NOTES ON THE FINE SELMER GROUPS[*]

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Abstract. In this paper, we study the fine Selmer group attached to a Galois module defined over a commutative complete Noetherian ring with finite residue field of characteristic p. Namely, we are interested in its properties upon taking residual representation and within field extensions. In particular, we will show that the variation of the fine Selmer group in a cyclotomic $\mathbb{Z}_p$-extension is intimately related to the variation of the class groups in the cyclotomic tower.

Key words. Fine Selmer groups, admissible p-adic Lie extensions, Auslander regular ring, pseudo-null.

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1. Introduction. Let F be a finite extension of $\mathbb{Q}$, p a prime and F^{cyc} the cyclotomic $\mathbb{Z}_p$-extension of F. Denote $K(F^{\mathrm{cyc}})$ to be the maximal unramified pro-p extension of F^{cyc} at which every prime of F^{cyc} above p splits completely. Set $\Gamma = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. Iwaswa has proven that $\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})$ is a finitely generated torsion $\mathbb{Z}_p[\![\Gamma]\!]$-module. He further conjectured that it is in fact a finitely generated $\mathbb{Z}_p$-module (see [Iw1, Iw2]). Conjectures parallel to this conjecture of Iwasawa have been formulated for the fine Selmer groups attached to elliptic curves by Coates and Sujatha [CS, Conjecture A], and for the fine Selmer groups attached to modular forms and Hida families by Sujatha and Jha [JhS, Jh, Conjecture A, Conjecture 1]. Following their footsteps, we make an analogous conjecture for the fine Selmer groups attached to a Galois module defined over a commutative complete Noetherian ring with finite residue field of characteristic p. We then show that this generalized conjecture turns out to be a consequence of the original conjecture of Iwasawa (see Theorem 3.1 and 3.5). We remark that such an implication has been established for elliptic curves by Coates and Sujatha (see [CS, Theorem 3.4, Corollary 3.5]) and our approach is mainly inspired by their work. The proof adopted by them made use of descent technique and relied on the observation that the extension carved out by all the p-power torsion points of the elliptic curve in question is a p-adic Lie extension. It is not difficult to extend their argument to a p-adic representation defined over a ring of integers of some finite extension of $\mathbb{Q}_p$. However, since the Galois modules we considered are general, the extensions carved out by these modules need not be p-adic Lie extensions, and so their argument does not carry over. Therefore, our proof takes a different route, which we will explain in a while.

In the second part of the article, we will investigate the question of pseudo-nullity of the fine Selmer groups over an admissible p-adic Lie extension of dimension strictly greater than one. A somewhat related question in this direction was first considered by Greenberg [Gr], where he conjectured that the Galois group of the maximal abelian unramified pro-p-extension of the compositum of all $\mathbb{Z}_p$-extensions $\tilde{F}$ of F is a pseudo-null $\mathbb{Z}_p[\![\mathrm{Gal}(\tilde{F}/F)]\!]$-module. Now if T is the Tate module of all the p-power roots of unity, the dual fine Selmer group is precisely $\mathrm{Gal}(K(F_\infty)/F_\infty)$, where we denote $K(F_\infty)$ to be the maximal unramified pro-p extension of F_∞ at which every prime of F_∞ above p splits completely. Hachimori and Sharifi [HSh] has constructed

several classes of admissible p-adic Lie extensions F_∞ of F of dimension > 1 such that $\mathrm{Gal}(K(F_\infty)/F_\infty)$ is not pseudo-null as a $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-module. On the other hand, Coates and Sujatha have conjectured that pseudo-nullity for the fine Selmer groups of elliptic curves should hold for any admissible p-adic extensions of dimension > 1 (see [CS, Conjecture B]). This was further extrapolated and formulated for fine Selmer groups of modular forms and Hida families by Jha [Jh, Conjecture B, Conjecture 2]. In this paper, we will establish two results pertaining to the pseudo-nullity property of the fine Selmer group of a Galois module. The first result (Theorem 5.4) shows that the pseudo-nullity property can be lifted, namely, if the fine Selmer group of the residual representation of a given Galois representation is pseudo-null, then so is the fine Selmer group of the Galois representation itself. Such a result has also been proved in [Jh, Theorem 10][1], and here we give a slightly different proof. The second result (Theorem 5.7) is concerned on the descent property of the pseudo-nullity of the fine Selmer groups, where we show that if $F_\infty' \subseteq F_\infty$ are two admissible p-adic Lie extensions of dimension ≥ 2 with $\mathrm{Gal}(F_\infty/F_\infty')$ being a solvable uniform pro-p group, then the fine Selmer group over F_∞ is pseudo-null whenever the fine Selmer group over F_∞' is pseudo-null. We note that in this result, the extensions F_∞ and F_∞' themselves need not be solvable extensions of F.

We briefly summarize the approach towards the investigation of the fine Selmer group in this paper. Namely, we observe that the structure of the dual fine Selmer group is intimately linked with a certain inverse limit of second cohomology groups. This latter group turns out to behave well upon taking residual (see Lemma 2.1) and descent (cf. Lemma 2.2), which is key to our examination of the relationship between the said fine Selmer groups.

We now give an outline of the paper. In Section 2, we introduce the fine Selmer groups and discuss some of their basic properties. In Section 3, we will study the variation of the fine Selmer groups over the cyclotomic $\mathbb{Z}_p$-extension as mentioned in the first paragraph of the introduction. In Section 4, we collect some results on the ranks of modules over a completed group algebra which will be applied in Section 5 for the discussion of the pseudo-nullity of the fine Selmer groups. In Section 6, we discuss some numerical examples coming from class groups, and an elliptic curve and its associated Hida deformation to illustrate the results of the paper. In Section 7, we will make some complementary remark on the torsionness of the fine Selmer group. Finally, we will provide proofs to certain results on the structure of the completed group algebra in the Appendix.

2. Fine Selmer group. Throughout the paper, p will denote a prime number. If M is a pro-p group or a discrete p-primary group, we denote the Pontryagin dual of

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[1]Jha's theorem also deals with the converse direction which we will not address in this paper.

M by $M^\vee = \mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$. Let F be a number field, i.e., a finite extension of $\mathbb{Q}$. Let S denote a finite set of primes of F containing the primes above p and the infinite primes, which we shall fix once and for all. Let F_S denote the maximal algebraic extension of F unramified outside S. For any algebraic (possibly infinite) extension $\mathcal{L}$ of F contained in F_S, we write $G_S(\mathcal{L}) = \mathrm{Gal}(F_S/\mathcal{L})$. Let R be a commutative complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, where k is finite of characteristic p. Let T denote a finitely generated R-module. One verifies easily that $T \cong \varprojlim_i T/\mathfrak{m}^i T$, and we shall endow T with the $\mathfrak{m}$-adic topology. We assume further that T has a continuous R-linear G_F-action, and is unramified outside S. This group action on T induces a continuous group homomorphism

$$\rho : G_S(F) \longrightarrow \mathrm{Aut}_R(T).$$

We will write $W = T^\vee(1)$, where "(1)" denotes the Tate twist. Let v be a prime in S. For each finite extension L of F contained in F_S, we define

$$K_v^i(W/L) = \bigoplus_{w|v} H^i(L_w, W) \quad (i = 0, 1),$$

where w runs over the (finite) set of primes of L above v. If $\mathcal{L}$ is an infinite extension of F contained in F_S, we define

$$K_v^i(W/\mathcal{L}) = \varinjlim_{L} K_v^i(W/L),$$

where the direct limit is taken over all finite extensions L of F contained in $\mathcal{L}$ under the restriction maps.

For any algebraic (possibly infinite) extension $\mathcal{L}$ of F contained in F_S, the fine Selmer group of W over $\mathcal{L}$ (with respect to S) is defined to be

$$R_S(W/\mathcal{L}) = \ker\left(H^1(G_S(\mathcal{L}), W) \longrightarrow \bigoplus_{v \in S} K_v^1(W/\mathcal{L}) \right).$$

We shall write $Y_S(T/\mathcal{L})$ for the Pontryagin dual $R_S(W/\mathcal{L})^\vee$ of the fine Selmer group. It follows from the Poitou-Tate sequence that we have the following exact sequence

$$0 \longrightarrow Y_S(T/\mathcal{L}) \longrightarrow \varprojlim_{L} H^2(G_S(L), T) \longrightarrow \left(\bigoplus_{v \in S} K_v^0(W/\mathcal{L}) \right)^\vee \longrightarrow W(\mathcal{L})^\vee \longrightarrow 0,$$

where the inverse limit is taken over all finite extensions L of F contained in $\mathcal{L}$ under the corestriction maps and $W(\mathcal{L})$ denote $W^{\mathrm{Gal}(F_S/\mathcal{L})}$. From now on, we shall write $H_S^2(\mathcal{L}/F, T) = \varprojlim_{L} H^2(G_S(L), T)$.

Examples: (a) If $T = \mathbb{Z}_p(1)$, then we have $W = \mathbb{Q}_p/\mathbb{Z}_p$. In this case, one checks easily that $Y_S(\mathbb{Z}_p(1)/F) = \mathrm{Cl}_S(F)[p^\infty]$, where $\mathrm{Cl}_S(F)$ is the S-class group of F.

(b) If E is an elliptic curve over F, then the fine Selmer group for $W = E[p^\infty]$ has been studied in [CS] and is related to the classical Selmer group via the following exact sequence

$$0 \longrightarrow R_S(E[p^\infty]/F) \longrightarrow \mathrm{Sel}_{p^\infty}(E/F) \longrightarrow \bigoplus_{v|p} H^1(F_v, E[p^\infty]).$$

Note that in this instance, the fine Selmer group is independent of the set S as long as it contains all the primes above p, the infinite primes and the primes at which E has bad reduction.

We mention certain basic properties on the structure of the dual fine Selmer group. Let F_∞ be a p-adic Lie extension of F contained in F_S. Set $G = \mathrm{Gal}(F_\infty/F)$. It is known that the ring $R[\![G]\!]$ is Noetherian (cf. [LSh, Proposition 3.0.1], see also Proposition A.1(a) in this paper). By [LSh, Proposition 4.1.3], we have that $H^2_S(F_\infty/F, T)$ is finitely generated over $R[\![G]\!]$. Therefore, it follows from the Poitou-Tate sequence that $Y_S(T/F_\infty)$ is finitely generated over $R[\![G]\!]$. We end the section with two lemmas on the behavior of $H^2_S(F_\infty/F, T)$ under residue and descent.

LEMMA 2.1. *Let x be a nonzero and nonunital element of R. Write $\bar{R} = R/xR$ and $\bar{T} = T/xT$. Suppose that either p is odd or F has no real places. Then one has an isomorphism*

$$H^2_S(F_\infty/F, T)/x \cong H^2_S(F_\infty/F, \bar{T})$$

of $\bar{R}[\![\mathrm{Gal}(F_\infty/F)]\!]$-modules.

