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A COMBINATORIAL ALGORITHM FOR COMPUTING HIGHER ORDER LINKING NUMBERS*

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Abstract. We develop the intersection theory at relative chain-cochain level, and apply it along with the use of Seifert disks for an oriented link to give a combinatorial algorithm to compute Massey's higher order linking numbers and therefore Milnor numbers.

Key words. Linking number, higher order linking number, cohomology operation, knots, links, Seifert disks, intersection product, cup product, Massey product.

AMS subject classifications. 57M25.

1. Introduction. It is well-known that there are many different methods of computing the linking number of an oriented 2-component link in S^3 (see for example [9, pp.132–135]), and perhaps the simplest is the combinatorial formula which counts the number of signed crossings of one component going under another component in a diagram of the link. It is much more subtle to compute higher-order linking numbers, and it has been a folklore to use the intersection theory in the process, which was first suggested by W. Massey. In [7] Massey introduced the higher-order linking as an application of his higher-order cohomology operations defined in terms of suitable cochains [6], and he calculated the third-order linking numbers for some 3-component links by shifting from cohomology and cup product to homology and intersection theory via duality theorems for manifolds. Later several works along this direction ([1],[10], [8], [3]) gave methods in various forms for computing the higher-order linking numbers, but a formal derivation of the formulae which are used for the computation has not yet been given, and a concrete algorithm for the computation is hence lacking. In this paper we will complete Massey's original approach by developing systematically the intersection theory at the relative chain-cochain level in the simplicial category, and use it to derive recursive combinatorial formulae for computing all higher-order linking numbers. The formulae are algorithmic in the sense that the computation of an n-th order linking number requires the construction of certain surfaces from the assumption that the $(n-1)$-st order linking numbers are zero. From a diagram of a link, these formulae give rise to a combinatorial algorithm for computing higher-order linkings by using Seifert disks of the link to facilitate the construction of the intersection of Seifert surfaces of the link, which is necessary for the inductive step.

The paper is organized into 5 sections. Section 2 gives preliminaries for the definition and results in section 3 on the intersection product at the relative chain-cochain level. In section 4 explicit formulas for the Massey higher-order linking numbers in terms of intersection product of relative chains are presented, and a geometric topology interpretation for the relative chains and their intersection product is given. A combinatorial algorithm for computing Massey third-order linking number is given in section 5, and to demonstrate its use we apply the algorithm on two examples.

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2. Preliminaries. We work in the piecewise linear category. Most material in this section is in the area of combinatorial topology and can be found in [5], [4] and [11]. Let M be a closed oriented n-manifold and N a closed submanifold of M, and let (K, L) be a fixed triangulation of (M, N) with a given ordering of the vertices (this means a partial ordering of the vertices of K such that the vertices of each simplex are totally ordered). Consider the barycentric subdivision (K', L') of (K, L), and for a p-simplex σ of K, denote by $D(\sigma)$ the dual $(n-p)$-cell of σ with orientation given by the transverse orientation of σ. Explicitly, $D(\sigma)$ is the subcomplex of K' whose vertices are barycenters of simplexes of K having σ as a face. The collection $K^* = \{D(\sigma)|\ \sigma \text{ is a simplex of } K\}$ forms a cell decomposition (or block decomposition) of K'. Note that $L^* = N(L')$ as sub-complexes of K', the latter being the regular neighborhood of L' in K'. Clearly L^* is a sub-cell-complex of K^*. Let $C_q(K^*)((C_q(L^*)$ respectively) be the free abelian group generated by the q-cells $D(\sigma)$ of K^* (L^*, respectively). The boundary operator $\partial : C_q(K^*) \to C_{q-1}(K^*)$ is defined by setting $\partial D(\sigma)$ to be the subcomplex of K' whose vertices are barycenters of simplexes of K having σ as a proper face. If σ is a p-simplex of K, then $D(\sigma)$ is an $(n-p)$-cell in K^*, and $\partial D(\sigma)$ is a union of $D(\tau)$ for some $(p+1)$-simplex τ of K, so $\partial D(\sigma) \,\epsilon\, C_{n-p-1}(K^*)$ is a sum of $(n-p-1)$-cells in K^*. If $a = \sum a_i\sigma_i \,\epsilon\, C_p(K)$, then let $D(a) = \sum a_i D(\sigma_i) \,\epsilon\, C_{n-p}(K^*)$. We have the chain complex $\{C_q(K^*), \partial\}$ and the sub-chain complex $\{C_q(L^*), \partial\}$, and the boundary operator ∂ induces the boundary operator on the relative chain complex $\overline{\partial} : C_q(K^*, L^*) \to C_{q-1}(K^*, L^*)$. Recall that the subdivision operator $sd : (K, L) \to (K', L')$ induces a chain map $sd_\# : C_p(K, L) \to C_p(K', L')$, and a simplicial map $\theta : (K', L') \to (K, L)$ which gives rise to a chain homotopy inverse $\theta_\# : C_p(K', L') \to C_p(K, L)$ of $sd_\#$ is defined as follows. For any vertex b of K', b is the barycenter of a unique simplex σ of K. Define $\theta(b)$ to be an arbitrary chosen vertex of σ, and extend θ piecewise linearly over K'. We shall use the convention that $\theta(b)$ is the last vertex in the ordering of vertices of σ. It is clear that $\theta_\# \circ sd_\# = id$ on $C_p(K, L)$, and dually we have the cochain maps on relative cochain complexes $sd^\# : C^p(K', L') \to C^p(K, L)$ and $\theta^\# : C^p(K, L) \to C^p(K', L')$, and we have $sd^\# \circ \theta^\# = id$ on $C^p(K, L)$.

Consider first the absolute case where $L = \phi$. For a p-simplex σ of K, let $u_\sigma \,\epsilon\, C^p(K)$ be the dual of σ satisfying

$$u_\sigma(\tau) = \begin{cases} 1 & \text{if} \quad \tau = \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

for any simplex τ of K. Then $B = \{u_\sigma \mid \sigma \text{ is a } p\text{-simplex of } K\}$ is a basis for $C^p(K)$. Let $\xi \,\epsilon\, Z_n(K)$ be the n-cycle representing the fundamental class of K. It is known [11] that $sd_\#(\xi) \cap \theta^\#(u_\sigma) = D(\sigma)$ for any simplex σ of K, here $D(\sigma)$ is considered as an element of $C_{n-p}(K')$, and the map $\phi : C^p(K) \to C_{n-p}(K^*)$ given by $\phi(u_\sigma) = D(\sigma)$ and extends linearly is an isomorphism satisfying $\phi(\delta\mu_\sigma) = (-1)^{p+1}\partial D(\sigma)$, from which follows the Poincare duality. In [11, p. 508] , the intersection product is defined to be the pairing : $C_p(K) \otimes C_q(K^*) \xrightarrow{\bullet} C_{p+q-n}(K')$ (with $p + q \geq n$) given by the composition

$$C_p(K) \otimes C_q(K^*) \xrightarrow{1\otimes\phi^{-1}} C_p(K) \otimes C^{n-q}(K) \xrightarrow{sd_\#\otimes 1} C_p(K') \otimes C^{n-q}(K)$$
$$\xrightarrow{1\otimes\theta^\#} C_p(K') \otimes C^{n-q}(K') \xrightarrow{\cap} C_{p+q-n}(K').$$

Remark. It follows from the definition that for any p-simplex σ and any $(n-q)$-simplex τ of K, $\sigma \cdot D(\tau) = sd_\#(\sigma) \cap \theta^\#(u_\tau)$, and by the topological definition of the cap product (see, for example [1], page 204), that $\sigma \cdot D(\tau)$ is an $(n-p)$-simplex in K' whose underlying set $|\sigma \cdot D(\tau)|$ satisfies $|\sigma \cdot D(\tau)| \subset |sd_{\#}(\sigma)|$, and $|\sigma \cdot D(\tau)| = |sd_{\#}(\sigma) \cap \theta^{\#}(u_\tau)| \subset |sd_{\#}(\xi) \cap \theta^{\#}(u_\tau)| = |D(\tau)|$, so $|\sigma \cdot D(\tau)|$ is contained in the intersection of $|sd_{\#}(\sigma)|$ and $|D(\tau)|$. Conversely, given a simplex η of K' such that $|\eta| \subset |sd_{\#}(\sigma)| \cap |D(\tau)|$, we have $|\eta| \subset |D(\tau)| = |sd_{\#}(\xi) \cap \theta^{\#}(u_\tau)| = |[sd_{\#}(\xi)\lambda_p, \theta^{\#}(u_\tau)](sd_{\#}(\xi)\rho_{n-p})|$, and since $|\eta| \subset |sd_{\#}(\sigma)|$, it follows that $|\eta| \subset |[sd_{\#}(\sigma)\lambda_p, \theta^{\#}(u_\tau)](sd_{\#}(\sigma)\rho_{n-p})| = |sd_{\#}(\sigma) \cap \theta^{\#}(u_\tau) = |\sigma \cdot D(\tau)|$. Thus $|\sigma \cdot D(\tau)|$ is equal to the intersection of $|sd_{\#}(\sigma)| = |\sigma|$ and $|D(\tau)|$, which justifies the term "intersection" for the pairing defined. An example for the case $n = 3, p = 2$, and $q = 2$ is given in the Appendix, which will be used later for the combinatorial algorithm in section 5 to "see" the sense of direction of the intersection curve of two Seifert disks.

