FUNCTION FIELDS OF ALGEBRAIC TORI REVISITED[*]

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Abstract. Let K/k be a finite Galois extension and $\pi = \mathrm{Gal}(K/k)$. An algebraic torus T defined over k is called a π-torus if $T \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \simeq \mathbb{G}_{m,K}^n$ for some integer n. The set of all algebraic π-tori defined over k under the stably birational equivalence forms a semigroup, denoted by $T(\pi)$. We will give a complete proof of the following theorem due to Endo and Miyata [EM3]. Theorem. Let π be a finite group. Then $T(\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$ where $\Omega_{\mathbb{Z}\pi}$ is a maximal $\mathbb{Z}$-order in $\mathbb{Q}\pi$ containing $\mathbb{Z}\pi$ and $C(\Omega_{\mathbb{Z}\pi})$ is the locally free class group of $\Omega_{\mathbb{Z}\pi}$, provided that π is isomorphic to one of the following four types of groups : C_n (n is any positive integer), D_m (m is any odd integer ≥ 3), $C_{q^f} \times D_m$ (m is any odd integer ≥ 3, q is an odd prime number not dividing m, $f \geq 1$, and $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle$ for any prime divisor p of m), Q_{4m} (m is any odd integer ≥ 3, $p \equiv 3 \pmod 4$ for any prime divisor p of m).

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1. Introduction. In [EM3], Endo and Miyata investigated the classification of the function fields of algebraic tori. An additional paper was planned, which would contain a complete proof of $(1') \Rightarrow (2)$ of Theorem 3.3 (see [EM3, page 187]). This article was announced in [EM3, p.189, line -14]. Unfortunately the plan didn't materialize. The present article may be regarded as a supplement to the papers [EM3] and the paper [Sw5]. We thank Prof. Richard G. Swan who detected a mistake in [EM3, p.96] and in the first version of this paper (see the remark at the end of Section 4). He also showed us how we could simplify the proof of Theorem 4.4 in a revised version of the first version.

To begin with, we recall some definitions and terminology. Let k be a field, $k \subset L$ be a field extension. The field L is said to be rational over k (in short, k-rational) if, for some n, $L \simeq k(X_1, \ldots, X_n)$ over k where $k(X_1, \ldots, X_n)$ is the rational function field of n variables over k. Two field extensions $k \subset L_1, L_2$ are stably isomorphic over k if, $L_1(X_1, \ldots, X_m) \simeq L_2(Y_1, \ldots, Y_n)$ over k where $X_1, \ldots, X_m$ are algebraically independent over L_1 and $Y_1, \ldots, Y_n$ are algebraically independent over L_2. In particular, a field extension $k \subset L$ is stably k-rational if $L(X_1, \ldots, X_m)$ is k-rational where $X_1, \ldots, X_m$ are some elements algebraically independent over L. When k is an infinite field, a field extension L over k is said to be retract k-rational if there is a k-algebra A contained in L such that (i) L is the quotient field of A, (ii) there exist a non-zero polynomial $f \in k[X_1, \ldots, X_n]$ (where $k[X_1, \ldots, X_n]$ is the polynomial ring) and k-algebra morphisms $\varphi \colon A \to k[X_1, \ldots, X_n][1/f]$ and $\psi \colon k[X_1, \ldots, X_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$ (see [Sa]). When V is an irreducible algebraic variety defined over k, V is k-rational (resp. stably k-rational, retract k-rational) if so is the function field $k(V)$ over k. If V_1 and V_2 are irreducible varieties over k, V_1 and V_2 are stably birational equivalent over k if so are their function fields over k.

An algebraic torus T defined over a field k is an affine algebraic group defined

over k such that $T \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k}) \simeq \mathbb{G}^n_{m,\bar{k}}$ for some integer n where $\bar{k}$ is the algebraic closure of k and $\mathbb{G}_{m,K}$ is the 1-dimensional multiplicative group defined over a field K (containing the base field k) [On; Sw4, page 36; Vo]. By [On, Proposition 1.2.1], for any algebraic torus T over k, there is a finite separable extension field K of k satisfying that $T \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \simeq \mathbb{G}^n_{m,K}$; such a field K is called a splitting field of T.

Let k be a field, π be a finite group. We will say that the field k admits a π-extension if there is a Galois field extension K/k such that $\pi = \mathrm{Gal}(K/k)$.

DEFINITION 1.1. Let π be a finite group, k be a field admitting a π-extension. An algebraic torus T over k is called a π-torus if it has a splitting field K which is Galois over k with $\mathrm{Gal}(K/k) = \pi$.

Let π be a finite group. Recall that a finitely generated $\mathbb{Z}\pi$-module M is called a π-lattice if it is torsion-free as an abelian group.

If T is a π-torus over a field k with $\pi = \mathrm{Gal}(K/k)$, then its character module $\mathrm{Hom}(T \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K), \mathbb{G}_{m,K})$ is a π-lattice. Conversely, every π-lattice M is isomorphic to the character module of some algebraic π-torus T over k (as $\mathbb{Z}\pi$-modules) [On; Sw4, page 36].

DEFINITION 1.2. Let K/k be a finite Galois field extension with $\pi = \mathrm{Gal}(K/k)$. Let $M = \bigoplus_{1 \le i \le n} \mathbb{Z} \cdot e_i$ be a π-lattice. We define an action of π on $K(M) = K(x_1, \ldots, x_n)$, the rational function field of n variables over K, by $\sigma \cdot x_j = \prod_{1 \le i \le n} x_i^{a_{ij}}$ if $\sigma \cdot e_j = \sum_{1 \le i \le n} a_{ij} e_i \in M$, for any $\sigma \in \pi$ (note that π acts on K also). The fixed field is denoted by $K(M)^\pi$.

Let K/k be a finite Galois extension with $\pi = \mathrm{Gal}(K/k)$. There is a duality between the category of algebraic π-tori defined over k and the category of π-lattices. In fact, if T is a π-torus and M is its character module, then the function field of T is isomorphic to $K(M)^\pi$ (see [On; Sw4, page 36]). Thus the study of rationality problems of π-tori is reduced to that of π-lattices.

DEFINITION 1.3 ([EM1; EM3, page 86]). Let $\pi = \mathrm{Gal}(K/k)$ be a finite group where K/k is a Galois extension. Define an equivalence relation in the category of π-lattices: Two π-lattices M and N are equivalent, denoted by $M - N$, if the fields $K(M)^\pi$ and $K(N)^\pi$ are stably isomorphic over k, i.e. $K(M)^\pi(X_1, \ldots, X_m) \simeq K(N)^\pi(Y_1, \ldots, Y_n)$ for some algebraically independent elements X_i, Y_j.

Let π be a finite group. Define a commutative monoid $T(\pi)$ as follows. As a set, $T(\pi)$ is the set of all equivalence classes $[M]$ under the equivalence relation "$-$" defined above (note that $[M]$ is the equivalence class containing the π-lattice M); the monoid operation is defined by $[M] + [N] = [M \oplus N]$.

THEOREM 1.4 ([EM3, page 95, Theorem 3.3; page 187]). *Let π be a finite group. Then the following statements are equivalent:*

(1) *π is isomorphic to*

    (i) *a cyclic group C_n where n is any positive integer, or*

    (ii) *a dihedral group D_m of order $2m$ where m is an odd integer ≥ 3, or*

    (iii) *a direct product $C_{q^f} \times D_m$ where q is an odd prime number, $f \ge 1$, m an odd integer ≥ 3, $\gcd\{q, m\} = 1$ and for any prime divisor p of m, $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle$, or*

    (iv) *$Q_{4m} = \langle \sigma, \tau : \sigma^{2m} = \tau^4 = 1, \sigma^m = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$, the generalized quaternion group of order $4m$, where $m \ge 3$ is an odd integer and $p \equiv 3$ (mod 4) for any prime divisor p of m.*

(2) $T(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$ where $\Omega_{\mathbb{Z}\pi}$ is a maximal $\mathbb{Z}$-order in $\mathbb{Q}\pi$ containing $\mathbb{Z}\pi$.

(3) $T(\pi)$ is a finite group.

The purpose of this article is to give a proof of Theorem 1.4 supplementing the proof outlined in [EM3]. Note that the definition of the locally free class groups $C(\mathbb{Z}\pi)$, $C(\Omega_{\mathbb{Z}\pi})$ and the associated group $C^q(\mathbb{Z}\pi)$ may be found in Definition 2.11 and Definition 2.12; the isomorphisms $T(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ and $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$ are described in Definition 2.13.

The main ideas of the proof of $(1) \Rightarrow (2)$ in Theorem 1.4 will be explained at the beginning of Section 5.

The organization of this paper is as follows. In Section 2, the notions of flabby, coflabby, invertible, permutation π-lattices are recalled. Some fundamental results are summarized also. Section 3 contains the definitions of twisted group rings, denoted by $S \circ G$, and crossed-product orders, denoted by $(S \circ G)_f$ (where f is a 2-cocycle of G). Some results of Rosen's Ph.D. dissertation (unpublished in the journals) will be quoted for the convenience of the reader. We suggest the reader to consult similar results in Lee's paper [Lee]. Section 4 is a first step of the proof of $(1) \Rightarrow (2)$ of Theorem 1.4; this section contains a devissage theorem for $[M]^{fl}$ where M is an invertible π-lattice (for the definition of $[M]^{fl}$, see Definition 2.1). Section 5 is devoted to the proof of $(1) \Rightarrow (2)$ of Theorem 1.4. In the final section, we compute $C(\Omega_{\mathbb{Z}\pi})$ when π are the groups in Theorem 1.4. As a consequence, we deduce a result about D_{p^c}-tori; the case of D_p-tori was proved by Hoshi, Kang and Yamasaki by a different method [HKY]. On the other hand, some properties of π-lattices proved by Colliot-Thélène and Sansuc, when π is the Klein four-group or the quaternion group of order 8 [CTS, pages 186-187], will be generalized to the situation when π is some dihedral 2-group or quaternion groups of order 8, 16, 32, etc.; see Proposition 6.6 and Proposition 6.8.

Terminology and notations. Throughout the paper, we denote by k a field and by π a finite group. We denote by $\mathbb{Z}\pi$ the integral group ring of the group π.

Denote by C_n (resp. D_n) the cyclic group of order n (resp. the dihedral group of order $2n$). The group Q_{4n} denotes the generalized quaternion group of order $4n$ where $n \geq 2$, i.e. $Q_{4n} = \langle \sigma, \tau : \sigma^{2n} = \tau^4 = 1, \sigma^n = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$.

If q is a prime power, $\mathbb{F}_q$ denotes the finite field with q elements. For any positive integer $n \geq 2$, ζ_n denotes a primitive n-th root of unity and $\Phi_n(X) \in \mathbb{Z}[X]$ the n-th cyclotomic polynomial. $(\mathbb{Z}/n\mathbb{Z})^\times$ is the group of units of the ring $\mathbb{Z}/n\mathbb{Z}$. A commutative integral domain R is called a DVR if it is a discrete rank-one valuation ring. If R is a ring, $M_n(R)$ denotes the matrix ring of all $n \times n$ matrices over R. If M is an A-module where A is a ring, we denote by $M^{(n)}$ the direct sum of n copies of the module M.

When π is a finite group and Λ is a $\mathbb{Z}$-order satisfying that $\mathbb{Z}\pi \subset \Lambda \subset \mathbb{Q}\pi$ (see [CR1, page 524, Definition 23.2] for the definition of an order), a locally free Λ-lattice of rank n is a finitely generated Λ-module M such that $M_P \simeq (\mathbb{Z}_P \otimes_{\mathbb{Z}} \Lambda)^{(n)}$ for any non-zero prime ideal P of $\mathbb{Z}$ where $\mathbb{Z}_P$ is the localization of $\mathbb{Z}$ at the prime ideal P and $M_P = \mathbb{Z}_P \otimes_{\mathbb{Z}} M$ [CR2, page 382, Defintion 55.28]. A projective ideal over $\mathbb{Z}\pi$ is a left ideal $\mathcal{A}$ of $\mathbb{Z}\pi$ such that $\mathcal{A}$ is also a projective $\mathbb{Z}\pi$-module. By [Sw1], $\mathcal{A}$ is a projective ideal over $\mathbb{Z}\pi$ if and only if it is a locally free $\mathbb{Z}\pi$-lattice of rank one.

We remind the reader that the definitions of $T(\pi)$, $T^g(\pi)$, $C(\Lambda)$ and $[M]^{fl}$, are given in Definition 1.3, Definition 2.8, Definition 2.11 and Definition 2.1; the monoid F_π is defined in the paragraph before Definition 2.1.

2. Preliminaries. In this section we recall some notions related to π-lattices and summarize several results in [EM3] and [Sw4].

A π-lattice M is called a permutation lattice if M has a $\mathbb{Z}$-basis permuted by π; M is called an invertible lattice if it is a direct summand of some permutation lattice. A π-lattice M is called a flabby lattice (or a flasque lattice) if $H^{-1}(\pi', M) = 0$ for any subgroup π' of π; it is called coflabby (or a coflasque lattice) if $H^1(\pi', M) = 0$ for any subgroup π' of π. For details, see [EM3; CTS; Sw4].

Two π-lattices M_1 and M_2 are similar, denoted by $M_1 \sim M_2$, if $M_1 \oplus P_1 \simeq M_2 \oplus P_2$ for some permutation π-lattices P_1 and P_2. The flabby class monoid F_π is the monoid whose elements consist of flabby π-lattices under the above similarity relation. A typical element in F_π is $[M]$, the equivalence class containing M where M is a flabby π-lattice; the monoid operation on F_π is defined by $[M_1] + [M_2] = [M_1 \oplus M_2]$. Thus F_π becomes an abelian monoid and $[P]$ is the identity element of it where P is any permutation π-lattice [Sw4, page 33].

DEFINITION 2.1. Let π be a finite group, M be any π-lattice. Then M has a flabby resolution, i.e. there is an exact sequence of π-lattices: $0 \to M \to P \to E \to 0$ where P is a permutation lattice and E is a flabby lattice [EM3, Lemma 1.1]. The class $[E] \in F_\pi$ is uniquely determined by the lattice M [Sw4, Lemma 8.7]. We define $[M]^{fl} = [E] \in F_\pi$. Sometimes we will say that $[M]^{fl}$ is permutation or invertible if the class $[E]$ contains a permutation or invertible lattice.

Be aware that the equivalence relation $M - N$ in Definition 1.3 is different from the above similarity relation $M_1 \sim M_2$. The monoids $T(\pi)$ in Definition 1.3 and the above monoid F_π are isomorphic through the following lemma.

LEMMA 2.2. Let π be a finite group.
(1) If M and N are π-lattices. Then $M - N$ if and only if $[M]^{fl} = [N]^{fl}$.
(2) Define a monoid homomorphism $\Phi : T(\pi) \to F_\pi$ by $\Phi([M]) = [M]^{fl}$. Then Φ is an isomorphism.

Proof. Let $\pi = \mathrm{Gal}(K/k)$. By the same idea in the proof of [Le, Theorem 1.7], it is not difficult to show that $K(M)^\pi$ and $K(N)^\pi$ are stably isomorphic over k if and only if there exist exact sequences of π-lattices $0 \to M \to E \to P \to 0$ and $0 \to N \to E \to Q \to 0$ where E is some π-lattice and P, Q are permutation π-lattices. The latter condition is equivalent to $[M]^{fl} = [N]^{fl}$ by [Sw4, Lemma 8.8].

For the isomorphism of Φ, note that Φ is well-defined by (1). If $\Phi([M]) = 0$, then $[M]^{fl} = 0$. Thus $K(M)^\pi$ is stably k-rational by the following Lemma 2.3. Hence $M - \mathbb{Z}$ where $\mathbb{Z}$ is the π-lattice with trivial π actions. Thus Φ is injective. On the other hand, if E is any flabby π-lattice, let $E^0 = \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$ be its dual lattice. Take a flabby resolution of E^0, $0 \to E^0 \to P \to F \to 0$ as in Definition 2.1. We get an exact sequence $0 \to F^0 \to P^0 \to E \to 0$. Thus $[E] = [F^0]^{fl} = \Phi([F^0])$ and Φ is surjective. $\square$

The following lemma is due to Endo and Miyata [EM2, Theorem 1.2], Voskresenskii and Saltman.

LEMMA 2.3. (e.g. [Le, Theorem 1.7; Sa, Theorem 3.14]) Let K/k be a finite Galois field extension, $\pi = \mathrm{Gal}(K/k)$, M be a π-lattice. Then (i) $K(M)^\pi$ is stably k-rational if and only if $[M]^{fl} = 0$ in F_π, and (ii) $K(M)^\pi$ is retract k-rational if and only if $[M]^{fl}$ is an invertible lattice.

LEMMA 2.4. Let π be a finite group, and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of π-lattices.

(1) (e.g. [Le, Proposition 1.5]) *If M'' is invertible, then $[M]^{fl} = [M']^{fl} + [M'']^{fl}$.*

(2) ([EM3, Lemma 2.2]) *If all the Sylow subgroups of π are cyclic and $\widehat{H}^0(\pi', M') = 0$ for any subgroup $\pi' \subset \pi$, then $[M]^{fl} = [M']^{fl} + [M'']^{fl}$.* (Note that $\widehat{H}^0(\pi', M')$ denotes the Tate cohomology.)