Proof. Since we are assuming that either p is odd or F has no real places, we have $H^i_S(F_\infty/F, -) = 0$ for $i \geq 3$. In particular, $H^2_S(F_\infty/F, -)$ is right exact. Therefore, from the exact sequence

$$T \xrightarrow{\ x\ } T \longrightarrow \bar{T} \longrightarrow 0,$$

we obtain

$$H^2_S(F_\infty/F, T) \xrightarrow{\ x\ } H^2_S(F_\infty/F, T) \longrightarrow H^2_S(F_\infty/F, \bar{T}) \longrightarrow 0.$$

The required isomorphism is now immediate. $\square$

LEMMA 2.2. *Let F'_∞ be another p-adic extension of F such that $F \subseteq F'_\infty \subseteq F_\infty$ Suppose that either p is odd or F has no real places. Then one has an isomorphism*

$$H^2_S(F_\infty/F, T)_{\mathrm{Gal}(F_\infty/F'_\infty)} \cong H^2_S(F'_\infty/F, T)$$

of $R[\![\mathrm{Gal}(F'_\infty/F)]\!]$-modules.

Proof. By [LSh, Theorem 3.1.8], there is a spectral sequence

$$H_i(\mathrm{Gal}(F_\infty/F'_\infty), H^{-j}_S(F_\infty/F, T)) \Longrightarrow H^{-i-j}_S(F'_\infty/F, T).$$

The required isomorphism follows from reading off the $(0, -2)$-term. $\square$

REMARK. If T is free over R, one can deduce Lemmas 2.1 and 2.2 from [FK, Proposition 1.6.5(iii)].

3. Fine Selmer groups over a cyclotomic $\mathbb{Z}_p$-extension. In this section, we examine the variation of the fine Selmer groups over a cyclotomic $\mathbb{Z}_p$-extension. We retain the notation and assumptions from the previous section. *We shall also always assume that the number field F has no real primes when $p = 2$.* Denote F^{cyc} to be the cyclotomic $\mathbb{Z}_p$-extension of F and write $\Gamma = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. As mentioned in the previous section, the dual fine Selmer group $Y_S(T/F^{\mathrm{cyc}})$ is a finitely generated $R[\![\Gamma]\!]$-module. In fact, we conjecture that the following stronger statement should hold.

CONJECTURE A. *For any number field F, $Y_S(T/F^{\mathrm{cyc}})$ is a finitely generated R-module.*

We mention several well known cases of the above conjecture. For any extension $\mathcal{L}$ of F^{cyc} contained in F_S, we denote $K(\mathcal{L})$ to be the maximal unramified pro-p extension of $\mathcal{L}$ where every prime of $\mathcal{L}$ above p splits completely. Since $F^{\mathrm{cyc}} \subseteq \mathcal{L}$, it follows that every finite prime of $\mathcal{L}$ splits completely in $K(\mathcal{L})$. Therefore, in the case when $T = \mathbb{Z}_p(1)$, the dual of the fine Selmer group $Y_S(\mathbb{Z}_p(1)/F^{\mathrm{cyc}})$ is precisely $\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})$. In this context, this is equivalent to the conjecture made by Iwasawa [Iw1, Iw2]. We shall call this conjecture the Iwasawa μ-invariant conjecture for F^{cyc}. Currently, the Iwasawa μ-invariant conjecture is only proved in the case when F is abelian over $\mathbb{Q}$ (see [FW, Sin]).

This conjecture has also been considered for an elliptic curve (see [CS, Conjecture A]), for a Galois representation attached to a cuspidal eigenform which is ordinary at p (see [Jh, JhS, Conjecture A]) and a Galois representation attached to a Hida family (see [Jh, Conjecture 1]).

We now state the main theorem of this section which is a natural extension of [CS, Corollary 3.5].

THEOREM 3.1. *Let F be a number field. Suppose the Iwasawa μ-invariant conjecture holds for L^{cyc} for any finite extension L of F. Then $Y_S(T/F^{\mathrm{cyc}})$ is a finitely generated R-module.*

As one will see from the proof (see also Theorem 3.5), one only requires that the Iwasawa μ-invariant conjecture holds for a particular extension L of F. We should mention that our method of proof does not allow us to deduce the finite generation of $Y_S(T/F^{\mathrm{cyc}})$ from the validity of the Iwasawa μ-invariant conjecture for F, and one has to assume that the Iwasawa μ-invariant conjecture holds for an extension L of F in general. In preparation for the proof of Theorem 3.1, we first prove three lemmas. Recall that we write $W(\mathcal{L}) = W^{\mathrm{Gal}(F_S/\mathcal{L})}$ for any $F \subseteq \mathcal{L} \subseteq F_S$. Similarly, for each $v \in S$, we write $W(\mathcal{L}) = W^{\mathrm{Gal}(\bar{F}_v/\mathcal{L})}$ for any $F_v \subseteq \mathcal{L} \subseteq \bar{F}_v$.

LEMMA 3.2. *Let F be a number field. Suppose that L is a finite extension of F contained in F_S. If $Y_S(T/L^{\mathrm{cyc}})$ is a finitely generated R-module, then $Y_S(T/F^{\mathrm{cyc}})$ is a finitely generated R-module.*

Proof. Let $\triangle = \mathrm{Gal}(L^{\mathrm{cyc}}/F^{cyc})$. Consider the following commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & R_S(W/F^{\mathrm{cyc}}) & \longrightarrow & H^1(G_S(F^{\mathrm{cyc}}), W) & \longrightarrow & \bigoplus_{v \in S} K_v^1(W/F^{\mathrm{cyc}}) \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
0 & \longrightarrow & R_S(W/L^{\mathrm{cyc}})^{\triangle} & \longrightarrow & H^1(G_S(L^{\mathrm{cyc}}), W)^{\triangle} & \longrightarrow & \left(\bigoplus_{v \in S} K_v^1(W/L^{\mathrm{cyc}}) \right)^{\triangle}
\end{array}
$$

where the vertical maps are the natural restriction maps. Now $\ker \alpha$ is contained in $H^1(\triangle, W(L^{\mathrm{cyc}}))$ which can be easily seen to be a cofinitely generated R-module. The conclusion is now immediate. $\square$

LEMMA 3.3. (*Compare with [CS, Lemma 3.8]*) *Suppose that F contains μ_p and suppose that M is a finite trivial $G_S(F^{\mathrm{cyc}})$-module which is killed by p. Then we have an isomorphism*

$$
Y_S(M/F^{\mathrm{cyc}}) \cong (\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})/p) \otimes_{\mathbb{Z}_p} M
$$

of abelian groups. In particular, $Y_S(M/F^{\mathrm{cyc}})$ is finite if and only if $\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})$ is finitely generated over $\mathbb{Z}_p$.

Proof. Since $pM = 0$, we have $N := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mu_{p^\infty}) = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mu_p)$. As the group $G_S(F^{\mathrm{cyc}})$ acts trivially on M and $\mu_p \subseteq F$, it also acts trivially on $N = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mu_p)$. Therefore, we have

$$R_S(N/F^{\mathrm{cyc}}) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})/p, N),$$

noting that every finite prime of F^{cyc} splits completely in $K(F^{\mathrm{cyc}})$. On the other hand, one has the following adjunction isomorphism

$$\Big(\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})/p \otimes_{\mathbb{Z}_p} M \Big)^\vee \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})/p, N).$$

(Here we are identifying $M^\vee$ with N using the assumption that $F(\mu_p) = F$ and $pM = 0$.) Combining both equalities and taking dual, we obtain the required isomorphism. The second assertion is immediate from the first. $\square$

LEMMA 3.4. (*Compare with [CS, Lemma 3.2]*) $Y_S(T/F^{cyc})$ *is finitely generated over R if and only if $H_S^2(F^{cyc}/F, T)$ is finitely generated over R.*

Proof. From the Poitou-Tate sequence, we have the following exact sequence

$$0 \longrightarrow Y_S(T/F^{\mathrm{cyc}}) \longrightarrow H_S^2(F^{cyc}/F, T) \longrightarrow \Big(\bigoplus_{v \in S} K_v^0(W/F^{\mathrm{cyc}}) \Big)^\vee.$$

For each $v \in S$, one has

$$K_v^0(W/F^{\mathrm{cyc}}) \cong \bigoplus_{w|v} W(F_w^{cyc}).$$

Since every prime splits finitely in F^{cyc}/F, the direct sum is a finite sum of cofinitely generated R-modules, and so it is a cofinitely generated R-module. (Note that the direct sum needs not be a finite sum if $p = 2$ and F has real primes but our standing assumption is that if $p = 2$, then F has no real primes, so this situation will not occur under our assumption.) The conclusion of the lemma then follows. $\square$

We can now prove our theorem.

Proof of Theorem 3.1. By Lemma 3.2, we may replace F, if necessary, so that F contains μ_{2p} (in particular, F has no real primes) and that $G_S(F)$ acts trivially on $T/\mathfrak{m}T$. By Lemma 3.4, it then suffices to show that $H_S^2(F^{\mathrm{cyc}}/F, T)$ is finitely generated over R. Choose a set of generators $x_1, ..., x_d$ of $\mathfrak{m}$ such that they form a basis for $\mathfrak{m}/\mathfrak{m}^2$. By a repeated application of Lemma 2.1, we have an isomorphism $H_S^2(F^{\mathrm{cyc}}/F, T)/\mathfrak{m} \cong H_S^2(F^{\mathrm{cyc}}/F, T/\mathfrak{m}T)$. By Nakayama lemma, we are reduced to showing that $H_S^2(F^{\mathrm{cyc}}/F, T/\mathfrak{m}T)$ is finite. By another application of Lemma 3.4, this is equivalent to showing that $Y((T/\mathfrak{m}T)/F^{\mathrm{cyc}})$ is finite. This latter assertion is an immediate consequence of Lemma 3.3 and the validity of the Iwasawa μ-conjecture. $\square$

Upon a finer examination of our proof of Theorem 3.1, one actually shows something more precise. Let $\bar{\rho} : G_S(F) \longrightarrow \mathrm{Aut}_k(T/\mathfrak{m}T)$ be the residual representation of ρ. Denote $F(\mu_{2p}, T/\mathfrak{m}T)$ to be $F_S^{\ker \bar{\rho}}(\mu_{2p})$. Note that this is a finite Galois extension of F.

