3,\mathbb{C})$ *is either* $Ag_{\alpha,\ell}(\lambda)$ *of rank 1 type* (4.8) *or* $Am_{\alpha,\ell}(\lambda)$ *of rank 2 type* (4.9)*, and they have the following properties:*
(1) the determinant is $\pm[(\lambda^3+\alpha^3)/(\lambda^3-\alpha^3)]^{rank(R)}$*, independent of* ℓ*;*
(2) $Am_{\alpha,\ell}(\lambda) = \frac{\lambda^3+\alpha^3}{\lambda^3-\alpha^3} Ag_{\alpha,\ell}(-\lambda)$*;*
(3) $A\, g_{\alpha,\ell}(\lambda) = g_{\alpha,\ell A^{-1}}(\lambda)\, A$*.*

Finally, we consider the τ-reality condition. Recall $g(\lambda) \in \Lambda_I^\sigma GL(3,\mathbb{C})$ satisfies τ-reality condition iff

$$g(\lambda) = \tau(g(\bar{\lambda}^{-1})) = T\overline{g(\bar{\lambda}^{-1})}T^{-1}.$$

If g has a pole at α , it must also have a pole at $\bar{\alpha}^{-1}$. So $|\alpha|$ must be 1 if the above simple element $Ag_{\alpha,\ell}(\lambda)$ or $Am_{\alpha,\ell}(\lambda)$ satisfies τ-reality condition. We will treat hyperbolic case ($\tau = \tau_1$) and elliptic case ($\tau = \tau_2$) separately.

(1) For hyperbolic case, we first consider the rank 2 type $Am_{\alpha,\ell}(\lambda)$, then

$$\bar{A}\bar{m}_{\alpha,\ell}(\bar{\lambda}^{-1}) m^t_{\alpha,\ell}(-\lambda) A^t = I. \tag{4.10}$$

We divide our computations into the following steps:
(i) Evaluate LHS of (4) at $\lambda = \infty$ to get

$$\bar{A}(I - 3\bar{\alpha}^{-1}\mathrm{diag}(\bar{R}))A^t = I,$$

where diag(M) is the diagonal part of a matrix M. This implies that

$$\begin{pmatrix} \frac{\bar{d}}{2\bar{b}_1\bar{c}_1-1} & & \\ & \bar{d}^{-1}(2\bar{b}_1\bar{c}_1-1) & \\ & & \pm1 \end{pmatrix}\begin{pmatrix} d & & \\ & d^{-1} & \\ & & \pm1 \end{pmatrix} = I.$$

It follows that

$$|d|^2 = (2\bar{b}_1\bar{c}_1 - 1). \tag{4.11}$$

(ii) Compute the residues of LHS of (4) at $\lambda = -\alpha$ and $\lambda = \bar{\alpha}^{-1}$ to get

$$\overline{Am_{\alpha,\ell}(-\bar{\alpha}^{-1})}R^t = 0,$$

i.e.

$$m_{\alpha,\ell}(-\bar{\alpha}^{-1})\bar{R}^t = 0. \tag{4.12}$$

Residues of LHS of (4) at $\lambda = -\epsilon^2\alpha,\ -\epsilon^4\alpha$ give the same condition (4.12). From (4.12), we have:

$$\left[I + \frac{2}{-\alpha^3-\alpha^3}\begin{pmatrix} \frac{\alpha^3(b_1c_1-1)}{2b_1c_1-1} & \frac{-\alpha\lambda^2 c_1{}^2}{2b_1c_1-1} & \frac{\alpha^2\lambda c_1}{2b_1c_1-1} \\ \alpha^2\lambda b_1^2 & \alpha^3(1-b_1c_1) & -\alpha\lambda^2 b_1 \\ -\alpha\lambda^2 b_1 & \alpha^2\lambda c_1 & 0 \end{pmatrix}\right]\overline{\begin{pmatrix} \frac{b_1c_1-1}{2b_1c_1-1} & \frac{-c_1{}^2}{2b_1c_1-1} & \frac{c_1}{2b_1c_1-1} \\ b_1^2 & (1-b_1c_1) & -b_1 \\ -b_1 & c_1 & 0 \end{pmatrix}}^{t} = 0$$

i.e.

$$\begin{pmatrix} \frac{b_1c_1}{2b_1c_1-1} & \frac{c_1{}^2}{2b_1c_1-1} & \frac{c_1}{2b_1c_1-1} \\ b_1^2 & b_1c_1 & b_1 \\ b_1 & c_1 & 1 \end{pmatrix}\begin{pmatrix} \overline{\frac{b_1c_1-1}{2b_1c_1-1}} & \overline{b_1^2} & \overline{-b_1} \\ \overline{\frac{-c_1{}^2}{2b_1c_1-1}} & \overline{(1-b_1c_1)} & \overline{c_1} \\ \overline{\frac{c_1}{2b_1c_1-1}} & \overline{-b_1} & 0 \end{pmatrix} = 0$$

i.e.

$$(b_1, c_1, 1)\begin{pmatrix} \overline{b_1c_1-1} & \overline{b_1^2} & \overline{-b_1} \\ \overline{-c_1}^2 & \overline{(1-b_1c_1)} & \overline{c_1} \\ \overline{c_1} & \overline{-b_1} & 0 \end{pmatrix} = 0.$$

Direct computation implies that:

$$c_1 = \bar{b}_1. \tag{4.13}$$

Combine (4.11) and (4.13), we get that $Am_{\alpha,\ell}(\lambda) \in \Lambda_I^{\sigma,\tau_1} GL(3,\mathbb{C})$ iff

$$|\alpha| = 1, \quad c_1 = \bar{b}_1, \quad |d|^2 = (2\bar{b}_1\bar{c}_1 - 1) = (2|b_1|^2 - 1) > 0.$$

Then $Am_{\alpha,\ell}(\lambda)$ can be written as

$$Am_{\alpha,\ell}(\lambda) = \begin{pmatrix} d & & \\ & d^{-1} & \\ & & \pm 1 \end{pmatrix}\left[I + \frac{2}{\lambda^3 - \alpha^3}\begin{pmatrix} \frac{\alpha^3(|b|^2-1)}{2|b|^2-1} & \frac{-\alpha\lambda^2\bar{b}^2}{2|b|^2-1} & \frac{\alpha^2\lambda\bar{b}}{2|b|^2-1} \\ \alpha^2\lambda b^2 & \alpha^3(1-|b|^2) & -\alpha\lambda^2 b \\ -\alpha\lambda^2 b & \alpha^2\lambda\bar{b} & 0 \end{pmatrix}\right]. \tag{4.14}$$

If we consider the rank 1 type $Ag_{\alpha,\ell}(\lambda)$, by the similar computation, we will get a contradiction when we compute the $(3,3)$-entry at $\lambda = \infty$. So there is no rank 1 type simple element in $\Lambda_I^{\sigma} GL(3,\mathbb{C})$ satisfying the τ_1-reality condition.

(2) For elliptic case, if we consider the rank 1 type $Ag_{\alpha,\ell}(\lambda)$, then we will induce the same contradiction as hyperbolic case. If we consider the rank 2 type, we will get

$$|d|^2 = (2\bar{b}_1\bar{c}_1 - 1), \quad c_1 = -\bar{b}_1,$$

i.e. $|d|^2 = -(2|b_1|^2 + 1)$, which is also a contradiction. So there is no simple element with 3 poles in $\Lambda_I^{\sigma} GL(3,\mathbb{C})$ satisfying the τ_2-reality condition.

When $g(\lambda) \in \Lambda_I^{\sigma,\tau} GL(3,\mathbb{C})$ has a pole at α with $|\alpha| \neq 1$, (σ,τ)-reality condition implies that it also has the same type of poles at $\{\epsilon^2\alpha, \epsilon^4\alpha, \bar{\alpha}^{-1}, \epsilon^2\bar{\alpha}^{-1}, \epsilon^4\bar{\alpha}^{-1}\}$. So we will simply try the product of two simple elements in $\Lambda_I^{\sigma} GL(3,\mathbb{C})$ with poles at $\{\alpha, \epsilon^2\alpha, \epsilon^4\alpha\}$ and $\{\bar{\alpha}^{-1}, \epsilon^2\bar{\alpha}^{-1}, \epsilon^4\bar{\alpha}^{-1}\}$ respectively, i.e., we try $h(\lambda) = A_1 g_{\alpha,\ell_1}(\lambda) A_2 g_{\bar{\alpha}^{-1},\ell_2}(\lambda)$. Considering the Proposition 4.2 property (3), we have

$$\begin{aligned} h(\lambda) &= A_1 g_{\alpha,\ell_1}(\lambda) A_2 g_{\bar{\alpha}^{-1},\ell_2}(\lambda) \\ &= A_1 A_2 \; g_{\alpha,\ell_1 A_2}(\lambda) g_{\bar{\alpha}^{-1},\ell_2}(\lambda) \\ &= \tilde{A}_1 g_{\alpha,\tilde{\ell}_1}(\lambda) g_{\bar{\alpha}^{-1},\ell_2}(\lambda). \end{aligned}$$