LEMMA 2.5. *Let π be a finite group, M be a π-lattice.*

(1) (e.g. [Le, Proposition 1.2]) *M is invertible if and only if, for any coflabby π-lattice C, any short exact sequence $0 \to C \to E \to M \to 0$ splits.*

(2) ([EM4, Lemma 1.1]) *There is a short exact sequence of π-lattices $0 \to M \to C \to P \to 0$ such that C is coflabby and P is permutation.*

THEOREM 2.6 ([EM3, Theorem 1.5]). *Let π be a finite group. Then all the flabby π-lattices (resp. all the coflabby π-lattices) are invertible if and only if all the Sylow subgroups of π are cyclic.*

In the literature there are two different definitions for metacyclic groups. A finite group is metacyclic if it is an extension of a cyclic group by another cyclic group (see [Is, page 160]). Thus we will call a metacyclic group π a split metacyclic group if π has a cyclic normal subgroup π_0 such that π/π_0 is cyclic and $\gcd\{|\pi_0|, |\pi/\pi_0|\} = 1$. It is known that π is split metacyclic if and only if all the Sylow subgroups of π are cyclic [Is, page 160, Theorem 5.16]. However, in Zassenhaus's monograph [Za, page 174], a finite group π is called metacyclic group if π' and π/π' are cyclic groups where π' is the commutator subgroup of π (see [Za, page 175, Theorem 11] also).

THEOREM 2.7 ([EM4, Theorem 2.1]). *Let π be a finite group. Then the π-lattices which are both flabby and coflabby are necessarily invertible if and only if all the p-Sylow subgroups of π are cyclic for odd prime p, and all the 2-Sylow subgroups of π are cyclic or dihedral (including the Klein four-group).*

DEFINITION 2.8 ([EM3, page 86]). Let π be a finite group. Define $T^g(\pi) = \{[M] \in T(\pi) : \text{There exists } [N] \in T(\pi) \text{ such that } [M] + [N] = 0 \text{ in } T(\pi)\}$. It follows that $T^g(\pi)$ is a subgroup of $T(\pi)$; it is the maximal subgroup of $T(\pi)$. By Jacobinski's Theorem [Ja, Proposition 5.8], $T^g(\pi)$ is finitely generated.

If M is a π-lattice and $[M] \in T^g(\pi)$, then there is an invertible π-lattice E such that $[M] = [E]$ in $T^g(\pi)$ (see [EM3, Lemma 1.6]).

Now we turn to R-orders in a separable K-algebra Σ where R is a Dedekind domain with quotient field K [CR1, page 523]. Recall that a separable K-algebra Σ is a finite-dimensional semi-simple algebra over K such that the center K' of Σ is an étale K-algebra, i.e. $K' = \prod_{1 \leq i \leq t} K_i$ where each K_i is a finite separable field extension of K. An R-order Λ is a subring of Σ, $R \subset \Lambda \subset \Sigma$ satisfying that $K\Lambda = \Sigma$ and Λ is a finitely generated R-module.

DEFINITION 2.9. Let R be a Dedekind domain with quotient field K, Λ be an R-order in a separable K-algebra Σ. A Λ-lattice M is a left Λ-module which is finitely generated and projective as an R-module [CR1, page 524]. Two Λ-lattices M and N are in the same genus if $M_P \simeq N_P$ for any prime ideal P of R where $M_P = R_P \otimes_R M$ and R_P is the localization of R at the prime ideal P [CR1, pages 642–643].

THEOREM 2.10 (Jacobinski, Roiter [CR2, page 660, Theorem 31.28]). *Let R be a Dedekind domain whose quotient field is a global field K. Let Σ be a separable K-algebra and Λ be an R-order in Σ. Let M and N be Λ-lattices in the same genus*

and F be a faithful Λ-lattice. Then there is a Λ-lattice F' such that F, F' are in the same genus and $M \oplus F \simeq N \oplus F'$. In particular, if M and N are Λ-lattices in the same genus, then $M \oplus \Lambda \simeq N \oplus \mathcal{A}$ for some projective ideal $\mathcal{A}$ over Λ.

DEFINITION 2.11. Let π be a finite group, and let Λ be a $\mathbb{Z}$-order with $\mathbb{Z}\pi \subset \Lambda \subset \mathbb{Q}\pi$. Recall the definition of locally free Λ-modules given at the end of Section 1. We define $K_0^{lf}(\Lambda)$, the Grothendieck group of the category of locally free Λ-modules, as follows: $K_0^{lf}(\Lambda)$ is the abelian group with generators $[P]$ for locally free Λ-modules P and relations $[P] = [P'] + [P'']$ if $0 \to P' \to P \to P'' \to 0$ is an exact sequence of locally free Λ-modules. Taking ranks provides a surjective group homomorphism $\varphi : K_0^{lf}(\Lambda) \to \mathbb{Z}$. The kernel of φ is called the locally free class group of Λ, and is denoted by $C(\Lambda)$. The group $C(\Lambda)$ is also called the projective class group of Λ.

When $\Lambda = \mathbb{Z}\pi$, the locally free class group $C(\mathbb{Z}\pi)$ may be defined via $K_0(\mathbb{Z}\pi)$, the usual Grothendieck group of the category of finitely generated projective $\mathbb{Z}\pi$-modules: $C(\mathbb{Z}\pi)$ is a subgroup of $K_0(\mathbb{Z}\pi)$ defined as $C(\mathbb{Z}\pi) = \{[\mathcal{A}] - [\mathbb{Z}\pi] \in K_0(\mathbb{Z}\pi) : \mathcal{A}$ is a projective ideal over $\mathbb{Z}\pi\}$, because a locally free $\mathbb{Z}\pi$-module of rank n is isomorphic to a direct sum of $\mathbb{Z}\pi^{(n-1)}$ and some projective ideal over $\mathbb{Z}\pi$ by [Sw1].

Note that $C(\Lambda)$ is a finite group [CR2, page 51, Proposition 39.13]. For details of $C(\Lambda)$, see [EM2, page 397; CR2, page 50, page 219, page 230].

DEFINITION 2.12. Let π be a finite group and $\Omega_{\mathbb{Z}\pi}$ be a maximal $\mathbb{Z}$-order in $\mathbb{Q}\pi$ satisfying $\mathbb{Z}\pi \subset \Omega_{\mathbb{Z}\pi} \subset \mathbb{Q}\pi$. It is known that the natural map $\varphi_1 : C(\mathbb{Z}\pi) \to C(\Omega_{\mathbb{Z}\pi})$ defined by $\varphi_1([\mathcal{A}] - [\mathbb{Z}\pi]) = [\Omega_{\mathbb{Z}\pi} \otimes_{\mathbb{Z}\pi} \mathcal{A}] - [\Omega_{\mathbb{Z}\pi}]$ is surjective (see [CR2, page 290, Theorem 49.25]). Define $\widetilde{C}(\mathbb{Z}\pi) = \mathrm{Ker}(\varphi_1)$. From the definition, $\widetilde{C}(\mathbb{Z}\pi) = \{[\mathcal{A}] - [\mathbb{Z}\pi] \in C(\mathbb{Z}\pi) : (\Omega_{\mathbb{Z}\pi} \otimes_{\mathbb{Z}\pi} \mathcal{A}) \oplus \Omega_{\mathbb{Z}\pi} \simeq \Omega_{\mathbb{Z}\pi} \oplus \Omega_{\mathbb{Z}\pi}\}$ (see [EM3]).

In [CR2, page 234] and other literature, $\widetilde{C}(\mathbb{Z}\pi)$ is written as $D(\mathbb{Z}\pi)$; but we choose sticking to the notation in [EM2; EM3].

On the other hand, a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ may be regarded as a π-lattice. Thus we may define $\varphi_2 : C(\mathbb{Z}\pi) \to T^g(\pi)$ defined by $\varphi_2([\mathcal{A}] - [\mathbb{Z}\pi]) = [\mathcal{A}] \in T^g(\pi)$. It is easy to check that φ_2 is a well-defined morphism of abelian groups. Define $C^q(\mathbb{Z}\pi) = \mathrm{Ker}(\varphi_2)$. Clearly $C^q(\mathbb{Z}\pi) = \{[\mathcal{A}] - [\mathbb{Z}\pi] \in C(\mathbb{Z}\pi) : \mathcal{A}$ is a projective ideal over $\mathbb{Z}\pi$ satisfying $[\mathcal{A}]^{fl} = 0$ in $F_\pi\}$.

Finally define $\widetilde{C}^q(\mathbb{Z}\pi) = \{[\mathcal{A}] - [\mathbb{Z}\pi] \in C^q(\mathbb{Z}\pi) : \mathcal{A}$ is a projective ideal over $\mathbb{Z}\pi$ satisfying $\mathcal{A} \oplus P \simeq \mathbb{Z}\pi \oplus P$ for some permutation π-lattice $P\}$ (see [EM2, page 698]).

It can be shown that $\widetilde{C}^q(\mathbb{Z}\pi)$ is a subgroup of $\widetilde{C}(\mathbb{Z}\pi)$. In fact, Oliver proves that $\widetilde{C}^q(\mathbb{Z}\pi) = \widetilde{C}(\mathbb{Z}\pi)$ for any finite group π (see [Ol, Theorem 5]). Hence $\widetilde{C}(\mathbb{Z}\pi) \subset C^q(\mathbb{Z}\pi)$. It follows that there is always a surjective map $C(\Omega_{\mathbb{Z}\pi}) \to C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$.

We remark that there exists a meta-cyclic group π such that $\widetilde{C}(\mathbb{Z}\pi) \neq C^q(\mathbb{Z}\pi)$ ([EM2, page 709, Example 4.3]). However, it is shown [EM2] that $\widetilde{C}(\mathbb{Z}\pi) = C^q(\mathbb{Z}\pi)$ for many groups π, including those groups listed in Theorem 1.4 (1) (see Section 5).

DEFINITION 2.13. Let π be a finite group satisfying $C^q(\mathbb{Z}\pi) = \widetilde{C}(\mathbb{Z}\pi)$.

Let φ_1 and φ_2 be the group homomorphisms in Definition 2.12. Since $0 \to C^q(\mathbb{Z}\pi) \to C(\mathbb{Z}\pi) \xrightarrow{\varphi_2} T^g(\pi)$ is left-exact, we will write $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ when the canonical injection $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \xrightarrow{\bar{\varphi}_2} T^g(\pi)$ is an isomorphism.

Similarly, when $C^q(\mathbb{Z}\pi) = \widetilde{C}(\mathbb{Z}\pi)$ and $c : T^g(\pi) \to T(\pi)$ is the inclusion map, then the composite map $c \cdot \bar{\varphi}_2 \cdot \bar{\varphi}_1^{-1} : C(\Omega_{\mathbb{Z}\pi}) \to C(\mathbb{Z}\pi)/\widetilde{C}(\mathbb{Z}\pi) = C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \to T^g(\pi) \to T(\pi)$ is well-defined. We will write $T(\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$ if the injection $c \cdot \bar{\varphi}_2 \cdot \bar{\varphi}_1^{-1}$ is surjective and hence is an isomorphism.

THEOREM 2.14. *Let π be a finite group.*

(1) ([EM3, Proposition 3.1]) $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ *if and only if, for any invertible π-lattice M, there is a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$.*

(2) ([EM3, Theorem 3.2]) *If π is a p-group, then $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$.*

3. Twisted group rings.

DEFINITION 3.1 ([CR1, page 183, page 599–600; CR2, page 291]). Let K be a field, L/K be a finite Galois field extension with $G = \mathrm{Gal}(L/K)$, and $f : G \times G \to L^\times$ be a 2-cocycle of G where $L^\times = L \backslash \{0\}$ is the multiplicative group of L. The crossed product algebra, denoted by $(L \circ G)_f$, is defined by

$$(L \circ G)_f = \bigoplus_{\sigma \in G} L \cdot u_\sigma, \quad u_\sigma \cdot u_\tau = f(\sigma, \tau) u_{\sigma\tau}, \quad u_\sigma \cdot \alpha = \sigma(\alpha) \cdot u_\sigma$$

where $\alpha \in L$, $\sigma, \tau \in G$. The K-algebra $(L \circ G)_f$ is a central simple K-algebra.

Suppose that R is a Dedekind domain with quotient field K and K is a number field. Let S be the integral closure of R in L. Let $h : G \times G \to U(S)$ be a 2-cocycle where $U(S)$ is the group of units in S. Then we may define the crossed-product order, denoted by $(S \circ G)_h$, as follows:

$$(S \circ G)_h = \bigoplus_{\sigma \in G} S \cdot u_\sigma, \quad u_\sigma \cdot u_\tau = h(\sigma, \tau) u_{\sigma\tau}, \quad u_\sigma \cdot \alpha = \sigma(\alpha) \cdot u_\sigma$$

where $\alpha \in S$, $\sigma, \tau \in G$. The R-algebra $(S \circ G)_h$ is an R-order in $(L \circ G)_h$.

THEOREM 3.2 (Williamson, Harada [CR1, page 600, Theorem 28.12; Re, page 375, Theorem 40.15]). *Let the notations be the same as in Definition 3.1. The crossed-product R-order $(S \circ G)_h$ is a hereditary order if and only if S/R is tamely ramified, i.e. for any ramified prime ideal Q of S over R, if $\mathrm{char}\, S/Q = p > 0$, then $p \nmid e(Q, L/K)$ where $e(Q, L/K)$ is the ramification index of Q in L/K.*

DEFINITION 3.3. The twisted group algebra and the twisted group ring are special cases of the crossed-product algebra and the crossed-product order when the 2-cocycle f, h are the trivial one, i.e. $f(\sigma, \tau) = h(\sigma, \tau) = 1$ for all $\sigma, \tau \in G$.

For emphasis, we repeat the definition of a twisted group ring and denote it by $S \circ G$ (see [CR1, page 589]). Recall that S is a Dedekind domain whose quotient field L is an algebraic number field, G is a finite subgroup of $\mathrm{Aut}(S)$ with $R = S^G = \{a \in S : \sigma(a) = a$ for any $\sigma \in G\}$. Then

$$S \circ G = \bigoplus_{\sigma \in G} S \cdot u_\sigma, \quad (a u_\sigma) \cdot (b u_\tau) = (a \cdot \sigma(b)) \cdot u_{\sigma\tau}$$

where $a, b \in S$, $\sigma, \tau \in G$.

THEOREM 3.4. (1) ([Re, page 374, Theorem 40.14; CR1, page 591, Theorem 28.5]) *The twisted group ring $S \circ G$ is a maximal R-order if and only if S/R is unramified.*

(2) (Rosen [Ro; Re, page 373, Theorem 40.13]) *The twisted group ring $S \circ G$ is a hereditary R-order if and only if S/R is tamely ramified.*

REMARK. Theorem 3.4 (2) was obtained first. Then Theorem 3.2 came out as the generalization of Theorem 3.4 (2).

DEFINITION 3.5. Let $\Lambda = S \circ G$ be a twisted group ring and L be the quotient field of S with $R = S^G$. We will endow on S a Λ-module structure by defining $(a u_\sigma) \cdot \alpha = a \cdot \sigma(\alpha)$ for any $a, \alpha \in S$, any $\sigma \in G$. The field L can be given a Λ-module structure by the same way. If J is a fractional S-ideal of L such that $\sigma(J) \subset J$ for any $\sigma \in G$, then J becomes a Λ-submodule of L; such an ideal J is called an ambiguous ideal [CR1, page 596; Ro].

THEOREM 3.6 (Rosen). *Let $\Lambda = S \circ G$ be a twisted group ring and $R = S^G$.*

(1) ([Ro, Proposition 3; CR1, page 596]) *Let $Q_1, Q_2, \ldots, Q_t$ be all the ramified primes of S over R and $e_i = e_i(Q_i, S/R)$ be the ramification index of Q_i for $1 \leq i \leq t$. For each $1 \leq i \leq t$, let $\{Q_i^{(j)} : 1 \leq j \leq g_i\}$ be the set of G-orbits of Q_i (i.e. $Q_i^{(j)} = \sigma(Q_i)$ for some $\sigma \in G$, and $Q_i^{(1)}, \ldots, Q_i^{(g_i)}$ are distinct prime ideals of S). Define $J_i = \prod_{1 \leq j \leq g_i} Q_i^{(j)}$. As Λ-modules, any ambiguous ideal J is isomorphic to, $J_1^{a_1} \cdots J_t^{a_t} I$ where $0 \leq a_i < e_i$ and I is some ideal of R.*

(2) ([Ro, Theorem 3.2 and Theorem 3.3]) *Assume that S/R is tamely ramified. Then Λ is a left hereditary ring. The ambiguous ideals $J_1^{a_1} J_2^{a_2} \cdots J_t^{a_t} I$ in (1) are indecomposable projective Λ-modules. If M is a Λ-module such that M is a finitely generated torsion-free R-module, then M is isomorphic to a direct sum of these ambiguous ideals $J_1^{a_1} J_2^{a_2} \cdots J_t^{a_t} I$.*

REMARK. In the above theorem, it is possible that some of J_i (where $1 \leq i \leq t$) are isomorphic to each other. See [Ro] for the uniqueness statement.