It follows from the fact (see, e.g. [11]) that for $a \,\epsilon\, C_p(K)$ and $b \,\epsilon\, C_q(K^*)$,

$$\partial(a \cdot b) = (-1)^{n-q}(\partial a) \cdot b + (-1)^{n+1} a \cdot (\partial b),$$

there is an induced intersection product on the homology classes : For $[a] \,\epsilon\, H_p(K)$ and $[b] \,\epsilon\, H_q(K^*)$, $[a] \cdot [b] = [a \cdot b] \,\epsilon\, H_{p+q-n}(K')$.

We will use the following fact of the functional property of cap product (see, e.g. [2])

LEMMA 1. *For any map $f : X \to Y$, and any $a \,\epsilon\, C_{p+q}(X)$ and $b \,\epsilon\, C^p(Y)$,*

$$f_{\#}(a \cap f^{\#}(b)) = f_{\#}(a) \cap b.$$

The same formula holds for $f : (X, A) \to (Y, B)$, and any $a \,\epsilon\, C_{p+q}(X, A)$ and $b \,\epsilon\, C^p(Y, B)$.

3. Intersection product of relative chains. Given the simplicial complex pair (K, L), let $N(L)$ be a regular neighborhood of L in K. Every $g \,\epsilon\, C^p(K - \overset{\circ}{N}(L))$ extends to a unique $\widetilde{g} \,\epsilon\, C^p(K)$ satisfying $\widetilde{g}(\triangle) = 0$ for any p-simplex $\triangle$ whose support is contained in $N(L)$. So we may consider $C^p(K - \overset{\circ}{N}(L))$ as a sub-cochain complex of $C^p(K)$, of $C^p(K, N(L))$, and of $C^p(K, L)$. Considering $C^p(K - \overset{\circ}{N}(L))$ as a sub-cochain complex of $C^p(K)$, we have

THEOREM 2. *The map $\bar{\phi} = \phi\Big|_{C^p(K - \overset{\circ}{N}(L))} : C^p(K - \overset{\circ}{N}(L)) \to C_{n-p}(K^*, L^*)$ given by $\bar{\phi}(u_\sigma) = \overline{D(\sigma)}$ for any p-simplex σ of $K - \overset{\circ}{N}(L)$, is an isomorphism satisfying $\bar{\phi}(\delta u_\sigma) = (-1)^{p+1}\, \bar{\partial}\, \overline{D(\sigma)}$ $(= (-1)^{p+1}\, \overline{\partial D(\sigma)})$.*

Proof. Since $\bigcup_\sigma D(\sigma) = K^* - L^*$, where the union ranges over all p-simplexes σ of $K - \overset{\circ}{N}(L)$, the collection $\{D(\sigma)|\ \sigma$ is a p-simplex of $K - \overset{\circ}{N}(L)\}$ spans $C_{n-p}(K^* - L^*)$ freely. Now $D(\sigma) = \phi(u_\sigma)$ and $\phi : C^p(K) \to C_{n-p}(K^*)$ is an isomorphism, the restriction

$$\bar{\phi} = \phi\Big|_{C^p(K - \overset{\circ}{N}(L))} : C^p(K - \overset{\circ}{N}(L)) \to C_{n-p}(K^* - L^*)$$

is an isomorphism. Since

$$C_{n-p}(K^*) = C_{n-p}(K^* - L^*) \oplus C_{n-p}(L^*),$$

the result follows from the natural identification

$$C_{n-p}(K^* - L^*) \cong \frac{C_{n-p}(K^*)}{C_{n-p}(L^*)} = C_{n-p}(K^*, L^*),$$

in which $D(\sigma)$ in $C_{n-p}(K^* - L^*)$ is identified with $\overline{D(\sigma)}$ in $C_{n-p}(K^*, L^*)$. □

Remark. If $C_p(K - \overset{\circ}{N}(L))$ is considered as a sub-chain complex of $C_p(K, L)$, then for any p-simplex σ of $K - \overset{\circ}{N}(L)$, $\bar{\phi}(\mu_\sigma) = sd_\#(\xi\Big|_{K-\overset{\circ}{N}(L)}) \cap \theta^\#(u_\sigma)$, where ξ is the fundamental class of K.

Thus we have proved the following version of the Alexander duality:

Corollary 3. *The isomorphism $\overline{\phi}$ induces an isomorphism (still denoted $\overline{\phi}$)*

$$\overline{\phi} : H^p(K - \overset{\circ}{N}(L)) \to H_{n-p}(K^*, L^*).$$

The intersection product at the relative chain-cochain level is the pairing

$$C_p(K, L) \otimes C_q(K^*, L^*) \xrightarrow{\bullet} C_{p+q-n}(K', L')$$

(with $p + q \geq n$) given by the composition

$$\begin{aligned}
&C_p(K, L) \otimes C_q(K^*, L^*) \overset{1\otimes\phi^{-1}}{\longrightarrow} C_p(K, L) \otimes C^{n-q}(K - \overset{\circ}{N}(L)) \\
&\overset{sd_\#\otimes 1}{\longrightarrow} C_p(K', L') \otimes C^{n-q}(K - \overset{\circ}{N}(L)) \overset{1\otimes\theta^\#}{\longrightarrow} C_p(K', L') \otimes C^{n-q}((K - \overset{\circ}{N}(L))') \\
&\overset{1\otimes i}{\longrightarrow} C_p(K', L') \otimes C^{n-q}(K', L') \overset{\cap}{\longrightarrow} C_{p+q-n}(K', L'),
\end{aligned}$$

where $i : C^{n-q}((K - \overset{\circ}{N}(L))') \longrightarrow C^{n-q}(K', L')$ is the naturally induced mapping.

Remark. Similar to the absolute case, for simplexes σ and τ of K, the underlying set $|\overline{\sigma} \cdot \overline{D(\tau)}|$ is the intersection of $|sd_\#(\sigma)| = |\sigma|$ and $|D(\tau)|$ mod $|L'| = |L|$, and in general, for any $\bar{a} \,\epsilon\, C_p(K, L)$ and any $\bar{b} \,\epsilon\, C_q(K^*, L^*)$, $|\bar{a} \cdot \bar{b}|$ is the intersection of $|sd_\#(a)|$ and $|D(b)|$ mod $|L'|$. Furthermore, $|\theta_\#(\overline{\sigma} \cdot \theta_\# D(\tau))|$ is the intersection of $|\sigma|$ and $|\theta_\#(D(\tau))|$ mod $|L|$.

Correspondingly there is an induced intersection product on relative homology classes : For $[a] \,\epsilon\, H_p(K, L)$ and $[b] \,\epsilon\, H_q(K^*, L^*)$, $[a] \cdot [b] = [a \cdot b] \,\epsilon\, H_{p+q-n}(K', L')$.

Recall that for a topological space X, the epimorphism $\partial_\# : C_0(X) \to Z$ defined by $\partial_\#(\sum m_i p_i) = \sum m_i$, where p_i are points in X, satisfies the property $\partial_\#(c \cap \eta) = [c, \eta]$ $(= \eta(c))$ for any $c \,\epsilon\, C_p(X)$ and $\eta \,\epsilon\, C^p(X)$. If X is path connected, then $\partial_\#(B_0(X)) = 0$. One can similarly define $\partial_\# : C_0(X, A) \to Z$, which satisfies $\partial_\#(\bar{c} \cap \bar{\eta}) = [\bar{c}, \bar{\eta}]$ for any $\bar{c} \,\epsilon\, C_p(X, A)$ and $\bar{\eta} \,\epsilon\, C^p(X, A)$.

It follows from the definition that for any p-simplex σ of K and $(n - q)$-simplex τ of $K - \overset{\circ}{N}(L)$, $\overline{\sigma} \cdot \overline{D(\tau)} = sd_\#(\sigma) \cap \theta^\#(u_\tau)$. In general, for any $\overline{a} \,\epsilon\, C_p(K, L)$ and $v \,\epsilon\, C^{n-q}(K - \overset{\circ}{N}(L))$, $\overline{a} \cdot \overline{\phi}(v) = sd_\#(a) \cap \theta^\#(v)$. In particular, if $p + q = n$, then $\partial_\#(\overline{a} \cdot \overline{\phi}(v)) = [sd_\#(a), \theta^\#(v)]$, here $\theta^\#(v) \,\epsilon\, C^p((K - \overset{\circ}{N}(L))')$ is identified with $i(\theta^\#(v))$ in $C^p(K', L')$, where $i : C^p((K - \overset{\circ}{N}(L))') \to C^p(K', L')$ is the naturally induced map.

PROPOSITION 4. *For* $\overline{a} \,\epsilon\, C_n(K, L)$ *and* $\beta \,\epsilon\, C^p(K)$, $\overline{a} \cap \beta = \theta_{\#}(sd_{\#}(\overline{a}) \cap \theta^{\#}(\beta))$.