Without loss of generality, we set

$$h(\lambda) = A g_{\alpha,\ell_1}(\lambda) g_{\bar{\alpha}^{-1},\ell_2}(\lambda) = (h_{ij})_{1\le i,j\le 3} \tag{4.15}$$

here

$$h_{11} = d\frac{[\lambda^3(2b_1c_1-1)+\alpha^3][\lambda^3\bar{\alpha}^3(2b_2c_2-1)+1]+4\lambda^3|\alpha|^2b_2c_1(2b_2c_2-1)(b_2c_1+|\alpha|^2)}{(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_1c_1-1)(2b_2c_2-1)},$$

$$h_{12} = d\frac{2\lambda^2\bar{\alpha}^2c_2^2[\lambda^3(2b_1c_1-1)+\alpha^3]+(2b_2c_2-1)[2\lambda^2\alpha c_1^2(2b_2c_2-1+\lambda^3\bar{\alpha}^3)+4\lambda^2\alpha^2\bar{\alpha}c_1c_2]}{(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_1c_1-1)(2b_2c_2-1)},$$

$$h_{13} = d\frac{2\lambda\bar{\alpha}c_2[\lambda^3(2b_1c_1-1)+\alpha^3]+(2b_2c_2-1)[4\lambda^4\alpha\bar{\alpha}^2b_2c_1^2+2\lambda\alpha^2c_1(\lambda^3\bar{\alpha}^3+1)]}{(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_1c_1-1)(2b_2c_2-1)},$$

$$h_{21} = \frac{2\lambda\alpha^2b_1^2[\lambda^3\bar{\alpha}^3(2b_2c_2-1)+1]+(2b_2c_2-1)[2\lambda\bar{\alpha}b_2^2(2\alpha^3b_1c_1-\alpha^3+\lambda^3)+4\lambda^4\alpha\bar{\alpha}^2b_1b_2]}{d(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_2c_2-1)},$$

$$h_{22} = \frac{4\lambda^3|\alpha|^4b_1^2c_2^2+(2b_2c_2-1)\{[\alpha^3(2b_1c_1-1)+\lambda^3](2b_2c_2-1+\lambda^3\bar{\alpha}^3)+4\lambda^3\alpha\bar{\alpha}^2b_1b_2\}}{d(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_2c_2-1)},$$

$$h_{23} = \frac{4\lambda^2\alpha^2\bar{\alpha}b_1^2c_2+(2b_2c_2-1)[2\lambda^2\bar{\alpha}^2b_2(2\alpha^3b_1c_1-\alpha^3+\lambda^3)+2\lambda^2\alpha b_1(\lambda^3\bar{\alpha}^3+1)]}{d(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_2c_2-1)},$$

$$h_{31} = \pm\frac{2\lambda^2\alpha b_1[\lambda^3\bar{\alpha}^3(2b_2c_2-1)+1]+(2b_2c_2-1)[4\lambda^2\alpha^2\bar{\alpha}b_2^2c_1+2\lambda^2\bar{\alpha}^2b_2(\lambda^3+\alpha^3)]}{(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_2c_2-1)},$$

$$h_{32} = \pm\frac{4\lambda^4\alpha\bar{\alpha}^2b_1c_2^2+(2b_2c_2-1)[2\lambda\alpha^2c_1(\lambda^3\bar{\alpha}^3+2b_2c_2-1)+2\lambda\bar{\alpha}c_2(\lambda^3+\alpha^3)]}{(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_2c_2-1)},$$

$$h_{33} = \pm\frac{4\lambda^3|\alpha|^2b_1c_2+(2b_2c_2-1)[4\lambda^3|\alpha|^4b_2c_1+(\lambda^3+\alpha^3)(\lambda^3\bar{\alpha}^3+1)]}{(\lambda^3-\alpha^3)(\lambda^3\bar{\alpha}^3-1)(2b_2c_2-1)},$$

all these entries of $h(\lambda)$ are computed by Maple. It satisfies τ-reality condition iff

$$\tau(h(\bar{\lambda}^{-1})) = h(\lambda),$$

i.e.

$$\bar{A}\bar{g}_{\alpha,\ell_1}(\bar{\lambda}^{-1})\bar{g}_{\bar{\alpha}^{-1},\ell_2}(\bar{\lambda}^{-1})\ P_{12}\,T\ g^t_{\bar{\alpha}^{-1},\ell_2}(-\lambda)^t g^t_{\alpha,\ell_1}(-\lambda)A^t = P_{12}\ T. \tag{4.16}$$

Step 1: Evaluate LHS of (4.16) at $\lambda=\infty$ to get

$$\bar{A}(I-3\bar{\alpha}^{-1}\mathrm{diag}(\bar{R}_1))(I-3\alpha\mathrm{diag}(\bar{R}_2))\ P_{12}\,T A^t = \ P_{12}\,T.$$

This implies that

$$\begin{pmatrix} \frac{-\bar{d}}{2\bar{b}_1\bar{c}_1-1} & & \\ & \bar{d}^{-1}(1-2\bar{b}_1\bar{c}_1) & \\ & & \mp 1\end{pmatrix}\begin{pmatrix} \frac{-1}{2\bar{b}_2\bar{c}_2-1} & & \\ & (1-2\bar{b}_2\bar{c}_2) & \\ & & -1\end{pmatrix}\ P_{12}\,T\ \begin{pmatrix} d & & \\ & d^{-1} & \\ & & \pm 1\end{pmatrix}$$
$$=\ P_{12}\,T\ .$$

It follows that

$$|d|^2 = (2\bar{b}_1\bar{c}_1-1)(2\bar{b}_2\bar{c}_2-1). \tag{4.17}$$

Step 2: Compute the residue of LHS of (4.16) at $\lambda=-\alpha$ and $\lambda=\bar{\alpha}^{-1}$ to get

$$\overline{Ag_{\alpha,\ell_1}(-\bar{\alpha}^{-1})g_{\bar{\alpha}^{-1},\ell_2}(-\bar{\alpha}^{-1})}\ P_{12}\,T\ g^t_{\bar{\alpha}^{-1},\ell_2}(\alpha)R_1^t = 0.$$

Since $Ag_{\alpha,\ell_1}(-\bar{\alpha}^{-1})$ is invertible, we have

$$\overline{g_{\bar{\alpha}^{-1},\ell_2}(-\bar{\alpha}^{-1})}\ P_{12}\,T\ g^t_{\bar{\alpha}^{-1},\ell_2}(\alpha)R_1^t = 0. \tag{4.18}$$

Residues of LHS of (4.16) at $\lambda = -\epsilon^2\alpha,\ -\epsilon^4\alpha$ give the same condition (4.18). Compute the residue of LHS of (4.16) at $\lambda = -\bar{\alpha}^{-1}$ and $\lambda = \alpha$ to get

$$\overline{Ag_{\alpha,\ell_1}(-\alpha)g_{\bar{\alpha}^{-1},\ell_2}(-\alpha)}\ P_{12}\,T\ R_2^t g_{\alpha,\ell_1}^t(\bar{\alpha}^{-1}) = 0.$$

But $g_{\alpha,\ell_1}(\bar{\alpha}^{-1})$ is invertible, so

$$\overline{Ag_{\alpha,\ell_1}(-\alpha)g_{\bar{\alpha}^{-1},\ell_2}(-\alpha)}\ P_{12}\,T\ R_2^t = 0. \tag{4.19}$$

Step 3: Consider (4.19). (4.4) implies $\ker Ag_{\alpha,\ell_1}(-\alpha)$ is spanned by $(c_1, b_1, 1)^t$. So (4.19) implies that

$$g_{\bar{\alpha}^{-1},\ell_2}(-\alpha)\ P_{12}\,T\ \begin{pmatrix}\bar{b}_2\\ \bar{c}_2\\ 1\end{pmatrix} \parallel \begin{pmatrix}c_1\\ b_1\\ 1\end{pmatrix}, \tag{4.20}$$

where $\parallel$ means two vectors are parallel. Tedious, but direct computation implies that

$$g_{\bar{\alpha}^{-1},\ell_2}(-\alpha)\begin{pmatrix}\bar{b}_2\\ \bar{c}_2\\ -H/2\end{pmatrix} \parallel \begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix}, \tag{4.21}$$

where

$$\begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix} = \begin{pmatrix}\bar{b}_2 - \frac{2c_2}{(2b_2c_2-1)(1+|\alpha|^6)}(|b_2|^2 + |\alpha|^4|c_2|^2 + \frac{H}{2}|\alpha|^2)\\ \bar{c}_2 - \frac{2b_2}{1+|\alpha|^6}(-|\alpha|^2|b_2|^2 + |c_2|^2 - \frac{H}{2}|\alpha|^4)\\ -\frac{H}{2} - \frac{2}{1+|\alpha|^6}(|\alpha|^4|b_2|^2 - |\alpha|^2|c_2|^2 - \frac{H}{2})\end{pmatrix}.$$