4. A devissage theorem. If M is a $\mathbb{Z}\pi$-module, we will write $(M)_0 = M/t(M)$ where $t(M)$ is the torsion submodule of M.

First recall Swan's Theorem 5.1 in [Sw5].

THEOREM 4.1 ([Sw5, Theorem 5.1 and Corollary 5.2]). *Let $\pi = \langle \sigma \rangle \simeq C_m$ be a cyclic group of order m. Suppose that $n \mid m$ and M is an invertible π-lattice. Then*

$$[M/(\sigma^n - 1)M]^{fl} = \sum_{d|n} [(M/\Phi_d(\sigma)M)_0]^{fl}, \ and$$

$$[(M/\Phi_n(\sigma)M)_0]^{fl} = \sum_{d|n} \mu\left(\frac{n}{d}\right) [M/(\sigma^d - 1)M]^{fl}$$

where μ is the Möbius function.

Examining the proof of the above Theorem 4.1 in [Sw5], we find that such a result is valid if σ belongs to the center of the group π. We record this observation as follows.

THEOREM 4.2. *Let π be a finite group and $\sigma \in \pi$ such that σ belongs to the center of π. Suppose that $\langle \sigma \rangle \simeq C_m$ and $n \mid m$. If M is an invertible π-lattice, then*

$$[M/(\sigma^n - 1)M]^{fl} = \sum_{d|n} [(M/\Phi_d(\sigma)M)_0]^{fl}, \ and$$

$$[(M/\Phi_n(\sigma)M)_0]^{fl} = \sum_{d|n} \mu\left(\frac{n}{d}\right) [M/(\sigma^d - 1)M]^{fl}$$

where μ is the Möbius function.

DEFINITION 4.3. In Theorem 4.4 we will consider a finite group π which contains a cyclic normal subgroup $\langle \sigma \rangle$ satisfying that, for any element $\lambda \in \pi$, $\lambda \sigma \lambda^{-1} = \sigma$ or

σ^{-1} depending on λ; thus we will write $\lambda\sigma\lambda^{-1} = \sigma^{\epsilon(\lambda)}$ where $\epsilon(\lambda) \in \{1, -1\}$. Note that $\epsilon(\lambda_1\lambda_2) = \epsilon(\lambda_1)\epsilon(\lambda_2)$ for any $\lambda_1, \lambda_2 \in \pi$. Some sample examples of such groups π are : $\pi = \langle \sigma, \tau : \sigma^m = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \simeq D_m$, $\pi = \langle \sigma, \rho, \tau : \sigma^m = \rho^n = \tau^2 = 1, \tau\rho = \rho\tau, \sigma\rho = \rho\sigma, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \simeq C_n \times D_m$, and $\pi = \langle \sigma, \tau : \sigma^{2m} = \tau^4 = 1, \sigma^m = \tau^2, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \simeq Q_{4m}$,

Let π be the group as above and M be a π-lattice. Define another π-lattice M^* as follows. As an abelian group, $M^* = M$. If $x \in M^*$, $\lambda \in \pi$, denote the scalar product of λ and x in M^* by $\lambda * x$; define $\lambda * x := \epsilon(\lambda) \cdot (\lambda \cdot x)$ where $\lambda \cdot x$ is the scalar product in M.

It is easy to verify that $(M_1 \oplus M_2)^* \simeq M_1^* \oplus M_2^*$ and $(\mathbb{Z}\pi)^* \simeq \mathbb{Z}\pi$ (Reason: If $\sum_{\lambda \in \pi} n_\lambda\lambda$ is a general element in $\mathbb{Z}\pi$, then a $\mathbb{Z}\pi$-module isomorphism from $\mathbb{Z}\pi$ to $(\mathbb{Z}\pi)^*$ is given by sending $\sum_{\lambda \in \pi} n_\lambda\lambda$ to $\sum_{\lambda \in \pi} \epsilon(\lambda)n_\lambda\lambda$).

In particular, if $\mathcal{A}$ is a projective ideal over $\mathbb{Z}\pi$, $\mathcal{A}^*$ is also a projective ideal over $\mathbb{Z}\pi$. However, if M is an invertible π-lattice, it is not necessary that M^* should be invertible (when $\pi = D_m$, $\mathbb{Z}^*$ is not invertible).

THEOREM 4.4. *Let π be a group in Definition 4.3 with $\langle \sigma \rangle \simeq C_m$ where m is an integer ≥ 1 (note that this includes the case of Q_{4m} where $\langle \sigma \rangle \simeq C_{2m}$, i.e. regarding $2m$ as m). Let M be an invertible π-lattice and M^* be the π-lattice defined in Definition 4.3. Suppose that $n \mid m$.*
(i) *If $n = 1$ or $n = 2$ (when m is an even integer), then*

$$[M/(\sigma^n - 1)M]^{fl} = \sum_{d|n}[(M/\Phi_d(\sigma)M)_0]^{fl}, \text{ and}$$

$$[(M/\Phi_n(\sigma)M)_0]^{fl} = \sum_{d|n}\mu\left(\frac{n}{d}\right)[M/(\sigma^d - 1)M]^{fl}$$

where μ is the Möbius function.
(ii) *If $n \geq 3$, then*

$$[M/(\sigma^n - 1)M]^{fl} = [(M/\Phi_1(\sigma)M)_0]^{fl} + \sum_{d|n,d\geq 3}[(M^*/\Phi_d(\sigma)M^*)_0]^{fl} \text{ (n is odd)},$$

$$[M/(\sigma^n - 1)M]^{fl} = [(M/\Phi_1(\sigma)M)_0]^{fl} + [(M/\Phi_2(\sigma)M)_0]^{fl} \tag{1}$$
$$+ \sum_{d|n,d\geq 3}[(M^*/\Phi_d(\sigma)M^*)_0]^{fl} \text{ (n is even), and}$$

$$[(M^*/\Phi_n(\sigma)M^*)_0]^{fl} = \sum_{d|n}\mu\left(\frac{n}{d}\right)[M/(\sigma^d - 1)M]^{fl} \tag{2}$$

where μ is the Möbius function. In particular,

$$[\mathbb{Z}\pi/\langle\sigma^n - 1\rangle]^{fl} = \sum_{d|n}[\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle]^{fl}, \text{ and}$$

$$[\mathbb{Z}\pi/\langle\Phi_n(\sigma)\rangle]^{fl} = \sum_{d|n}\mu\left(\frac{n}{d}\right)[\mathbb{Z}\pi/\langle\sigma^d - 1\rangle]^{fl}.$$

Proof. For the case $n = 1$ or 2, we may regard the lattices $M/(\sigma^n - 1)M$, $(M/\Phi_d(\sigma)M)_0$ (where d divides n) as π'-lattices where $\pi' = \pi/\langle\sigma^2\rangle$. In this situation, σ belongs to the center of π'. Thus we may apply Theorem 4.2.

From now on, we will assume that $n \geq 3$.

We will follow the proof of [Sw5, Theorem 5.1 and Corollary 5.2] with necessary modifications. The Formulae (1) and (2) are equivalent. It suffices to prove Formula (2). As to the case $M = \mathbb{Z}\pi$, recall that $(\mathbb{Z}\pi)^* \simeq \mathbb{Z}\pi$; thus the formulae for $\mathbb{Z}\pi$ are consequences of Formulae (1) and (2).

Note that $M/(\sigma^n - 1)M$ is torsion-free, because M is invertible (see [Sw5, Lemma 5.3]).

Step 1. Let $n \mid m$ and $f(X) \cdot g(X) \mid X^n - 1$. Let $f(\sigma) \cdot \mathbb{Z}\pi$ be the right ideal of $\mathbb{Z}\pi$ generated by $f(\sigma)$. For any $\lambda \in \pi$, since $\lambda \cdot \sigma \cdot \lambda^{-1} = \sigma^i$ for some $i \geq 1$ with $\gcd\{i, m\} = 1$, it follows that $\lambda \cdot f(\sigma) \cdot \lambda^{-1} = f(\sigma^i)$. For any d with $d \mid m$, we have $\Phi_d(X^i) = \Phi_d(X) \cdot h(X)$ for some $h(X) \in \mathbb{Z}[X]$ because $\gcd\{i, d\} = 1$. As $f(X)$ is a product of cyclotomic polynomials $\Phi_d(X)$ with $d \mid n$, it follows that $f(\sigma^i) \in f(\sigma) \cdot \mathbb{Z}\pi$.

In summary, the right ideal $f(\sigma) \cdot \mathbb{Z}\pi$ is also a left ideal. We write it simply as $\langle f(\sigma) \rangle$. Hence $\mathbb{Z}\pi/\langle f(\sigma) \rangle$, $\mathbb{Z}\pi/\langle f(\sigma)g(\sigma) \rangle$, $\mathbb{Z}\pi/\langle g(\sigma) \rangle$ may be regarded as two-sided $\mathbb{Z}\pi$-modules.

In [Sw5, Lemma 4.2], the exact sequence

$$0 \to \mathbb{Z}\pi/\langle f(\sigma) \rangle \to \mathbb{Z}\pi/\langle f(\sigma)g(\sigma) \rangle \to \mathbb{Z}\pi/\langle g(\sigma) \rangle \to 0$$

is considered when π is a cyclic group. Note that the map $\mathbb{Z}\pi/\langle f(\sigma) \rangle \to \mathbb{Z}\pi/\langle f(\sigma)g(\sigma) \rangle$ is the multiplication map by $g(\sigma)$. A similar exact sequence as above when π is the dihedral group is not available because the multiplication map by $g(\sigma)$ is not a morphism of $\mathbb{Z}\pi$-modules. We will give a modification of it in Step 2.

For another modification, in [Sw5, pages 246–247], a sequence of integers e_k is defined and polynomials $g_k(X), h_k(X) \in \mathbb{Z}[X]$ are constructed such that the exact sequences

$$0 \to \mathbb{Z}\pi/\langle g_k(\sigma) \rangle \to \mathbb{Z}\pi/\langle h_k(\sigma) \rangle \to \mathbb{Z}\pi/\langle \sigma^{e_k} - 1 \rangle \to 0 \qquad (3)$$

satisfy the conditions

$$h_{2k}(X) = h_{2k+1}(X) \quad \text{and} \quad g_{2k-1}(X) = g_{2k}(X).$$

In fact, it is defined that $g_k(X) = f_{k-1}(X)$ if k is even, $g_k(X) = f_k(X)$ if k is odd, and $h_k(X) = f_k(X)$ if k is even, $h_k(X) = f_{k-1}(X)$ if k is odd (see the definition of $f_k(X)$ in [Sw5, page 246]).

We will find a modification of the above construction in Step 3. A concrete case will be illustrated in Step 4 to exemplify the ideas of Step 2 and Step 3. The general case will be proved in Step 6.

Step 2. Recall $\langle \sigma \rangle \simeq C_m$ where $m \geq 3$.

For each $d \mid m$ with $d \geq 3$, define $\widetilde{\Phi}_d = \sigma^{-\varphi(d)/2} \Phi_d(\sigma) \in \mathbb{Z}\pi$ where $\varphi(d)$ is the Euler φ-function. Define $\widetilde{\Phi}_1 = \sigma - 1$ and, if m is an even integer, define $\widetilde{\Phi}_2 = \sigma + 1 \in \mathbb{Z}\pi$.

Note that $\widetilde{\Phi}_d$ belongs to the center of $\mathbb{Z}\pi$ if $d \geq 3$.

To simplify the notation, we will write (d) for the two-sided ideal $\langle \widetilde{\Phi}_d \rangle = \langle \Phi_d(\sigma) \rangle$; thus $(d_1)(d_2) \cdots (d_t)$ is just the principal ideal generated by $\widetilde{\Phi}_{d_1} \cdot \widetilde{\Phi}_{d_2} \cdots \widetilde{\Phi}_{d_t}$.

Step 3. Let $p_1, p_2, \ldots, p_r$ be the distinct prime divisors of n; if n is an even integer, we choose to write $p_r = 2$, the last prime divisor of n. Define a sequence of integers

$d_0, d_1, \ldots, d_{2^r-1}$ as in [Sw5, page 246]. Define $d_0 = 1$, $d_1 = p_1$. If $d_0, d_1, \ldots, d_{2^{s-1}-1}$ have been defined for $s \geq 1$ (using $p_1, \ldots, p_{s-1}$), define $d_k = p_s d_{2^s-k-1}$ for $2^{s-1} \leq k \leq 2^s - 1$. Thus $d_{2^{s-1}}, \ldots, d_{2^s-1}$ are $p_s d_{2^{s-1}-1}, \ldots, p_s d_1, p_s d_0$ respectively.

Define $e_k = n/d_k$ for $0 \leq k \leq 2^r - 1$.

Consider the exact sequence of π-lattices

$$0 \to \langle \sigma^{e_{2^r-1}} - 1 \rangle / \langle \sigma^n - 1 \rangle \to \mathbb{Z}\pi / \langle \sigma^n - 1 \rangle \to \mathbb{Z}\pi / \langle \sigma^{e_{2^r-1}} - 1 \rangle \to 0. \qquad (4)$$

We designate $I^{(2^r-1)}/J^{(2^r-1)} = \langle \sigma^{e_{2^r-1}} - 1 \rangle / \langle \sigma^n - 1 \rangle$ and define $I^{(2^r-1)} = \langle \sigma^{e_{2^r-1}} - 1 \rangle$ and $J^{(2^r-1)} = \langle \sigma^n - 1 \rangle$. Then we will construct two-sided ideals $I^{(k)}$ and $J^{(k)}$ inductively where $1 \leq k \leq 2^r - 2$ so that we have the exact sequences

$$0 \to I^{(k)}/J^{(k)} \to \mathbb{Z}\pi / J^{(k)} \to \mathbb{Z}\pi / \langle \sigma^{e_k} - 1 \rangle \to 0 \qquad (5)$$

satisfying the conditions

$$J^{(2k)} = J^{(2k-1)} \quad \text{and} \quad I^{(2k)}/J^{(2k)} \simeq I^{(2k+1)}/J^{(2k+1)}. \qquad (6)$$

Assume that we have these exact sequences. Suppose M is an invertible π-lattice. Tensor it with these exact sequences (4) and (5) over $\mathbb{Z}\pi$. We get an exact sequence of $\mathbb{Z}\pi$-modules

$$\mathrm{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi / \langle \sigma^{e_k} - 1 \rangle, M) \to (I^{(k)}/J^{(k)}) \otimes_{\mathbb{Z}\pi} M \to M/J^{(k)}M \to M/(\sigma^{e_k} - 1)M \to 0.$$

Note that $\mathrm{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi / \langle \sigma^{e_k} - 1 \rangle, M) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Tor}_1^{\mathbb{Q}\pi}(\mathbb{Q}\pi / \langle \sigma^{e_k} - 1 \rangle, \mathbb{Q} \otimes_{\mathbb{Z}} M)$. Since $\mathbb{Q}\pi$ is a semi-simple algebra, it follows that $\mathrm{Tor}_1^{\mathbb{Q}\pi}(\cdot, \cdot) = 0$. Thus the kernel of $\{(I^{(k)}/J^{(k)}) \otimes_{\mathbb{Z}\pi} M \to M/J^{(k)}M\}$ is torsion. Now we may apply [Sw5, Lemma 5.4] because $M/(\sigma^{e_k} - 1)M$ is torsion-free.

Define $N_k = ((I^{(k)}/J^{(k)}) \otimes_{\mathbb{Z}\pi} M)_0$, $N_k' = (M/J^{(k)}M)_0$. Thus we obtain exact sequences of π-lattices

$$0 \to N_k \to N_k' \to M/(\sigma^{e_k} - 1)M \to 0$$

where $1 \leq k \leq 2^r - 1$.

Since $M/(\sigma^{e_k} - 1)M$ is invertible, apply Lemma 2.4. We find

$$[N_k']^{fl} = [N_k]^{fl} + [M/(\sigma^{e_k} - 1)M]^{fl}.$$

The conditions (6) ensure us to conclude that

$$[N_1]^{fl} = \sum (-1)^k [M/(\sigma^{e_k} - 1)M]^{fl} \qquad (7)$$

where $0 \leq k \leq 2^r - 1$. Because $\mu(d_k) = (-1)^k$ (see [Sw5, page 247]), we get $[N_1]^{fl} = \sum \mu(d_k)[M/(\sigma^{n/d_k} - 1)M]^{fl}$ where d_k runs over the square-free divisors of n. This finishes the proof of Formula (2).

Note that N_1 may be regarded as a module over $\mathbb{Z}\pi / \langle \Phi_n(\sigma) \rangle$. More precisely, we will show that N_1 is isomorphic to $(M^*/\Phi_n(\sigma)M^*)_0$ in Step 5 and Step 6.

Step 4. As an illustration, consider the case $m = n = pqr$ where p, q, r are distinct prime numbers; if m is even, we require that $r = 2$.

By Step 3, $d_0, d_1, d_2, \ldots, d_7$ are $1, p, qp, q, rq, rqp, rp, r$ respectively. Thus $e_0, e_1, \ldots, e_7$ are $pqr, qr, r, pr, p, 1, q, pq$.

Define $I^{(7)} = \langle \sigma^{pq} - 1 \rangle = (1)(p)(q)(pq)$, $J^{(7)} = \langle \sigma^{pqr} - 1 \rangle = \prod_{d|pqr}(d)$ (remember the notation $(d_1)(d_2) \cdots (d_t)$ in Step 2).