Proof. By Lemma 1, we have

$$\begin{aligned} \theta_{\#}(sd_{\#}(\overline{a}) \cap \theta^{\#}(\beta)) &= \theta_{\#}(sd_{\#}(\overline{a})) \cap \beta \\ &= \overline{a} \cap \beta, \end{aligned}$$

since $\theta_{\#} \circ sd_{\#} = id$. □

PROPOSITION 5. *Let* $\alpha \,\epsilon\, C^p(K)$, $\beta \,\epsilon\, C^q(K)$, *and* $T \,\epsilon\, C_{p+q}(K)$. *Then* $(\alpha \cup \beta)(T) = \partial_{\#}(\theta_{\#}(\theta_{\#}(T \cdot \phi(\alpha)) \cdot \phi(\beta)))$.

Proof. The result is obtained from the following direct computation using Proposition 3.0.4 and the definition of intersection product.

$$\begin{aligned} (\alpha \cup \beta)(T) &= \partial_{\#}(T \cap (\alpha \cup \beta)) = \partial_{\#}((T \cap \alpha) \cap \beta) \\ &= \partial_{\#}(\theta_{\#}(sd_{\#}(T \cap \alpha) \cap \theta^{\#}(\beta)) \\ &= \partial_{\#}(\theta_{\#}((T \cap \alpha) \cdot (\phi(\beta))) \\ &= \partial_{\#}(\theta_{\#}(\theta_{\#}(sd_{\#}(T) \cap \theta^{\#}(\alpha)) \cdot \phi(\beta))) \\ &= \partial_{\#}(\theta_{\#}(\theta_{\#}(T \cdot \phi(\alpha)) \cdot \phi(\beta))). \end{aligned}$$

□

Note that the above result extends to the case where $\alpha \,\epsilon\, C^p(K - \overset{\circ}{N}(L))$, $\beta \,\epsilon\, C^q(K - \overset{\circ}{N}(L))$, and $T \,\epsilon\, C_{p+q}(K - \overset{\circ}{N}(L))$, which is the case we will consider later. In this case, for the right hand side T may be considered as in $C_{p+q}(K, L)$, and α as in $C^p(K)$, so $T \cap \alpha \,\epsilon\, C_q(K, L)$ and the intersection product $(T \cap \alpha) \cdot \overline{\phi}(\beta)$ is defined.

The deformation retract from $(|K'|, |N(L')|)$ to $(|K'|, |L'|)$ of the underlying spaces is homotopically equivalent to a simplicial map $r : (|K'|, |N(L')|) \to (|K'|, |L'|)$ via simplicial approximation of the deformation retract, and it induces an isomorphism $r_* : H_q(K', N(L')) \to H_q(K', L')$ as simplicial homology groups ([4, Theorem 3.6.6, p.120]), which is the inverse of the inclusion-induced isomorphism $j_* : H_q(K', L') \to H_q(K', N(L'))$.

LEMMA 6. *The inclusion map* $i : (K^*, L^*) \to (K', N(L'))$ *induces an isomorphism* $i_* : H_q(K^*, L^*) \to H_q(K', N(L'))$.

Proof. By [4, Theorem 3.8.8, p. 131], the inclusion map $i : K^* \to K'$ induces an isomorphism $i_* : H_q(K^*) \to H_q(K')$. Now since L^* is a block dissection of $N(L')$, the inclusion map $i : L^* \to N(L')$ induces an isomorphism $i_* : H_q(L^*) \to H_q(N(L'))$. The result follows from the long exact sequences of $H_*(K^*, L^*)$ and $H_*(K', N(L'))$, and by applying the Five Lemma:

$$\begin{array}{ccccccccc} H_q(L^*) & \longrightarrow & H_q(K^*) & \longrightarrow & H_q(K^*, L^*) & \longrightarrow & H_{q-1}(L^*) & \longrightarrow & H_{q-1}(K^*) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ H_q(N(L')) & \longrightarrow & H_q(K') & \longrightarrow & H_q(K', N(L')) & \longrightarrow & H_{q-1}(N(L')) & \longrightarrow & H_{q-1}(K') \end{array}$$

□

We now restrict to the case where $p = 1$, $q = 2$, and $n = 3$, and consider the following diagram

$$\begin{array}{ccc}
H^1(K-\mathring{N}(L))\otimes H^1(K-\mathring{N}(L)) & \stackrel{\cup}{\longrightarrow} & H^2(K-\mathring{N}(L)) \\
\Big\downarrow \overline{\phi}\otimes\overline{\phi} & & \Big\downarrow \overline{\phi} \\
H_2(K^*,L^*)\otimes H_2(K^*,L^*) & & H_1(K^*,L^*) \\
\Big\downarrow (\theta_*\circ r_*\circ i_*)\otimes id & & \Big\downarrow r_*\circ i_* \\
H_2(K,L)\otimes H_2(K^*,L^*) & \stackrel{\bullet}{\longrightarrow} & H_1(K',L')
\end{array}$$

Proposition 7. *The above diagram commutes. i.e. for any* $[\alpha]$, $[\beta]\ \epsilon\ H^1(K-\mathring{N}(L))$, $(r_*\circ i_*)([\ \overline{\phi}(\alpha\cup\beta)\ ]) = ((\theta_*\circ r_*\circ i_*)([\ \overline{\phi}(\alpha)\ ]))\cdot[\overline{\phi}(\beta)]$

Proof. For any $[\alpha]$, $[\beta]$ in $H^1(K-\mathring{N}(L))$,

$$\begin{aligned}
((r_*\ \circ\ i_*)\circ\overline{\phi})([\alpha]\cup[\beta]) &= (r_*\circ i_*)(\overline{\phi}([\alpha\cup\beta])) \\
&= (r_*\circ i_*)([\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}]) \\
&= [\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}],
\end{aligned}$$

Considering α and β as in $Z^1(K,L)$, then $\theta^\#(\alpha\cup\beta)\in Z^1(K',L')$, and we may consider $[\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}]$ as in $H_1(K',L')$. Then $i_*([\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}]) = j_*([\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}])$, or equivalently $(r_*\ \circ\ i_*)([\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}]) = [\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}]$, since $j_* = r_*^{-1}$.

Now as an element in $H_1(K',L')$, $[\overline{sd_\#(\xi)\cap\theta^\#(\alpha\cup\beta)}] = [\overline{sd_\#(\xi)\cap(\theta^\#(\alpha)\cup\theta^\#(\beta))}] = [\overline{(sd_\#(\xi)\cap\theta^\#(\alpha))\cap\theta_\#(\beta)}]$.

Using the property $sd^\#\circ\theta^\# = id$ and by lemma 2.0.1 we obtain

$$sd_\#(\xi)\cap\theta^\#(\alpha) = sd_\#(\xi\cap(sd^\#\circ\theta^\#)(\alpha)) = sd_\#(\xi\cap\alpha),$$

so we have

$$[\overline{(sd_\#(\xi)\cap\theta^\#(\alpha))\cap\theta_\#(\beta)}] = [\overline{sd_\#(\xi\cap\alpha)\cap\theta^\#(\beta)}] = [\overline{(\xi\cap\alpha)\cdot(sd_\#(\xi)\cap\theta^\#(\beta))}]$$

It follows from Lemma 2.0.1 and the fact that $\theta_\#\circ sd_\# = id$ that

$$\begin{aligned}
[\overline{(\xi\cap\alpha)\cdot(sd_\#(\xi)\cap\theta^\#(\beta))}] &= [\overline{(\theta_\#(sd_\#(\xi)\cap\theta^\#(\alpha))\cdot(sd_\#(\xi)\cap\theta^\#(\beta))}] \\
&= \theta_*([\overline{sd_\#(\xi)\cap\theta^\#(\alpha)}])\cdot[\overline{sd_\#(\xi)\cap\theta^\#(\beta)}]
\end{aligned}$$

Note that here $\overline{sd_\#(\xi)\cap\theta^\#(\alpha)}$ is considered as in $C_2(K',L')$. The last homology class is equal to

$$\begin{aligned}
&((\theta_*\circ r_*\circ i_*)([\overline{sd_\#(\xi)\cap\theta^\#(\alpha)}]))\cdot[\overline{sd_\#(\xi)\cap\theta^\#(\beta)}] \\
&= ((\theta_*\circ r_*\circ i_*)[\overline{\phi}(\alpha)])\cdot[\overline{\phi}(\beta)] \\
&= (\theta_*\circ r_*\circ i_*\otimes id)([\bar{\phi}(\alpha)],[\bar{\phi}(\beta)]).
\end{aligned}$$

□

4. Linkings in S^3. Let $K = S^3$ with the usual orientation and a fixed triangulation, $L = \coprod K_i$ an oriented link in S^3, $T_i = \partial N(K_i)$, and $\mu_i \in Z_1(S^3 - \mathring{N}(L))$ be the 1-cycle representing the meridian of K_i (with $lk(K_i, \mu_i) = 1$). For each i, let $[\mu_i^*] \in H^1(S^3 - \mathring{N}(L)) = Hom(H_1(S^3 - \mathring{N}(L), Z)$ be the dual of $[\mu_i] \in H_1(S^3 - \mathring{N}(L))$ and let F_i be a spanning surface of K_i in S^3. It is well known that the isomorphism

$$H^1(S^3 - \mathring{N}(L)) \to H_2(S^3, L)$$

given by the cap product with the fundamental class $[S^3 - \mathring{N}(L)]$, takes $[\mu_i^*]$ to $[F_i]$, that is, $[S^3 - \mathring{N}(L)] \cap [\mu_i^*] = [F_i]$. On the other hand, the Alexander duality in Corollary 3.0.3

$$\overline{\phi} : H^1(S^3 - \mathring{N}(L)) \to H_2((S^3)^*, L^*),$$

gives

$$\begin{aligned}\overline{\phi}([\mu_i^*]) &= [sd_{\#}(S^3 - \mathring{N}(L)) \cap \theta^{\#}(\mu_i^*)] \\ &= sd_*([S^3 - \mathring{N}(L)]) \cap \theta^*([\mu_i^*]) \\ &= \theta_*^{-1}([S^3 - \mathring{N}(L)]) \cap \theta^*([\mu_i^*]).\end{aligned}$$