By (4.21), we have

$$b_1 = \frac{-b_2(-|\alpha|^2|b_2|^2 + |c_2|^2 - \frac{H}{2}|\alpha|^4) + \frac{1}{2}(1+|\alpha|^6)\bar{c}_2}{-|\alpha|^4|b_2|^2 + |\alpha|^2|c_2|^2 + \frac{H}{4}(1-|\alpha|^6)}, \tag{4.22}$$

$$c_1 = \frac{-c_2(|b_2|^2 + |\alpha|^4|c_2|^2 + \frac{H}{2}|\alpha|^2) + \frac{1}{2}(2b_2c_2-1)(1+|\alpha|^6)\bar{b}_2}{(2b_2c_2-1)[-|\alpha|^4|b_2|^2 + |\alpha|^2|c_2|^2 + \frac{H}{4}(1-|\alpha|^6)]}. \tag{4.23}$$

Step 4: Consider (4.18). σ_1-reality condition of $g_{\bar{\alpha}^{-1},\ell_2}(\lambda)$ implies

$$g_{\bar{\alpha}^{-1},\ell_2}(-\lambda)\ P_{12}\ g_{\bar{\alpha}^{-1},\ell_2}^t(\lambda) = P_{12}.$$

Compute residue at $\lambda = \bar{\alpha}^{-1}$ to get

$$g_{\bar{\alpha}^{-1},\ell_2}(-\bar{\alpha}^{-1})\ P_{12}\ R_2^t = 0,$$

which implies that

$$\ker g_{\bar{\alpha}^{-1},\ell_2}(-\bar{\alpha}^{-1}) = \mathrm{Span}_{\mathbb{C}}\begin{pmatrix}c_2\\ b_2\\ 1\end{pmatrix}.$$

So (4.18) implies that

$$P_{12}\ Tg_{\bar{\alpha}^{-1},\ell_2}^*(\alpha)\begin{pmatrix}\bar{b}_1\\ \bar{c}_1\\ 1\end{pmatrix} \parallel \begin{pmatrix}c_2\\ b_2\\ 1\end{pmatrix}.$$

It is equivalent to (4.20).

Conversely, if the product of two simple element $h(\lambda)$ satisfies (4.17), (4.22) and (4.23), $h(\lambda)$ satisfies the τ-reality condition.

We conclude the above computations into the following theorem.

THEOREM 4.3. *(1) Take any $\alpha \in \mathbf{S}^1$ and $b \in \mathbb{C}$ so that $|b|^2 > \frac{1}{2}$. Then $c = \bar{b}$ and $|d|^2 = (2|b|^2 - 1) \Leftrightarrow Am_{\alpha,\ell}(\lambda) \in \Lambda_I^{\sigma,\tau_1} GL(3,\mathbb{C})$.*
(2) There is no simple element with 3 simple poles in $\Lambda_I^{\sigma,\tau_2} GL(3,\mathbb{C})$.
(3) Take any $\alpha \in \mathbb{C}^\times/\mathbf{S}^1$, $b_2, c_2 \in \mathbb{C}$, $2b_2c_2 \neq 1$, define

$$\ell_1 = \mathbb{C}\cdot(b_1, c_1, 1), \quad \ell_2 = \mathbb{C}\cdot(b_2, c_2, 1), \quad \bar{\ell}_2 \; P_{12} \; T \; g^t_{\bar{\alpha}^{-1},\ell_2}(-\alpha) = \ell_1 P_{12}, \tag{4.24}$$

$$A = \mathrm{diag}(d, d^{-1}, 1), \quad |d|^2 = (2b_1c_1 - 1)(2b_2c_2 - 1).$$

Here (4.24) *is equivalent to* (4.22)-(4.23). *Then $Ag_{\alpha,\ell_1}(\lambda) g_{\bar{\alpha}^{-1},\ell_2}(\lambda) \in \Lambda_I^{\sigma,\tau} GL(3,\mathbb{C})$.*

REMARK 4.4. It would be interesting to consider the converse question: whether any rational element with 6 poles must take the above form? More generally, it would be interesting to know whether any rational element can be factorized into product of the above simple ones? Such factorization problem has been treated by Uhlenbeck [42] in the case of unitary group. We will answer these questions in a subsequent paper.

REMARK 4.5. The formula (4.17) implies $(2b_1c_1-1)(2b_2c_2-1)$ decides $|d|^2$. Since $|d|^2$ is real positive, there is an important restriction that the choice of α, b_2, c_2 must keep

$$(2b_1c_1 - 1)(2b_2c_2 - 1) > 0.$$

Substitute $b_1,\ c_1$ into (4.17), we get

$$|d|^2 = (2b_1c_1 - 1)(2b_2c_2 - 1) = \frac{\Psi_{\alpha,b_2,c_2}}{(-|\alpha|^4|b_2|^2 + |\alpha|^2|c_2|^2 + \frac{H}{4}(1-|\alpha|^6))^2},$$

here

$$\begin{aligned}\Psi_{\alpha,b_2,c_2} =& \frac{1}{4}|2b_2c_2 - 1|^2|\alpha|^{12} - |c_2|^4|\alpha|^{10} - H|c_2|^2|\alpha|^8 + (-2Re(b_2c_2) - \frac{1}{2})|\alpha|^6 \\ & - H|b_2|^2|\alpha|^4 - |b_2|^4|\alpha|^2 + \frac{1}{4}|2b_2c_2 - 1|^2.\end{aligned} \tag{4.25}$$

It is easy to see $(2b_1c_1-1)(2b_2c_2-1)$ is real and only need to choose α, b_2, c_2 such that $\Psi_{\alpha,b_2,c_2} > 0$, then $(2b_1c_1 - 1)(2b_2c_2 - 1) > 0$. For convenience, we will always assume α, b_2, c_2 satisfy this restriction when we choose a simple element $h(\lambda)$ afterward.

5. Dressing actions of simple elements. In this section we use the simple element to give the dressing actions on definite affine spheres.

Let us review the technique of dressing action (the idea of dressing was dated from [45] but see [23] or [40] for an elementary introduction). Let $G = SL(3,\mathbb{C})$, $g(\lambda) \in \Lambda_I^{\sigma,\tau} G$, and $E(z,\bar{z},\lambda) \in \Lambda_{\mathbb{C}^\times}^{\sigma,\tau} G$ is the frame of a definite affine sphere. Assume we can do the following factorization for each fixed $(z,\bar{z})$:

$$g(\lambda)E(z,\bar{z},\lambda) = \tilde{E}(z,\bar{z},\lambda)\tilde{g}(z,\bar{z},\lambda), \tag{5.1}$$

with $\tilde{E}(z,\bar{z},\lambda) \in \Lambda^{\sigma,\tau}_{\mathbb{C}^\times} G$, $\tilde{g}(z,\bar{z},\lambda) \in \Lambda^{\sigma,\tau}_I G$. Then $\tilde{E}(z,\bar{z},\lambda)$ will also be the frame of a new definite affine sphere. We sketch the proof here. By Theorem 3.2, it suffices to prove that $\tilde{E}^{-1}\tilde{E}_z$ and $\tilde{E}^{-1}\tilde{E}_{\bar{z}}$ are linear in λ and λ^{-1} respectively. But

$$\begin{aligned}\tilde{E}^{-1}\tilde{E}_z &= \tilde{g}(z,\bar{z},\lambda)E^{-1}E_z\tilde{g}(z,\bar{z},\lambda)^{-1} - \tilde{g}(z,\bar{z},\lambda)_z\tilde{g}(z,\bar{z},\lambda)^{-1}\\ &= \tilde{g}(z,\bar{z},\lambda)(u_0+\lambda u_1)\tilde{g}(z,\bar{z},\lambda)^{-1} - \tilde{g}(z,\bar{z},\lambda)_z\tilde{g}(z,\bar{z},\lambda)^{-1}.\end{aligned}$$

On the left hand side $\tilde{E}^{-1}\tilde{E}_z$ is holomorphic in $\lambda \in \mathbb{C}^\times$; on the right it has a simple pole at ∞. So

$$\tilde{E}^{-1}\tilde{E}_z = \tilde{u}_0 + \lambda\tilde{u}_1,$$

where $\tilde{u}_0$, $\tilde{u}_1$ are independent of λ. Similarly, $\tilde{E}^{-1}\tilde{E}_{\bar{z}}$ is linear in $\frac{1}{\lambda}$. This completes the proof.

Furthermore, $g * E := \tilde{E}$ defines a group action of $\Lambda^{\sigma,\tau}_I G$ on the frames of the definite affine spheres, which is called the *dressing action*.

Remark 5.1. We can consider the dressing action of $g \in \Lambda^{\sigma,\tau}_I GL(3,\mathbb{C})$ on $\Lambda^{\sigma,\tau}_{\mathbb{C}^\times} SL(3,\mathbb{C})$, since a scaling by $(\det g)^{-1/3}$ can make g lies in $SL(3,\mathbb{C})$ by Proposition 4.2 and scaling does not affect the dressing action. What's more, the differences between rank 1 type $Ag_{\alpha,\ell}(\lambda)$ and rank 2 type $Am_{\alpha,\ell}(\lambda)$, and between $A = diag(d,d^{-1},1)$ and $diag(d,d^{-1},-1)$ are also scalings. Without loss of generality, we only consider one type of simple element and choose $A = diag(d,d^{-1},1)$.