By Formula (5), because $e_6 = q$, define $I^{(6)} = \langle \sigma^q - 1 \rangle = (1)(q)$.

We will find $J^{(6)}$ such that $I^{(6)}/J^{(6)} \simeq I^{(7)}/J^{(7)}$ (to ensure the validity of Formula (6)). Define $J^{(6)} = (1)(q)(r)(pr)(qr)(pqr)$. Since $\widetilde{\Phi}_p, \widetilde{\Phi}_{pq}$ belong to the center of $\mathbb{Z}\pi$, the multiplication by $\widetilde{\Phi}_p \widetilde{\Phi}_{pq}$ is indeed a $\mathbb{Z}\pi$-isomorphism from $I^{(6)}/J^{(6)}$ to $I^{(7)}/J^{(7)}$.

We remark that it is not mysterious at all to find $J^{(6)}$. It is "equivalent" to find the product of cyclotomic polynomials $(X^{pqr} - 1)(X^{pq} - 1)^{-1}(X^q - 1) = (X^n - 1)(X^{e_7} - 1)^{-1}(X^{e_6} - 1)$.

Now we turn to $I^{(5)}$ and $J^{(5)}$ using the formulae (5) and (6).

By Formula (6), define $J^{(5)} = J^{(6)} = (1)(q)(r)(pr)(qr)(pqr)$. By Formula (5), define $I^{(5)} = \langle \sigma^{e_5} - 1 \rangle = (1)$. We get the exact sequence $0 \to I^{(5)}/J^{(5)} \to \mathbb{Z}\pi/J^{(5)} \to \mathbb{Z}\pi/\langle \sigma^{e_5} - 1 \rangle \to 0$ automatically.

Proceed as before. Define $I^{(4)} = \langle \sigma^{e_4} - 1 \rangle = (1)(p)$. We will find $J^{(4)}$ such that $I^{(4)}/J^{(4)} \simeq I^{(5)}/J^{(5)}$. Define $J^{(4)} = (1)(p)(q)(r)(pr)(qr)(pqr)$. Then the multiplication by $\widetilde{\Phi}_p$ gives the isomorphism $I^{(5)}/J^{(5)} \simeq I^{(4)}/J^{(4)}$.

Define $J^{(3)} = J^{(4)} = (1)(p)(q)(r)(pr)(qr)(pqr)$, $I^{(3)} = \langle \sigma^{e_3} - 1 \rangle = (1)(p)(r)(pr)$.

Define $I^{(2)} = \langle \sigma^{e_2} - 1 \rangle = (1)(r)$, $J^{(2)} = (1)(q)(r)(qr)(pqr)$. The multiplication by $\widetilde{\Phi}_p \widetilde{\Phi}_{pr}$ gives the isomorphism $I^{(2)}/J^{(2)} \simeq I^{(3)}/J^{(3)}$.

Define $J^{(1)} = J^{(2)} = (1)(q)(r)(qr)(pqr)$, $I^{(1)} = \langle \sigma^{e_1} - 1 \rangle = (1)(q)(r)(qr)$.

Note that $I^{(1)}/J^{(1)}$ is a module over $\mathbb{Z}\pi/\langle \Phi_{pqr}(\sigma) \rangle$. Moreover, $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ do not appear in the multiplication maps associated to the isomorphisms $I^{(7)}/J^{(7)} \simeq I^{(6)}/J^{(6)}$, $I^{(5)}/J^{(5)} \simeq I^{(4)}/J^{(4)}$ and $I^{(3)}/J^{(3)} \simeq I^{(2)}/J^{(2)}$.

Step 5. Continue the discussion of Step 4 for the case $m = pqr$. When M is an invertible π-lattice, we will show that $(I^{(1)}/J^{(1)}) \otimes_{\mathbb{Z}\pi} M \simeq M^*/\Phi_{pqr}(\sigma) \cdot M^*$ where $I^{(1)}$, $J^{(1)}$ are the two-sided ideals as in Step 4.

Recall that $I^{(1)} = \langle \sigma^{e_1} - 1 \rangle$ where $e_1 = n/p$ (in the general case, $e_1 = n/p_1$). We may regard $I^{(1)}/J^{(1)}$ as a two-sided cyclic $\mathbb{Z}\pi$-module with a suitable generator u.

Case 1. The order of σ is an odd integer m.

Define $\Psi_1 = \sigma - \sigma^{-1} \in \mathbb{Z}\pi$. Note that the two-sided ideals $\Psi_1 \cdot \mathbb{Z}\pi$ and $(\sigma - 1) \cdot \mathbb{Z}\pi$ are identical to each other. For, from $1 + \sigma + \cdots + \sigma^{m-1} = 1 + (1+\sigma)(\sigma + \sigma^3 + \cdots + \sigma^{m-2})$ (because m is odd), multiply both sides by $1 - \sigma$. We get $1 - \sigma \in \langle 1 - \sigma^2 \rangle$.

Also note that, for any $\lambda \in \pi$, $\lambda \Psi_1 \lambda^{-1} = \epsilon(\lambda) \cdot \Psi_1$.

Since $I^{(1)} = \langle \sigma^{e_1} - 1 \rangle = (1)(q)(r)(qr)$, we may define a generator u of $I^{(1)}/J^{(1)}$ by $u := \Psi_1 \widetilde{\Phi}_q \widetilde{\Phi}_r \widetilde{\Phi}_{qr} \in I^{(1)}/J^{(1)}$. In $\mathbb{Z}\pi$, note that $\lambda u \lambda^{-1} = \epsilon(\lambda)u$.

Define a map $\psi : M^* \to (I^{(1)}/J^{(1)}) \otimes_{\mathbb{Z}\pi} M$ by $\psi(x) = u \otimes x$ where $x \in M^*$. By the definition of M^*, it is easy to verify that ψ is a $\mathbb{Z}\pi$-morphism. Thus ψ is a surjective $\mathbb{Z}\pi$-morphism. When $M = (\mathbb{Z}\pi)^{(a)}$ (a free module), it is easy to see that ψ is a $\mathbb{Z}\pi$-isomorphism. In general, take a right exact sequence $(\mathbb{Z}\pi)^{(a)} \to (\mathbb{Z}\pi)^{(b)} \to M \to 0$. By the Three-Lemma, we find an isomorphism $\widetilde{\psi} : M^*/\Phi_{pqr}(\sigma)M^* \simeq (I^{(1)}/J^{(1)}) \otimes_{\mathbb{Z}\pi} M$.

Case 2. The order of σ is an even integer m.

Note that e_1 is an even integer because we require that $r = 2$ (in the general case, $p_r = 2$). Thus $I^{(1)} = \langle \sigma^{e_1} - 1 \rangle = \langle \sigma^{e_1/2} - \sigma^{-e_1/2} \rangle$

Now define the generator u of $I^{(1)}/J^{(1)}$ by $u := \sigma^{e_1/2} - \sigma^{-e_1/2}$. In $\mathbb{Z}\pi$, note that $\lambda u \lambda^{-1} = \epsilon(\lambda)u$ for any $\lambda \in \pi$.

Define a map $\psi : M^* \to (I^{(1)}/J^{(1)}) \otimes_{\mathbb{Z}\pi} M$ by $\psi(x) = u \otimes x$ where $x \in M^*$. As before, it is not difficult to show that we have an isomorphism $\widetilde{\psi} : M^*/\Phi_n(\sigma)M^* \simeq (I^{(1)}/J^{(1)}) \otimes_{\mathbb{Z}\pi} M$.

Step 6. Now consider the general case where n is any integer ≥ 3 with $n \mid m$. We keep the notations in Step 3; in particular, recall the prime divisors of n and the integers d_k, e_k when $0 \leq k \leq 2^r - 1$.

We will define monic polynomials $F_k \in \mathbb{Z}[X]$ (where $1 \leq k \leq 2^r - 1$). Using these integral polynomials, define $J^{(k)} = \langle F_k(\sigma) \rangle$ the principal two-sided ideal generated by $F_k(\sigma)$. We also define $I^{(k)} = \langle \sigma^{e_k} - 1 \rangle$. Then these $I^{(k)}$ and $J^{(k)}$ satisfy the conditions (4), (5), (6) in Step 3.

First of all, define $E_k(X) = X^{e_k} - 1 \in \mathbb{Z}[X]$ where $0 \leq k \leq 2^r - 1$. We will write E_k for $E_k(X)$ in the sequel.

For $1 \leq k \leq 2^r - 1$, define $F_k \in \mathbb{Z}[X]$ as follows. Define $F_{2^r-1} = E_0$, $F_1 = F_2, F_3 = F_4, \ldots, F_{2k-1} = F_{2k}, \ldots F_{2^r-3} = F_{2^r-2}$; define $F_{2k} = E_{2k}E_{2k+1}^{-1}E_{2k+2}E_{2k+3}^{-1} \cdots E_{2^r-1}^{-1}E_0$ if $2 \leq 2k \leq 2^r - 2$.

For $1 \leq k \leq 2^r - 1$, define $G_k \in \mathbb{Z}[X]$ which will be used in the multiplication isomorphisms of $I^{(2k)}/J^{(2k)}$ and $I^{(2k+1)}/J^{(2k+1)}$. Define $G_2 = G_3, G_4 = G_5, \ldots, G_{2k} = G_{2k+1}, \ldots, G_{2^r-2} = G_{2^r-1}$; define $G_{2k+1} = E_{2k+1}^{-1}E_{2k+2}E_{2k+3}^{-1}E_{2k+4} \ldots E_{2^r-1}^{-1}E_0$ if $1 \leq 2k + 1 \leq 2^r - 1$.

It is clear that $F_k = E_k G_k$ for $1 \leq k \leq 2^r - 1$.

Furthermore, all of these F_k and G_k are monic polynomials in $\mathbb{Z}[X]$ and each of them divides $X^n - 1$. Just compare the definitions of F_k and G_k with those of $f_k(X)$ in [Sw5, Lemma 5.6]: $f_k(X) = E_0 E_1^{-1} E_2 E_3^{-1} \cdots E_k^{(-1)^k}$. It is known that $f_k(X)$ is a monic polynomial dividing $X^n - 1$ (see [Sw5, Lemma 5.6]). It is not difficult to apply the same method in the proof of Lemma 5.6 of [Sw5] to prove the same results for F_k and G_k.

Define $I^{(k)} = \langle E_k(\sigma) \rangle = \langle \sigma^{e_k} - 1 \rangle$ and $J^{(k)} = \langle F_k(\sigma) \rangle$ for $1 \leq k \leq 2^r - 1$.

When $0 \leq k \leq 2^r - 1$, it is routine to verify that $d_{2k+1}/d_{2k} = p_1^{(-1)^k}$. Note that $G_{2k}(\sigma) = G_{2k+1}(\sigma)$. Now it is easy to show that (i) if k is an even integer, the isomorphism of $I^{(2k+1)}/J^{(2k+1)}$ to $I^{(2k)}/J^{(2k)}$ is given by the multiplication by $\prod_{t|e_{2k+1}} \tilde{\Phi}_{tp_1}$ (because $(X^{e_{2k+1}} - 1) \prod_{t|e_{2k+1}} \Phi_{tp_1}(X) = X^{e_{2k}} - 1$), and (ii) if k is an odd integer, the isomorphism of $I^{(2k)}/J^{(2k)}$ to $I^{(2k+1)}/J^{(2k+1)}$ is given by the multiplication by $\prod_{t|e_{2k}} \tilde{\Phi}_{tp_1}$ (because $(X^{e_{2k}} - 1) \prod_{t|e_{2k}} \Phi_{tp_1}(X) = X^{e_{2k+1}} - 1$). Hence the isomorphism of Formula (6) in Step 3 is proved.

Note that $G_1 = \langle \Phi_n(X) \rangle$. If we write $I^{(1)} = \prod_{1 \leq i \leq t}(d_i)$, then $J^{(1)} = (n) \cdot \prod_{1 \leq i \leq t}(d_i)$. It follows that $I^{(1)}/J^{(1)}$ is a module over $\mathbb{Z}\pi/\langle \Phi_n(\sigma) \rangle$. Now we may show that $I^{(1)}/J^{(1)} \otimes_{\mathbb{Z}\pi} M \simeq M^*/\Phi_n(\sigma)M^*$ by the same method as in Step 5. $\square$

LEMMA 4.5. *Let ζ be a primitive m-th root of unity in $\mathbb{C}$ where m is an odd integer ≥ 3. Let $\pi = \langle \tau \rangle \simeq C_2$ act on $\mathbb{Z}[\zeta]$ by $\tau \cdot \zeta = \zeta^{-1}$. Then $\mathbb{Z}[\zeta] \simeq (\mathbb{Z}\pi)^{(\phi(m)/2)}$ as π-lattices.*

Proof. Step 1. Denote by $\Phi_d(X)$ the d-th cyclotomic polynomial. Define $\Psi_d(X) = (X^d - 1)/(X - 1)$.

Write $m = \prod_{1 \leq i \leq t} p_i^{a_i}$ where $p_1, \ldots, p_t$ are distinct prime numbers and $a_i \geq 1$. We will show that the coefficient of $X^{\phi(m)/2}$ in $\Phi_m(X)$ is an odd integer. This is

obvious if m is an odd prime power. From now on we may assume that $t \geq 2$ in the above prime decomposition of m.

Define a monic polynomial $F(X) \in \mathbb{Z}[X]$ by $\Psi_m(X) = F(X) \prod_{1 \leq i \leq t} \Psi_{p^{a_i}}(X)$. It follows that $F(1) = 1$. Since $\Phi_m(X)$ is a factor of $F(X)$, thus $\Phi_m(1) = 1$ or -1.

From $\Phi_m(X) = X^{\phi(m)} \Phi_m(X^{-1})$, we find that $\Phi_m(X)$ is "symmetric" with respect to the term $X^{\phi(m)/2}$. Because $\Phi_m(1) = 1$ or -1, thus the coefficient of $X^{\phi(m)/2}$ in $\Phi_m(X)$ is an odd integer.

Step 2. Let c be the coefficient of $X^{\phi(m)/2}$ in $\Phi_m(X)$. It follows that $c\zeta^{\phi(m)/2} \in \sum_{0 \leq j \leq (\phi(m)/2)-1} \mathbb{Z} \cdot \zeta^j + \sum_{(\phi(m)/2)+1 \leq j \leq \phi(m)} \mathbb{Z} \cdot \zeta^j$.

Let $\mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at the prime ideal $(2) = 2\mathbb{Z}$. Hence we have $c\zeta^{\phi(m)/2} \in \sum_{0 \leq j \leq (\phi(m)/2)-1} \mathbb{Z}_{(2)} \cdot \zeta^j + \sum_{(\phi(m)/2)+1 \leq j \leq \phi(m)} \mathbb{Z}_{(2)} \cdot \zeta^j$. It follows that $1 \in \sum_{1 \leq j \leq \phi(m)/2} (\mathbb{Z}_{(2)} \cdot \zeta^{-j} + \mathbb{Z}_{(2)} \cdot \zeta^j)$ and $\mathbb{Z}_{(2)}[\zeta] = \oplus_{1 \leq j \leq \phi(m)/2} (\mathbb{Z}_{(2)} \cdot \zeta^{-j} + \mathbb{Z}_{(2)} \cdot \zeta^j)$.

Step 3. The action of π induces an action on $\mathbb{Z}_{(2)}[\zeta]$. From Step 2, we find that $\mathbb{Z}_{(2)}[\zeta] \simeq (\mathbb{Z}_{(2)}\pi)^{(\phi(m)/2)}$, a free module over $\mathbb{Z}_{(2)}\pi$. Thus $Ext^1_{\mathbb{Z}_{(2)}\pi}(\mathbb{Z}_{(2)}[\zeta], \cdot)$ is a zero functor.

By Reiner's Theorem [Sw2, page 74, Theorem 4.19], $\mathbb{Z}[\zeta]$ is a direct sum of $\mathbb{Z}$, $\mathbb{Z}_-$, or $\mathbb{Z}\pi$. If $\mathbb{Z}$ or $\mathbb{Z}_-$ is a direct summand of $\mathbb{Z}[\zeta]$, then $\mathbb{Z}_{(2)}$ or $(\mathbb{Z}_{(2)})_-$ is also a direct summand of $\mathbb{Z}_{(2)}[\zeta]$. By [Sw1], neither $\mathbb{Z}_{(2)}$ nor $(\mathbb{Z}_{(2)})_-$ is a projective module over $\mathbb{Z}_{(2)}\pi$ (by counting the ranks). Hence $Ext^1_{\mathbb{Z}_{(2)}\pi}(\mathbb{Z}_{(2)}[\zeta], \cdot)$ would not be a zero functor. This leads to a contradiction. Thus the only direct summands of $\mathbb{Z}[\zeta]$ are $\mathbb{Z}\pi$'s, i.e. $\mathbb{Z}[\zeta]$ is a free module. $\square$

The following theorem is a variant of the above Theorem 4.4 for the group $C_{q^f} \times D_m$.