THEOREM 3.5. *Let L be a finite extension of F such that $F(\mu_{2p}, T/\mathfrak{m}T)$ is contained in a finite p-extension of L. Then the Iwasawa μ-invariant conjecture holds for L if and only if $Y_S(T/L^{\mathrm{cyc}})$ is finitely generated over R.*

Proof. Let L' be a finite p-extension of L that contains $F(\mu_{2p}, T/\mathfrak{m}T)$. The proof of Theorem 3.1 essentially proved the equivalence over L'. To establish the equivalence for L, we need to show that the finite generation property is preserved in a finite p-extension. By [Iw1, Theorem 3], the Iwasawa μ-invariant conjecture holds for L^{cyc} if and only if the Iwasawa μ-invariant conjecture holds for L'^{cyc}. It remains to show the same for the case of $Y_S(T/L^{\mathrm{cyc}})$. It is not difficult, by making use of the commutative diagram in Lemma 3.2, to show that the map

$$Y_S(T/L'^{\mathrm{cyc}})_\triangle \longrightarrow Y_S(T/L^{\mathrm{cyc}})$$

has kernel and cokernel which are finitely generated over R. Here $\triangle = \mathrm{Gal}(L'/L)$. It follows from this observation that $Y_S(T/L^{\mathrm{cyc}})$ is finitely generated over R if and only if $Y_S(T/L'^{\mathrm{cyc}})_\triangle$ is finitely generated over R. Since $\triangle$ is a p-group, $R[\triangle]$ is local with a unique maximal (two-sided) ideal $\mathfrak{M} = \mathfrak{m}R[\triangle] + I_\triangle$, where $I_\triangle$ is the augmentation ideal (see [NSW, Proposition 5.2.16(iii)]). It is easy to see from this that

$$Y_S(T/L'^{\mathrm{cyc}})/\mathfrak{M} \cong Y_S(T/L'^{\mathrm{cyc}})_\triangle/\mathfrak{m}Y_S(T/L'^{\mathrm{cyc}})_\triangle.$$

Therefore, Nakayama's lemma for R-modules tells us that $Y_S(T/L'_{\mathrm{cyc}})_\triangle$ is finitely generated over R if and only if $Y_S(T/L'^{\mathrm{cyc}})/\mathfrak{M}$ is finite. On the other hand, Nakayama's lemma for $R[\triangle]$-modules tells us that $Y_S(T/L'^{\mathrm{cyc}})/\mathfrak{M}$ is finite if and only if $Y_S(T/L'^{\mathrm{cyc}})$ is finitely generated over $R[\triangle]$. But since $\triangle$ is finite, the latter is equivalent to saying that $Y_S(T/L'^{\mathrm{cyc}})$ is finitely generated over R. Hence we conclude that $Y_S(T/L^{\mathrm{cyc}})$ is finitely generated over R if and only if $Y_S(T/L'^{\mathrm{cyc}})$ is finitely generated over R. The proof of the theorem is now completed. $\square$

One expects that the conjecture is invariant under isogeny. However, at present, we can only establish this partially in the following proposition. We will state this proposition in a more general context. Suppose that R' is another commutative Noetherian local ring with maximal ideal $\mathfrak{m}'$ and finite residue field of characteristic p, and suppose that T' is a finitely generated R'-module with a continuous R-linear $G_S(F)$-action. We then have the following proposition.

PROPOSITION 3.6. *Suppose that $F(T/\mathfrak{m}T, T'/\mathfrak{m}'T', \mu_{2p})$ is a finite p-extension of F. Then $Y_S(T/F^{\mathrm{cyc}})$ is finitely generated over R if and only if $Y_S(T'/F^{\mathrm{cyc}})$ is finitely generated over R'.*

Proof. As seen in the proof of Theorem 3.5, the finite generation property is preserved in a finite p-extension. Therefore, one is reduced to showing that $Y_S(T/L^{\mathrm{cyc}})$ is finitely generated over R if and only if $Y_S(T'/L^{\mathrm{cyc}})$ is finitely generated over R', where $L := F(T/\mathfrak{m}T, T'/\mathfrak{m}'T, \mu_{2p})$. Since L is clearly a finite p-extension of $F(T/\mathfrak{m}T, \mu_{2p})$ (resp., $F(T'/\mathfrak{m}'T, \mu_{2p})$), Theorem 3.5 applies to imply the assertion that the Iwasawa μ-invariant conjecture holds for L if and only if $Y_S(T/F^{\mathrm{cyc}})$ is finitely generated over R (resp., $Y_S(T'/F^{\mathrm{cyc}})$ is finitely generated over R'). The proposition then follows. $\square$

We end the section with the following remarks.

REMARKS. (a) In the case when T is the Tate module of an elliptic curve and the prime p is odd, the field extension $F(\mu_{2p}, T/\mathfrak{m}T)$ is precisely $F(E[p])$. One can

easily see that Theorem 3.5 recovers [CS, Theorem 3.4] and Proposition 3.6 recovers the observation made after [CS, Lemma 3.8].

(b) One can give alternative proofs to [Jh, Theorem 3] and [JhS, Theorem 8] by appealing to Lemma 3.4 and Lemma 2.1.

(c) The results here can be extended easily to the case when the ring R is a commutative Noetherian semi-local ring, complete with respect to its Jacobson radical $J(R)$, and that $R/J(R)$ is a finite ring of order a power of p. In this case, the ring R has finitely many maximal ideals $\mathfrak{m}_1, ..., \mathfrak{m}_r$ and is isomorphic to

$$R_{\mathfrak{m}_1} \times \cdots \times R_{\mathfrak{m}_r},$$

where each $R_{\mathfrak{m}_i}$ is commutative complete Noetherian local with finite residue field of characteristic p (see [M, Theorem 8.15]). Of course, there is still some work to be done after applying the said theorem, namely, one still needs to show that each $R_{\mathfrak{m}_i}$ is $\mathfrak{m}_i$-adic complete but this can be easily verified). Now every $R[G_S(F)]$-module T decomposes canonically as

$$T_{\mathfrak{m}_1} \times \cdots \times T_{\mathfrak{m}_r},$$

and it is easy to see that the decomposition is compatible with the Galois action. The results in this section can then be applied to each $R_{\mathfrak{m}_i}$-module $T_{\mathfrak{m}_i}$.

4. Ranks of Iwasawa modules. In this section, we establish some algebraic preliminaries to facilitate further discussion of the fine Selmer groups. We will prove certain formulas on the rank of modules over a completed group ring. Much of the materials considered here originates from [BHo, HO, Ho]. As a start, we shall prove a general lemma. Let Λ be a (not neccessarily commutative) Noetherian ring which has no zero divisors. Then it admits a skew field of fractions $K(\Lambda)$ which is flat over Λ (see [GW, Chapters 6 and 10] or [Lam, Chapter 4, §9 and §10]). If M is a finitely generated Λ-module, we define the Λ-rank of M to be

$$\mathrm{rank}_\Lambda M = \dim_{K(\Lambda)} K(\Lambda) \otimes_\Lambda M.$$

Clearly, one has $\mathrm{rank}_\Lambda M = 0$ if and only if $K(\Lambda) \otimes_\Lambda M = 0$. The following lemma will be useful in the discussion in this section.

LEMMA 4.1. *Let Λ be a Noetherian ring which has no zero divisors. Suppose Ω is a quotient of Λ such that it is also a Noetherian ring which has no zero divisors. Let M be a finitely generated Λ-module which has a finite free resolution of finite length. Then we have*

$$\mathrm{rank}_\Lambda M = \sum_{i \geq 0} (-1)^i \, \mathrm{rank}_\Omega \mathrm{Tor}_i^\Lambda(\Omega, M).$$

Proof. (Compare with proof of [Ho, Theorem 1.1]) Let

$$0 \longrightarrow \Lambda^{n_d} \longrightarrow \cdots \longrightarrow \Lambda^{n_0} \longrightarrow M$$

be a resolution of M which exists by the assumptions of the lemma. Then the groups $\mathrm{Tor}_i^\Lambda(\Omega, M)$ can be computed by the homology of the complex

$$\Omega^{n_d} \longrightarrow \cdots \longrightarrow \Omega^{n_0}.$$

This in turn implies that each $\mathrm{Tor}_i^\Lambda(\Omega, M)$ is finitely generated over Ω and the sum on the right hand side is a finite sum whose value coincides with

$$\sum_{i=0}^{d} (-1)^i n_i.$$

But this latter quantity is precisely $\mathrm{rank}_\Lambda M$. $\square$

We record another lemma. Write $M^+ = \mathrm{Hom}_\Lambda(M, \Lambda)$.

LEMMA 4.2. *Let Λ be a Auslander regular ring (see [V1, Definition 3.3]) with no zero divisors. Let M be a finitely generated Λ-module. Then the following are equivalent.*

(a) *The canonical map $\phi : M \longrightarrow M^{++}$ is zero.*

(b) *$K(\Lambda) \otimes_\Lambda M = 0$, where $K(\Lambda)$ is the skew field of Λ.*

(c) *$\mathrm{Hom}_\Lambda(M, \Lambda) = 0$.*

Proof. The equivalence of (a) and (c) follows from [V1, Remark 3.7]. Suppose that $K(\Lambda) \otimes_\Lambda M = 0$. Let $f \in \mathrm{Hom}_\Lambda(M, \Lambda)$ and $x \in M$. Then since $K(\Lambda) \otimes_\Lambda M = 0$, there exists $\lambda \in \Lambda \setminus \{0\}$ such that $\lambda x = 0$. This in turn implies that $\lambda f(x) = f(\lambda x) = 0$. Since Λ has no zero divisor, we have $f(x) = 0$. This shows that $\mathrm{Hom}_\Lambda(M, \Lambda) = 0$ and the implication (b)$\Rightarrow$(c). Conversely, suppose that $\mathrm{Hom}_\Lambda(M, \Lambda) = 0$. By [V1, Proposition 2.5] and the Auslander condition, the canonical map $\phi : M \longrightarrow M^{++}$ has kernel and cokernel which are $R[\![H]\!]$-torsion. Therefore, ϕ induces an isomorphism

$$K(\Lambda) \otimes_\Lambda M \xrightarrow{\sim} K(\Lambda) \otimes_\Lambda M^{++}.$$

Now if $\phi = 0$, then it will follow immediately that $K(\Lambda) \otimes_\Lambda M = 0$. This establishes (a)$\Rightarrow$(b). $\square$

We now apply the above discussion to the context of a completed group ring. Let R be a complete regular local ring with finite residue field of characteristic p, where p is a prime. Let H be a compact pro-p p-adic Lie group without p-torsion. It is well known that $R[\![H]\!]$ is a Auslander regular ring (see [V1], see also Theorem A.1). In particular, the ring $R[\![H]\!]$ is Noetherian local and has finite projective dimension. Therefore, it follows that every finitely generated $R[\![H]\!]$-module admits a finite free resolution of finite length. In the case that either the ring R has characteristic zero or H is a uniform pro-p group, the ring $R[\![H]\!]$ has no zero divisors (see Theorem A.1 and remarks after it), and therefore admits a skew field which enable one to define the notion of a rank as above. Now suppose that R has characteristic p and H is a compact pro-p p-adic Lie group without p-torsion. We then define the $R[\![H]\!]$-rank of a finitely generated $R[\![H]\!]$-module M by

$$\mathrm{rank}_{R[\![H]\!]} M = \frac{\mathrm{rank}_{R[\![H_0]\!]} M}{|H : H_0|},$$

where H_0 is an open normal uniform pro-p subgroup of H. We will see below that this is integral and independent of the choice of H_0.

LEMMA 4.3. *Let H be a compact pro-p p-adic Lie group without p-torsion. Let M be a finitely generated $R[\![H]\!]$-module. Then $H_i(H, M)$ is finitely generated over R for each i and we have the equality*

$$\mathrm{rank}_{R[\![H]\!]} M = \sum_{i \geq 0} (-1)^i \, \mathrm{rank}_R H_i(H, M)$$

In particular, the definition of $R[\![H]\!]$-rank is well-defined and integral.