Remark 5.2. Since

$$\Lambda^{\sigma,\tau}_I GL(3,\mathbb{C}) \cap \Lambda^{\sigma,\tau}_{\mathbb{C}^\times} SL(3,\mathbb{C}) = \{diag(e^{i\theta},e^{-i\theta},1) \mid \theta \in \mathbb{R}\}, \tag{5.2}$$

the factorization (5.1) will not be unique and the ambiguity lies in (5.2). By Theorem 3.2, we can eliminate the ambiguity by requiring $d > 0$ in the matrix A, without changing the geometric property of dressing action. We also know that

$$\Lambda^{\sigma}_I GL(3,\mathbb{C}) \cap \Lambda^{\sigma}_{\mathbb{C}^\times} SL(3,\mathbb{C}) = \{diag(d,d^{-1},1) \mid d \in \mathbb{C}^\times\}.$$

Lemma 5.3. *Let $Ag_{\alpha,\ell}(\lambda) \in \Lambda^{\sigma}_I GL(3,\mathbb{C})$ as in (4.8), and $E(\lambda) \in \Lambda^{\sigma}_{\mathbb{C}^\times} SL(3,\mathbb{C})$. If $\tilde{\ell} := \ell E(\alpha) \nsubseteq \Delta$, then we have*

$$\tilde{E}(\lambda) := Ag_{\alpha,\ell} \cdot E(\lambda) \cdot g^{-1}_{\alpha,\tilde{\ell}}\tilde{A}^{-1} \tag{5.3}$$

lies in $\Lambda^{\sigma}_{\mathbb{C}^\times} SL(3,\mathbb{C})$ for any $\tilde{A} = diag\{\tilde{d},\tilde{d}^{-1},1\}$, with arbitrary $\tilde{d} \in \mathbb{C}^\times$.

Proof. Since $\tilde{E}(\lambda)$ satisfied the σ-reality condition and holomorphic in $\mathbb{C}^\times$ except for possible simple poles coming from the poles of $Ag_{\alpha,\ell}$ and $g^{-1}_{\alpha,\tilde{\ell}}\tilde{A}^{-1}$, we only need to prove that residues of $\tilde{E}(\lambda)$ are zero at both α and $-\alpha$. But

$$\sigma_1(Ag_{\alpha,\ell}(\lambda)) = Ag_{\alpha,\ell}(-\lambda) \Longleftrightarrow P_{12} = (Ag_{\alpha,\ell}(\lambda))\ P_{12}\ A(g_{\alpha,\ell}(-\lambda))^t,$$

whose residue is zero at α implies that $\ell\ P_{12}\ (g_{\alpha,\ell}(-\alpha))^t = 0$, or equivalently $(g_{\alpha,\ell}(-\alpha))\ P_{12}\ \ell^t = 0$. These two equations are also true for $\tilde{\ell}$. Therefore $(b_1,c_1,1)E(\alpha) \in \tilde{\ell}$ and the special form of R in (4.7) imply that

$$Res_\alpha\tilde{E} = 2\alpha\ R\ E(\alpha)\ P_{12}\ g_{\alpha,\tilde{\ell}}(-\alpha)^t\tilde{A}^t\ = 0,$$

and $E(-\alpha)\ P_{12}\ (\tilde{b}_1, \tilde{b}_2, 1)^t \in [E(-\alpha)\ P_{12}\ E(\alpha)^t]\ell^t = P_{12}\ \ell^t$ implies that

$$Res_{-\alpha}\tilde{E} = -2\alpha\ Ag_{\alpha,\ell}(-\alpha)\ E(-\alpha)\ P_{12}\ \tilde{R}^t\ P_{12} = 0,$$

The proof is completed once we notice that $\det\tilde{E} = 1$ by Proposition 4.2. □

When we impose the τ-reality condition in the following, the above $\tilde{d}$ will be uniquely determined by requiring positiveness (see Remark 5.2). Now we begin to compute the dressing actions of two types of elements in Theorem 4.3.

For the first type $|\alpha| = 1$. We consider the simple element $Am_{\alpha,\ell}$ for hyperbolic case($\tau = \tau_1$) first. Let $E(\lambda) \in \Lambda^{\sigma,\tau_1}_{\mathbb{C}^\times} SL(3,\mathbb{C})$, by Lemma 5.3, we get

$$\tilde{E}(\lambda) := Am_{\alpha,\ell} \cdot E(\lambda) \cdot m^{-1}_{\alpha,\tilde{\ell}} \tilde{A}^{-1} \in \Lambda^{\sigma}_{\mathbb{C}^\times} SL(3,\mathbb{C}).$$

here $\tilde{\ell} = \ell E(-\alpha)$. Since $Am_{\alpha,\ell}$ satisfies τ-reality condition, (4.13) implies $\ell P_{12} = \bar{\ell}$. For $\tilde{\ell}$, we also have

$$\tilde{\ell} P_{12} = \ell E(-\alpha) P_{12} = \bar{\ell}\bar{E}(-\alpha) = \bar{\tilde{\ell}}.$$

So we only need to choose $|\tilde{d}|^2 = 2|\tilde{b}|^2 - 1$ to get $\tilde{A}m_{\alpha,\tilde{\ell}_1}(\lambda) \in \Lambda^{\sigma,\tau_1}_I GL(3,\mathbb{C})$. Then we have $\tilde{E} \in \Lambda^{\sigma,\tau_1}_{\mathbb{C}^\times} SL(3,\mathbb{C})$. This leads to the following Theorem:

THEOREM 5.4. *Given any hyperbolic affine sphere r, there is a family of hyperbolic affine spheres r_λ with the local affine frames $E(z,\bar{z},\lambda) \in \Lambda^{\sigma,\tau_1}_{\mathbb{C}^\times} SL(3,\mathbb{C})$. We normalize this affine frames at (0,0), i.e., $E(0,0,\lambda) = I$. Pick any simple element $Am_{\alpha,\ell} \in \Lambda^{\sigma,\tau_1}_I GL(3,\mathbb{C})$. If $\ell E(z,\bar{z},-\alpha) \nsubseteq \Delta$ and $2|\tilde{b}|^2 - 1 > 0$ while $\tilde{b}$ is determined by $\tilde{\ell} := \ell E(z,\bar{z},-\alpha) \parallel (\tilde{b}, \bar{\tilde{b}}, 1)$. We define $\tilde{A} = diag\{\tilde{d}, \tilde{d}^{-1}, 1\}$ and $\tilde{d} = \sqrt{2|\tilde{b}|^2 - 1}$, and get the factorization:*

$$Am_{\alpha,\ell} \cdot E(z,\bar{z},\lambda) = \tilde{E}(z,\bar{z},\lambda) \cdot \tilde{A}(z,\bar{z}) m_{\alpha,\tilde{\ell}(z,\bar{z})} \in \Lambda^{\sigma,\tau_1}_{\mathbb{C}^\times} SL(3,\mathbb{C}) \times \Lambda^{\sigma,\tau_1}_I GL(3,\mathbb{C}).$$

From the new affine frame $\tilde{E}(z,\bar{z},\lambda)$, we can get explicit formula of new definite affine sphere:

$$\hat{r} = \frac{-H(\lambda^3 + \alpha^3) r e^{\psi} - 4\alpha^3 (ln\phi)_{\bar{z}} r_z - 4\lambda^3 (ln\phi)_z r_{\bar{z}}}{(\lambda^3 + \alpha^3) e^{\psi}}, \tag{5.4}$$

where $\phi := (b,\ \bar{b},\ 1) r_{-\alpha}$ is a scalar solution of (3.7) with parameter $-\alpha$. This new definite affine sphere has new affine metric $\tilde{d}^2 \cdot e^{\psi}$ and the same affine cubic form (and the same affine mean curvature).

There is no similar transformations for elliptic affine spheres due to Theorem 4.3.