THEOREM 4.6. *Let* $\pi = \langle \sigma, \rho, \tau : \rho^{q^f} = \sigma^m = \tau^2 = 1, \tau\rho\tau^{-1} = \rho, \tau\sigma\tau^{-1} = \sigma^{-1}, \sigma\rho = \rho\sigma \rangle \simeq C_{q^f} \times D_m$ *where* q *is an odd prime number,* $f \geq 1$, m *is an odd integer with* $m \geq 3$ *and* $\gcd\{q, m\} = 1$. *Suppose that* M *is an invertible* π-*lattice and* M^* *is the* π-*lattice defined in Definition 4.3. Then*

$$[M]^{fl} = [(M/\Phi_1(\sigma)M)_0]^{fl} + \sum_{d|m, d \geq 3} [(M^*/\Phi_d(\sigma)M^*)_0]^{fl} \ (m \text{ is odd}),$$

$$[M]^{fl} = [(M/\Phi_1(\sigma)M)_0]^{fl} + [(M/\Phi_2(\sigma)M)_0]^{fl}$$
$$+ \sum_{d|m, d \geq 3} [(M^*/\Phi_d(\sigma)M^*)_0]^{fl} \ (m \text{ is even}),$$

$$[(M^*/\Phi_m(\sigma)M^*)_0]^{fl} = \sum_{d|m} \mu\left(\frac{m}{d}\right) [M/(\sigma^d - 1)M]^{fl},$$

$$[(M^*/\Phi_m(\sigma)M^*)_0]^{fl} = [(M^*/\langle \Phi_{q^f}(\rho), \Phi_m(\sigma) \rangle M^*)_0]^{fl} + [M']^{fl}$$

where M' *is a lattice over* $\mathbb{Z}\pi/\pi'$ *for some normal subgroup* π' *with* $\pi' \neq \{1\}$.

Proof. The first two formulae follow from Theorem 4.4. It remains to prove the last formula.

Step 1. Consider the exact sequence $0 \to \mathbb{Z}\pi/\langle \Phi_{q^f}(\rho), \Phi_m(\sigma) \rangle \to \mathbb{Z}\pi/\langle \Phi_m(\sigma) \rangle \to \mathbb{Z}\pi/\langle \rho^{q^{f-1}} - 1, \Phi_m(\sigma) \rangle \to 0$ where the first map is defined to be the multiplication map by $\rho^{q^{f-1}} - 1$.

Tensor it with M^* over $\mathbb{Z}\pi$. Because M is an invertible lattice, it is not difficult to verify that the following is an exact sequence of π-lattices

$$0 \to (M^*/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle M^*)_0 \to (M^*/\Phi_m(\sigma)M^*)_0$$
$$\to (M^*/\langle\rho^{q^{f-1}} - 1, \Phi_m(\sigma)\rangle M^*)_0 \to 0$$

We remark that, for any π-lattice E, the lattice $(E^*/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle E^*)_0$ is isomorphic to the sublattice $(\rho^{q^{f-1}} - 1)g(\sigma) \cdot E$ of E where $g(X) \in \mathbb{Z}[X]$ is defined by $X^m - 1 = \Phi_m(X)g(X)$.

Write $M' := (M^*/\langle\rho^{q^{f-1}} - 1, \Phi_m(\sigma)\rangle M^*)_0$. M' may be regarded as a lattice over $\mathbb{Z}\pi/\langle\rho^{q^{f-1}}\rangle$.

In the next step we will show that $\hat{H}^0(\pi', (M^*/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle M^*)_0) = 0$ for all subgroups π' of π. Assume this result. We may apply Part (2) of Lemma 2.4 and obtain the desired formula

$$[(M^*/\Phi_m(\sigma)M^*)_0]^{fl} = [(M^*/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle M^*)_0]^{fl} + [M']^{fl}.$$

Step 2. We will show that, for any subgroup π' of π, $\hat{H}^0(\pi', N) = 0$ where $N := (M^*/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle M^*)_0$. Note that $\Phi_{q^f}(\rho) \cdot N = \Phi_m(\sigma) \cdot N = 0$. We may also assume that $\pi' \neq \{1\}$ and $N \neq 0$.

Since M is an invertible lattice, there is a π-lattice M'' such that $M \oplus M'' \simeq \oplus_{1 \leq i \leq s}\mathbb{Z}\pi/\pi_i$ where π_i's are some subgroups of π. If $\hat{H}^0(\pi', (\mathbb{Z}\pi/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle \otimes_{\mathbb{Z}\pi} (\mathbb{Z}\pi/\pi_i)^*)_0) = 0$ for all i, then $\hat{H}^0(\pi', N) = 0$. In other words, the problem is reduced to the case $M = \mathbb{Z}\pi/\pi_0$ where π_0 is a subgroup of π.

We claim that either $\pi_0 = \{1\}$ or $\pi_0 = \langle\sigma^i\tau\rangle$ for some integer i.

In fact, if $\pi_0 \cap \langle\rho\rangle \neq \{1\}$, then $\rho^{q^t} \in \pi_0$ for some $0 \leq t \leq f - 1$. Thus we have $(\rho^{q^t} - 1) \cdot N = 0$. But we also have $\Phi_{q^f}(\rho) \cdot N = 0$. Since N is torsion-free, this leads to the conclusion that $N = 0$, which has been ruled out at the beginning of this step. Similarly, it can be shown that $\pi_0 \cap \langle\sigma\rangle = \{1\}$.

In summary, we may assume that $M = \mathbb{Z}\pi$ or $M = \mathbb{Z}\pi/\langle\sigma^i\tau\rangle$. The situation of $M = \mathbb{Z}\pi/\langle\sigma^i\tau\rangle$ is similar to that of $M = \mathbb{Z}\pi/\langle\tau\rangle$. Thus we consider only the cases $M = \mathbb{Z}\pi$ and $M = \mathbb{Z}\pi/\langle\tau\rangle$.

Denote by ξ_1 and ξ_2 the images of ρ and σ in N respectively. Note that ξ_1 (resp. ξ_2) is a primitive q^f-th root of unity (resp. a primitive m-th root of unity). We will write $N = \mathbb{Z}[\xi_1, \xi_2, \tau]$ or $N = \mathbb{Z}[\xi_1, \xi_2]$ in the sequel.

For each prime divisor p of $| \pi' |$, choose a p-Sylow subgroup π'_p of π'. Since $\hat{H}^0(\pi', N) \to \oplus_p \hat{H}^0(\pi'_p, N)$ is injective, it suffices to show that $\hat{H}^0(\pi'_p, N) = 0$, i.e. without loss of generality, we may assume that π' is a p-group.

Subcase 1. $p = q$ and $\pi' = \langle\rho'\rangle \simeq C_{q^s}$.

Regard N as a π'-lattice. Then $N \simeq \mathbb{Z}[\xi_1]^{(a)}$ for some integer a. Let $\zeta_{q^s} \in \langle\xi_1\rangle$ be a primitive q^s-th root of unity. Regard $\mathbb{Z}[\xi_1]$ as a module over $\mathbb{Z}[\zeta_{q^s}]$; it is isomorphic to a direct sum of a free module and an ideal. Thus $N \simeq \mathbb{Z}[\xi_1]^{(a)} \simeq \mathbb{Z}[\zeta_{q^s}]^{(b)} \oplus I$ where b is some integer and I is a non-zero ideal of $\mathbb{Z}[\zeta_{q^s}]$. We will show that $\hat{H}^0(\pi', \mathbb{Z}[\zeta_{q^s}]) = 0 = \hat{H}^0(\pi', I)$.

Extend the action of ρ' to $\mathbb{Q}(\zeta_{q^s})$ and $\mathbb{Q}I(= \mathbb{Q}(\zeta_{q^s}))$. The characteristic polynomial of this linear map ρ' is $\Phi_{q^s}(X)$. Since $\Phi_{q^s}(1) \neq 0$, there is no vector in $\mathbb{Q}(\zeta_{q^s})$, which is left fixed by ρ'. Hence $\hat{H}^0(\pi', \mathbb{Z}[\zeta_{q^s}]) = 0$ and $\hat{H}^0(\pi', I) = 0$.

Subcase 2. p is a divisor of m and $\mid \pi' \mid = p^s$.

Regard N as a π'-lattice. Then $N \simeq \mathbb{Z}[\xi_2]^{(c)}$ for some integer c. Let $\zeta_{p^s} \in \langle \xi_2 \rangle$ be a primitive p^s-th root of unity. The remaining proof is similar to that of Subcase 1.

Subcase 3. $p = 2$ and $\pi' = \langle \sigma^i \tau \rangle$.

Because m is odd, $\sigma^i \tau$ is conjugate to τ. Thus $\hat{H}^0(\langle \sigma^i \tau \rangle, N) \simeq \hat{H}^0(\langle \tau \rangle, N)$. Hence we may assume that $\pi' = \langle \tau \rangle$ and it remains to show that $\hat{H}^0(\langle \tau \rangle, N) = 0$.

Suppose that $M = \mathbb{Z}\pi/\langle \tau \rangle$ (the situation for $M = \mathbb{Z}\pi$ is similar).

Write $N = \oplus_{i,j} \mathbb{Z}\xi_1^i \xi_2^j$ where $\tau \cdot \xi_1^i \xi_2^j = -\xi_1^i \xi_2^{-j}$ and $0 \leq i \leq \phi(q^f) - 1, 0 \leq j \leq \phi(m) - 1$. We will show that $N \simeq (\mathbb{Z}\pi')^{(d)}$ for some integer d. Assume this. From $\hat{H}^0(\langle \tau \rangle, \mathbb{Z}\pi') = 0$, it follows that $\hat{H}^0(\langle \tau \rangle, N) = 0$.

The proof that $N \simeq (\mathbb{Z}\pi')^{(d)}$ is similar to that of Lemma 4.5. In fact, let $\mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at the prime ideal $(2) = 2\mathbb{Z}$. By the same arguments as in Step 1 and Step 2 of the proof of Lemma 4.5, we find that $\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} N = \oplus_{0 \leq i \leq \phi(q^f)-1} \oplus_{1 \leq j \leq \phi(m)/2} (\mathbb{Z}_{(2)} \xi_1^i \xi_2^{-j} + \mathbb{Z}_{(2)} \xi_1^i \xi_2^j)$. Hence $\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} N \simeq (\mathbb{Z}_{(2)}\pi')^{(d)}$ for some integer d. Then apply Reiner's Theorem to N as in Step 3 of the proof of Lemma 4.5. We find that $N \simeq (\mathbb{Z}\pi')^{(d)}$. $\square$

REMARK. R. G. Swan kindly pointed out an error in the first version of this article. The exact sequence $0 \to \mathbb{Z}\pi/\langle f(\sigma) \rangle \to \mathbb{Z}\pi/\langle f(\sigma)g(\sigma) \rangle \to \mathbb{Z}\pi/\langle g(\sigma) \rangle \to 0$ in [Sw5, pages 246-247] should be taken carefully; it is not available if π is a non-abelian group.

We remark also that the five short exact sequences in the middle of page 96 of [EM3] were meant to be the exact sequences $0 \to \mathbb{Z}\pi/\langle g_k(\sigma) \rangle \to \mathbb{Z}\pi/\langle h_k(\sigma) \rangle \to \mathbb{Z}\pi/\langle \sigma^{e_k} - 1 \rangle \to 0$ of Formula (3) in Step 1 of the proof of Theorem 4.4. However, as pointed out before, when π is a non-abelian group (e.g. $\pi = D_m, C_n \times D_m, Q_{4m}$), there seems no obvious reason why the maps in these exact sequences should be $\mathbb{Z}\pi$-morphisms.

5. Proof of Theorem 1.4. We will devote this section to proving "$(1) \Rightarrow (2)$ of Theorem 1.4". The goal is to show that $T(\pi) = T^g(\pi)$, $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ and $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$.

The key idea for the proof of $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ is as follows. We use Theorem 2.14 (1) to prove $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$, i.e. for any invertible π-lattice M, we will find a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$. For this purpose, we apply Theorem 4.1, Theorem 4.4, etc. and reduce the question to the situation of $(M/\Phi_d(\sigma)M)_0$ or $(M^*/\Phi_d(\sigma)M^*)_0$ for any $d \mid n$ where $\langle \sigma \rangle \simeq C_n$ (the case $\pi = C_n \times D_m$ requires some more efforts). Since $(M/\Phi_d(\sigma)M)_0$ or $(M^*/\Phi_d(\sigma)M^*)_0$ is a torsion-free module over $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$, it is important to understand the structure of torsion-free modules over the $\mathbb{Z}$-order $\Lambda_d := \mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$. In most situations, Λ_d is a Dedekind domain, a twisted group ring or a maximal $\mathbb{Z}$-order. Thus the results in Section 3 are applicable. The final blow is to use Theorem 2.10, i.e. Jacobinski-Roiter's Theorem, to find the projective ideal $\mathcal{A}_d$ for $(M/\Phi_d(\sigma)M)_0$ or $(M^*/\Phi_d(\sigma)M^*)_0$.

Before the proof, we recall some basic facts of maximal orders [CR1, Section 26; Re].

DEFINITION 5.1. Let K be a field, Σ be a finite-dimensional separable algebra over K. Let R be a Dedekind domain with quotient field K, and Λ be a maximal R-order in Σ. A finitely generated left Λ-module M is called a Λ-lattice if it is a projective R-module.

THEOREM 5.2 ([CR1, page 565]). *Let the notations be the same as in Definition 5.1, let Λ be a maximal R-order and K be the quotient field of R. Then*

(1) *Λ is a left and right hereditary ring.*

(2) *Every left Λ-lattice is Λ-projective.*

(3) *A left Λ-lattice M is indecomposable if and only if KM is a simple module over $K\Lambda$.*

THEOREM 5.3 ([Re, page 176, Theorem 18.4]). *Let R be a DVR with quotient field K, Λ be an R-order such that $K\Lambda$ is a central simple algebra over a field containing K. Then Λ is a maximal order if and only if Λ is a hereditary ring and $\mathrm{rad}(\Lambda)$ is a maximal two-sided ideal where $\mathrm{rad}(\Lambda)$ is the Jacobson radical of Λ.*

Proof of (1) $\Rightarrow$ (2) of Theorem 1.4. First we show that $T(\pi) = T^g(\pi)$.

Note that π is a group such that all Sylow subgroups of π are cyclic. Hence any coflabby π-lattice is invertible by Theorem 2.6. On the other hand, if M is any π-lattice, by Lemma 2.5 (2), there is an exact sequence $0 \to M \to C \to P \to 0$ where C is coflabby (and hence invertible) and P is permutation. Apply Lemma 2.4 (1). We get $[C]^{fl} = [M]^{fl} + [P]^{fl} = [M]^{fl}$.

In conclusion, for any π-lattice M with $[M] \in T(\pi)$, there is an invertible π-lattice C such that $[M] = [C]$ in $T(\pi)$ by Lemma 2.2. Thus $T(\pi) = T^g(\pi)$.

We will use [EM2, page 707, Theorem 4.2] to show that $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$, i.e. $C^q(\mathbb{Z}\pi) = \widetilde{C}^q(\mathbb{Z}\pi) = \widetilde{C}(\mathbb{Z}\pi)$ (see Definition 2.13).

We remark first that [EM2, page 707, line 6] contains a misprint which should be corrected as follows: If $\pi = C \rtimes P$ where $C = \langle \sigma \rangle \simeq C_n$ is normal in π and $m \mid n$, the natural map $\mu_m : P \to \mathrm{Aut}(C/\langle \sigma^m \rangle)$ is induced from the action of P on C and $C/\langle \sigma^m \rangle$. Define $D_m = \mathrm{Ker}(\mu_m)$. In particular, $P_n = \{\lambda \in P : \lambda \sigma \lambda^{-1} = \sigma\}$.

Return to the group π in (1) of Theorem 1.4. We may write $\pi = C \rtimes P$ where C is cyclic, $P = \{1\}$, C_2 or C_4. Thus the assumptions (c) or (d) of [EM2, Theorem 4.2] are fulfilled. We conclude that $C^q(\mathbb{Z}\pi) = \widetilde{C}^q(\mathbb{Z}\pi) = \widetilde{C}(\mathbb{Z}\pi)$. Hence $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$.

By Theorem 2.14, to show that $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$, it suffices to show that, for any invertible π-lattice M, there is a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$. This is what we will prove in the sequel. The ideas of the proof have been explained at the beginning of this section. Once it is proved, the proof of (1) $\Rightarrow$ (2) is finished.

Case 1. $\pi = \langle \sigma \rangle \simeq C_n$.

W remark that this case is proved also in [Sw5, Theorem 2.10] using some ideas in [EM3]. In the following, we give a slightly different proof.

Step 1. For any $d \mid n$, note that $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle \simeq \mathbb{Z}[\zeta_d]$ where ζ_d is some primitive d-th root of unity.

Moreover, by Theorem 4.1, we find that

$$[\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = \sum_{e \mid d} \mu\left(\frac{d}{e}\right) [\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle]^{fl}.$$

Since $\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle \simeq \mathbb{Z}\pi/\pi'$ where $\pi' = \langle \sigma^e \rangle$, it follows that $\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle$ is a permutation π-lattice and therefore $[\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle]^{fl} = 0$. Thus $[\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = 0$.

Step 2. Suppose that M is an invertible π-lattice.