Proof. Suppose first that either the ring R has characteristic zero or H is a uniform pro-p group. Then the conclusion follows from applying Lemma 4.1 (taking $\Lambda = R[\![H]\!]$ and $\Omega = R$) and observing that $H_i(H, M) \cong \operatorname{Tor}_i^{R[\![H]\!]}(R, M)$ for all i. Now suppose that R has characteristic p and H is a compact pro-p p-adic Lie group without p-torsion. Fix a finite free $R[\![H]\!]$-resolution

$$0 \longrightarrow R[\![H]\!]^{n_d} \longrightarrow \cdots \longrightarrow R[\![H]\!]^{n_0} \longrightarrow M$$

of M. Then the groups $H_i(H, M) = \operatorname{Tor}_i^{R[\![H]\!]}(R, M)$ can be computed by the homology of the complex

$$R^{n_d} \longrightarrow \cdots \longrightarrow R^{n_0}.$$

This in turn implies that

$$\sum_{i=0}^{d} (-1)^i \operatorname{rank}_R H_i(H, M) = \sum_{i=0}^{d} (-1)^i n_i.$$

Let H_0 be any open normal uniform pro-p subgroup of H. The above free $R[\![H]\!]$-resolution is also a free $R[\![H_0]\!]$-resolution for M. Therefore, the groups $H_i(H_0, M) = \operatorname{Tor}_i^{R[\![H_0]\!]}(R, M)$ can be computed by the homology of the complex

$$R^{|H:H_0|n_d} \longrightarrow \cdots \longrightarrow R^{|H:H_0|n_0}$$

which gives

$$\sum_{i=0}^{d} (-1)^i \operatorname{rank}_R H_i(H_0, M) = |H : H_0| \sum_{i=0}^{d} (-1)^i n_i.$$

On the other hand, the sum on the left is precisely $\operatorname{rank}_{R[\![H_0]\!]} M$ by an application of Lemma 4.1 (taking $\Lambda = R[\![H_0]\!]$ and $\Omega = R$). Hence, we have

$$\frac{\operatorname{rank}_{R[\![H_0]\!]} M}{|H : H_0|} = \sum_{i=0}^{d} (-1)^i n_i.$$

The sum on the right is clearly integral and independent of H_0. Therefore, we have proved the proposition. $\square$

We may now define the notion of torsion modules over $R[\![H]\!]$ via the following lemma.

LEMMA 4.4. *Let R be a regular local ring with finite residue field of characteristic p. Let H be a compact pro-p p-adic Lie group without p-torsion. Let M be a finitely generated $R[\![H]\!]$-module. Then $\operatorname{rank}_{R[\![H]\!]} M = 0$ if and only if $\operatorname{Hom}_{R[\![H]\!]}(M, R[\![H]\!]) = 0$.*

Proof. If the regular local ring R has characteristic zero or H is a uniform pro-p group, then this lemma follows from lemma 4.2. For the exceptional case, we let H_0 be an open uniform normal subgroup of H. By the R-analog of [NSW, Proposition 5.4.17], we have

$$\operatorname{Hom}_{R[\![H]\!]}(M, R[\![H]\!]) \cong \operatorname{Hom}_{R[\![H_0]\!]}(N, R[\![H_0]\!]).$$

Therefore, it follows that $\operatorname{Hom}_{R\llbracket H\rrbracket}(M, R\llbracket H\rrbracket) = 0$ if and only if $\operatorname{Hom}_{R\llbracket H_0\rrbracket}(M, R\llbracket H_0\rrbracket) = 0$. On the other hand, it is clear from the definition that $\operatorname{rank}_{R\llbracket H\rrbracket} M = 0$ if and only if $\operatorname{rank}_{R\llbracket H_0\rrbracket} M = 0$. Thus, we may then apply the above discussion to obtain the required equivalence. $\square$

In view of the above lemma, we will say that a finitely generated $R\llbracket H\rrbracket$-module M is *torsion* if either of the two equivalent statements hold. The following lemma is a relative version of Lemma 4.3.

LEMMA 4.5. *Let H be a compact pro-p p-adic Lie group without p-torsion. Let U be a closed normal subgroup of H such that H/U is also a compact pro-p p-adic Lie group without p-torsion. Let M be a finitely generated $R\llbracket H\rrbracket$-module. Then $H_i(U, M)$ is finitely generated over $R\llbracket H/U\rrbracket$ for each i and we have the equality*

$$
\begin{aligned}
\operatorname{rank}_{R\llbracket H\rrbracket} M &= \sum_{i \geq 0} (-1)^i \operatorname{rank}_{R\llbracket H/U\rrbracket} H_i(U, M) \\
&= \operatorname{rank}_{R\llbracket H/U\rrbracket} M_U + \sum_{i \geq 1} (-1)^i \operatorname{rank}_{R\llbracket H/U\rrbracket} H_i(U, M).
\end{aligned}
$$

Proof. If the ring R has characteristic zero or H is a uniform pro-p group, the conclusion follows from applying Lemma 4.1 (taking $\Lambda = R\llbracket H\rrbracket$ and $\Omega = R\llbracket H/U\rrbracket$) and observing that $H_i(U, M) \cong \operatorname{Tor}_i^{R\llbracket H\rrbracket}(R\llbracket H/U\rrbracket, M)$ for all i. In general, one has the equality

$$
\begin{aligned}
\operatorname{rank}_{R\llbracket H\rrbracket} M &= \sum_{i \geq 0} (-1)^i \operatorname{rank}_R H_i(H, M) \\
&= \sum_{i,j \geq 0} (-1)^{i+j} \operatorname{rank}_R H_i(H/U, H_j(U, M)) \\
&= \sum_{j \geq 0} (-1)^j \operatorname{rank}_{R\llbracket H/U\rrbracket} H_i(U, M),
\end{aligned}
$$

where the first and third equalities follow from Lemma 4.3 and the second equality is a consequence of the spectral sequence

$$
H_i(H/U, H_j(U, M)) \Longrightarrow H_{i+j}(H, M).
$$

$\square$

It is immediate from the equality in Lemma 4.5 that if

$$
\operatorname{rank}_{R\llbracket H\rrbracket} M = \sum_{i \geq 1} (-1)^i \operatorname{rank}_{R\llbracket H/U\rrbracket} H_i(U, M) = 0,
$$

then $\operatorname{rank}_{R\llbracket H/U\rrbracket} M_U = 0$. The converse is true if we assume further that U is solvable, and this is the content of the next theorem.

THEOREM 4.6. *Let H be a compact pro-p p-adic Lie group without p-torsion. Let U be a closed normal subgroup of H such that U is solvable and H/U is a compact pro-p p-adic Lie group without p-torsion. Let M be a finitely generated $R\llbracket H\rrbracket$-module. Then M_U is a torsion $R\llbracket H/U\rrbracket$-module if and only if M is a torsion $R\llbracket H\rrbracket$-module and $\sum_{i \geq 1} (-1)^i \operatorname{rank}_{R\llbracket H/U\rrbracket} H_i(U, M) = 0$.*

Proof. The above discussion already establishes one direction. Now suppose that M_U is a torsion $R[\![H/U]\!]$-module. The required conclusion will follow once we show that M is a torsion $R[\![H]\!]$-module. This will follow from the next lemma which is a slight refinement of the final theorem in [BHo] and [HO, Lemma 2.6]. □

LEMMA 4.7. *Let H be a compact p-adic Lie group without p-torsion. Suppose that N is a closed normal subgroup of H with the property that N is a solvable uniform pro-p group and H/N has no p-torsion. Let M be a finitely generated $R[\![H]\!]$-module. If M_N is a torsion $R[\![H/N]\!]$-module, then M is a torsion $R[\![H]\!]$-module.*

To prove this lemma, we need two more lemmas.

LEMMA 4.8. *Let H be a uniform pro-p group, and let N be a closed normal subgroup of H with the property that N is a solvable uniform pro-p group and H/N has no p-torsion. Then there exists a closed normal subgroup N_0 of H satisfying all the following properties.*

(i) $N_0 \cong \mathbb{Z}_p^r$ for some $r > 0$.

(ii) H/N_0 is uniform with $\dim H/N_0 < \dim H$.

(iii) $N_0 \subseteq N$.

Proof. If N is abelian, then one may take $N_0 = N$. Now if N is not abelian, we write $N^0 = N$ and $N^{(n+1)} = \overline{[N^{(n)}, N^{(n)}]}$. Then $N^{(m+1)} = 0$ but $N^{(m)} \neq 0$ for some $m \geq 1$. Note that $N^{(m)}$ is abelian. Set

$$N_0 := \{h \in H \mid h^{p^j} \in N^{(m)} \text{ for some } j\}$$

The proof of statement (3) of the first proposition in [BHo, §4] shows that N_0 satisfies (i) and (ii). To see that N_0 satisfies (iii), one applies a similar argument as in the last paragraph of the proof of [HO, Lemma 2.6]. □

LEMMA 4.9. *Let H be a uniform pro-p group, and let N be a closed normal subgroup of H with the property that $N \cong \mathbb{Z}_p^r$ and H/N has no p-torsion. Let M be a finitely generated $R[\![H]\!]$-module. If M_N is a torsion $R[\![H/N]\!]$-module, then M is a torsion $R[\![H]\!]$-module.*

Proof. We prove this lemma by the method of contradiction, following the argument given in the last theorem in [BHo]. Suppose that M is a finitely generated $R[\![H]\!]$-module with $R[\![H]\!]$-rank $s > 0$. Recall that $R[\![H]\!]$ is Auslander regular and has no zero-divisors (cf. Theorem A.1). We shall first show that there is map $M \longrightarrow R[\![H]\!]^s$ with $R[\![H]\!]$-torsion kernel and cokernel. Let $K(H)$ denote the skew field of $R[\![H]\!]$. Write $M^+ = \mathrm{Hom}_{R[\![H]\!]}(M, R[\![H]\!])$. Then by [V1, Proposition 2.5], there is a map $M \longrightarrow M^{++}$ with $R[\![H]\!]$-torsion kernel and cokernel (the torsionness comes from the definition of Auslander regularity). Choose $f_1, ..., f_s \in M^+$ such that they form a basis for $K(H) \otimes_{R[\![H]\!]} M^+$. Then these elements give rise to a map $R[\![H]\!]^s \longrightarrow M^+$ which clearly has $R[\![H]\!]$-torsion kernel and cokernel. Taking $R[\![H]\!]$-dual, we obtain a map $M^{++} \longrightarrow R[\![H]\!]^s$ with $R[\![H]\!]$-torsion kernel and cokernel. Combining this with the above canonical map, we obtain the required map.

Projecting onto any (fixed) factor of $R[\![H]\!]^s$, we obtain a $R[\![H]\!]$-homomorphism $\phi : M \longrightarrow R[\![H]\!]$. It is not difficult to see that ϕ is nontrivial. Denote $I(N)$ to be augmentation ideal of the ring $R[\![N]\!]$, and denote J to be the two-sided ideal $I(N)R[\![H]\!] = R[\![H]\!]I(N)$. Since J is a closed ideal of $R[\![H]\!]$, we have $\cap_{n \geq 0} J^n = 0$. Therefore, we can find an m such that $\phi(M) \subseteq J^m$ and $\phi(M) \nsubseteq J^{m+1}$. Then

$M' = (\phi(M) + J^{m+1})/J^{m+1}$ is a nontrivial submodule of J^m/J^{m+1}. We also note that N acts trivially on M' and J^m/J^{m+1}. Hence we have $M' = (M')_N$ being a quotient of M_N. On the other hand, by a similar argument to that in the last paragraph of [BHo], we have that J^m/J^{m+1} is a free $R[\![H/N]\!]$-module with positive $R[\![H/N]\!]$-rank. Since M' is a nontrivial submodule of a free $R[\![H/N]\!]$-module, M' must also have positive $R[\![H/N]\!]$-rank which in turn implies that M_N has positive $R[\![H/N]\!]$-rank and this contradicts the hypothesis of the lemma. $\square$

We can now prove Lemma 4.7.