Proof. Since $\ell E(z,\bar{z},-\alpha) \parallel (\tilde{b}, \bar{\tilde{b}}, 1)$, we get

$$\begin{aligned}&(b,\ \bar{b},\ 1)\ \left(\tfrac{1}{-\alpha}\sqrt{-2H} e^{\frac{-\psi}{2}} (r_{-\alpha})_z,\quad -\alpha\sqrt{-2H} e^{\frac{-\psi}{2}} (r_{-\alpha})_{\bar{z}},\quad -H r_{-\alpha}\right)\\ &= \left(\tfrac{1}{-\alpha}\sqrt{-2H} e^{\frac{-\psi}{2}} \phi_z,\quad -\alpha\sqrt{-2H} e^{\frac{-\psi}{2}} \phi_{\bar{z}},\quad -H\phi\right)\end{aligned}$$

where $\phi := (b,\ \bar{b},\ 1) r_{-\alpha}$ is a scalar solution of (3.7) with parameter $-\alpha$. The third column of $\tilde{E}(z,\bar{z},\lambda)$ gives new hyperbolic affine sphere, so does the affine transformation of it by $m_{\alpha,\ell}(\lambda)^{-1}\, A^{-1}$:

$$\begin{aligned}\hat{r} &= m_{\alpha,\ell}(\lambda)^{-1}\, A^{-1}\, (\tilde{E}(\lambda))_3\\ &= (E(\lambda)\, m_{\alpha,\tilde{\ell}}(\lambda)^{-1}\, \tilde{A}^{-1})_3\\ &= \left(\tfrac{1}{\lambda}\sqrt{-2H}e^{\frac{-\psi}{2}}r_z,\quad \lambda\sqrt{-2H}e^{\frac{-\psi}{2}}r_{\bar{z}},\quad -Hr\right)\cdot\frac{1}{\lambda^3+\alpha^3}\cdot\begin{pmatrix}2\alpha^3\lambda\frac{\sqrt{-2H}e^{-\frac{\psi}{2}}\phi_{\bar{z}}}{H\phi}\\ 2\lambda^2\frac{\sqrt{-2H}e^{-\frac{\psi}{2}}\phi_z}{H\phi}\\ \lambda^3+\alpha^3\end{pmatrix}\\ &= \frac{-H(\lambda^3+\alpha^3)re^{\psi}-4\alpha^3(ln\phi)_{\bar{z}}r_z-4\lambda^3(ln\phi)_z r_{\bar{z}}}{(\lambda^3+\alpha^3)e^{\psi}}.\end{aligned}$$

Recall the discussion in the beginning of this section, $\tilde{E}^{-1}\tilde{E}_z$ be expressed as:

$$\begin{aligned}\tilde{E}^{-1}\tilde{E}_z &= \tilde{A}m_{\alpha,\tilde{\ell}}(u_0+\lambda u_1)(\tilde{A}m_{\alpha,\tilde{\ell}})^{-1}-(\tilde{A}m_{\alpha,\tilde{\ell}})_z(\tilde{A}m_{\alpha,\tilde{\ell}})^{-1}.\\ &= \tilde{u}_0+\lambda\tilde{u}_1.\end{aligned}$$

We can get $\tilde{u}_1$ by

$$\begin{aligned}\tilde{u}_1 &= \lim_{\lambda\to\infty}\frac{\tilde{E}^{-1}\tilde{E}_z}{\lambda}\\ &= \lim_{\lambda\to\infty}\tilde{A}m_{\alpha,\tilde{\ell}}\; u_1\; (\tilde{A}m_{\alpha,\tilde{\ell}})^{-1}\\ &= \tilde{A}\; u_1\; P_{12}\tilde{A}P_{12}\\ &= \begin{pmatrix}0 & 0 & \frac{\sqrt{-2H}}{2}\tilde{d}e^{\frac{\psi}{2}}\\ U\tilde{d}^{-2}e^{\psi} & 0 & 0\\ 0 & \frac{\sqrt{-2H}}{2}\tilde{d}e^{\frac{\psi}{2}} & 0\end{pmatrix}.\end{aligned}$$

Compare $\tilde{u}_1$ with the coefficient matrix of λ in (3.7), it is easy to see that this new definite affine sphere has new affine metric $\tilde{d}^2\cdot e^{\psi}$ and the same affine cubic form (and the same affine mean curvature). □

The formula (5.4) is similar to the classical Tzitzéica transformation of indefinite affine spheres.

Now we consider the dressing action of the second type $h(\lambda) = Ag_{\alpha,\ell_1}(\lambda)g_{\bar{\alpha}^{-1},\ell_2}(\lambda)\in\Lambda_I^{\sigma,\tau}GL(3,\mathbb{C})$ in Theorem 4.3 on $E\in\Lambda_{\mathbb{C}^\times}^{\sigma,\tau}SL(3,\mathbb{C})$. We just need to apply Lemma 5.3 twice:

$$\hat{E}(\lambda):=g_{\bar{\alpha}^{-1},\ell_2}(\lambda)\cdot E(\lambda)\cdot g^{-1}_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\lambda)\in\Lambda^{\sigma}_{\mathbb{C}^\times}SL_3\mathbb{C}$$

$$\tilde{E}(\lambda):=Ag_{\alpha,\ell_1}(\lambda)g_{\bar{\alpha}^{-1},\ell_2}(\lambda)\cdot E(\lambda)\cdot g^{-1}_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\lambda)g^{-1}_{\alpha,\tilde{\ell}_1}(\lambda)\tilde{A}^{-1}\in\Lambda^{\sigma}_{\mathbb{C}^\times}SL_3\mathbb{C}$$

where

$$\ell_1=\mathbb{C}\begin{pmatrix}b_1\\ c_1\\ 1\end{pmatrix}^t,\ell_2=\mathbb{C}\begin{pmatrix}b_2\\ c_2\\ 1\end{pmatrix}^t,$$

$$A=diag(d,d^{-1},1),$$

and $\tilde{\ell}_1, \tilde{\ell}_2$ satisfying

$$\tilde{\ell}_2 = \ell_2 E(\bar{\alpha}^{-1}),$$
$$\tilde{\ell}_1 = \ell_1 \hat{E}(\alpha) = \ell_1 g_{\bar{\alpha}^{-1},\ell_2}(\alpha) E(\alpha) g^{-1}_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\alpha),$$

or

$$\ell_2 = \tilde{\ell}_2 E^{-1}(\bar{\alpha}^{-1}), \tag{5.5}$$

$$\ell_1 = \tilde{\ell}_1 g_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\alpha) E^{-1}(\alpha) g^{-1}_{\bar{\alpha}^{-1},\ell_2}(\alpha). \tag{5.6}$$

Since $h(\lambda)$ satisfies τ-reality condition, from (4.20), we have

$$\bar{\ell}_2\ P_{12}\ T\ g^t_{\bar{\alpha}^{-1},\ell_2}(-\alpha) = \ell_1 P_{12},$$

or

$$\ell_1 g_{\bar{\alpha}^{-1},\ell_2}(\alpha) = \bar{\ell}_2 T. \tag{5.7}$$

Substitute ℓ_1, ℓ_2 in (5.5)-(5.6) into (5.7), and we get

$$\tilde{\ell}_1 g_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\alpha) = \bar{\tilde{\ell}}_2 T. \tag{5.8}$$

Compare (5.8) with (4.24) , we can find that the relationship between $\tilde{\ell}_1$ and $\tilde{\ell}_2$ satisfies (4.24) in Theorem 4.3. Only need to choose $\tilde{d}$ satisfies (4.17), we get $\tilde{h}(\lambda) = \tilde{A} g_{\alpha,\tilde{\ell}_1}(\lambda) g_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\lambda) \in \Lambda_I^{\sigma,\tau} GL(3,\mathbb{C})$. Then we have $\tilde{E} \in \Lambda^{\sigma,\tau}_{\mathbb{C}^\times} SL(3,\mathbb{C})$.

Summarizing the above computations, we have the following theorem.

THEOREM 5.5. *Given any definite affine sphere r, there is a family of definite affine spheres r_λ with the local affine frames $E(z,\bar{z},\lambda) \in \Lambda^{\sigma,\tau}_{\mathbb{C}^\times} SL(3,\mathbb{C})$. We normalize this affine frames at (0,0), i.e., $E(0,0,\lambda) = I$. Pick $h(\lambda) = A g_{\alpha,\ell_1}(\lambda) g_{\bar{\alpha}^{-1},\ell_2}(\lambda) \in \Lambda_I^{\sigma,\tau} GL(3,\mathbb{C})$. If both $\tilde{\ell}_2 := \ell_2 E(\bar{\alpha}^{-1}) \parallel (\tilde{b}_2, \tilde{c}_2, 1)$ and $\tilde{\ell}_1 = \ell_1 g_{\bar{\alpha}^{-1},\ell_2}(\alpha) E(\alpha) g^{-1}_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\alpha) \parallel (\tilde{b}_1, \tilde{c}_1, 1)$ are not in Δ and $(2\tilde{b}_1\tilde{c}_1 - 1)(2\tilde{b}_2\tilde{c}_2 - 1) > 0$, we define $\tilde{A} = diag\{\tilde{d}, \tilde{d}^{-1}, 1\}$ and $\tilde{d} = \sqrt{(2\tilde{b}_1\tilde{c}_1 - 1)(2\tilde{b}_2\tilde{c}_2 - 1)}$, and get the factorization:*

$$A g_{\alpha,\ell_1}(\lambda) g_{\bar{\alpha}^{-1},\ell_2}(\lambda) \cdot E(z,\bar{z},\lambda) = \tilde{E}(z,\bar{z},\lambda) \cdot \tilde{A}(z,\bar{z}) g_{\alpha,\tilde{\ell}_1(z,\bar{z})}(\lambda) g_{\bar{\alpha}^{-1},\tilde{\ell}_2(z,\bar{z})}(\lambda), \tag{5.9}$$

here $\tilde{E}(z,\bar{z},\lambda) \in \Lambda^{\sigma,\tau}_{\mathbb{C}^\times} SL(3,\mathbb{C})$, $\tilde{A}(z,\bar{z}) g_{\alpha,\tilde{\ell}_1(z,\bar{z})}(\lambda) g_{\bar{\alpha}^{-1},\tilde{\ell}_2(z,\bar{z})}(\lambda) \in \Lambda_I^{\sigma,\tau} GL(3,\mathbb{C})$.