Apply Theorem 4.1. We get

$$[M]^{fl} = [M/(\sigma^n - 1)M]^{fl} = \sum_{d|n}[(M/\Phi_d(\sigma)M)_0]^{fl}.$$

Since $(M/\Phi_d(\sigma)M)_0$ is a torsion-free module over $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle \simeq \mathbb{Z}[\zeta_d]$, it follows that $(M/\Phi_d(\sigma)M)_0 \simeq F_d \oplus J_d$ where F_d is a free module over $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$, J_d is a non-zero ideal of $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$. Note that $[F_d]^{fl} = 0$ because $[\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle]^{fl} = 0$ by Step 1.

Thus we get $[M]^{fl} = \sum_{d|n}[J_d]^{fl}$.

Step 3. Each J_d is locally isomorphic to $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$. Hence J_d and $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$ belong to the same genus. Apply Theorem 2.10. We find a projective ideal $\mathcal{A}_d$ over $\mathbb{Z}\pi$ such that $J_d \oplus \mathbb{Z}\pi \simeq \mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle \oplus \mathcal{A}_d$. Hence $[J_d]^{fl} = [\mathcal{A}_d]^{fl}$ because $[\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle]^{fl} = 0$ by Step 1.

Step 4. By Swan's Theorem [CR1, page 677, Corollary 32.12], $\bigoplus_{d|n} \mathcal{A}_d \simeq F \oplus \mathcal{A}$ for some projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ and some free $\mathbb{Z}\pi$-module F. Hence $[M]^{fl} = \sum_{d|n}[J_d]^{fl} = \sum_{d|n}[\mathcal{A}_d]^{fl} = [F \oplus \mathcal{A}]^{fl} = [\mathcal{A}]^{fl}$ as expected.

Step 5. We rephrase the above result as a form which will be used in the sequel.

Let $\pi \simeq C_n$, M be a π-lattice. Then there is a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$.

As before, by Lemma 2.5 (2), find an exact sequence $0 \to M \to N \to P \to 0$ where N is coflabby and P is permutation. Apply Lemma 2.4 (1). We get $[N]^{fl} = [M]^{fl} + [P]^{fl} = [M]^{fl}$.

By Theorem 2.6 any coflabby π-lattice is invertible; thus N is invertible. Since N is invertible, it is possible to find a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[N]^{fl} = [\mathcal{A}]^{fl}$ from the above Steps 1,2,3 and 4. It follows that $[M]^{fl} = [\mathcal{A}]^{fl}$.

Case 2. $\pi = \langle\sigma, \tau : \sigma^m = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle \simeq D_m$ where m is an odd integer with $m \geq 3$.

Subcase 2.1 $m = p^c$ where p is an odd prime number and $c \geq 1$.

Step 1. We use similar methods as in Case 1 and apply Theorem 4.4. However, if $d \mid p^c$, $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$ is no longer isomorphic to $\mathbb{Z}[\zeta_d]$.

When $d = 1$, $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$ is isomorphic to $\mathbb{Z}\pi'$ where $\pi' \simeq C_2$. Since $(M^*/\Phi_d(\sigma)M^*)_0$ is a torsion-free module over $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$ (where M is an invertible π-lattice), we may apply Step 5 of Case 1. Thus there is a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi'(\simeq \mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle)$ such that $[(M/\Phi_d(\sigma)M)_0]^{fl} = [\mathcal{A}]^{fl}$. Note that $\mathcal{A}$ and $\mathbb{Z}\pi'$ are in the same genus as π'-lattices and also as π-lattices. Apply Theorem 2.10 to find a projective ideal $\mathcal{B}$ over $\mathbb{Z}\pi$ such that $\mathcal{A} \oplus \mathbb{Z}\pi \simeq \mathbb{Z}\pi' \oplus \mathcal{B}$. Hence $[(M/\Phi_d(\sigma)M)_0]^{fl} = [\mathcal{B}]^{fl}$. Done.

When $d \mid p^c$ and $d > 1$, we will show that $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$ is isomorphic to a twisted group ring.

Step 2. For any $d \mid p^c$ such that $d > 1$, write $\Lambda_d := \mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle = S_d \circ H$ where $S_d = \mathbb{Z}[\zeta_d]$, $H = \langle\tau\rangle$ and $\zeta_d \in S_d$ is the image of σ in $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$. Define $R_d = S_d^{\langle\tau\rangle}$; note that $\tau \cdot \zeta_d = \zeta_d^{-1}$.

The only prime ideal which ramifies in $\mathbb{Z}[\zeta_{p^c}]$ over $\mathbb{Z}$ is the prime ideal lying over $p\mathbb{Z}$ and $p\mathbb{Z}[\zeta_{p^c}] = Q^{p^{c-1}(p-1)}$ where $Q = (1 - \zeta_{p^c})\mathbb{Z}[\zeta_{p^c}]$ (see, for example, [CR1, page 96]). Thus, for any $d \mid p^c$ (with $d > 1$), S_d is tamely ramified over R_d.

By Theorem 3.6, Λ_d is a left hereditary ring and the ambiguous ideals are isomorphic to IS_d or IQ_d where I is an ideal in R_d and $Q_d = Q \cap S_d$ where $Q = (1-\zeta_{p^c})\mathbb{Z}[\zeta_{p^c}]$

Note that IS_d and S_d belong to the same genus because they are locally isomorphic; similarly for IQ_d and Q_d.

Step 3. For any $d \mid p^c$, applying Theorem 4.4, we find that $[\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = 0$ because $\mathbb{Z}\pi/\langle \sigma^{e'} - 1 \rangle \simeq \mathbb{Z}[\pi/\pi']$ where $\pi' = \langle \sigma^{e'} \rangle$ if $e' \mid d$. In conclusion, $[\Lambda_d]^{fl} = 0$.

On the other hand, if $d \mid p^c$, define $\pi'' = \langle \sigma^d, \tau \rangle$ and $N = \mathbb{Z}[\pi/\pi'']$. It is not difficult to verify that $(N^*/\Phi_d(\sigma)N^*)_0 \simeq Q_d$ as π-lattices.

Apply Theorem 4.4 to $N = \mathbb{Z}[\pi/\pi'']$. For any $e' \mid d$, since $N/(\sigma^{e'} - 1)N \simeq \mathbb{Z}[\pi/\pi'] \otimes_{\mathbb{Z}\pi} \mathbb{Z}[\pi/\pi'']$ (where $\pi' = \langle \sigma^{e'} \rangle$) is a permutation π-lattice by [Sw5, Lemma 5.3], it follows that $[Q_d]^{fl} = 0$.

Step 4. For any $d \mid p^c$ (with $d > 1$), we will prove in Step 6 that $\Lambda_d \simeq S_d \oplus Q_d$ as Λ_d-modules, and hence as π-lattices.

Assume the above claim. By Step 3, $[\Lambda_d]^{fl} = [Q_d]^{fl} = 0$, it follows that $[S_d]^{fl} = 0$.

Let M be an invertible π-lattice. Apply Theorem 4.4 as in Step 3 of Case 1. Since $(M^*/\Phi_d(\sigma)M^*)_0$ is a torsion-free module over $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle = \Lambda_d$, we may apply results in Step 2. Thus $(M^*/\Phi_d(\sigma)M^*)_0$ is a direct sum of $S_d I_i$ ($1 \le i \le u$) and $Q_d I_j$ ($1 \le j \le v$) where I_i, I_j are ideals in R_d. In particular, $(M^*/\Phi_d(\sigma)M^*)_0$ and $S_d^{(u)} \oplus (Q_d)^{(v)}$ are in the same genus. Applying Theorem 2.10, we find a projective ideal $\mathcal{A}_d$ over $\mathbb{Z}\pi$ such that $(M^*/\Phi_d(\sigma)M^*)_0 \oplus \mathbb{Z}\pi \simeq (S_d^{(u)} \oplus (Q_d)^{(v)}) \oplus \mathcal{A}_d$. Thus $[(M^*/\Phi_d(\sigma)M^*)_0]^{fl} = [\mathcal{A}_d]^{fl}$.

Step 5. From Step 1 and Step 4, we find $[M]^{fl} = \sum_{d|n} [\mathcal{A}_d]^{fl}$ (the reader may compare the present situation with Step 3 of Case 1).

Use the same argument as in Step 4 of Case 1. It is easy to find a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ satisfying $[M]^{fl} = [\mathcal{A}]^{fl}$.

Step 6. It remains to show that $\Lambda_d \simeq S_d \oplus Q_d$.

Recall that $\Lambda_d = S_d \circ H = S_d + S_d u_\tau$ such that $u_\tau^2 = 1$, $u_\tau \cdot \alpha = \tau(\alpha) \cdot u_\tau$ for any $\alpha \in S_d$. Remember that ζ_d is the image of σ in Λ_d, and $S_d = \mathbb{Z}[\zeta_d]$, $Q_d = (\zeta_d - \zeta_d^{-1})S_d$.

Define $w = \zeta_d + \zeta_d^2 + \ldots + \zeta_d^{(d-1)/2} \in \Lambda_d$ and $e = -\tau(w)(1 + u_\tau) \in \Lambda_d$. Note that w, $\tau(w)$ are units in S_d and $1 + w + \tau(w) = 0$.

It is not difficult to show that $e^2 = e$. It follows that $\Lambda_d = \Lambda_d e \oplus \Lambda_d(1 - e)$.

Note that $\Lambda_d e = \Lambda_d(1 + u_\tau) = S_d(1 + u_\tau) \simeq S_d$. On the other hand, from $1 - e = -(1 - u_\tau)w$, we find that $\Lambda_d(1 - e) = S_d(1 - u_\tau)w \simeq Q_d$. Hence the result.

Subcase 2.2 There exist distinct odd prime numbers p_1 and p_2 such that $p_1 p_2 \mid m$.

Step 1. Let m' be an odd integer such that $p_1 p_2 \mid m'$ for two distinct prime numbers. Then $\mathbb{Z}[\zeta_{m'}]$ is unramified over $\mathbb{Z}[\zeta_{m'} + \zeta_{m'}^{-1}]$ by [Wa, page 16, Proposition 2.15]. This observation will play a crucial role in the subsequent proof.

Step 2. Let M be any invertible π-lattice. We will find a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$. The proof is similar to that of Subcase 2.1.

Apply Theorem 4.4. We may reduce the problem to the case $(M^*/\Phi_d(\sigma)M^*)_0$ for any $d \mid m$.

If $d = 1$, then Step 5 of Case 1 takes care of this situation (see Step 1 of Subcase 2.1).

If $d > 1$, note that $(M^*/\Phi_d(\sigma)M^*)_0$ is a torsion-free module over $\Lambda_d :=$ $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle = S_d \circ H$ where $S_d = \mathbb{Z}[\zeta_d]$, $H = \langle \tau \rangle$ and ζ_d is the image of σ in $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$. The group H acts on S_d by $\tau \cdot \zeta_d = \zeta_d^{-1}$. Define $R_d = S_d^{\langle \tau \rangle} = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$.

Suppose that d has two distinct prime divisors. Then S_d/R_d is unramified by Step 1. Thus Λ_d is hereditary; moreover, $S_d I$ are the only ambiguous ideals of S_d (where I runs over some ideals in R_d) by Theorem 3.6. We conclude that $(M^*/\Phi_d(\sigma)M^*)_0$ and $S_d^{(u)}$ belong to the same genus. It can be proved that $[S_d]^{fl} = 0$ as in Step 3 and Step 6 of Subcase 2.1. Thus we may find a projective ideal $\mathcal{A}_d$ over $\mathbb{Z}\pi$ satisfying $[(M^*/\Phi_d(\sigma)M^*)_0]^{fl} = [\mathcal{A}_d]^{fl}$ as Step 4 of Subcase 2.1.

On the other hand if $d = p^c$ for some odd prime number p and some $c \geq 1$, then the ideal $(1 - \zeta_{p^c})S_d$ is the only ramified prime ideal of S_d over R_d; moreover, S_d is tamely ramified over R_d. The remaining proof is the same as that of Subcase 2.1. Done.

Case 3. $\pi = \langle \sigma, \rho, \tau : \rho^{q^f} = \sigma^m = \tau^2 = 1, \tau^{-1}\rho\tau = \rho, \tau^{-1}\sigma\tau = \sigma^{-1}, \sigma\rho = \rho\sigma \rangle \simeq C_{q^f} \times D_m$ where q is an odd prime number, $f \geq 1$, m is an odd integer ≥ 3, $\gcd\{q, m\} = 1$, and $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle$ for any prime divisor p of m.

We will solve this case by induction on the order of the group π.

Step 1. Suppose that M be an invertible π-lattice. Apply Theorem 4.6.

When $d \mid m$ and $d \neq m$, the lattice $(M^*/\Phi_d(\sigma)M^*)_0$ may be regarded as a lattice over $\mathbb{Z}\pi/\langle\sigma^d\rangle$. Replacing this lattice by a coflabby lattice (applying Lemma 2.5) and using the induction hypothesis, we find a projective ideal $\mathcal{B}_d$ over $\mathbb{Z}\pi/\langle\sigma^d\rangle$ such that $[(M^*/\Phi_d(\sigma)M^*)_0]^{fl} = [\mathcal{B}_d]^{fl}$. By Theorem 2.10, find a projective ideal $\mathcal{A}_d$ over $\mathbb{Z}\pi$ such that $\mathcal{B}_d \oplus \mathbb{Z}\pi \simeq \mathbb{Z}\pi/\langle\sigma^d\rangle \oplus \mathcal{A}_d$. Hence $[(M^*/\Phi_d(\sigma)M^*)_0]^{fl} = [\mathcal{A}_d]^{fl}$. Similarly, for the lattice M' in Theorem 4.6, by induction on the order of the group π, we also can find a projective ideal $\mathcal{A}'$ over $\mathbb{Z}\pi$ such that $[M']^{fl} = [\mathcal{A}']^{fl}$.

By Theorem 4.6, $[M]^{fl} = [N]^{fl} + [\mathcal{A}]^{fl}$ where $N := (M^*/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle M^*)_0$ and $\mathcal{A}$ is a projective ideal over $\mathbb{Z}\pi$.

In summary, we have reduced the question from $[M]^{fl}$ to $[N]^{fl}$ where N is a torsion-free module over Λ with $\Lambda := \mathbb{Z}\pi/\langle\Phi_{q^f}(\rho), \Phi_m(\sigma)\rangle$.

Step 2. Denote by ζ_{q^f} and ζ_m the images of ρ and σ in Λ respectively. Note that ζ_{q^f} (resp. ζ_m) is a primitive q^f-th root of unity (resp. a primitive m-th root of unity). In the sequel we will write $\Lambda = \mathbb{Z}[\zeta_{q^f}, \zeta_m, \tau]$. Note that $\tau^{-1} \cdot \zeta_{q^f} \cdot \tau = \zeta_{q^f}$, $\tau^{-1}\zeta_m\tau = \zeta_m^{-1}$ and $\mathbb{Z}[\zeta_{q^f}, \zeta_m] = \mathbb{Z}[\zeta_{q^f}] \otimes_\mathbb{Z} \mathbb{Z}[\zeta_m]$, because $\gcd\{q, m\} = 1$.

Λ can be written as a twisted group ring: $\Lambda = S \circ H$ where $S = \mathbb{Z}[\zeta_{q^f}, \zeta_m]$ and $H = \langle\tau\rangle$. Define $R = S^{\langle\tau\rangle} = \mathbb{Z}[\zeta_{q^f}][\zeta_m + \zeta_m^{-1}]$.

Assume that $p_1 p_2 \mid m$ for two distinct odd prime numbers (the case $m = p^c$ for an odd prime number will be considered in Step 3).

Then S/R is unramified as in Step 1 of Subcase 2.2. Hence Λ is hereditary; moreover, SI are the only ambiguous ideals of S (where I runs over some ideals in R) by Theorem 3.6. It follows that N and $S^{(u)}$ are in the same genus for some integer u. The remaining proof is the same as in the proof of Step 2 of Subcase 2.2.

Step 3. Suppose that $m = p^c$ for some odd prime number p and some $c \geq 1$.

Use the assumption that $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle\bar{p}\rangle$. This assumption is equivalent to the fact that $p \cdot \mathbb{Z}[\zeta_{q^f}]$ is a prime ideal in $\mathbb{Z}[\zeta_{q^f}]$ (just think of the decomposition of $p \cdot \mathbb{Z}[\zeta_{q^f}]$ and the Frobenius automorphism associated to a prime ideal Q of $\mathbb{Z}[\zeta_{q^f}]$ such that $p \in Q$).

Return to $\Lambda = S \circ H$ with $H = \langle\tau\rangle$, $R = S^{\langle\tau\rangle}$.

We claim that the principal ideal $\langle 1 - \zeta_{p^c}\rangle = (1 - \zeta_{p^c})S$ is a prime ideal. For, $S/\langle 1 - \zeta_{p^c}\rangle = \mathbb{Z}[\zeta_{p^c}, \zeta_{q^f}]/\langle 1 - \zeta_{p^c}\rangle \simeq \mathbb{F}_p[X]/\Phi_{q^f}(X)$ is a field because $p \cdot \mathbb{Z}[\zeta_{q^f}]$ is a prime ideal (see, for example, [Wa, page 15, Proposition 2.14]).

Note that Λ is tamely ramified. Define $Q = \langle 1 - \zeta_{p^c} \rangle$. As in Subcase 2.1, it can be shown that $[\Lambda]^{fl} = [S]^{fl} = [Q]^{fl} = 0$. The remaining proof is the same and is omitted.