Proof of Lemma 4.7. We proceed by induction on the dimension of H. When $\dim H = 1$, the assertion of the lemma can be deduced from the classical result of Iwasawa. Now suppose $\dim H > 1$. If $N = \mathbb{Z}_p^r$, then we are done by Lemma 4.9. Else by Lemma 4.8, we can find a closed normal subgroup N_0 of H such that (i) $N_0 \cong \mathbb{Z}_p^r$ for some $r > 0$, (ii) H/N_0 is uniform with $\dim H/N_0 < \dim H$ and (iii) $N_0 \subseteq N$. Viewing $M_N = (M_{N_0})_{N/N_0}$, we may apply our induction hypothesis to deduce that M_{N_0} is a torsion $R[\![H/N_0]\!]$-module. Now applying Lemma 4.9 again, we obtain the required conclusion. $\square$

The following is an immediate corollary of the above. Alternatively, one can prove this directly by the rank calculation.

COROLLARY 4.10. *Let H be a compact pro-p p-adic Lie group without p-torsion. Let U be a closed normal subgroup of H such that $U \cong \mathbb{Z}_p$ and H/U is a compact pro-p p-adic Lie group without p-torsion. Let M be a finitely generated $R[\![H]\!]$-module. Then M_U is a torsion $R[\![H/U]\!]$-module if and only if M is a torsion $R[\![H]\!]$-module and $H_1(U, M)$ is a torsion $R[\![H/U]\!]$-module.*

We mention another corollary of Lemma 4.7 which can be proved via a similar argument as in [HO, Lemma 2.5].

COROLLARY 4.11. *Let H be a compact p-adic Lie group without p-torsion. Suppose that N is a closed normal subgroup of H with the property that N is a solvable uniform pro-p group and H/N has no p-torsion. Let M be a finitely generated $R[\![H]\!]$-module. Then we have*

$$\mathrm{rank}_{R[\![G]\!]} M \leq \mathrm{rank}_{R[\![H/N]\!]} M_N.$$

We now mention another variant of Howson's result which can be viewed as a slight generalization of [Ho, Corollary 1.10] (see also [Jh, Lemma 5]).

PROPOSITION 4.12. *Let H be a compact p-adic Lie group without p-torsion, and let $x \in R$ be a nonzero element such that $\bar{R} := R/xR$ is also a complete regular local ring. Let M be a finitely generated $R[\![H]\!]$-module. Then*

$$\mathrm{rank}_{\bar{R}[\![H]\!]} M/xM = \mathrm{rank}_{\bar{R}[\![H]\!]} M[x] + \mathrm{rank}_{R[\![H]\!]} M,$$

where $M[x]$ is the submodule of M killed by x.

Proof. Using the free resolution of $\bar{R}[\![G]\!]$

$$0 \longrightarrow R[\![G]\!] \overset{x}{\longrightarrow} R[\![G]\!] \longrightarrow \bar{R}[\![G]\!] \longrightarrow 0,$$

one can calculate that

$$\operatorname{Tor}^i_{R\llbracket H\rrbracket}(\bar{R}\llbracket H\rrbracket, M) = \begin{cases} M/xM & \text{if } i = 0, \\ M[x] & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The conclusion is now immediate from the above calculations and Lemma 4.1. ∎

COROLLARY 4.13. *Retaining the above assumptions. Let M be a finitely generated $R\llbracket H\rrbracket$-module. Then M/xM is a torsion $\bar{R}\llbracket H\rrbracket$-module if and only if $M[x]$ is a torsion $\bar{R}\llbracket H\rrbracket$-module and M is a torsion $R\llbracket H\rrbracket$-module.*

5. On the pseudo-nullity of fine Selmer groups. In this section, we will investigate certain properties of the fine Selmer groups over p-adic Lie extensions of dimension strictly larger than one. As in the previous section, we shall assume that our ring R is a complete regular local ring with finite residue field of characteristic p. Let G be a compact pro-p p-adic Lie group without p-torsion. Then $R\llbracket G\rrbracket$ is a Auslander regular ring. We say that a finitely generated torsion $R\llbracket G\rrbracket$-module is *pseudo-null* if $\operatorname{Ext}^1_{R\llbracket G\rrbracket}(M, R\llbracket G\rrbracket) = 0$. The following fundamental lemma will be crucial in our discussion.

LEMMA 5.1. *Let R be a complete regular local ring with finite residue field of characteristic p and let G be a compact pro-p p-adic Lie group without p-torsion. Suppose that H is a closed normal subgroup of G with $G/H \cong \mathbb{Z}_p$. Let M be a compact $R\llbracket G\rrbracket$-module which is finitely generated over $R\llbracket H\rrbracket$. Then M is a pseudo-null $R\llbracket G\rrbracket$-module if and only if M is a torsion $R\llbracket H\rrbracket$-module.*

Proof. This essentially follows from [V2], which we shall explain (see also [HSh, Lemma 3.1]). By the standard theory of compact p-adic Lie groups (for instances, see [DSMS]), one can find open subgroups H_0 of H and G_0 of G such that H_0 and G_0 are uniform pro-p groups and $G_0/H_0 \cong \mathbb{Z}_p$. By [NSW, Proposition 5.4.17] (the same conclusion with a similar proof holds if we replace $\mathbb{Z}_p$ by R), we have

$$\operatorname{Ext}^i_{R\llbracket G\rrbracket}(N, R\llbracket G\rrbracket) \cong \operatorname{Ext}^i_{R\llbracket G_0\rrbracket}(N, R\llbracket G_0\rrbracket)$$

for any $R\llbracket G\rrbracket$-module N and all i, and a similar statement holds for H and H_0. Therefore, we are reduced to showing the lemma under the assumptions that G and H are uniform pro-p groups.

We now write $\Gamma = G/H$. There is a natural group homomorphism $\phi : \Gamma \longrightarrow \operatorname{Aut}(H)$. Suppose that

$$\operatorname{im} \phi \subseteq \{f \in \operatorname{Aut}(H) \mid f(h)h^{-1} \in H^p \text{ for all } h \in H\}. \qquad (*)$$

Then the conclusion of the lemma follows from an R-analog of [V2, Example 2.3] and [V2, Proposition 5.4] in this instance. In general, since H is a uniform pro-p group by our assumption, H^p is an open characteristic subgroup of H. Therefore, the map ϕ induces a continuous group homomorphism $\bar{\phi} : \Gamma \longrightarrow \operatorname{Aut}(H/H^p)$. Since H/H^p is finite, we have $\Gamma_1 := \ker \bar{\phi} \cong \mathbb{Z}_p$. Let G_1 be the open normal subgroup of G containing H such that $G_1/H = \Gamma_1$. By another application of the R-analog of [NSW, Proposition 5.4.17], we are reduced to showing that M is a pseudo-null $R\llbracket G_1\rrbracket$-module if and only if M is a torsion $R\llbracket H\rrbracket$-module. However, in this case, the natural map $\phi_1 : \Gamma_1 \longrightarrow \operatorname{Aut}(H)$ clearly satisfies $(*)$ by our choice of Γ_1, and so [V2, Example 2.3, Proposition 5.4] can be applied. ∎

REMARK. In the case when $H \cong \mathbb{Z}_p$, the lemma follows directly from the R-analog of [V2, Example 2.3] and [V2, Proposition 5.4].

We return to arithmetic. As before, p will denote a prime. For the remainder of the paper, we shall assume further that the number field F has no real primes when $p = 2$. Following [CS], we say that F_∞ is a S-admissible p-adic extension of F if (i) $\mathrm{Gal}(F_\infty/F)$ is compact pro-p p-adic Lie group without p-torsion, (ii) F_∞ contains F^{cyc} and (iii) F_∞ is contained in F_S. Write $G = \mathrm{Gal}(F_\infty/F)$ and $H = \mathrm{Gal}(F_\infty/F^{\mathrm{cyc}})$. From now on, we shall assume that our Galois module T is a free R-module of finite rank with a continuous R-linear $G_S(F)$-action. The following lemma is also considered in [CS, Lemma 3.2].

LEMMA 5.2. *Let F_∞ be a S-admissible p-adic Lie extension of F. Then the following statements are equivalent.*

(a) *$Y_S(T/F^{\mathrm{cyc}})$ is a finitely generated R-module.*

(b) *$H_S^2(F^{\mathrm{cyc}}/F, T)$ is a finitely generated R-module.*

(c) *$Y_S(T/F_\infty)$ is a finitely generated $R[\![H]\!]$-module.*

(d) *$H_S^2(F_\infty/F, T)$ is a finitely generated $R[\![H]\!]$-module.*

Proof. The equivalence of (a) and (b) is shown in Lemma 3.4. The equivalence of (c) and (d) can be shown by a similar argument. The equivalence of (b) and (d) follows from Lemma 2.2 and Nakayama lemma. $\square$

The following question has been studied by many before.

QUESTION B. Let F_∞ be a S-admissible p-adic Lie extension of F of dimension > 1, and suppose that $Y_S(T/F_\infty)$ is a finitely generated $R[\![H]\!]$-module. Is $Y_S(T/F_\infty)$ a pseudo-null $R[\![G]\!]$-module, or equivalently a torsion $R[\![H]\!]$-module?

This is precisely [CS, Conjecture B] when T is the Tate module of an elliptic curve. In this context, the conjecture has also been studied and verified for some elliptic curves in [Bh] and [Oc]. When T is the $R(1)$-dual of the Galois representation attached to a normalized eigenform ordinary at p, this is [Jh, Conjecture B]. In the case when T is the $R(1)$-dual of the Galois representation coming from a Λ-adic form, this is [Jh, Conjecture 2]. We note that in the case when T is the Tate module of all the p-power roots of unity, the dual fine Selmer group is precisely $\mathrm{Gal}(K(F_\infty)/F_\infty)$, where $K(F_\infty)$ is the maximal unramified pro-p extension of F_∞ at which every prime of F_∞ above p splits completely, as defined in Section 3. In this case, Hachimori and Sharifi [HSh] has constructed a class of admissible p-adic Lie extension F_∞ of F of dimension > 1 such that $\mathrm{Gal}(K(F_\infty)/F_\infty)$ is not pseudo-null. Despite so, they have speculated that the pseudo-nullity condition should hold for admissible p-adic extensions "coming from algebraic geometry" (see [HSh, Question 1.3] for details, and see also [Sh1, Conjecture 7.6] for a related assertion and [Sh2] for positive results in this direction).

Before continuing our discussion, we introduce the following hypothesis on our admissible extension F_∞.

(Dim$_S$): For each $v \in S$, the decomposition group of G at v, denoted by G_v, has dimension ≥ 2.