From the new affine frame $\tilde{E}(z,\bar{z},\lambda)$, we can get explicit formula of new definite affine sphere

$$\begin{aligned}
\tilde{r} &= N \frac{\alpha\lambda^3\tilde{b}_1\ \{-2\sqrt{-2H}e^{-\frac{\psi}{2}}[\lambda^3\bar{\alpha}^3(2\tilde{b}_2\tilde{c}_2 - 1) - 1]\ r_{\bar{z}} + 4\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}^2\tilde{c}_2^2\ r_z + 4H\bar{\alpha}\tilde{c}_2\ r\}}{(\lambda^3\bar{\alpha}^3 + 1)(2\tilde{b}_2\tilde{c}_2 - 1)(\lambda^3 + \alpha^3)} \\
&+ N \frac{\alpha^2\tilde{c}_2\ [4\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}\lambda^3\tilde{b}_2^2\ r_{\bar{z}} + 2\sqrt{-2H}e^{-\frac{\psi}{2}}(\lambda^3\bar{\alpha}^3 + 1 - 2\tilde{b}_2\tilde{c}_2)\ r_z + 4H\bar{\alpha}^2\lambda^3\tilde{b}_2\ r]}{(\lambda^3\bar{\alpha}^3 + 1)(\lambda^3 + \alpha^3)} \\
&+ N \frac{(-\lambda^3 + \alpha^3)[2\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}^2\lambda^3\tilde{b}_2\ r_{\bar{z}} - 2\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}\tilde{c}_2\ r_z + H(\lambda^3\bar{\alpha}^3 - 1)\ r]}{(\lambda^3\bar{\alpha}^3 + 1)(\lambda^3 + \alpha^3)}
\end{aligned}$$

here

$$N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{-\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & \sqrt{H/2} \end{pmatrix}$$

This new definite affine sphere has new affine metric $\tilde{d}^2 \cdot e^{\psi}$ *and the same affine cubic form (and the same affine mean curvature).*

Proof. By Remark 3.1, we have

$$\begin{aligned}
\hat{r} &= (N[Ag_{\alpha,\ell_1}(\lambda)g_{\bar{\alpha}^{-1},\ell_2}(\lambda)]^{-1}\tilde{E}(\lambda)N^{-1})_3 \\
&= (NE(\lambda)g_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\lambda)^{-1}g_{\alpha,\tilde{\ell}_1}(\lambda)^{-1}\tilde{A}_1^{-1})_3 \\
&= (NE(\lambda)\ P_{12}\ g_{\bar{\alpha}^{-1},\tilde{\ell}_2}(-\lambda)^t g_{\alpha,\tilde{\ell}_1}(-\lambda)^t\tilde{A}_1^t)_3 \\
&= N\,(\lambda\sqrt{-2H}e^{-\frac{\psi}{2}}r_{\bar{z}}, \frac{1}{\lambda}\sqrt{-2H}e^{-\frac{\psi}{2}}r_z, -Hr) \\
&\quad\cdot \begin{pmatrix} \frac{\lambda^3\bar{\alpha}^3(2\tilde{c}_1\tilde{c}_2-1)-1}{(\lambda^3\bar{\alpha}^3+1)(2\tilde{c}_1\tilde{c}_2-1)} & \frac{2\bar{\alpha}\lambda\tilde{c}_1^2}{\lambda^3\bar{\alpha}^3+1} & \frac{-2\bar{\alpha}^2\lambda^2\tilde{c}_1}{\lambda^3\bar{\alpha}^3+1} \\ \frac{-2\bar{\alpha}^2\lambda^2\tilde{c}_2^2}{(\lambda^3\bar{\alpha}^3+1)(2\tilde{c}_1\tilde{c}_2-1)} & \frac{\lambda^3\bar{\alpha}^3+1-2\tilde{c}_1\tilde{c}_2}{\lambda^3\bar{\alpha}^3+1} & \frac{2\bar{\alpha}\lambda\tilde{c}_2}{\lambda^3\bar{\alpha}^3+1} \\ \frac{2\bar{\alpha}\lambda\tilde{c}_2}{(\lambda^3\bar{\alpha}^3+1)(2\tilde{c}_1\tilde{c}_2-1)} & \frac{-2\bar{\alpha}^2\lambda^2\tilde{c}_1}{\lambda^3\bar{\alpha}^3+1} & \frac{\lambda^3\bar{\alpha}^3-1}{\lambda^3\bar{\alpha}^3+1} \end{pmatrix} \cdot \begin{pmatrix} \frac{-2\alpha\lambda^2\tilde{b}_1}{\lambda^3+\alpha^3} \\ \frac{2\alpha^2\lambda\tilde{b}_2}{\lambda^3+\alpha^3} \\ \frac{\lambda^3-\alpha^3}{\lambda^3+\alpha^3} \end{pmatrix} \\
&= N\frac{\alpha\lambda^3\tilde{b}_1\ \{-2\sqrt{-2H}e^{-\frac{\psi}{2}}[\lambda^3\bar{\alpha}^3(2\tilde{c}_1\tilde{c}_2-1)-1]\ r_{\bar{z}} + 4\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}^2\tilde{c}_2^2\ r_z + 4H\bar{\alpha}\tilde{c}_2\ r\}}{(\lambda^3\bar{\alpha}^3+1)(2\tilde{c}_1\tilde{c}_2-1)(\lambda^3+\alpha^3)} \\
&\quad + N\frac{\alpha^2\tilde{b}_2\ [4\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}\lambda^3\tilde{c}_1^2\ r_{\bar{z}} + 2\sqrt{-2H}e^{-\frac{\psi}{2}}(\lambda^3\bar{\alpha}^3+1-2\tilde{c}_1\tilde{c}_2)\ r_z + 4H\bar{\alpha}^2\lambda^3\tilde{c}_1\ r]}{(\lambda^3\bar{\alpha}^3+1)(\lambda^3+\alpha^3)} \\
&\quad + N\frac{(-\lambda^3+\alpha^3)[2\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}^2\lambda^3\tilde{c}_1\ r_{\bar{z}} - 2\sqrt{-2H}e^{-\frac{\psi}{2}}\bar{\alpha}\tilde{c}_2\ r_z + H(\lambda^3\bar{\alpha}^3-1)\ r]}{(\lambda^3\bar{\alpha}^3+1)(\lambda^3+\alpha^3)}.
\end{aligned}$$

∎

6. Basic examples. In this section, we will apply our results to construct some basic examples.

Example 6.1 (The **vacuum** solution). Assume $H = -2$ and $U = 1$ in the equation (3.5) for hyperbolic definite affine spheres. Then it admits the trivial solution $\psi = 0$ (also called the vacuum solution), and the corresponding surface is $x_1x_2x_3 = \frac{\sqrt{3}}{72}$. One can integrate (3.7) to obtain the whole family of frames. Note that the surface is independent of the parameter λ. So the associated family is really a family of parameterizations of the same affine sphere.

We may choose the following conformal parametrization of the vacuum definite affine sphere after certain affine transformation:

$$X = \frac{\sqrt{3}}{6}\begin{pmatrix} R(\lambda) \\ R(\epsilon^2\lambda) \\ R(\epsilon^4\lambda) \end{pmatrix},$$

where

$$\epsilon = e^{\frac{\pi i}{3}},\ R(\lambda) = \exp(\lambda z + \frac{1}{\lambda}\bar{z}).$$

Then

$$F(z,\bar z,\lambda)=\Big(\tfrac{1}{\lambda}\sqrt{-2H}e^{\frac{-\psi}{2}}X_z,\quad \lambda\sqrt{-2H}e^{\frac{-\psi}{2}}X_{\bar z},\quad -HX\Big)$$
$$=\frac{\sqrt{3}}{3}\begin{pmatrix} R(\lambda) & R(\lambda) & R(\lambda)\\ \epsilon^2R(\epsilon^2\lambda) & \epsilon^4R(\epsilon^2\lambda) & R(\epsilon^2\lambda)\\ \epsilon^4R(\epsilon^4\lambda) & \epsilon^2R(\epsilon^4\lambda) & R(\epsilon^4\lambda)\end{pmatrix}.$$