By Lemma 1 again, the last expression is equal to $\theta_*^{-1}([S^3 - \mathring{N}(L)] \cap {\theta^*}^{-1}(\theta^*([\mu_i^*]))$, which is equal to ${\theta_*}^{-1}([S^3 - \mathring{N}(L)] \cap [\mu_i^*]) = \theta_*^{-1}([F_i])$, and is equal to $i_*^{-1} \circ j_* \circ \theta_*^{-1}([F_i])$ as an element in $H_2((S^3)^*, L^*)$. This shows that $[F_i] = (\theta_* \circ r_* \circ i_*)(\overline{\phi}([\mu_i^*]))$. It follows that $(\theta_{\#} \circ r_{\#} \circ i_{\#})(\overline{\phi}([\mu_i^*])) = F_i + b_i$ for some $b_i \in B_2(S^3, L)$. So the underlying set $|F_i| = |(\theta_{\#} \circ r_{\#} \circ i_{\#})(\overline{\phi}([\mu_i^*]))| \cup |b_i|$.

As an example we apply the intersection theory developed in the previous section to obtain a formula for the ordinary (second-order) linking number of a link with two components.

Example. Let $L = K_1 \cup K_2$ be an oriented link of two components in the given specific order with meridians μ_1, μ_2 respectively. Then $lk(K_1, K_2) = ([\mu_1^*] \cup [\mu_2^*])([T_1])(= ([\mu_2^*] \cup [\mu_1^*])([T_2]) = -([\mu_2^*] \cup [\mu_1^*])([T_1]))$. By Proposition 3.0.5, we have

$$(\mu_1^* \cup \mu_2^*)(T_1) = \partial_\sharp(\theta_\sharp(\theta_\sharp(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot \overline{\phi}(\mu_2^*)).$$

We first note that since $|T_1| \subset S^3 - \mathring{N}(L)$, $|\partial_\sharp(\theta_\sharp(\theta_\sharp(T_1 \cdot \overline{\phi}(\mu_1^*))))| = |T_1| \cap |(\theta_{\#} \circ r_{\#} \circ i_{\#})(\overline{\phi}([\mu_i^*]))| \subset |T_1|$. It follows that

$$\begin{aligned}|\theta_\sharp(\theta_\sharp(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot \overline{\phi}(\mu_2^*))| &= |\theta_\sharp(T_1 \cdot \overline{\phi}(\mu_1^*))| \cap |(\theta_{\#} \circ r_{\#} \circ \circ i_{\#})(\overline{\phi}(\mu_2^*))| \\ &= |T_1| \cap |F_1 + b_1| \cap |F_2 + b_2| \\ &= |T_1| \cap (|F_1| \cup |b_1|) \cap (|F_2| \cup |b_2|) \\ &= (|T_1| \cap |F_1| \cap |F_2|) \cup (|T_1| \cap |F_1| \cap |b_2|) \\ &\quad \cup (|T_1| \cap |b_1| \cap |F_2|) \cup (|T_1| \cap |b_1| \cap |b_2|).\end{aligned}$$

Now $b_i = \partial\beta_i + \widetilde{b_i}$, $\beta_i \in C_3(S^3)$ and $\widetilde{b_i} \in C_2(L)$ for $i = 1, 2$, the second part in the above union can be written as $|T_1| \cap |F_1| \cap |b_2| = (|T_1| \cap |F_1| \cap |\partial\beta_2|) \cup (|T_1| \cap |F_1| \cap |\widetilde{b_2}|$.

Since $|T_1| \cap |F_1|$ is a circle in T_1 parallel to K_1, and $|\partial\beta_2|$ is homeomorphic to a closed surface in S^3, the algebraic intersection number of $|T_1| \cap |F_1|$ with $|\partial\beta_2|$ is 0. Clearly $|T_1| \cap |F_1| \cap |\widetilde{b_2}| = \phi$(the empty set). So the algebraic intersection number due to the second part in the union is 0. Similar arguments apply to show that the algebraic intersection number of the third and the fourth parts in the union are both 0. Thus, using # for the algebraic intersection number, we have

$$\begin{aligned}(\mu_1^* \cup \mu_2^*)(T_1) &= \partial_\sharp(\theta_\sharp(\theta_\sharp(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot \overline{\phi}(\mu_2^*)) \\ &= \#(|T_1| \cap |F_1| \cap |F_2|) \\ &= \#(|K_1| \cap |F_2|) \\ &= lk(K_1, K_2).\end{aligned}$$

Remark. One can also show that

$$(\mu_2^* \cup \mu_1^*)(T_2) = \partial_\sharp(\theta_\sharp(\theta_\sharp(T_2 \cdot F_2) \cdot F_1)) = lk(K_1, K_2).$$

Now we consider the third-order linking. Let $L = K_1 \cup K_2 \cup K_3$ be an oriented link of three components in the given specific order with meridians μ_1, μ_2, μ_3 respectively. Then

$$H^1(S^3 - \overset{\circ}{N}(L)) =< [\mu_1^*], [\mu_2^*], [\mu_3^*] >\cong Z^3.$$

Assume $lk(K_1, K_2) = lk(K_2, K_3) = lk(K_1, K_3) = 0$. Now since $H_2(S^3 - \overset{\circ}{N}(K_1 \cup K_2)) =< [T_1] > (=< [T_2] >) \cong Z$, and $([\mu_1^*] \cup [\mu_2^*])([T_1]) = (\mu_1^* \cup \mu_2^*)(T_1) = lk(K_1, K_2) = 0$, which implies that $[\mu_1^* \cup \mu_2^*] = 0$ in $H^2(S^3 - \overset{\circ}{N}(K_1 \cup K_2))$, which in turn implies that $[\mu_1^* \cup \mu_2^*] = 0$ in $H^2(S^3 - \overset{\circ}{N}(L))$. Thus we have $\mu_1^* \cup \mu_2^* = \delta\widetilde{C}_{12}$ for some $\widetilde{C}_{12} \,\epsilon\, C^1(S^3 - \overset{\circ}{N}(L))$. Similarly $\mu_2^* \cup \mu_3^* = \delta\widetilde{C}_{23}$ for some $\widetilde{C}_{23} \,\epsilon\, C^1(S^3 - \overset{\circ}{N}(L))$. Then the Massey third-order product of L, in the given order of components K_1, K_2, and K_3, is uniquely defined and is given by

$$< [\mu_1^*], [\mu_2^*], [\mu_3^*] >= [\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^*] \;\in\; H^2(S^3 - \overset{\circ}{N}(L)),$$

and by definition, Massey's third-order linking number of L, with its components in this order, is

$$< [\mu_1^*], [\mu_2^*], [\mu_3^*] > ([T_1]) = [\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^*]([T_1]).$$

Theorem 8. *Massey's third order linking number is given by*

$$\begin{aligned}&[\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^*]([T_1]) \\ &= \partial_\#(\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot C_{23}^*)) + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{12}^*) \cdot \overline{\phi}(\mu_3^*))) \\ &= \#\Big(|T_1| \cap |F_1| \cap |C_{23}|\Big) + \#\Big(|T_1| \cap |C_{12}| \cap |F_3|\Big),\end{aligned}$$

where $C_{12}^* = \overline{\phi}(\widetilde{C}_{12})$, $C_{23}^* = \overline{\phi}(\widetilde{C}_{23})$, *and* $C_{12} = (\theta \circ r \circ i)_\#(C_{12}^*)$, $C_{23} = (\theta \circ r \circ i)_\#(C_{23}^*)$.