LEMMA 5.3. *Let F_∞ be a S-admissible p-adic Lie extension of F and assume that $Y_S(T/F_\infty)$ is a finitely generated $R[\![H]\!]$-module. Then the following statements hold.*

(a) *If $H_S^2(F_\infty/F, T)$ is a pseudo-null $R[\![G]\!]$-module, so is $Y_S(T/F_\infty)$.*

(b) *Suppose that F_∞ satisfies* $(\mathbf{Dim}_S)$. *Then if $Y_S(T/F_\infty)$ is a pseudo-null $R[\![G]\!]$-module, so is $H_S^2(F_\infty/F, T)$.*

Proof. From the Poitou-Tate sequence, we have the following exact sequence

$$0 \longrightarrow Y_S(T/F_\infty) \longrightarrow H_S^2(F_\infty/F, T) \longrightarrow \Big(\bigoplus_{v \in S} K_v^0(W/F_\infty) \Big)^\vee.$$

Statement (a) is then immediate. It remains to show that statement (b) holds. For each $v \in S$, fix a prime w of F_∞ above v. By abuse of notation, we denote the prime of F^{cyc} below w by w. Write H_w to be the decomposition group of H at w. Then one sees that $K_v^0(W/F_\infty)^\vee$ is isomorphic to a finite sum of terms of the form

$$R[\![H]\!] \otimes_{R[\![H_w]\!]} W(F_{\infty,w})^\vee$$

which is clearly finitely generated over $R[\![H]\!]$. The assertion of statement (b) will follow once we show that the above term is a torsion $R[\![H]\!]$-module. By the assumption $(\mathbf{Dim}_S)$, the group H_w has dimension ≥ 1. It is then easy to see that $W(F_{\infty,w})^\vee$ is a finitely generated torsion $R[\![H_w]\!]$-module. The required conclusion then follows from observing that

$$\mathrm{Hom}_{R[\![H]\!]} \big(R[\![H]\!] \otimes_{R[\![H_w]\!]} W(F_{\infty,w})^\vee, R[\![H]\!] \big)$$
$$= R[\![H]\!] \otimes_{R[\![H_w]\!]} \mathrm{Hom}_{R[\![H_w]\!]} \big(W(F_{\infty,w})^\vee, R[\![H_w]\!] \big)$$
$$= 0.$$

$\square$

We can now state the first main result of this section which is a slight refinement of the implication $(3) \Rightarrow (1)$ in [Jh, Theorem 10].

THEOREM 5.4. *Let F_∞ be a S-admissible p-adic Lie extension of F and assume that $Y_S(T/F_\infty)$ is a finitely generated $R[\![H]\!]$-module. Suppose that there exists a prime ideal $\mathfrak{p}$ of R such that the ring $R/\mathfrak{p}$ is also regular local. Suppose also that F_∞ satisfies* $(\mathbf{Dim}_S)$. *If $Y_S\big((T/\mathfrak{p}T)/F_\infty\big)$ is a pseudo-null $R/\mathfrak{p}[\![G]\!]$-module, then $Y_S(T/F_\infty)$ is a pseudo-null $R[\![G]\!]$-module.*

Before proving the theorem, we first prove the following lemma.

LEMMA 5.5. *Let F_∞ be a S-admissible p-adic Lie extension of F. Suppose that $Y_S(T/F_\infty)$ is a finitely generated $R[\![H]\!]$-module. Let x be a nonzero and nonunital element of R, and suppose that $\bar{R} := R/xR$ is also regular local. Write $\bar{T} = T/xT$. Then $H_S^2(F_\infty/F, \bar{T})$ is a pseudo-null $\bar{R}[\![G]\!]$-module if and only if $H_S^2(F_\infty/F, T)$ is a pseudo-null $R[\![G]\!]$-module and $H_S^2(F_\infty/F, T)[x]$ is a pseudo-null $\bar{R}[\![G]\!]$-module.*

Proof. By Lemma 2.1, we have an isomorphism

$$H_S^2(F_\infty/F, T)/x \cong H_S^2(F_\infty/F, \bar{T})$$

of $\bar{R}[\![G]\!]$-modules. The lemma is now immediate from Lemma 5.1 and Corollary 4.13. $\square$

We now give the proof of Theorem 5.4.

Proof of Theorem 5.4. By Lemma 5.3(a), it suffices to show that $H_S^2(F_\infty/F, T)$ is a pseudo-null $R[\![G]\!]$-module. By the theory of regular local rings (for instances, see [M,

§14]), one can find a set of generators $x_1, ..., x_r$ of $\mathfrak{p}$ such that each intermediate ring $R/(x_1, ..., x_i)$ is also regular local for $i = 1, .., r$. By a repeated application of Lemma 5.5, we are reduced to showing that $H^2_S(F_\infty/F, T/\mathfrak{p}T)$ is a pseudo-null $R/\mathfrak{p}[\![G]\!]$-module which will follow from the hypothesis of the theorem and Lemma 5.3(b). $\square$

We record an immediate corollary of Theorem 5.4.

COROLLARY 5.6. *Let F_∞ be a S-admissible p-adic Lie extension of F and assume that $Y_S(T/F_\infty)$ is a finitely generated $R[\![H]\!]$-module. Suppose that $(\mathbf{Dim}_S)$ is satisfied. If $Y_S\big((T/\mathfrak{m}T)/F_\infty\big)$ is a pseudo-null $k[\![G]\!]$-module, then $Y_S(T/F_\infty)$ is a pseudo-null $R[\![G]\!]$-module.*

We now prove the following descent result for pseudo-nullity.

THEOREM 5.7. *Let F_∞ be a S-admissible p-adic Lie extension of F and assume that $Y_S(T/F^{\mathrm{cyc}})$ is a finitely generated R-module. Suppose that F'_∞ is another S-admissible p-adic Lie extension of F which satisfies the following properties.*

(i) F'_∞ is contained in F_∞.

(ii) F'_∞ satisfies $(\mathbf{Dim}_S)$.

(iii) The group $N := \mathrm{Gal}(F_\infty/F'_\infty)$ is a solvable uniform pro-p group and H/N has no p-torsion.

Then $Y_S(T/F'_\infty)$ is a pseudo-null $R[\![\mathrm{Gal}(F'_\infty/F)]\!]$-module if and only if $Y_S(T/F_\infty)$ is a pseudo-null $R[\![G]\!]$-module and $\sum_{i \geq 1}(-1)^i \mathrm{rank}_{R[\![H/N]\!]} H_i\big(N, H^2_S(F_\infty/F, T)\big) = 0$.

REMARK. Note that we do not require F'_∞ and F_∞ to be solvable extensions of F.

Proof. By Lemmas 2.2, 5.1 and 5.3, it is equivalent to showing that $H^2_S(F'_\infty/F, T)$ is a torsion $R[\![\mathrm{Gal}(F'_\infty/F^{\mathrm{cyc}})]\!]$-module if and only if $H^2_S(F_\infty/F, T)$ is a torsion $R[\![H]\!]$-module and

$$\sum_{i \geq 1}(-1)^i \mathrm{rank}_{R[\![H/N]\!]} H_i\big(N, H^2_S(F_\infty/F, T)\big) = 0.$$

But this is immediate from Theorem 4.6. $\square$

We end the section with the following remark.

REMARK. The results here can be extended to the case when the ring R is a commutative Noetherian local domain which is finite flat over a regular local ring R_0. In this case, for a compact pro-p p-adic Lie group without p-torsion, the notion of torsion and pseudo-nullity over $R[\![G]\!]$ is defined similarly as in the regular case. It then follows from the flatness condition of R that one has the following isomorphisms

$$R \otimes_{R_0} \mathrm{Ext}^i_{R_0[\![G]\!]}(M, R_0[\![G]\!]) \cong \mathrm{Ext}^i_{R[\![G]\!]}(R \otimes_{R_0} M, R[\![G]\!]) \cong \mathrm{Ext}^i_{R[\![G]\!]}(M^d, R[\![G]\!]),$$

where d is the R_0-rank of R, and a R-free module T is clearly still R_0-free. Therefore, the question of pseudo-nullity over $R[\![G]\!]$ is reduced to pseudo-nullity over $R_0[\![G]\!]$, and the results in this section apply.

6. Some examples. In this section, we will discuss some numerical examples of pseudo-nullity.

(a) The first example we consider is the case $T = \mathbb{Z}_p(1)$. Take $F = \mathbb{Q}(\mu_p)$. Let S be the set of prime(s) of F above p. By the theorem of Ferrero-Washington, the group $\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})$ is finitely generated over $\mathbb{Z}_p$. Now if $p < 1000$, it follows from [Sh2, Theorem 1.4] that $\mathrm{Gal}(K(F_\infty)/F_\infty)$ is a finitely generated $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F'_\infty)]\!]$-module (and hence a pseudo-null $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-module) for every S-admissible p-adic extension F_∞ of F which contains $F'_\infty := \mathbb{Q}(\mu_{p^\infty}, p^{-p^\infty})$. Note that $\mathbb{Q}(\mu_{p^\infty}, p^{-p^\infty})$ (and hence any F_∞ containing it) satisfies $(\mathbf{Dim}_S)$ by [HV, Lemma 3.9]. Write $N = \mathrm{Gal}(F_\infty/F'_\infty)$. It follows that $\mathrm{Gal}(K(F_\infty)/F_\infty)$ is finitely generated over $\mathbb{Z}_p[\![N]\!]$ if and only if $H^2_S(F_\infty/F, \mathbb{Z}_p(1))$ is finitely generated over $\mathbb{Z}_p[\![N]\!]$. Therefore in this case, we have

$$\mathrm{rank}_{\mathbb{Z}_p[\![H/N]\!]} H_i\big(N, H^2_S(F_\infty/F, \mathbb{Z}_p(1))\big) = 0$$

for all i and the equation of Lemma 4.5 is vacuous here. We mention in passing that if p is a regular prime, it will follow from an application of a classical result of Iwasawa that $\mathrm{Gal}(K(F_\infty)/F_\infty) = 0$ for every S-admissible p-adic extension F_∞ of F (for instance, see [Oc, Section 4]).

(b) The next example comes from [Bh, Example 23] which is also considered in [CS, Example 4.8]. Let E be the elliptic curve $150A1$ of Cremona's table which is given by

$$y^2 + xy = x^3 - 3x - 3,$$

Take $p = 5$ and $F = \mathbb{Q}(\mu_5)$. Let S be the set of primes of F lying above $2, 3, 5$ and ∞. The elliptic curve E has good ordinary reduction at the unique prime of F above 5 and split multiplicative reduction at the unique primes of F above 2 and 3. It was shown in [Bh, Example 23] that $Y(T_5E/F_\infty)$ is a pseudo-null $\mathbb{Z}_5[\![\mathrm{Gal}(F_\infty/F)]\!]$-module for the S-admissible 5-adic extension $F_\infty = \mathbb{Q}(E[5^\infty], 3^{5^{-\infty}})$. Since $\mathbb{Q}(E[5^\infty])$ satisfies $(\mathbf{Dim}_S)$ (cf. [C, Lemma 2.8]), so does $\mathbb{Q}(E[5^\infty], 3^{5^{-\infty}})$. Applying Theorem 5.7, we have that $Y(T_5E/\mathcal{L})$ is a pseudo-null $\mathbb{Z}_5[\![\mathrm{Gal}(\mathcal{L}/F)]\!]$-module when $\mathcal{L}$ is one of the following S-admissible 5-adic extensions:

$$\mathbb{Q}(E[5^\infty], 2^{5^{-\infty}}, 3^{5^{-\infty}}), \quad \mathbb{Q}(E[5^\infty], 3^{5^{-\infty}}, 5^{5^{-\infty}}), \quad \mathbb{Q}(E[5^\infty], 2^{5^{-\infty}}, 3^{5^{-\infty}}, 5^{5^{-\infty}})),$$

$$L_\infty(E[5^\infty], 2^{5^{-\infty}}, 3^{5^{-\infty}}), \quad L_\infty(E[5^\infty], 3^{5^{-\infty}}, 5^{5^{-\infty}}), \quad L_\infty(E[5^\infty], 2^{5^{-\infty}}, 3^{5^{-\infty}}, 5^{5^{-\infty}}),$$

where L_∞ is any $\mathbb{Z}_5^r$-extension of F for $1 \leq r \leq 3$.