Normalize this frame in a neighborhood of 0, we get

$$E(z,\bar z,\lambda):=F^{-1}(0,0,\lambda)F(z,\bar z,\lambda)=exp(\lambda P_{132}\cdot z+\frac{1}{\lambda}P_{132}^t\cdot \bar z), \tag{6.1}$$

where

$$P_{132}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

Example 6.2 (The one 'soliton' solution). Choose simple element $Am_{\alpha,\ell}$ with $\alpha=-i,\ell=\mathbb{C}(-\frac{1}{2}+i,-\frac{1}{2}-i,1)$, and $E(z,\bar z,\lambda)$ is the normalized frame of vacuum definite affine sphere in (6.1). Then we have the factorization:

$$Am_{\alpha,\ell}(\lambda)E(\lambda)=\tilde E(\lambda)\tilde Am_{\alpha,\tilde\ell}(\lambda),$$

where

$$\tilde\ell=\ell E(-\alpha)=\mathbb{C}(\tilde b,\bar{\tilde b},1),$$

$$\tilde b=\frac{(-\frac{3}{4}-\frac{\sqrt3}{2}+\frac{3}{2}i+\frac{3\sqrt3}{4}i)e^{y-\sqrt3x}+(-\frac34+\frac{\sqrt3}{2}+\frac32 i-\frac{3\sqrt3}{4}i)e^{y+\sqrt3x}}{(\frac32-\sqrt3)e^{y-\sqrt3x}+(\frac32+\sqrt3)e^{y+\sqrt3x}},$$

$$\tilde d=\sqrt{2|\tilde b|^2-1}=\sqrt{-\frac{3[(4\sqrt3-7)e^{4\sqrt3x}-4e^{2\sqrt3x}-4\sqrt3-7]}{[(2\sqrt3-3)e^{2\sqrt3x}-(2\sqrt3+3)]^2}}.$$

By Theorem 5.4, we can get the solution of Tzitzéica equation (3.5) (new affine metric):

$$h=e^{\tilde\psi}=\tilde d^2=\frac{3[(7-4\sqrt3)e^{4\sqrt3x}+4e^{2\sqrt3x}+4\sqrt3+7]}{[(2\sqrt3-3)e^{2\sqrt3x}-(2\sqrt3+3)]^2}. \tag{6.2}$$

Recall the one soliton type solution of Tzitzéica equation in [26]:

$$h=1-2(ln\tau)_{z\bar z},\ with\ \tau=1-e^{kz+\frac{3\bar z}{k}+s}. \tag{6.3}$$

Here we adopt the usual notation τ for τ-function, not the same as the reality condition before. After some computation, we can see our solution (6.2) coincide with (6.3) with $k=\sqrt3$ and $s=\ln(\frac{2\sqrt3-3}{3+2\sqrt3})$. We give the picture of new solution in Figure 1.

Recall a complete hyperbolic affine sphere in [24]:

$$\mathbb{R}^2_{++}\begin{pmatrix}x\\y\end{pmatrix}\to\vec r=\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}=\begin{pmatrix}-y(\frac32\ln(x)+3\ln(y)+x)\\ x^{-\frac32}y^{-2}\sqrt{1+x}\\ y\end{pmatrix}, \tag{6.4}$$

which is asymptotic to the boundary of the cone obtained by the homogenization of the epigraph of the exponential function. To compare with this hyperbolic affine sphere, we use the conformal parametrization in [29]:

$$\vec{r'} = \begin{pmatrix} \frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3}x)(\sinh^2(\sqrt{3}x) + 3y)e^y \\ -\frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3}x)\sqrt{1+\sinh^2(\sqrt{3}x)}e^{-2y} \\ -\frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3}x)e^y \end{pmatrix}.$$

with the affine metric, affine mean curvature and affine cubic form:

$$g = \frac{3}{2}(\operatorname{csch}(\sqrt{3}x)^2 + \frac{2}{3})(dz\ d\bar{z}), \quad H = -2, \quad C = I \cdot (dz^3 + d\bar{z}^3). \tag{6.5}$$

We only need to make a shift of variable x by : $x \to x - \frac{s}{2\sqrt{3}}$ and choose $\lambda = i$ in our case. Then we get an definite affine sphere with the same invariants with (6.4), i.e. one of the family of definite affine sphere (5.4) is a complete hyperbolic affine sphere. Since the specific formula is too long to place in this paper, we give the picture of this affine sphere instead.

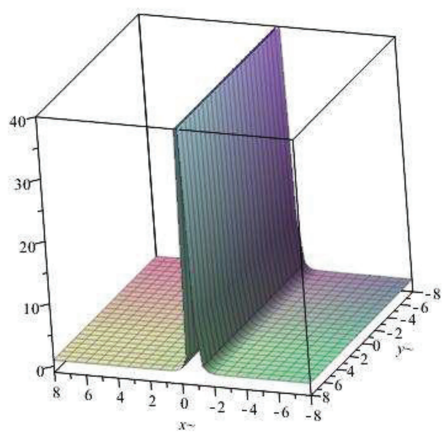

FIG. 1. $-8 < x < 8, -8 < y < 8$

EXAMPLE 6.3 (The two 'soliton' solution). There are two cases.
(1) Choose simple element $Am_{\alpha,\ell}$, where $\alpha = i, \ell = \mathbb{C}(1+i, 1-i, 1)$, and $E(z, \bar{z}, \lambda)$ is (6.1). Then we have the factorization

$$Am_{\alpha,\ell}(\lambda)E(\lambda) = \tilde{E}(\lambda)\tilde{A}m_{\alpha,\tilde{\ell}}(\lambda),$$

where

$$\tilde{\ell} = \ell E(-\alpha) = \mathbb{C}(\tilde{b}, \bar{\tilde{b}}, 1),$$

$$\tilde{b} = \frac{(-3i+\sqrt{3})e^{\sqrt{3}x-y} + (-3i-\sqrt{3})e^{-\sqrt{3}x-y} - 6e^{-2y}}{-2(\sqrt{3}e^{\sqrt{3}x-y} - \sqrt{3}e^{-\sqrt{3}x-y} + 3e^{2y})},$$

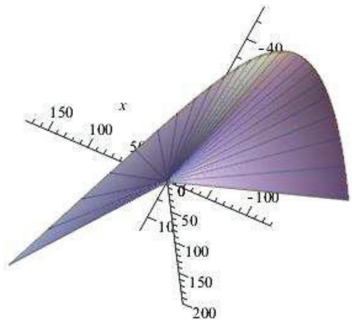

FIG. 2. $-2 < x < 2, -2 < y < 2$

$$\tilde{d} = \sqrt{2|\tilde{b}|^2 - 1} = \sqrt{\frac{3(3e^{4y} - 4e^{-2y} - 4\sqrt{3}e^{\sqrt{3}x+y} + 4\sqrt{3}e^{-\sqrt{3}x+y} + e^{2\sqrt{3}x-2y} + e^{-2\sqrt{3}x-2y})}{(\sqrt{3}e^{\sqrt{3}x-y} - \sqrt{3}e^{-\sqrt{3}x-y} + 3e^{2y})^2}}.$$

By Theorem 5.4, we can get the solution of Tzitzéica equation (3.5) (new affine metric):

$$\begin{aligned} h &= e^{\tilde{\psi}} = \tilde{d}^2 \\ &= \frac{3(3e^{4y} - 4e^{-2y} - 4\sqrt{3}e^{\sqrt{3}x+y} + 4\sqrt{3}e^{-\sqrt{3}x+y} + e^{2\sqrt{3}x-2y} + e^{-2\sqrt{3}x-2y})}{(\sqrt{3}e^{\sqrt{3}x-y} - \sqrt{3}e^{-\sqrt{3}x-y} + 3e^{2y})^2}. \end{aligned}$$

Recall the two soliton type solution of Tzitzéica equation in [26]:

$$h = 1 - 2(ln\tau)_{z\bar{z}},$$

with

$$\tau = 1 + e^{k_1 z + \frac{3\bar{z}}{k_1} + s_1} + e^{k_2 z + \frac{3\bar{z}}{k_2} + s_2} + \frac{(k_1 - k_2)^2(k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)} e^{(k_1+k_2)z + (\frac{3}{k_1} + \frac{3}{k_2})\bar{z} + s_1 + s_2} \tag{6.6}$$

Then we get the τ-function of the new solution h:

$$\tau = 1 + \frac{\sqrt{3}}{3} e^{\sqrt{3}x - 3y} - \frac{\sqrt{3}}{3} e^{-\sqrt{3}x - 3y}.$$

It coincide with the τ-functions (6.6) with parameters $k_1 = \frac{\sqrt{3}-3i}{2}$, $k_2 = \frac{-\sqrt{3}-3i}{2}$, $s_1 = ln(\frac{\sqrt{3}}{3})$, $s_2 = ln(-\frac{\sqrt{3}}{3})$.

We give the picture of this solution in Figure 3. Notice in Figure 3, the solution goes below the red plane (z=0) only in the very small region, which means the metric become negative in a curve. If we choose $H = -2$ and $U = 1$, it is easy to see that Tzitzéica equation (3.5) equals to

$$h_{z\bar{z}} h - h_z h_{\bar{z}} - h^3 + 1 = 0. \tag{6.7}$$

and the negative parts are also the solution of (6.7). From geometric point of view, if we fix the affine mean curvature $H = -2$, h being positive means the affine metric is positive definite, then it gives a local hyperbolic affine sphere; h being negative means the affine metric is negative definite, it gives a local elliptic affine sphere.