Proof. By Proposition 3.0.5 and its remark, we obtain

$$\begin{aligned}
&[\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^*]([T_1]) \\
&= (\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^*)(T_1) \\
&= (\mu_1^* \cup \widetilde{C}_{23})(T_1) + (\widetilde{C}_{12} \cup \mu_3^*)(T_1) \\
&= \partial_\#(\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot \overline{\phi}(\widetilde{C}_{23}))) + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\widetilde{C}_{12}))) \cdot \overline{\phi}(\mu_3^*)) \\
&= \partial_\#(\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot C_{23}^*)) + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{12}^*) \cdot \overline{\phi}(\mu_3^*))).
\end{aligned}$$

Now the underlying set

$$\begin{aligned}
|\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot C_{23}^*)| &= |\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*))| \cap |(\theta_\# \circ r_\# \circ i_\#)(C_{23}^*)| \\
&= |T_1| \cap (|F_1| \cup |b_1|) \cap |C_{23}| \\
&= \Big(|T_1| \cap |F_1| \cap |C_{23}|\Big) \cup \Big(|T_1| \cap |b_1| \cap |C_{23}|\Big),
\end{aligned}$$

and similarly $|\theta_\#(\theta_\#(T_1 \cdot C_{23}^*) \cdot \overline{\phi}(\mu_1^*))| = \Big(|T_1| \cap |C_{12}| \cap (|F_3| \cup |b_3|)\Big)$, where b_1 and b_3 are as in the above Example. since $b_1 = \partial\beta_1 + \widetilde{b}_1$, where $\beta_1 \in C_3(S^3)$ and $\widetilde{b}_1 \in C_2(L)$, $|T_1| \cap |b_1| \cap |C_{23}| = (|T_1| \cap |\partial\beta_1| \cap |C_{23}|) \cup (|T_1| \cap |\widetilde{b}_1| \cap |C_{23}|)$. Now $|T_1| \cap |\partial\beta_1|$ consists of disjoint curves that bound in T_1, and $C_{23} \in C_2(S^3, L)$, it follows that $\#(|T_1| \cap |\partial\beta_1| \cap |C_{23}|) = 0$. Also $|T_1| \cap |\widetilde{b}_1| \cap |C_{23}| = \phi$(the empty set), since $|T_1| \cap |\widetilde{b}_1| = \phi$. Thus we have proved $\partial_\#(\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot C_{23}^*)) = \#\Big(|T_1| \cap |F_1| \cap |C_{23}|\Big)$. A similar arguments can be applied to show that $\partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{12}^*) \cdot \overline{\phi}(\mu_3^*))) = \#\Big(|T_1| \cap |C_{12}| \cap |F_3|\Big)$. □

Remark. It is clear that the third order linking number is independent of the choice of $b_i \in B_2(S^3, L)$ with which $(\theta \circ r \circ i)_\#(\overline{\phi}(\mu_i^*)) = F_i + b_i$. The same is true for the choices of $\widetilde{C}_{12}$ and $\widetilde{C}_{23}$ in $C^1(S^3 - \mathring{N}(L))$, which can be verified as follows. Assume $\widetilde{C}'_{12}$ and $\widetilde{C}'_{23}$ are another choices with which $\mu_1^* \cup \mu_2^* = \delta\widetilde{C}'_{12}$ and $\mu_2^* \cup \mu_3^* = \delta\widetilde{C}'_{23}$. For the first case we show that if $C'^*_{12} = \overline{\phi}(\widetilde{C}'_{12})$, then

$$\partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{12}^*) \cdot \overline{\phi}(\mu_3^*))) = \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C'^*_{12}) \cdot \overline{\phi}(\mu_3^*))).$$

Now $\delta(\widetilde{C}_{12} - \widetilde{C}'_{12}) = 0$ in $C^2(S^3 - \mathring{N}(L))$ implies $\partial(C_{12}^* - C'^*_{12}) = 0$ in $C_1((S^3)^*, L^*)$. Thus

$$\begin{aligned}
\partial(\theta_\#(T_1 \cdot (C_{12}^* - C'^*_{12}))) &= \theta_\#(\partial(T_1 \cdot (C_{12}^* - C'^*_{12})) \\
&= \theta_\#\big(-\partial T_1 \cdot (C_{12}^* - C'^*_{12}) + T_1 \cdot (\partial(C_{12}^* - C'^*_{12}))\big) \\
&= 0.
\end{aligned}$$

This shows that $\theta_\#(T_1 \cdot (C_{12}^*)$ and $\theta_\#(T_1 \cdot (C'^*_{12})$ are each a collection of the same arcs in $C_1(S^3, L)$, i.e. same arcs in S^3 rel L. But $T_1 = \partial N(K_1)$, so they are actually a collection of the same arcs in S^3. We then have

$$\begin{aligned}
\partial_\#\big(\theta_\#(\theta_\#(T_1 \cdot (C_{12}^* - C'^*_{12})) \cdot \overline{\phi}(\mu_3^*))\big) &= \theta_\#\big(\partial(\theta_\#(T_1 \cdot (C_{12}^* - C'^*_{12}))) \cdot \overline{\phi}(\mu_3^*)\big) \\
&= 0.
\end{aligned}$$

Similarly we can show $\partial_\#(\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot C_{23}^*) = \partial_\#(\theta_\#(\theta_\#(T_1 \cdot \overline{\phi}(\mu_1^*)) \cdot C_{23}'^*)$.

The geometric topology meaning of C_{12} and C_{23} are as follows. By Proposition 3.0.7 we have, on one hand,

$$(r_* \circ i_*)([\overline{\phi}(\mu_1^* \cup \mu_2^*)] = \theta_*(r_*(i_*([F_1]))) \cdot [F_2] = [\theta_\#(r_\#(i_\#(F_1))) \cdot F_2].$$

On the other hand,

$$\begin{aligned}(r_* \circ i_*)([\overline{\phi}(\mu_1^* \cup \mu_2)] &= (r_* \circ i_*)([\overline{\phi}(\delta \widetilde{C}_{12})]) \\ &= (r_* \circ i_*)([\partial \overline{\phi}(\widetilde{C}_{12})]) \\ &= (r_* \circ i_*)([\partial C_{12}^*]) \\ &= [\partial((r \circ i)_\#(C_{12}^*))].\end{aligned}$$

Thus we have $\partial((r \circ i)_\#(C_{12}^*)) = \theta_\#(r_\#(i_\#(F_1))) \cdot F_2 + b_{12}'$, where $b_{12}' \,\epsilon\, B_1((S^3)', L')$. Applying $\theta_\#$ on both sides we obtain $\partial((\theta \circ r \circ i)_\#(C_{12}^*)) = \theta_\#((\theta_\#(r_\#(i_\#(F_1))) \cdot F_2 + b_{12}$, where $b_{12} \,\epsilon\, B_1(S^3, L)$. Whence $b_{12} = \partial\beta_{12} + \widetilde{b}_{12}$, with $\beta_{12} \,\epsilon\, C_2(S^3)$ and $\widetilde{b}_{12} \,\epsilon\, C_1(L)$. Now since $(\theta \circ r \circ i)_\#$ is the "retraction" from $((S^3)^*, L^*)$ to (S^3, L), C_{12} can be viewed as a 2-complex in (S^3, L) whose boundary, after adding to the boundary of some 2-complex in (S^3, L), is the intersection of F_1(which is identified with $(\theta \circ r \circ i)_\#(F_1)$) and F_2, rel L. One such candidate for C_{12} is any spanning surface for the new link components formed by the arcs and circles of $F_1 \cdot F_2$ and portions of components of L. Note that if C_{12}' is another choice of such spanning surface, then $\partial C_{12}' = \partial C_{12}$, so $C_{12}' - C_{12} = \bar{\partial}\bar{b}$ for some $\bar{b} \,\epsilon\, C_3((S^3)^*, L^*)$. By letting $\widetilde{C}_{12}' = \bar{\phi}^{-1}(C_{12}')$ and $v = \bar{\phi}^{-1}(\bar{b}) \,\epsilon\, C^0(S^3 - \overset{\circ}{N}(L))$, then $\widetilde{C}_{12}' - \widetilde{C}_{12} = \bar{\phi}^{-1}(C_{12}' - C_{12}) = \bar{\phi}^{-1}(\bar{\partial}\bar{b})) = (\bar{\phi}^{-1}\bar{\partial})(\bar{\phi}(v)) = \delta v$. Thus the third-order linking is independent of the choice of C_{12}. Similar choice can be made for C_{23}, which is any spanning surface for the new link components formed by arcs and circles of the intersections of F_2 and F_3 rel L. Therefore C_{12} and C_{23} are constructed from the Seifert surfaces F_1, F_2, F_3, and their boundaries L, and in general they are each a collection of surfaces.

REMARK. The first term in Theorem 4.0.8 can be interpreted as $lk(K_1, \partial C_{23})$.