Write $K_\infty = \mathbb{Q}(E[5^\infty])$. Applying Theorem 5.7 in this direction, we have

$$\mathrm{rank}_{\mathbb{Z}_5[\![\mathrm{Gal}(K_\infty/F^{\mathrm{cyc}})]\!]} Y(T_5E/K_\infty)$$
$$= \mathrm{rank}_{\mathbb{Z}_5[\![\mathrm{Gal}(K_\infty/F^{\mathrm{cyc}})]\!]} H_1\big(\mathrm{Gal}(F_\infty/K_\infty), H^2_S(F_\infty/F, T_5E)\big).$$

Unfortunately, we do not know how to show that this latter quantity is zero which will then verify the pseudo-nullity for $Y(T_5E/L_\infty)$. What we do know at present is that this quantity is either zero or two (cf. [CS, Example 4.8]).

(c) We now discuss our final numerical example which has also been considered in [Jh, p. 362]. Let E be the elliptic curve $79A1$ of Cremona's table which is given by

$$y^2 + xy + y = x^3 + x^2 - 2x.$$

Take $p = 3$ and $F = \mathbb{Q}(\mu_3)$. Let S be the set of primes of F lying above $3, 79$ and ∞. The elliptic curve E has good ordinary reduction at the unique prime of F above 3 and non split multiplicative reduction at the two primes of F above 79. It was shown in [Jh, p. 362] that $Y(T_3E/F_\infty)$ is a pseudo-null $\mathbb{Z}_3[\![\mathrm{Gal}(F_\infty/F)]\!]$-module when $F = \mathbb{Q}(\mu_{3^\infty}, 79^{-3^\infty})$. Again noting that $\mathbb{Q}(\mu_{3^\infty}, 79^{-3^\infty})$ satisfies ($\mathbf{Dim}_S$) (cf. [HV, Lemma 3.9]), one can apply Theorem 5.7 to conclude that $Y(T_3E/\mathcal{L})$ is a pseudo-null $\mathbb{Z}_3[\![\mathrm{Gal}(\mathcal{L}/F)]\!]$-module when $\mathcal{L}$ is one of the following S-admissible 3-adic extensions:

$$\mathbb{Q}(\mu_{3^\infty}, 3^{-3^\infty}, 79^{-3^\infty}), \quad L_\infty(79^{-3^\infty}), \quad L_\infty(3^{-3^\infty}, 79^{-3^\infty}).$$

Here L_∞ is the unique $\mathbb{Z}_3^2$-extension of F.

Now the residual representation on the Tate module of the elliptic curve $79A1$ is irreducible for $p = 3$. Hence there exists a complete Noetherian local domain R which is finite flat over $\Lambda = \mathbb{Z}_3[\![X]\!]$ and a free R-module T of rank 2 with a continuous $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action which is the R-dual of the Galois representation attached to the Hida family associated to the weight 2 newform corresponding to E (cf. [Hi], also see [SS, Section 1 and Remarks at the end of Section 6] for details on R and T; note that our T here is the $R(1)$-dual of the representation considered there). As mentioned above, the S-admissible 3-adic extension $\mathbb{Q}(\mu_{3^\infty}, 79^{-3^\infty})$ satisfies ($\mathbf{Dim}_S$), and therefore any S-admissible 3-adic extension containing $\mathbb{Q}(\mu_{3^\infty}, 79^{-3^\infty})$ will also satisfy ($\mathbf{Dim}_S$). Applying Theorem 5.4 (and noting the remarks made at the end of Section 5), we have that $Y(T/\mathcal{L})$ is a pseudo-null $R[\![\mathrm{Gal}(\mathcal{L}/F)]\!]$-module when $\mathcal{L}$ is one of the following S-admissible 3-adic extensions:

$$\mathbb{Q}(\mu_{3^\infty}, 79^{-3^\infty}), \quad \mathbb{Q}(\mu_{3^\infty}, 3^{-3^\infty}, 79^{-3^\infty}), \quad L_\infty(79^{-3^\infty}), \quad L_\infty(3^{-3^\infty}, 79^{-3^\infty}).$$

(Alternatively, one can deduce this by combining Jha's observation and Theorem 5.7.)

7. Complement: On torsionness of fine Selmer group. As before, p will denote a prime. Let F be a number field, where we assume that it has no real primes when $p = 2$. We also assume that our Galois module T is a free R-module of finite rank with a continuous R-linear $G_S(F)$-action, where R is a complete regular local ring with finite residue field of characteristic p. The following is a weaker version of the conjecture made in Section 3.

CONJECTURE A′. *For any number field F and a S-admissible p-adic Lie extension F_∞ of F, $Y_S(T/F_\infty)$ is a finitely generated torsion $R[\![G]\!]$-module.*

It is clear that Conjecture A′ will follow from Conjecture A and Lemma 5.2. In particular, by Theorem 3.1, Conjecture A′ is a consequence of the Iwasawa μ-conjecture. We now record the following lemma (compare with [CS, Lemma 3.1]). We write $A = T \otimes_R R^\vee$.

LEMMA 7.1. *Let F_∞ be a S-admissible p-adic Lie extension of F. Then the following statements are equivalent.*

(a) $H_S^2(F_\infty/F, T)$ *is a torsion $R[\![G]\!]$-module.*

(b) $Y_S(T/F_\infty)$ *is a torsion $R[\![G]\!]$-module.*

(c) $H^2(G_S(F_\infty), A) = 0$.

REMARK. In view of Statement (c), Conjecture A′ is also sometimes called the "weak Leopoldt conjecture for A" (over F_∞).

Proof. The equivalence of (a) and (b) follows from a similar argument as in Lemma 3.4. We will now establish the equivalence of (a) and (c). Consider the following general version of the spectral sequence of Jannsen [Ja, Theorem 1]

$$E_2^{i,j} = \mathrm{Ext}_{R[\![G]\!]}^i \left(H^j(G_S(F_\infty), A)^\vee, R[\![G]\!] \right) \Longrightarrow H_S^{i+j}(F_\infty/F, T).$$

(One can obtain this spectral sequence by combining the middle two derived isomorphisms in [FK, 1.6.12(4)].) Since the spectral sequence is bounded (as $R[\![G]\!]$ has finite projective dimension), it follows that $E_m^{r,s}$ must stabilize for large enough m. In particular, we have that $E_\infty^{i,j}$ is a subquotient of $E_2^{i,j}$. By the Auslander regularity of $R[\![G]\!]$, the terms $E_2^{i,j}$, and hence $E_\infty^{i,j}$, are torsion $R[\![G]\!]$-modules for $i \neq 0$. Since $H_S^2(F_\infty/F, T)$ has a finite filtration with factors $E_\infty^{i,2-i}$ for $i = 0, 1, 2$, this in turn yields $\mathrm{rank}_{R[\![G]\!]} H_S^2(F_\infty/F, T) = \mathrm{rank}_{R[\![G]\!]} E_\infty^{0,2}$. On the other hand, one sees easily that the edge map $E_\infty^{0,2} \to E_2^{0,2}$ is injective and has cokernel isomorphic to a subquotient of

$$\mathrm{Ext}_{R[\![G]\!]}^2(H^1(G, A)^\vee, R[\![G]\!]) \oplus \mathrm{Ext}_{R[\![G]\!]}^3(H^0(G, A)^\vee, R[\![G]\!]).$$

As observed above, these are torsion over $R[\![G]\!]$. Therefore, we may conclude that

$$\mathrm{rank}_{R[\![G]\!]} H_S^2(F_\infty/F, T) = \mathrm{rank}_{R[\![G]\!]} \mathrm{Hom}_{R[\![G]\!]} \left(H^2(G_S(F_\infty), A)^\vee, R[\![G]\!] \right).$$

Therefore, it follows that $H_S^2(F_\infty/F, T)$ is a torsion $R[\![G]\!]$-module if and only if $\mathrm{Hom}_{R[\![G]\!]} \left(H^2(G_S(F_\infty), A)^\vee, R[\![G]\!] \right)$ is a torsion $R[\![G]\!]$-module. Since $H^2(G_S(F_\infty), A)^\vee$ is reflexive by [SS, Proposition 3.5(ii)] and [V1, Proposition 3.11(i)], the latter statement holds if and only if $H^2(G_S(F_\infty), A) = 0$. $\square$

Combining the preceding lemma with Theorem 4.6 and Corollary 4.13, we have the following results for the torsion property of the dual fine Selmer groups which are analogous to the pseudo-nullity results obtained in Section 5.

PROPOSITION 7.2. *Let F_∞ be a S-admissible p-adic Lie extension of F. Suppose that F_∞' is another S-admissible p-adic Lie extension of F which satisfies the following properties.*

(i) F_∞' is contained in F_∞.

(ii) The group $N := \mathrm{Gal}(F_\infty/F_\infty')$ is a solvable uniform pro-p group and G/N has no p-torsion.

Then $Y_S(T/F_\infty')$ is a torsion $R[\![\mathrm{Gal}(F_\infty'/F)]\!]$-module if and only if $Y_S(T/F_\infty)$ is a torsion $R[\![G]\!]$-module and $\sum_{i \geq 1}(-1)^i \mathrm{rank}_{R[\![G/N]\!]} H_i \left(N, H_S^2(F_\infty/F, T) \right) = 0$.

PROPOSITION 7.3. *Let F_∞ be a S-admissible p-adic Lie extension of F. Let x be a nonzero and nonunital element of R, and suppose that $\bar{R} := R/xR$ is also regular local. Write $\bar{T} = T/xT$. Then $Y_S(\bar{T}/F_\infty)$ is a torsion $\bar{R}[\![G]\!]$-module if and only if $Y_S(T/F_\infty)$ is a torsion $R[\![G]\!]$-module and $H_S^2(F_\infty/F, T)[x]$ is a torsion $\bar{R}[\![G]\!]$-module.*

REMARK. Proposition 7.2 can be viewed as a somewhat analogous statement for Selmer groups as in [HV, Theorem 2.8] and [HO, Theorem 2.3].

We end by briefly mentioning some known cases of Conjecture A′. In the case when $T = \mathbb{Z}_p(1)$ and $F_\infty = F^{\mathrm{cyc}}$, this follows from a classical theorem of Iwasawa [Iw1, Theorem 5]. For a general admissible extension F_∞, one can deduce the conjecture from the cyclotomic case using a limit argument with statement (c) of Lemma 7.1 (see [OcV, Theorem 6.1] for details).

When T is the Tate module of an elliptic curve E which does not have potentially supersingular reduction at any primes above p, it is in fact conjectured that the Pontryagin dual of the classical Selmer group is a torsion $\mathbb{Z}_p[\![G]\!]$-module (see [HO, Maz, Sch]). Conjecture A′ is then a consequence of this more general conjecture.

In the case when E is an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at p and F is an abelian extension of $\mathbb{Q}$, it follows from a deep theorem of Kato [K] that the dual Selmer group over F^{cyc} is a torsion $\mathbb{Z}_p[\![\Gamma]\!]$-module. In particular, conjecture A′ holds in this case. Note that one cannot apply a limit argument as in [OcV, Theorem 6.1] in this situation to the case of a general admissible p-adic Lie extension F_∞, since in general a finite extension L of F contained in F_∞ need not be abelian over $\mathbb{Q}$ and therefore, Kato's theorem does not apply. However, one may still apply Proposition 7.2 to obtain cases of Conjecture A′ over solvable admissible p-adic Lie extension of F.