Step 4. Finally we remark that we cannot replace the prime power q^f in the assumption by an odd integer n with $\gcd\{n, m\} = 1$, because $\mathbb{F}_p[X]/\Phi_n(X)$ is not an integral domain if $q_1 q_2 \mid n$ for two distinct odd prime numbers q_1, q_2 different from p. Thus $\langle 1 - \zeta_{p^c} \rangle$ is not a prime ideal of S if $\mathbb{F}_p[X]/\Phi_n(X)$ is not a field. See [EM3, page 188] for further investigation.

Case 4. $\pi = \langle \sigma, \tau : \sigma^{2m} = \tau^4 = 1, \sigma^m = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle \simeq Q_{4m}$ where m is an odd integer ≥ 3 such that $p \equiv 3 \pmod 4$ for any prime divisor p of m.

Step 1. We follow the method in Cases 1,2,3.

Let M be an invertible π-lattice.

Suppose $(\sigma^n - 1) \cdot M = 0$ for some $n \mid 2m$. If $n = 1$ or 2, we may regard M as a lattice over $\mathbb{Z}\pi''$ where $\pi'' = \pi/\langle \sigma^n \rangle$ and $\pi'' \simeq C_2$ or C_4. Apply Case 1.

From now on, we assume that $n \geq 3$ and apply Theorem 4.4.

For any $d \mid n$, consider $(M^*/\Phi_d(\sigma)M^*)_0$ which is a module over $\Lambda_d := \mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$.

If $d = 1$, then $\Lambda_d \simeq \mathbb{Z}\pi'$ where $\pi' \simeq C_2$. If $d = 2$, then $\Lambda_d \simeq \mathbb{Z}[\sqrt{-1}]$; thus $\mathbb{Z}\pi/\langle \sigma^2 \rangle \to \Lambda_d$ is surjective (note that $\mathbb{Z}\pi/\langle \sigma^2 \rangle \simeq \mathbb{Z}\pi'$ where $\pi' \simeq C_4$). Apply the methods in Step 1 of Subcase 2.1 and Step 2 of Case 3 to settle these two cases.

Now consider the case $d \geq 3$.

Write ζ_d and u_τ to be the images of σ and τ in Λ_d respectively. Define $S_d = \mathbb{Z}[\zeta_d]$.

If $d \mid m$, from $\tau^2 = \sigma^m$, we find $u_\tau^2 = 1$ in S_d. Thus $\Lambda_d \simeq S_d \circ H$ where $H = \langle \tau' \rangle \simeq C_2$. The proof is the same as Case 2.

If $d \mid 2m$, $d \nmid m$ and $d \geq 3$, since $\tau^2 = \sigma^m$, $(\sigma^m)^2 = \sigma^{2m} = 1$, we find $u_\tau^2 = -1$ in S_d.

Thus $\Lambda_d = S_d + S_d \cdot u_\tau$ where $u_\tau^2 = -1$. Note that Λ_d may be regarded as a crossed-product order $\Lambda_d = (S_d \circ H)_f$ where $H = \langle \tau' \rangle \simeq C_2$, $f : H \times H \to U(S_d)$ is defined as $f(1,1) = f(1,\tau') = f(\tau',1) = 1$, $f(\tau',\tau') = -1$. Define $R_d = S_d^{\langle \tau' \rangle}$ where $\tau' \cdot \zeta_d = \zeta_d^{-1}$.

Step 2. From now on till the end of the proof, we assume $d \mid 2m$, $d \nmid m$ and $d \geq 3$. In Step 3 and Step 4, we will show that Λ_d is a maximal R_d-order in $(L \circ H)_f$ where $L = \mathbb{Q}(\zeta_d)$.

Assume this fact. Then Λ_d is hereditary and any Λ_d-lattice is a direct sum of indecomposable projective Λ_d-modules by Theorem 5.2.

Let K be the quotient field of R_d. Since $K\Lambda_d$ is a central simple K-algebra of degree 2, it is either a central division K-algebra or is isomorphic to $M_2(K)$. We will show that $K\Lambda_d$ is a central division K-algebra. Otherwise, the 2-cocycle f is a 2-coboundary. Equivalently, -1 belongs to the image of the norm map from L to K. This implies that there exist $a, b \in K$ satisfying that $-1 = a^2 + (\zeta_d + \zeta_d^{-1})ab + b^2$, which is impossible because K is a real field.

Since $K\Lambda_d$ is a division ring, it is obvious that Λ_d is an indecomposable projective Λ_d-module by Theorem 5.2.

We may prove $[\Lambda_d]^{fl} = 0$ as in Case 2. By Theorem 5.2, $(M^*/\Phi_d(\sigma)M^*)_0$ is in the same genus as $\Lambda_d^{(u)}$. Apply Theorem 2.10 as in Case 2. Thus there is a projective ideal $\mathcal{A}_d$ over $\mathbb{Z}\pi$ such that $[(M^*/\Phi_d(\sigma)M^*)_0]^{fl} = [\mathcal{A}_d]^{fl}$. The remaining proof is the same as before.

Step 3. We will show that Λ_d is a maximal order if $p_1 p_2 \mid d$ for two distinct odd prime numbers p_1 and p_2. The situation when $d = p^c$ or $2p^c$ (where p is an odd prime number and $c \geq 1$) will be taken care of in Step 4.

Since $p_1 p_2 \mid d$, S_d is unramified over R_d. Thus S_d is a Galois extension of R_d relative to $H = \langle \tau' \rangle \simeq C_2$ in the sense of [AG, pages 395–402]. Thus $\Lambda_d = (S_d \circ H)_f$ is a central separable R_d-algebra by [AG, page 402, Theorem A.12]. Hence Λ_d is a maximal order in $K\Lambda_d$ (where K is the quotient field of R_d) by [AG, page 386, Proposition 7.1].

Step 4. When $d = 2p^c$ for some odd prime number p, we will show that Λ_d is a maximal order.

Note that $\zeta_{2p^c} = -\zeta_{p^c}$. For simplicity, we write Λ for Λ_d throughout this step, i.e. $\Lambda = S + S \cdot u_\tau$ where $S = \mathbb{Z}[\zeta_{p^c}]$, $\tau \cdot \zeta_{p^c} = \zeta_{p^c}^{-1}$, $u_\tau^2 = -1$ and $R = S^{\langle \tau \rangle} = \mathbb{Z}[\zeta_{p^c} + \zeta_{p^c}^{-1}]$.

Note that to be a maximal order is a local property [Re, page 132, Corollary 11.2]. We will check it at all localizations of R.

Let P' be a prime ideal of R such that $p \notin P'$. Write $\Lambda_{P'}$, $S_{P'}$ and $R_{P'}$ for the localizations of Λ, S and R at P'. It follows that $S_{P'}$ is unramified over $R_{P'}$. Hence $\Lambda_{P'}$ is a central separable algebra as in Step 3. Thus $\Lambda_{P'}$ is a maximal order.

On the other hand, let P be the prime ideal of R with $p \in P$. We will show that Λ_P is also a maximal order.

Since S_P is tamely ramified over R_P, Λ_p is a hereditary order by Theorem 3.2.

We will apply Theorem 5.3 to show that Λ_P is a maximal order. Let K be the quotient field of R. It is clear that $K\Lambda = K\Lambda_P$ is a central simple K-algebra. It remains to verify that $\mathrm{rad}(\Lambda_P)$ is a maximal two-sided ideal of Λ_P where $\mathrm{rad}(\Lambda_P)$ is the Jacobson radical of Λ_P.

Consider $R_P \subset S_P \subset \Lambda_P$. It is clear that $\mathrm{rad}(S_P) = (1 - \zeta_{p^c})S_P$. Thus $(1 - \zeta_{p^c})(1 - \zeta_{p^c}^{-1}) \in R_P \cap \mathrm{rad}(S_P) = \mathrm{rad}(R_P)$, since S_P is integral over R_P. It follows that $(1 - \zeta_{p^c})(1 - \zeta_{p^c}^{-1}) \in \mathrm{rad}(\Lambda_P)$ because $\mathrm{rad}(R_P)\Lambda_P \subset \mathrm{rad}(\Lambda_P)$ by [Re, page 82, Theorem 6.15]. On the other hand, $(1 - \zeta_{p^c})(1 - \zeta_{p^c}^{-1}) = -\zeta_{p^c}^{-1}(1 - \zeta_{p^c})^2$. We find that $(1 - \zeta_{p^c})^2 \in \mathrm{rad}(\Lambda_P)$. Hence $1 - \zeta_{p^c} \in \mathrm{rad}(\Lambda_P)$ (because, for any simple module M over Λ_P, $(1 - \zeta_{p^c})^2 M = 0$ implies $(1 - \zeta_{p^c})M = 0$). We conclude that the two-sided ideal $\langle 1 - \zeta_{p^c} \rangle = (1 - \zeta_{p^c})\Lambda_P$ is contained in $\mathrm{rad}(\Lambda_P)$.

We will show that $\Lambda_P / \langle 1 - \zeta_{p^c} \rangle$ is a field. Once it is proved, then $\langle 1 - \zeta_{p^c} \rangle = \mathrm{rad}(\Lambda_P)$ and $\Lambda_P / \mathrm{rad}(\Lambda_P)$ is a field. And therefore $\mathrm{rad}(\Lambda_P)$ is a maximal two-sided ideal of Λ_P.

It remains to show that $\Lambda_P / \langle 1 - \zeta_{p^c} \rangle$ is a field. Note that $p \equiv 3 \pmod 4$ by assumption; this is equivalent to $\mathbb{F}_p[X]/\langle X^2 + 1 \rangle$ is a field.

Note that $p \in \langle 1 - \zeta_{p^c} \rangle$. We have $u_\tau \cdot \zeta_{p^c} \cdot u_\tau^{-1} = \zeta_{p^c}^{-1}$. However, in $\Lambda_d / \langle 1 - \zeta_{p^c} \rangle$ we find that $\bar{u}_\tau \cdot \bar{\zeta}_{p^c} = \bar{\zeta}_{p^c}^{-1} \bar{u}_\tau = \bar{\zeta}_{p^c} \cdot \bar{u}_\tau + (\bar{\zeta}_{p^c}^{-1} - \bar{\zeta}_{p^c})\bar{u}_\tau = \bar{\zeta}_{p^c} \cdot \bar{u}_\tau$ because $\zeta_{p^c}^{-1} - \zeta_{p^c} = \zeta_{p^c}^{-1}(1 + \zeta_{p^c})(1 - \zeta_{p^c}) \in \langle 1 - \zeta_{p^c} \rangle$.

Thus $\Lambda_P / \langle 1 - \zeta_{p^c} \rangle$ is a commutative ring; furthermore, $\Lambda_P / \langle 1 - \zeta_{p^c} \rangle \simeq \mathbb{F}_p[\zeta_{p^c}, u_\tau]/\langle 1 - \zeta_{p^c} \rangle \simeq \mathbb{F}_p[X]/(X^2 + 1)$ which is a field by assumption.

A final remark. Since $\Lambda_P / \mathrm{rad}(\Lambda_P)$ is a field, it follows that $K\Lambda = K\Lambda_P$ is a division ring by [Re, page 189, Theorem 21.6], which has been proved in Step 2. $\square$

We explain briefly the strategy of the proof of Theorem 1.4. In [EM3], this theorem is proved by showing $(2) \Rightarrow (3) \Rightarrow (1') \Rightarrow (1)$ where $(1')$ is given as follows.

$(1')$ π is isomorphic to

(i') a cyclic group, or

(ii') $C \times H$ where $C = \langle \sigma \rangle$ is a cyclic group of order n, $H = \langle \rho, \tau : \rho^m = \tau^{2^d} = 1, \tau^{-1}\rho\tau = \rho^{-1} \rangle$ such that n and m are odd positive integers, $\gcd\{n, m\} = 1$, $d \geq 1$, and, for any prime divisor p of m, the principal ideal $p\mathbb{Z}[\zeta_{n \cdot 2^d}]$ is a prime ideal in $\mathbb{Z}[\zeta_{n \cdot 2^d}]$.

The implication of $(2) \Rightarrow (3)$ is easy, because $C(\mathbb{Z}\pi) \rightarrow C(\Omega_{\mathbb{Z}\pi})$ is surjective (see Definition 2.12).

The proof of $(3) \Rightarrow (1')$ was given in [EM3, pages 97–98, pages 188–189]. We will not repeat the proof.

For the remaining part of this section, we will give a proof of $(1) \Leftrightarrow (1')$.

Proof of $(1) \Leftrightarrow (1')$. For the proof of $(1) \Rightarrow (1')$, if $\pi \simeq C_n$ or D_m, it is trivial to see that π belongs to the class described in $(1')$. If $\pi = Q_{4m} = \langle \sigma, \tau : \sigma^{2m} = \tau^4 = 1, \sigma^m = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$, define $\rho = \sigma^2$. Then $\pi = \langle \rho, \tau : \rho^m = \tau^4 = 1, \tau^{-1}\rho\tau = \rho^{-1} \rangle$. The assumption on the prime divisor p of m in (iv) of (1) is equivalent to $p \cdot \mathbb{Z}[\sqrt{-1}]$ is a prime ideal in $\mathbb{Z}[\sqrt{-1}]$ (we have $d = 2$, $n = 1$ for (ii') of $(1')$ in this situation).

Now consider the case $\pi = C_{q^f} \times D_m$ in (iii) of (1). Take $n = q^f$ in (ii') of $(1')$. If p is a prime divisor of m, since $p \neq q$, the prime number p is unramified in $\mathbb{Q}(\zeta_{q^f})$. Let P be a prime ideal of $\mathbb{Z}[\zeta_{q^f}]$ with $p \in P$. Since $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle$, it follows that $\mathrm{Gal}(\mathbb{Q}(\zeta_{q^f})/\mathbb{Q}) = \langle \varphi \rangle$ where $\varphi(\zeta_{q^f}) = (\zeta_{q^f})^p$. On the other hand, φ induces the Frobenius automorphism of $\mathbb{Z}[\zeta_{q^f}]/P$ over $\mathbb{Z}/p\mathbb{Z}$. It follows that $P = p\mathbb{Z}[\zeta_{q^f}]$ and thus p remains prime in $\mathbb{Z}[\zeta_{q^f}]$ as expected.

For the proof of $(1') \Rightarrow (1)$, we first note that, if n is odd and $d \geq 1$, then $(\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times$ is a cyclic group if and only if $(n, 2^d) = (1, 2)$, $(1,4)$, or $(q^f, 2)$ where q is some odd prime number.

On the other hand, note that $\mathrm{Gal}(\mathbb{Q}(\zeta_{n \cdot 2^d})/\mathbb{Q}) \simeq (\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times$. If a prime number p remains prime in $\mathbb{Z}[\zeta_{n \cdot 2^d}]$, then $[\mathbb{Z}[\zeta_{n \cdot 2^d}]/p\mathbb{Z}[\zeta_{n \cdot 2^d}] : \mathbb{Z}/p\mathbb{Z}] = |(\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times|$. Thus the Frobenius automorphism of p generates $\mathrm{Gal}(\mathbb{Q}(\zeta_{n \cdot 2d})/\mathbb{Q})$. Hence $(\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times$ is cyclic and $(n, 2^d) = (1, 2)$, $(1,4)$, or $(q^f, 2)$.

When $(n, 2^d) = (1, 2)$, the group π in (ii') of $(1')$ is isomorphic to D_m.

When $(n, 2^d) = (1, 4)$, the group π in (ii') of $(1')$ is isomorphic to Q_{4m} such that $p \equiv 3 \pmod 4$ for any prime divisor p of m.

When $(n, 2^d) = (q^f, 2)$, the group π in (ii') of $(1')$ is isomorphic to $C_{q^f} \times D_m$. Since every prime divisor p of m remains prime in $\mathbb{Z}[\zeta_{q^f}]$, write $P = p\mathbb{Z}[\zeta_{q^f}]$ the prime ideal of $\mathbb{Z}[\zeta_{q^f}]$. Then the Frobenius automorphism of $\mathbb{Z}[\zeta_{q^f}]/P$ generates $\mathrm{Gal}(\mathbb{Q}(\zeta_{q^f})/\mathbb{Q}) \simeq (\mathbb{Z}/q^f\mathbb{Z})^\times$. Thus $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle$. $\square$

6. The maximal orders. Because of Theorem 1.4, we will determine $C(\Omega_{\mathbb{Z}\pi})$ when $\pi = C_n$, D_n, $C_n \times D_m$, Q_{4n}. Given a finite group π, there may be more than one maximal order containing $\mathbb{Z}\pi$. For our purpose, it is enough to select just one of them.

DEFINITION 6.1. Let K be an algebraic number field, A be a cental simple K-algebra, v be a place of K (finite or infinite). We say that A ramifies at v if $[K_v \otimes_K A] \neq 0$ in the Brauer group of K_v where K_v is the completion of K at v [Re, page 272].

DEFINITION 6.2. Let R be a Dedekind domain whose quotient field K is an algebraic number field, and let A be a central simple K-algebra. Define $I(R)$ to be the multiplicative group of fractional R-ideals in K, and define $P(R) = \{R\alpha : \alpha \in K\backslash\{0\}\}$. Recall that $C(R) = I(R)/P(R)$.