Next we consider the fourth-order linking. Let $L = K_1 \cup K_2 \cup K_3 \cup K_4$ be an oriented link of four components in the given specific order with meridians $\mu_1, \mu_2, \mu_3, \mu_4$ respectively. Assume all the second-order and all third-order Massey products vanish, i.e. $\mu_i^* \cup \mu_j^* = 0$ and $< \mu_i^*, \mu_j^*, \mu_k^* >= 0$ for all permutations (i, j) and (i, j, k) of $\{1, 2, 3, 4\}$. Then in particular we have $\mu_1^* \cup \mu_2^* = \delta\widetilde{C}_{12}$, $\mu_2^* \cup \mu_3^* = \delta\widetilde{C}_{23}$, and $\mu_3^* \cup \mu_4^* = \delta\widetilde{C}_{34}$ for some $\widetilde{C}_{12}, \widetilde{C}_{23}$, and $\widetilde{C}_{34} \,\epsilon\, C^1(S^3 - \overset{\circ}{N}(L))$; also $< \mu_1^*, \mu_2^*, \mu_3^* >= 0$ implies $\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^* = \delta\widetilde{C}_{123}$ and $< \mu_2^*, \mu_3^*, \mu_4^* >= 0$ implies $\mu_2^* \cup \widetilde{C}_{34} + \widetilde{C}_{23} \cup \mu_4^* = \delta\widetilde{C}_{234}$ for some $\widetilde{C}_{123}$ and $\widetilde{C}_{234} \,\epsilon\, C^1(S^3 - \overset{\circ}{N}(L))$. Then the Massey fourth-order product of L, in this specific order of components K_1, K_2, K_3 and K_4, is uniquely defined and is given by

$$< [\mu_1^*], [\mu_2^*], [\mu_3^*], [\mu_4^*] >= [\mu_1^* \cup \widetilde{C}_{234} + \widetilde{C}_{12} \cup \widetilde{C}_{34} + \widetilde{C}_{123} \cup \mu_4^*] \,\epsilon\, H^1(S^3 - \overset{\circ}{N}(L)),$$

and by definition, Massey's fourth-order linking number of L, with its components in this order, is

$$< [\mu_1^*], [\mu_2^*], [\mu_3^*], [\mu_4^*] > ([T_1]) = [\mu_1^* \cup \widetilde{C}_{234} + \widetilde{C}_{12} \cup \widetilde{C}_{34} + \widetilde{C}_{123} \cup \mu_4^*]([T_1]).$$

By a similar discussion as in the case of third-order linking, we obtain

Theorem 9. *Massey's fourth-order linking number is given by*

$$\begin{aligned}
&[\mu_1^* \cup \widetilde{C}_{234} + \widetilde{C}_{12} \cup \widetilde{C}_{34} + \widetilde{C}_{123} \cup \mu_4^*]([T_1]) \\
&= \partial_\#(\theta_\#(\theta_\#(T_1 \cdot \bar{\phi}(\mu_1^*)) \cdot C_{234}^*) + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{12}^*) \cdot C_{34}^*)) \\
&\quad + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{123}^*) \cdot \bar{\phi}(\mu_4^*))) \\
&= \#\Big(|T_1| \cap |F_1| \cap |C_{234}|\Big) + \#\Big(|T_1| \cap |C_{12}| \cap |C_{34}|\Big) \\
&\quad + \#\Big(|T_1| \cap |C_{123}| \cap |F_4|\Big),
\end{aligned}$$

where $C_{12}^* = \overline{\phi}(\widetilde{C}_{12}), C_{34}^* = \overline{\phi}(\widetilde{C}_{34}), C_{123}^* = \overline{\phi}(\widetilde{C}_{123}), C_{234}^* = \overline{\phi}(\widetilde{C}_{234})$, *and* $C_{12} = (\theta \circ r \circ i)_\#(C_{12}^*), C_{34} = (\theta \circ r \circ i)_\#(C_{34}^*), C_{123} = (\theta \circ r \circ i)_\#(C_{123}^*), C_{234} = (\theta \circ r \circ i)_\#(C_{234}^*)$.

Geometric interpretations of C_{123} and C_{234} can be made similar to that of C_{12} and C_{23} in the case of third-order linking. For C_{123}, we have

$$\begin{aligned}
&(r_* \circ i_*)([\overline{\phi}(\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^*)] \\
&= \theta_*(i_*([\bar{\phi}(\mu_1^*)])) \cdot [\bar{\phi}(\widetilde{C}_{23})] + \theta_*(i_*([\bar{\phi}(\widetilde{C}_{12})]) \cdot [\bar{\phi}(\mu_3^*)])] \\
&= [\theta_\#(i_\#(F_1)) \cdot C_{23}^* + \theta_\#(i_\#(C_{12}^*)) \cdot F_3],
\end{aligned}$$

and the left hand side is equal to

$$\begin{aligned}
(r_* \circ i_*)([\bar{\phi}(\delta\widetilde{C}_{123})] &= (r_* \circ i_*)([\partial\bar{\phi}(\widetilde{C}_{123})] \\
&= (r_* \circ i_*)([\partial C_{123}^*]) \\
&= [\partial(r \circ i)_\#(C_{123}^*)].
\end{aligned}$$

Thus we have $\partial(r \circ i)_\#(C_{123}^*) = \theta_\#(i_\#(F_1)) \cdot C_{23}^* + \theta_\#(i_\#(C_{12}^*)) \cdot F_3 + b'_{123}$, where $b'_{123} \epsilon B_1(S^{3'}, L')$. Applying $\theta_\#$ on both sides we obtain $\partial((\theta \circ r \circ i)_\#(C_{123}^*)) = \theta_\#((\theta_\#(r_\#(i_\#(F_1))) \cdot C_{23}^* + \theta_\#(\theta_\#(i_\#(C_{12}^*)) \cdot F_3) + b_{123}$, where $b_{123} \epsilon B_1(S^3, L)$. Since $b_{123} = \partial\beta_{123} + \widetilde{b}_{123}$, with $\beta_{123} \epsilon C_2((S^3)')$ and $\widetilde{b}_{123} \epsilon C_2(L')$, and since $(r \circ i)_\#$ is the "retraction" from $((S^3)^*, L^*)$ to (S^3, L), $C_{123} = (\theta \circ r \circ i)_\#(C_{123}^*)$ can be viewed as a 2-complex in (S^3, L) whose boundary, after attaching to the boundary of some 2-complex in (S^3, L), is the sum of the intersection of F_1(which is identified with $(\theta \circ r \circ i)_\#(F_1)$) and C_{23} and the intersection of C_{12} and F_3, rel L. One such candidate for C_{123} can be taken to be any spanning surface for the new link components formed by the arcs and circles of $F_1 \cdot C_{23}$ and $C_{12} \cdot F_3$ and portions of components of L. Similar choice can be made for C_{234}, which is any spanning surface for the new link components formed by arcs and circles of $F_2 \cdot C_{34}$ and $C_{23} \cdot F_4$ and portions of components of L. Thus C_{123} and C_{234} are reductively constructed from C_{12}, C_{23}, and C_{34}, which are constructed from the previous case of third-order linking.

Remark. A formula for Massey's linking number of order ≥ 5 can be obtained inductively when its linking numbers of order $\leq n-1$ all vanish, following the pattern in the third- and fourth-order linkings. For example, the fifth order linking number is given by

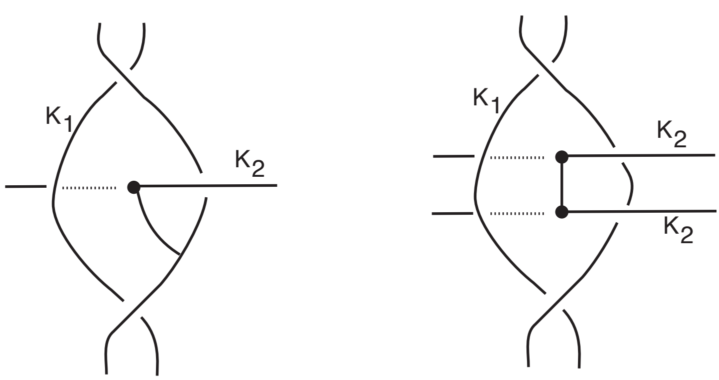
$$\begin{aligned}
&[\mu_1^* \cup \widetilde{C}_{2345} + \widetilde{C}_{12} \cup \widetilde{C}_{345} + \widetilde{C}_{123} \cup \widetilde{C}_{45} + \widetilde{C}_{1234} \cup \mu_5^*]([T_1]) \\
&= \partial_\#(\theta_\#(\theta_\#(T_1 \cdot \bar{\phi}(\mu_1^*)) \cdot C_{2345}^*) + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{12}^*) \cdot C_{345}^*)) \\
&\quad + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{123}^*) \cdot C_{45}^*)) + \partial_\#(\theta_\#(\theta_\#(T_1 \cdot C_{1234}^*) \cdot \bar{\phi}(\mu_5^*))) \\
&= \#\Big(|T_1| \cap |F_1| \cap |C_{2345}|\Big) + \#\Big(|T_1| \cap |C_{12}| \cap |C_{345}|\Big) \\
&\quad + \#\Big(|T_1| \cap |C_{123}| \cap |C_{45}|\Big) + \#\Big(|T_1| \cap |C_{234}| \cap |F_5|\Big),
\end{aligned}$$

where $\widetilde{C}_{12}, C_{12}, \widetilde{C}_{45}, C_{45}, \widetilde{C}_{123}, C_{123}, \widetilde{C}_{345}$ and C_{345} are similarly obtained as before, and $\widetilde{C}_{1234}$ and $\widetilde{C}_{2345}$ satisfy $\mu_1^* \cup \widetilde{C}_{234} + \widetilde{C}_{12} \cup \widetilde{C}_{34} + \widetilde{C}_{123} \cup \mu_4^* = \delta\widetilde{C}_{1234}$ and $\mu_2^* \cup \widetilde{C}_{345} + \widetilde{C}_{23} \cup \widetilde{C}_{45} + \widetilde{C}_{234} \cup \mu_5^* = \delta\widetilde{C}_{2345}$.

5. A combinatorial algorithm for computing Massey numbers. In this section we give an algorithm for computing the third-order linking number by first constructing ∂C_{12} and ∂C_{23}, which will be oriented links in R^3 and from which C_{12} and C_{23} can be constructed as the Seifert surfaces spanning ∂C_{12} and ∂C_{23}, respectively. We will see that this procedure can be generalized to the 4th and higher order linkings. The construction will be facilitated by the use of Seifert disks for F_1 and F_2, which we discuss in the following.