On the other hand, one can deduce conjecture A′ for elliptic curve over a S-admissible p-adic Lie extension of F containing $F(E[p^\infty])$ via [CS, Lemma 2.4]. Here one does not need any reduction hypothesis on E nor assumption on F.

Finally, we observe that under appropriate modification as noted in the Remark at the end of Section 5, we can extend [SS, Corollary 6.17] to a solvable admissible p-adic Lie extension.

Appendix A. On the structure of $R[\![G]\!]$. The purpose of this appendix is to prove certain results on the structure of the completed group algebra $R[\![G]\!]$. We believe results of such are well-known among experts, although they do not seem to have been written down properly anywhere except for the case $R = \mathbb{Z}_p$. Since these results are basic for the discussion in this paper, we have included their proofs here. The following is the main result of the appendix.

THEOREM A.1. (*a*) *If R is a commutative Noetherian local ring with finite residue field of characteristic p and G is a compact p-adic Lie group, then $R[\![G]\!]$ is Noetherian.*

(*b*) *If R is a commutative Noetherian local domain with finite residue field of characteristic p and G is a uniform pro-p group, then $R[\![G]\!]$ has no zero divisor.*

(*c*) *If R be a commutative complete regular local ring with finite residue field of characteristic p and G is a compact p-adic Lie group without p-torsion, then $R[\![G]\!]$ is an Auslander regular ring.*

Statement (a) and (b) are well-known theorems of Lazard [Laz] when $R = \mathbb{Z}_p$ or R is a finite field of order p. (see also [DSMS, Corollary 7.25, Corollary 7.26]). Statement (a) in this generality has been established in [Wil, Theorem 8.7.8] and [LSh, Proposition 3.0.1]. Neumann [Neu] has shown statement (b) when G is a pro-p p-adic Lie group without p-torsion and $R = \mathbb{Z}_p$ by a different approach. His method can be easily extended to the case when R is a regular local ring with characteristic zero. However, his method does not seem to apply to the case when R has characteristic p. Statement (c) is the theorem of Venjakob when $R = \mathbb{Z}_p$ or R is a finite field of order p [V1, Theorems 3.26, 3.30(b)]. In this appendix, we will give a uniform[2] approach to prove all three statements simultaneously.

We first review some facts which can be found in [DSMS]. For a finitely generated pro-p group G, we write $G^{p^i} = \langle g^{p^i} \,|\, g \in G \rangle$, that is, the group generated by the p^ith-powers of elements in G. The pro-p group G is said to be *powerful* if $G/\overline{G^p}$ is abelian for odd p, or if $G/\overline{G^4}$ is abelian for $p = 2$. We define the lower p-series by $P_1(G) = G$,

---

[2] literally and figuratively

and

$$P_{i+1}(G) = \overline{P_i(G)^p [P_i(G), G]}, \text{ for } i \geq 1.$$

It follows from [DSMS, Thm. 3.6] that if G is a powerful pro-p group, then $G^{p^i} = P_{i+1}(G)$. This in turn implies that a finitely generated pro-p group G is powerful if and only if $[G, G] \subseteq G^{p^2}$. It follows from the same theorem that for a finitely generated powerful pro-p group G, the p-power map

$$P_i(G)/P_{i+1}(G) \xrightarrow{\cdot p} P_{i+1}(G)/P_{i+2}(G)$$

is surjective for each $i \geq 1$. If the p-power maps are isomorphisms for all $i \geq 1$, we say that G is *uniformly powerful* (abrev. *uniform*). Note that in this case, we have an equality $|G : P_2(G)| = |P_i(G) : P_{i+1}(G)|$ for every $i \geq 1$. We now recall the following characterization of compact p-adic Lie groups due to Lazard [Laz] (see also [DSMS, Cor. 8.34]): a topological group G is a compact p-adic Lie group if and only if G contains a open normal uniform pro-p subgroup.

For the remainder of the appendix, G will always be a uniform pro-p group, unless otherwise stated. For $n \geq 0$, we shall write $G_n = P_n(G)$. Denote I_n to be the augmentation kernel of the map $R[\![G]\!] \twoheadrightarrow R[G/G_n]$ which is a closed two-sided ideal of $R[\![G]\!]$. Since the subgroups G_n form a basis of neighbourhood of 1 in G, we have $R[\![G]\!] = \varprojlim_n R[\![G]\!]/I_n$. Fix a minimal set of topological generators $a_1, a_2, ..., a_d$ of G. Write $b_i = a_i - 1$ for each i. For $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ (here, $\mathbb{N}$ is the set of natural numbers including 0) and any d-tuple $\mathbf{v} = (v_1, ..., v_d) \in \Lambda^d$, we write

$$\langle \alpha \rangle = \alpha_1 + \cdots + \alpha_d, \quad \mathbf{v}^\alpha = v_1^{\alpha_1} \cdots v_d^{\alpha_d}.$$

In particular, we write $\mathbf{b}^\alpha = b_1^{\alpha_1} \cdots b_d^{\alpha_d}$. We now examine the structure of $R[\![G]\!]$ as an R-module. For $n \geq 1$, we define

$$T_n = \{\alpha \in \mathbb{N}^d \mid \alpha_i < p^{n-1} \text{ for } i = 1, ..., d\}.$$

LEMMA A.2. *If G is uniform, then we have a direct sum decomposition*

$$R[\![G]\!] = I_n \oplus \bigoplus_{\alpha \in T_n} R\mathbf{b}^\alpha$$

of R-modules. Furthermore, we have $R\mathbf{b}^\alpha \cong R$ for each α and

$$I_n = I_{n+1} \oplus \bigoplus_{\alpha \in T_{n+1} \setminus T_n} R\mathbf{b}^\alpha$$

Proof. (Compare with [DSMS, Lemma 7.9]) Denote ϕ to be the canonical quotient $R[\![G]\!] \twoheadrightarrow R[G/G_n]$. By [DSMS, Theorem 3.6], every element of G/G_n can written as $a_1^{\alpha_1} \cdots a_d^{\alpha_d} G_n$ with $\alpha_i < p^{n-1}$. Hence $\{\phi(\mathbf{a}^\alpha) \mid \alpha \in T_n\}$ generates $R[G/G_n]$ as an R-module. Since G is uniform, we have $|G/G_n| = p^{(n-1)d}$, and so $\phi(R[\![G]\!])$ is a free R-module of rank $p^{(n-1)d}$. On the other hand, we have $|T_n| = p^{(n-1)d}$. Therefore, the generating set $\{\phi(\mathbf{b}^\alpha) \mid \alpha \in T_n\}$ is actually a free R-basis for this module. The assertions in the lemma are now immediate from this. $\square$

The next lemma is the key ingredient to the proof of our theorem. We write $\mathrm{gr}_{\{I_n\}} R[\![G]\!]$ for the graded ring $\bigoplus_{n \geq 0} I_n / I_{n+1}$ whose multiplication is given by

$$(\mathbf{b}^\alpha + I_{n+1})(\mathbf{b}^\beta + I_{m+1}) = \mathbf{b}^\gamma + I_{n+m+1},$$

where $\gamma_i = \alpha_i + \beta_i$. To see that this multiplication is well-defined, we note that by an induction argument, it suffices to show that $b_i b_j - b_j b_i \in I_2$. Now observe that

$$b_i b_j - b_j b_i = a_i a_j - a_j a_i = ([a_i, a_j] - 1) a_j a_i$$

Since G is powerful, we have that $[a_i, a_j] \in G_2$. This implies that $[a_i, a_j] - 1 \in I_2$ which in turn implies that $b_i b_j - b_j b_i \in I_2$, as required. We may now prove our lemma.

LEMMA A.3. *Let R be a complete Noetherian local ring with finite residue field of characteristic p. Let G be a uniform pro-p group. Then we have an R-algebra isomorphism given by*

$$\Phi : R[X_1, ..., X_d] \longrightarrow \mathrm{gr}_{\{I_n\}} R[\![G]\!]$$
$$X_j \mapsto b_j + I_2.$$

Proof. The above argument shows that the assignment $X_j \mapsto b_j + I_2$ give a well-defined R-algebra homomorphism. Surjectivity of Φ is an immediate consequence of Lemma A.2. It remains to show that Φ is injective. Let A_n be the R-submodule of $R[X_1, ..., X_d]$ generated by monomials of the form $X_1^{\alpha_1} \cdots X_d^{\alpha_d}$, where $\alpha_i < p^{n-1}$ for $i = 1, ..., d$. Then one has

$$\Phi(A_n) = \bigoplus_{j=1}^{n} I_j / I_{j+1} = \bigoplus_{\alpha \in T_n} R\mathbf{b}^\alpha,$$

where the second equality comes from Lemma A.2. Now A_n and $\Phi(A_n)$ are both free R-modules of rank $m := |T_n| = p^{(n-1)d}$. Choose R-isomorphisms $f : R^m \xrightarrow{\sim} A_n$ and $g : R^m \xrightarrow{\sim} \Phi(A_n)$. Then $g^{-1} \circ \Phi \circ f$ is a surjective endomorphism of R^m. By [M, Theorem 2.4], it follows that $g^{-1} \circ \Phi \circ f$ is an automorphism. In particular, this implies that $\ker \Phi|_{A_n} = 0$. Therefore, we have $\ker \Phi = \cup_n \ker \Phi|_{A_n} = 0$, hence proving the lemma. $\square$

We can now give a proof of Theorem A.1.

Proof of Theorem A.1. Statement (b) is an immediate consequence of Lemma A.3 and [DSMS, Proposition 7.27(i)]. Now let U be an open normal uniform pro-p subgroup of G. Since U is a subgroup of G with finite index, it follows that $R[\![G]\!]$ is Noetherian if $R[\![U]\!]$ is so. The latter is then an immediate consequence of Lemma A.3 and [DSMS, Proposition 7.27(ii)].

It remains to show statement (c). By (a), the ring $R[\![G]\!]$ is Noetherian. Therefore, the global dimension of $R[\![G]\!]$ coincides with its topological projective dimension. Let U be an open normal uniform pro-p subgroup of G. Since G has no p-torsion, we may apply a result of Serre (cf. [Se, Corollaire 1]) to conclude that $\mathrm{cd}_p(G) = \mathrm{cd}_p(U)$. It then follows from [Bru, Theorem 4.1] that $R[\![G]\!]$ has finite global dimension. It remains to verify the Auslander condition. By [NSW, Proposition 5.4.17], we have an isomorphism $\mathrm{Ext}^i_{R[\![G]\!]}(M, R[\![G]\!]) \cong \mathrm{Ext}^i_{R[\![U]\!]}(M, R[\![U]\!])$ of $R[\![U]\!]$-modules for any $R[\![G]\!]$-module M. Therefore, we are reduced to showing that $R[\![U]\!]$ is Auslander regular. By Lemma A.3, we have that the associated graded ring of $\mathrm{gr}_{\{I_n\}} R[\![U]\!]$ is a

commutative regular local ring, since it is a polynomial ring over a regular local ring. The conclusion will follow from an application of the theorem of Björk (see Remarks after [Bjö, Theorem 3.9] or [V1, Theorem 3.21]). (One still needs to verify the closure condition in the cited theorem but this follows immediately from the observation that the I_n's are two-sided closed ideals of $R[\![U]\!]$). □

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