So the solution $h = e^{\tilde{\psi}}$ can be extended to be a global solution of Tzitzéica equation (6.7) and we get a global mixed definite affine sphere. Some parts of it are hyperbolic type, the other are elliptic.

We also use the formula (5.4) to get the new definite affine sphere:

$$\tilde{X} = \frac{-H(\lambda^3+\alpha^3)Xe^{\psi} - 4\alpha^3(ln\phi)_{\bar{z}}X_z - 4\lambda^3(ln\phi)_z X_{\bar{z}}}{(\lambda^3+\alpha^3)e^{\psi}}$$

here

$$\phi = \ell F_\lambda^{-1}(0,0)X(-\alpha) = \frac{1}{2}\,\mathrm{e}^{2\,y} + \frac{1}{6}\,\mathrm{e}^{-\frac{1}{3}\sqrt{3}\left(\sqrt{3}y-3\,x\right)}\sqrt{3} - \frac{1}{6}\,\mathrm{e}^{-\frac{1}{3}\sqrt{3}\left(\sqrt{3}y+3\,x\right)}\sqrt{3}.$$

Since the specific formula of new definite affine sphere is too long to place in this paper, we give the picture of it in Figure 4 with $\lambda = e^{\frac{2}{3}\pi i}$ instead.

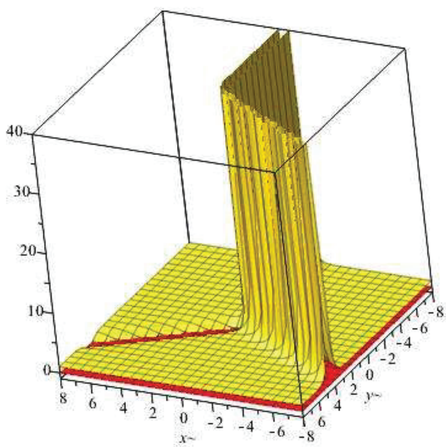

FIG. 3. $-8 < x < 8, -8 < y < 8$

REMARK 6.1. By computation, we find out that when $\mathrm{Re}(b) = -\frac{1}{2}$ the simple element with 3 simple poles will give the one soliton type solution, and other cases it gives two soliton type solution.

(2) Choose simple element $Ag_{\alpha,\ell_1}(\lambda)g_{\bar{\alpha}^{-1},\ell_2}(\lambda)$, where $\ell_2 = \mathbb{C}(b_2, c_2, 1)$, $\alpha \in \mathbb{C}^\times/S^1$ satisfy $\Psi_{\alpha,b_2,c_2} > 0$ (see Remark 4.5), and $E(z,\bar{z},\lambda)$ is (6.1). Then we have the following factorization:

$$Ag_{\alpha,\ell_1}(\lambda)g_{\bar{\alpha}^{-1},\ell_2}(\lambda)E(\lambda) = \tilde{E}(\lambda)\tilde{A}g_{\alpha,\tilde{\ell}_1}(\lambda)g_{\bar{\alpha}^{-1},\tilde{\ell}_2}(\lambda),$$

where

$$\tilde{l}_2 = l_2 E(\bar{\alpha}^{-1}) = \mathbb{C}(\tilde{b}_2, \tilde{c}_2, 1),$$

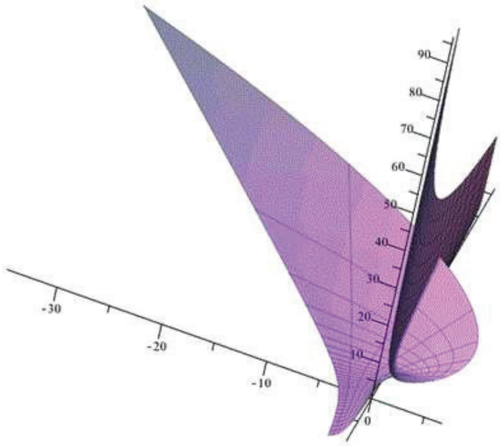

FIG. 4. $0 < x < 1.5, -0.8 < y < 3$

$$\begin{pmatrix} \tilde{b}_2 \\ \tilde{c}_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{(b_2+c_2+1)R(\bar{\alpha}^{-1})+(b_2+c_2\epsilon^4+\epsilon^2)R(\epsilon^2\bar{\alpha}^{-1})+(b_2+c_2\epsilon^2+\epsilon^4)R(\epsilon^4\bar{\alpha}^{-1})}{(b_2+c_2+1)R(\bar{\alpha}^{-1})+(b_2\epsilon^4+c_2\epsilon^2+1)R(\epsilon^2\bar{\alpha}^{-1})+(b_2\epsilon^2+c_2\epsilon^4+1)R(\epsilon^4\bar{\alpha}^{-1})} \\ \frac{(b_2+c_2+1)R(\bar{\alpha}^{-1})+(b_2\epsilon^2+c_2+\epsilon^4)R(\epsilon^2\bar{\alpha}^{-1})+(b_2\epsilon^4+c_2+\epsilon^2)R(\epsilon^4\bar{\alpha}^{-1})}{(b_2+c_2+1)R(\bar{\alpha}^{-1})+(b_2\epsilon^4+c_2\epsilon^2+1)R(\epsilon^2\bar{\alpha}^{-1})+(b_2\epsilon^2+c_2\epsilon^4+1)R(\epsilon^4\bar{\alpha}^{-1})} \\ 1 \end{pmatrix}, \qquad (6.8)$$

$$\tilde{A} = diag(\tilde{d}, \tilde{d}^{-1}, 1)$$

$$\tilde{d} = \sqrt{(2\tilde{b}_1\tilde{c}_1 - 1)(2\tilde{b}_2\tilde{c}_2 - 1)} = \sqrt{\frac{\Psi_{\alpha,\tilde{b}_2,\tilde{c}_2}}{(-|\alpha|^4|\tilde{b}_2|^2 + |\alpha|^2|\tilde{c}_2|^2 - \frac{1}{2}(1 - |\alpha|^6))^2}}.$$

By Theorem 5.5, we can claim that $h = e^{\tilde{\psi}} = d^2$ is the new solution of Tzitzéica equation (3.5). One can check this by Maple.

In particular, we choose $c_1 = \frac{1}{2} + \frac{\sqrt{3}i}{2}, c_2 = 0, \alpha = -\frac{1}{2}i, d = 1$. Then we can get the τ-function of the new solution h:

$$\begin{aligned}\tau =& 1 + (\frac{45}{26} - \frac{55\sqrt{3}i}{78})e^{\frac{15}{4}y - \frac{5}{4}\sqrt{3}x + \frac{9}{4}ix + \frac{3}{4}\sqrt{3}iy} + (\frac{45}{26} + \frac{55\sqrt{3}i}{78})e^{\frac{15}{4}y - \frac{5}{4}\sqrt{3}x - \frac{9}{4}ix - \frac{3}{4}\sqrt{3}iy} \\ &+ e^{\frac{15}{2}y - \frac{5}{2}\sqrt{3}x}.\end{aligned}$$

It coincide with the τ-functions of 2-soliton solutions (6.6) with parameters $k_1 = -\frac{\sqrt{3}}{4} - \frac{3i}{4}$, $k_2 = -\sqrt{3} - 3i$, $s_1 = ln(\frac{135-55\sqrt{3}i}{78})$, $s_2 = ln(\frac{135+55\sqrt{3}i}{78})$.

At last, we give the picture of this solution and the new definite affine sphere in Figure 5 and Figure 6 respectively.

REMARK 6.2. In particularly, we choose $\ell = \mathbb{C} \cdot (1, 1, 1)$ and any $\alpha \in \mathbb{C}^\times/S^1$. From (6.8), we can see that $\tilde{\ell}_1 = \tilde{\ell}_2 = \mathbb{C} \cdot (1, 1, 1)$, which means the dressing actions is trivial, and this special simple element is commutative with the special affine frames in this case.

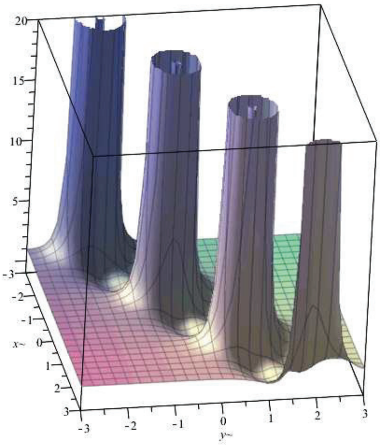

FIG. 5. $-3 < x < 3, -3 < y < 3, -1 < z < 20$

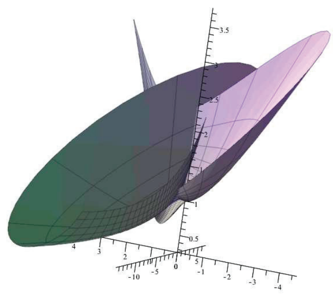

FIG. 6. $-0.4 < x < 0.8, -1.5 < y < 1.5$

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