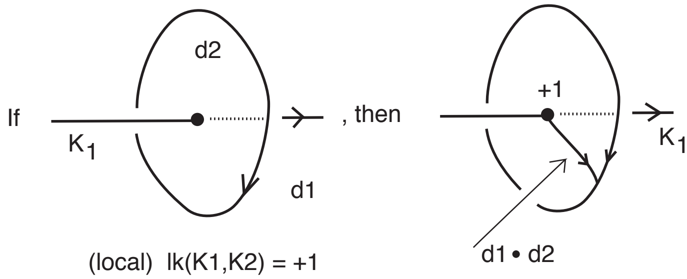
Let S^3 be given the usual right-hand orientation, and let K_1, K_2 be oriented knots with oriented Seifert surfaces F_1 and F_2 respectively. Then F_1 and F_2 are each a union of Seifert disks and half-twist bands. We may assume that they are in general position.

Consider $F_1 \cdot F_2$. By an isotopy we may assume that the Seifert disks of F_1 intersect only with Seifert disks of F_2 and vice versa. This can be done by pushing Seifert disks of F_1 away from the twist bands of F_2, and pushing Seifert disks of F_2 away from the twist bands of F_1. By using an innermost disk argument, we may also assume that no curve of intersection of any two Seifert disks is a circle. In this case, we say that F_1 and F_2 are in *normal position.* When F_1 and F_2 are in normal position, a component of $F_1 \cdot F_2$ is either a curve joining K_1 and K_2 or a curve joining two points in K_1 or two points in K_2. See the figure given below.



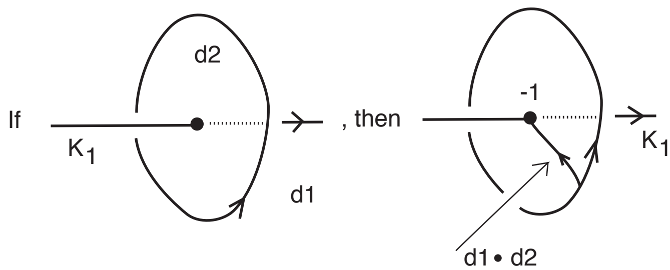
The intersection theory at chain-cochain level developed earlier can be applied to show that, a curve of intersection of a Seifert disk of F_1 and a Seifert disk of F_2 has the following orientation determined by that of F_1 and F_2: For $i = 1, 2$, let d_i be a Seifert disk of F_i and let α be a curve of $d_1 \cdot d_2$ (in this order). The orientation on α is the one satisfying the diagramatic convention depicted in the following figures.

(1)



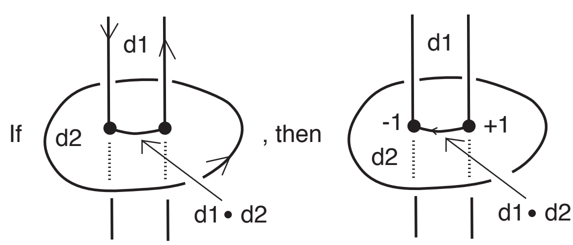
That is, if the local linking of K_1 and K_2 is $+1$, then the curve $d_1 \cdot d_2$ (in this order) goes "out" from the point of intersection of K_1 and d_2 labeled with $+1$ to the boundary of d_2. Note that the disk d_1 in the diagram can be on either side of K_1.

(2)



Thus if the local linking of K_1 and K_2 is -1, then the curve $d_1 \cdot d_2$ (in this order) goes "into" the point of intersection of K_1 and d_2 labeled with -1 from a boundary point of d_2.

(3)



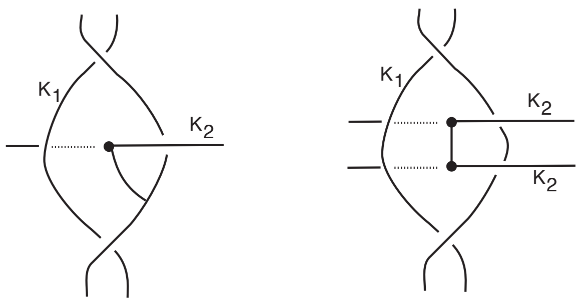
In this case $d_1 \cdot d_2$ goes from the point of intersection labeled with $+1$ to the point of intersection labeled with -1. Note that when considering $F_1 \cdot F_2$ we need only consider Seifert disks of F_2, and see how K_1 intersects these disks.

Now Massey's Third-order linking number is $< [\mu_1^*], [\mu_2^*], [\mu_3^*] > ([T_1]) = [\mu_1^* \cup \widetilde{C}_{23} + \widetilde{C}_{12} \cup \mu_3^*]([T_1]) = \partial_\#(\theta_\#(T_1 \cdot F_1) \cdot C_{23}) + \partial_\#(\theta_\#(T_1 \cdot C_{12}) \cdot F_3)$, where $C_{12} \epsilon C_2(S^3, K_1 \cup K_2)$ with $|\partial C_{12}| \subseteq |(F_1 \cdot F_2)| \cup |K_1 \cup K_2|$, and $C_{23} \epsilon C_2(S^3, K_2 \cup K_3)$ with $\partial |C_{23}| \subseteq |(F_2 \cdot F_3)| \cup |K_2 \cup K_3|$.

The following procedure gives the construction of ∂C_{12}, each component of which is a simple closed curve in R^3, from which C_{12} can be constructed as the Seifert surface spanning ∂C_{12}. The same procedure can be used to construct ∂C_{23} and therefore C_{23}. Assume F_1 and F_2 are in normal position. Practically we will only need the points of intersection of K_1 with Seifert disks of F_2 with their ± 1 labeling.

Note that $lk(K_1, K_2) = 0$ implies that the number of geometric intersections of K_1 with Seifert disks of F_2 is even, and the algebraic intersection number is 0.

(1) As a starting point, choose p to be any point of intersection of K_1 with a Seifert disk of F_2 that is labeled -1, there are two possibilities, see Figure 1(a) and 1(b).



(a) (b)

Figure 1

(2) Follow the orientation of K_1 until meeting the next Seifert disk of F_2. If the intersection point is labeled with -1, skip and continue traveling along K_1 until meeting the next Seifert disk of F_2.

Continue this process (repeatedly going back to (2) if the intersecting point encountered is labeled with -1) until an interesection point labeled with $+1$ is encountered. Such an intersection point exists since the algebraic intersection number of K_1 and Seifert disks of F_2 is 0. Then there is an oriented arc of $F_1 \cdot F_2$ contained in this Seifert disk of K_2.

(3) Follow the orientation of the oriented arc to reach either

(3.1) another point of intersection of K_1 with the same Seifert disk of F_2 labeled with -1, see Figure 2; or

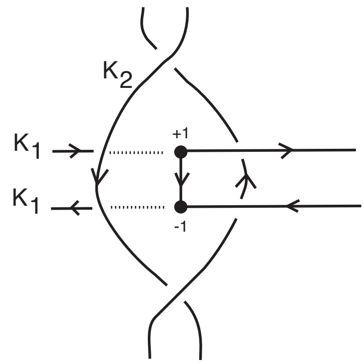


Figure 2

(3.2) a point in K_2, see Figure 3.

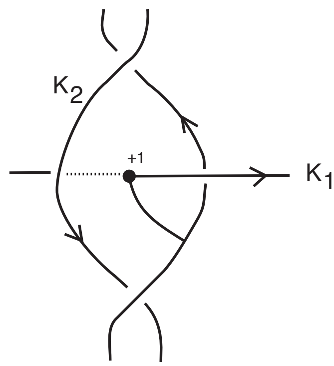


Figure 3

In case of (3.1), go back to (1) and continue the procedure, and proceeds with the condition that when a +1 intersection point is encountered, it needs be checked whether the point has been encountered previously, skip and continue if encountered previously.

In case of (3.2), follow the orientation of K_2, until it either

(3.2.1) reaches another intersection arc of $F_1 \cdot F_2$ lying in this Seifert disk of F_2, see Figure 4; or

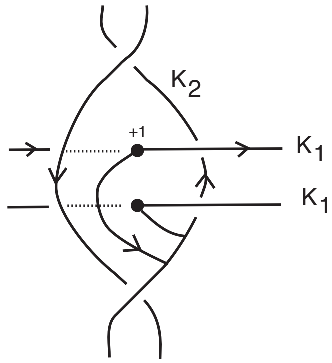


Figure 4

(3.2.2) reaches a boundary component of a twist band attached to this Seifert disk of F_2, see Figure 5.

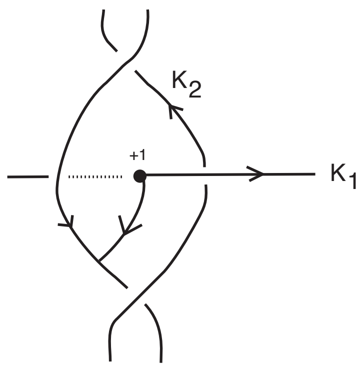


Figure 5

In the case of (3.2.1), check the orientation of the arc of intersection of $F_1 \cdot F_2$ encountered to see if it is possible to follow the orientation. If not (this means that the point of intersection for this arc is also labeled with $+1$), then skip and continue on K_2 and go back to (3.2), see Figure 6.

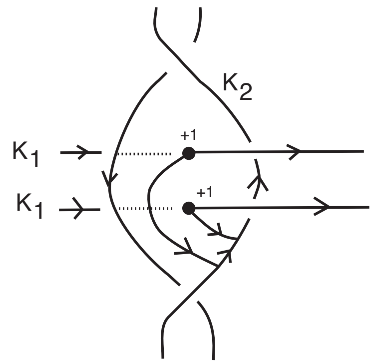


Figure 6

Otherwise follow the orientation of the intersection arc to reach to the point of intersection of K_1 with this Seifert disk of F_2, see Figure 7.

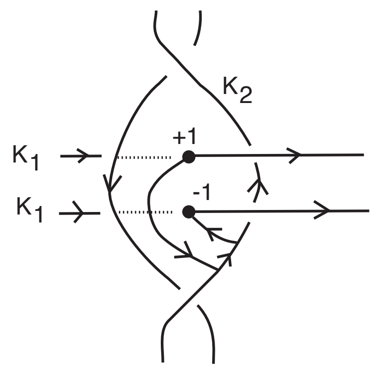


Figure 7

The point of intersection must be labeled -1 , so

(3.2.1.1) check if it is the point p started initially.

If no, go back to (1), and if yes, a simple loop is obtained; then check to see if there are other points of intersection of K_1 and Seifert disks of F_2 left undone, and if yes, choose any one of such points and go back to (1) or (3) depending on the label of the point is -1 or $+1$ respectively; otherwise all the points of intersection of K_1 and Seifert disks of F_2 have been accounted for, so ∂C_{12} is obtained and algorithm stops.

In the case of (3.2.2), follow the boundary component encountered to reach to a Seifert disk of F_2, which may be the one started with initially. Traveling along K_2 in the boundary of this Seifert disk of F_2. Then it either

(3.2.2.1) meets the boundary component of another twist band attached to this Seifert disk of F_2, and if so go back to (3.2.2), see Figure 8; or

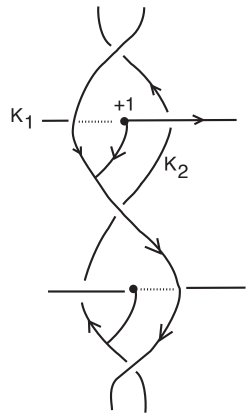


Figure 8

(3.2.2.2) meets an arc of intersection whose orientation cannot be followed, this is equivalent to the case that the point of intersection for this arc is labeled with +1 also, and if so then skip and continue to follow the orientation of K_2, i.e. go back to (3.2), see Figure 9; or

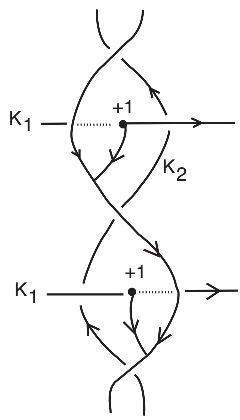


Figure 9

(3.2.2.3) meets an arc of intersection whose orientation can be followed, and this is equivalent to the case that the point of intersection for this arc is labeled with -1. Then go to (3.2.2.1) to check if the point is the point p started initially, and proceeds accordingly. See Figure 10

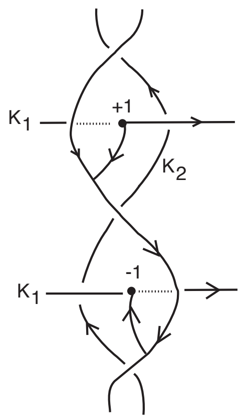
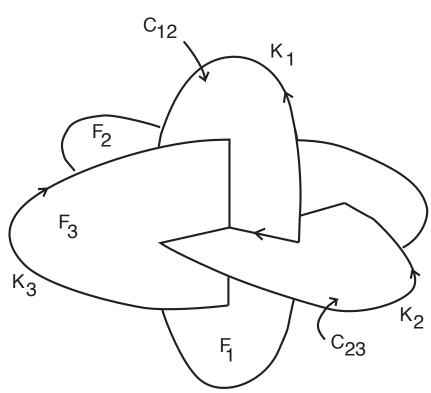


Figure 10

Remark. One can choose the starting point p to be a point of intersection of K_1 with a Seifert disk of F_2 labeled with $+1$ instead of -1, but then the procedure changes accordingly. In general, to obtain ∂C_{12} say, first find all the curves of intersection of $F_1 \cdot F_2$ (in this order) and start with any one such curve and following its orientation and the orientations of L and other curves of intersection, until all the curves of intersection have been encountered.

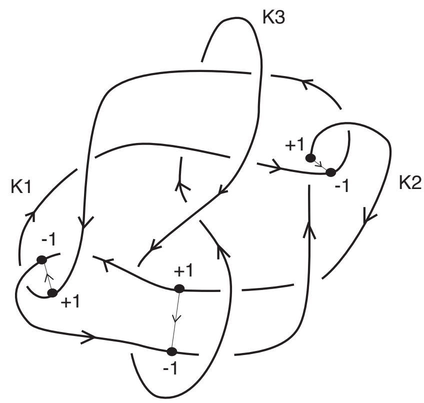
Example 1. Let $L = K_1 \cup K_2 \cup K_3$ be the Borromean rings with spanning surfaces F_1, F_2, F_3 as depicted in the figure given below.



Clearly the third-order linking number is

$$\begin{aligned}\#\Big(|T_1|\cap|F_1|\cap|C_{23}|\Big)+\#\Big(|T_1|\cap|C_{12}|\cap|F_3|\Big) &= lk(K_1,\partial C_{23})+\#\Big(|T_1|\cap|C_{12}|\cap|F_3|\Big)\\ &= 1+0\\ &= 1.\end{aligned}$$

EXAMPLE 2. Consider the link $L = K_1 \cup K_2 \cup K_3$ as depicted in the figure given below



Here again the Seiert surface F_i for K_i is a disk, and one computes $\#\Big(|T_1|\cap|F_1|\cap|C_{23}|\Big)+\#\Big(|T_1|\cap|C_{12}|\cap|F_3|\Big) = lk(K_1,\partial C_{23}) = -1$, see Figure 11;

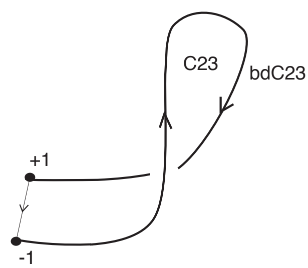


Figure 11

and $lk(K_1, \partial C_{23}) + \#\Big(|T_1| \cap |C_{12}| \cap |F_3|\Big) = -2$, see Figure 12.

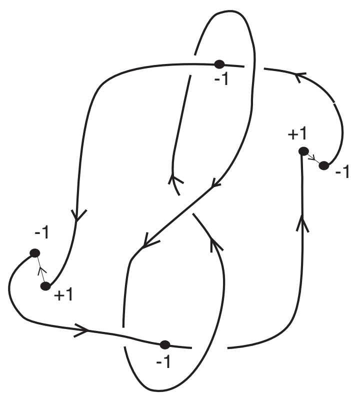
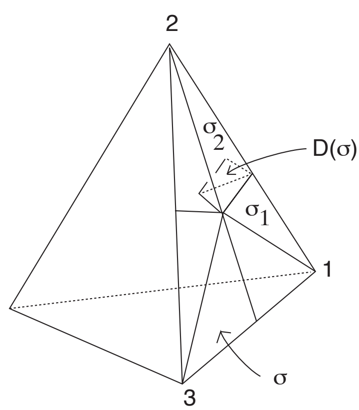


Figure 12

So the third-order linking number is -3, which indicates that the components of L are "more linked" than the Borromean rings.

6. Appendix. The intersection product has the following combinatorial and geometric topology interpretation, which is illustrated for the case $n = 3$, σ is the 2-simplex with vertices 1, 2, 3, and τ the 1-simplex joining 1 and 2, see the figure given below.



Then $\mu_\tau \,\epsilon\, C^1(K)$, and $D(\tau) \,\epsilon\, C_2(K^*)$, and by the topological definition of cap product we have

$$\begin{aligned}\sigma \cdot D(\tau) &= sd_\#(\sigma) \cap \theta^\#(u_\tau)\\ &= [sd_\#(\sigma)\lambda_1, \theta^\#(u_\tau)](sd_\#(\sigma)\rho_1),\end{aligned}$$

where λ_1 is the front first ace and ρ_1 is the back first face. Let $sd_{\#}(\sigma) = \sum_{i=1}^{6} \sigma_i$, so

$$\begin{aligned}\sigma \cdot D(\tau) &= \sum_{i=1}^{6} (\theta^{\#}(\mu_\tau)(\sigma_i \lambda_1))(\sigma_i \rho_1) \\ &= \sum_{i=1}^{6} \mu_\tau(\theta_{\#}(\sigma_i \lambda_1)) \\ &= \sigma_1 \rho_1,\end{aligned}$$

since

$$\mu_\tau(\theta_{\#}(\sigma_i \lambda_1)) = \begin{cases} \mu_\tau(\tau) = 1 & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

Notice the sense of direction of the 1-simplex $\sigma_i \rho_1$ of $\sigma \cdot D(\tau)$.

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