Define S to be the set of all infinite places of K ramified in A. Define $P_A(R) = \{R\alpha : \alpha \in K\backslash\{0\}, \alpha_v > 0$ for all v in $S\}$, the principal ray group (mod S). The ray class group (mod S), denoted by $C_A(R)$, is defined as $C_A(R) = I(R)/P_A(R)$ [Re, page 309]. If S is the empty set, then $C(R) \simeq C_A(R)$.

In general, the kernel of the surjective map $C_A(R) \to C(R)$ is the group $P(R)/P_A(R)$ which may be computed by the exact sequence $U(R) \to D \to P(R)/P_A(R) \to 0$, where $U(R)$ is the group of units of R, $D = \prod_{v \in S} K_v^\times/(K_v^\times)^+$; note that $K_v^\times$ is the multiplicative group of non-zero elements in K_v, $(K_v^\times)^+$ is the group of positive elements of $K_v^\times$, and $K_v^\times/(K_v^\times)^+ \simeq \mathbb{Z}/2\mathbb{Z}$. The map $U(R) \to D$ is defined by $\alpha \mapsto (\dots, \bar{\alpha}_v, \dots)$ where α_v is the image of α in K_v; the map of D to $P(R)/P_A(R)$ can be found in [Sw2, page 139].

THEOREM 6.3 (Swan [Re, page 313, Theorem 35.14]). *Let the notations be the same as in Definition 6.2. Let Λ be a maximal R-order in A. Then $C(\Lambda) \simeq C_A(R)$ under the reduced norm map.*

THEOREM 6.4. *Let π be a group, $\Omega_{\mathbb{Z}\pi}$ be a maximal $\mathbb{Z}$-order in $\mathbb{Q}\pi$ containing $\mathbb{Z}\pi$.*

(1) *If $\pi = C_n$, then*

$$C(\Omega_{\mathbb{Z}\pi}) \simeq \oplus_{d|n} C(\mathbb{Z}[\zeta_d]).$$

(2) *If $\pi = D_n$ where n is an integer ≥ 2, then*

$$C(\Omega_{\mathbb{Z}\pi}) \simeq \oplus_{d|n} C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]).$$

(3) *If $\pi = C_n \times D_m$ where $\gcd\{n, m\} = 1$ and m is an integer ≥ 2. For any $d \mid nm$, write $d = d_1 d_2$ where $d_1 \mid n$, $d_2 \mid m$.*
If m is odd, then

$$C(\Omega_{\mathbb{Z}\pi}) \simeq (\oplus_{d|n} C(\mathbb{Z}[\zeta_d])^{(2)}) \oplus (\oplus_{\substack{d|nm \\ d\nmid n}} C(\mathbb{Z}[\zeta_{d_1}, \zeta_{d_2} + \zeta_{d_2}^{-1}])).$$

If m is even, then

$$C(\Omega_{\mathbb{Z}\pi}) \simeq (\oplus_{d|n} C(\mathbb{Z}[\zeta_d])^{(4)}) \oplus (\oplus_{\substack{d|nm \\ d\nmid n, d\geq 3}} C(\mathbb{Z}[\zeta_{d_1}, \zeta_{d_2} + \zeta_{d_2}^{-1}])).$$

(4) *If $\pi = Q_{4n}$ where n is an integer ≥ 2, then*

$$C(\Omega_{\mathbb{Z}\pi}) \simeq \oplus_{d|n} (C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}])) \oplus (\oplus_{\substack{d|2n \\ d\nmid n, d\geq 3}} C_{A_d}(R_d))$$

where $C_{A_d}(R_d)$ is defined in Definition 6.2 with $R_d = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$, $K_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$, $L_d = \mathbb{Q}(\zeta_d)$, and A_d is the central simple K_d-algebra defined by $A_d = L_d + L_d \cdot u$ with $u^2 = -1$, $u\alpha = \tau(\alpha)u$ for any $\alpha \in L_d$, τ acts on L_d by $\tau \cdot \zeta_d = \zeta_d^{-1}$.

Proof. Case 1. $\pi = C_n$.

Write $\mathbb{Z}\pi = \mathbb{Z}[X]/\langle X^n - 1\rangle$. Note that $\mathbb{Z}[X]/\langle X^n - 1\rangle \hookrightarrow \prod_{d|n} \mathbb{Z}[X]/\langle \Phi_d(X)\rangle \hookrightarrow \mathbb{Q}[X]/\langle X^n - 1\rangle$. Hence $\Omega_{\mathbb{Z}\pi} \simeq \prod_{d|n} \mathbb{Z}[\zeta_d]$.

Case 2. $\pi = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle \simeq D_n$.

For any monic polynomial $f(X) \mid X^n - 1$, the right ideal $f(\sigma) \cdot \mathbb{Z}\pi$ is a two-sided ideal in $\mathbb{Z}\pi$ (see Step 1 in the proof of Theorem 4.4); write it as $\langle f(\sigma)\rangle$. By abusing the notation, we will also write $\langle f(\sigma)\rangle$ as the two-sided ideal in $\mathbb{Q}\pi$ generated by $f(\sigma)$.

It is not difficult to verify that $\mathbb{Z}\pi \hookrightarrow \prod_{d|n} \mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$ and $\mathbb{Q}\pi = \prod_{d|n} \mathbb{Q}\pi/\langle\Phi_d(\sigma)\rangle$.

When $d = 1$ or 2, $\mathbb{Q}\pi/\langle\Phi_d(\sigma)\rangle \simeq \mathbb{Q}C_2 \simeq \mathbb{Q} \times \mathbb{Q}$. It follows that the maximal $\mathbb{Z}$-order containing $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$ is $\mathbb{Z} \times \mathbb{Z}$. Thus the factor $\mathbb{Q}\pi/\langle\Phi_d(\sigma)\rangle$ (for $d = 1$ or 2) has no contribution to $C(\Omega_{\mathbb{Z}\pi})$.

When $d \geq 3$, write $\Lambda_d := \mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$, $A_d := \mathbb{Q}\pi/\langle\Phi_d(\sigma)\rangle$. We will show that A_d is a simple algebra whose center K_d is an algebraic number field. Let Γ_d be a maximal $\mathbb{Z}$-order with $\Lambda_d \subset \Gamma_d \subset A_d$. It follows that $C(\Omega_{\mathbb{Z}\pi}) \simeq \oplus_{\substack{d|n \\ d \geq 3}} C(\Gamma_d)$. We will show that $C(\Gamma_d) \simeq C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]$.

When $d \geq 3$, $A_d \simeq L_d \circ H$ where $L_d = \mathbb{Q}(\zeta_d)$, $H = \langle\tau\rangle \simeq C_2$ and $\tau(\zeta_d) = \zeta_d^{-1}$. Define $K_d = L_d^{\langle\tau\rangle} = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$. Note that $L_d \circ H \simeq M_2(K_d)$ is a matrix ring over K_d. Thus K_d has no ramified infinite place for A_d. Let R_d be the ring of algebraic integers of K_d. By Theorem 6.3, $C(\Gamma_d) \simeq C(R_d)$.

Case 3. $\pi = \langle\sigma', \rho, \tau : \sigma'^n = \rho^m = \tau^2 = 1, \tau^{-1}\sigma'\tau = \sigma', \tau^{-1}\rho\tau = \rho^{-1}, \sigma'\rho = \rho\sigma'\rangle \simeq C_n \times D_m$.

Define $\sigma = \sigma'\rho$ and consider $\Lambda_d := \mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$, $A_d := \mathbb{Q}\pi/\langle\Phi_d(\sigma)\rangle$ where $d \mid nm$ as in Case 2. The proof is almost the same as that in Case 2. Write $d = d_1 d_2$ where $d_1 \mid n$ and $d_2 \mid m$.

Case 3.1 m is odd.

If $d \mid n$, then Λ_d is isomorphic to the group ring of $C_2 = \langle\tau\rangle$ over $S_d = \mathbb{Z}[\zeta_d]$ (see Step 2 of Case 3 in the proof of Theorem 1.4). A maximal order containing Λ_d is $S_d \cdot (1 + \tau)/2 \oplus S_d \cdot (1 - \tau)/2$. Thus the contribution of Λ_d to $C(\Omega_{\mathbb{Z}\pi})$ is $C(\mathbb{Z}[\zeta_d]) \oplus C(\mathbb{Z}[\zeta_d])$.

Now consider Λ_d where $d \nmid n$. Then $A_d \simeq L_d \circ H$ where we keep the notations of H, L_d as in Case 2, but K_d should be replaced by $K_d = \mathbb{Q}(\zeta_{d_1}, \zeta_{d_2} + \zeta_{d_2}^{-1})$.

As in Case 2, $C(\Lambda_d) \simeq C(R_d)$.

Case 3.2 m is even.

The proof is almost the same as Case 3.1. Consider separately the three situations $d \mid n$, $d \nmid n$ (but $d_2 \neq 2$) and $d_2 = 2$.

For the last situation, if $d_2 = 2$, then Λ_d is isomorphic to the group ring of $C_2 = \langle\tau\rangle$ over $S_d = \mathbb{Z}[\zeta_d] = \mathbb{Z}[\zeta_{d/2}]$. This explains the reason why there is an extra summand $C(\mathbb{Z}[\zeta_{d/2}])^{(2)}$.

Case 4. $\pi = \langle\sigma, \tau : \sigma^{2n} = \tau^4 = 1, \sigma^n = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle \simeq Q_{4n}$.

As in Case 2, for $d \mid 2n$, consider $\Lambda_d := \mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle$.

If $d = 1$ or 2, Λ_d has no contribution to $C(\Omega_{\mathbb{Z}\pi})$ (note that, when $d = 2$ and n is odd, $\mathbb{Z}\pi/\langle\Phi_d(\sigma)\rangle \simeq \mathbb{Z}[\sqrt{-1}]$) .

It remains to consider Λ_d where $d \geq 3$.

Define $A_d := \mathbb{Q}\pi/\langle\Phi_d(\sigma)\rangle$ and Γ_d a maximal $\mathbb{Z}$-order with $\Lambda_d \subset \Gamma_d \subset A_d$. Write ζ_d for the image of σ in Λ_d and in A_d, and τ' for the image of τ.

If $d \mid n$ and $d \geq 3$, the relation $\tau^2 = \zeta_d^n = 1$ entails the consequence A_d is a matrix ring over K_d as in Case 2. Thus, as before, $C(\Gamma_d) \simeq C(R_d)$ where $R_d = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$.

It remains to consider the case $d \mid 2n$, $d \nmid n$ and $d \geq 3$.

In A_d, we have $\tau'^2 = -1$. It follows that $\Lambda_d = (S_d \circ H)_f$ where the notation is the same as in Step 1 of Case 4 in the proof of Theorem 1.4. Define $R_d = S_d^{\langle\tau'\rangle} = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$ and $K_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$.

Note that A_d is a central division K_d-algebra with $[A_d : K_d] = 4$; in fact, it is a totally definite quaternion algebra by Step 2 of Case 4 in the proof of Theorem 1.4. By Theorem 6.3, $C(\Lambda) \simeq C_{A_d}(R_d)$. $\square$

REMARK. A thorough study of all the maximal $\mathbb{Z}$-orders in $\mathbb{Q}\pi$ (when $\pi = D_n$ or Q_{4n}) can be found in [Sw3, pages 75-79].

Let h_m be the class number of $\mathbb{Q}(\zeta_m)$, h_m^+ be the class number of $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$. It is known that h_m^+ is a divisor of h_m [Wa, page 40, Theorem 4.14]. Recall the definition of π-tori in Definition 1.1.

THEOREM 6.5. *Let p be an odd prime number, $c \geq 1$, and K/k be a Galois extension with $\mathrm{Gal}(K/k) = D_{p^c}$. Then all the D_{p^c}-tori defined over k are stably rational over k if and only if $h_{p^c}^+ = 1$.*

Proof. Apply Theorem 6.4.

For any $c' \leq c$, the extension $\mathbb{Z}[\zeta_{p^{c'}} + \zeta_{p^{c'}}^{-1}] \hookrightarrow \mathbb{Z}[\zeta_{p^c} + \zeta_{p^c}^{-1}]$ has one fully ramified prime divisor. Thus $h_{p^c}^+ = 1$ implies $h_{p^{c'}}^+ = 1$ for any $c' \leq c$ [Wa, page 39, Proposition 4.11]. $\square$

REMARK. (1) The case of D_p-tori in the above theorem is proved by Hoshi, Kang and Yamasaki by a different method [HKY, Theorem 1.5].

(2) According to Washington [Wa, page 420], the calculation of h_m^+ is rather sophisticated. It is known that $h_m^+ = 1$ if $m \leq 66$; if the generalized Riemann hypothesis is assumed, then $h_m^+ = 1$ if $m \leq 161$ [Wa, page 421]. By [Mi], $h_{2^t}^+ = 1$ if $2^t = 128, 256$; so is $2^t = 512$ if the generalized Riemann hypothesis is assumed.

We turn to the situation of 2-groups such as D_n (the dihedral group of order $2n$ with $n \geq 2$) and Q_{4n} (the generalized quaternion group of order $4n$ with $n \geq 2$).

The following proposition is an easy consequence of Endo and Miyata's Theorems in [EM2], [EM3] and [EM4]. We record it just to keep the reader aware.

PROPOSITION 6.6. *Let $\pi = D_n$, the dihedral group of order $2n$ where $n = 2^t$ and $t \geq 1$. Then $C(\Omega_{\mathbb{Z}\pi}) \simeq T^g(\pi)$. Consequently, if $h_{2^t}^+ = 1$ (e.g. $1 \leq t \leq 8$) and M is a π-lattice, then M is both flabby and coflabby if and only if it is stably permutation, i.e. $M \oplus P_1 \simeq P_2$ where P_1 and P_2 are permutation lattices.*

Proof. Since π is a dihedral group, $C^q(\mathbb{Z}\pi) = \widetilde{C}(\mathbb{Z}\pi)$ by [EM2, page 709, Theorem 4.6]. Thus $C(\Omega_{\mathbb{Z}\pi}) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$. On the other hand, π is a 2-group, it follows that $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq T^g(\pi)$ by Theorem 2.14. Hence the result.

Suppose that $h_{2^t}^+ = 1$. Then $h_{2^s}^+ = 1$ for all $1 \leq s \leq t$ by the same arguments as in the proof of Theorem 6.5. It follows that $C(\Omega_{\mathbb{Z}\pi}) = 0$ according to Theorem 6.4. Hence $T^g(\pi) = 0$.

The condition $T^g(\pi) = 0$ is equivalent to $[M]^{fl} = 0$ for any invertible π-lattice M, which means that there is a short exact sequence $0 \to M \to P_1 \to P_2 \to 0$ for some permutation lattices P_1 and P_2. But this sequence splits because of Lemma 2.5; thus M is stably permutation. By Theorem 2.7, M is stably permutation if and only if it is flabby and coflabby.

Note that, by [Wa, page 421; Mi], $h_{2^t}^+ = 1$ if $1 \leq t \leq 8$. $\square$

LEMMA 6.7. *Let $\pi = Q_{4n}$ be the generalized quaternion group of order $4n$ where $n = 2^t$ with $t \geq 1$. If $\Omega_{\mathbb{Z}\pi}$ is a maximal order in $\mathbb{Q}\pi$ containing $\mathbb{Z}\pi$, then $C(\Omega_{\mathbb{Z}\pi}) \simeq \bigoplus_{d|2n} C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}])$. In particular, if $\pi = Q_8, Q_{16}, Q_{32}, Q_{64}$ or Q_{128}, then $C(\Omega_{\mathbb{Z}\pi}) = 0$.*

Proof. Apply Theorem 6.4. It suffices to determine $C_A(R)$ where $R = \mathbb{Z}[\zeta_{2n}+\zeta_{2n}^{-1}]$, $K = \mathbb{Q}(\zeta_{2n} + \zeta_{2n}^{-1})$ is the quotient field of R, and A is the central simple K-algebra defined by $A = L+Lu$ with $u^2 = -1$, $u\alpha = \tau(\alpha)u$ for any $\alpha \in L$ (τ acts on $L = \mathbb{Q}(\zeta_{2n})$ by $\tau(\zeta_{2n}) = \zeta_{2n}^{-1}$).

Use the exact sequence $U(R) \to D \to P(R)/P_A(R) \to 0$ in Definition 6.2. By Weber's Theorem [Ha, Satz 6, page 29; CR2, page 272] the map $U(R) \to D$ is surjective. Thus $P_A(R) = P(R)$ and $C_A(R) \simeq C(R) \simeq C(\mathbb{Z}[\zeta_{2n} + \zeta_{2n}^{-1}])$.

By [Wa, page 421], $h_{2^s}^+ = 1$ if $1 \leq s \leq 6$. Hence $C(\Omega_{\mathbb{Z}\pi}) = 0$ if $\pi = Q_8, Q_{16}, Q_{32}, Q_{64}$ or Q_{128}. $\square$

PROPOSITION 6.8. *If $\pi \simeq Q_8, Q_{16}, Q_{32}, Q_{64}$ or Q_{128}, then $C(\Omega_{\mathbb{Z}\pi}) = 0 = T^g(\pi)$. It follows that an invertible π-lattice is always stably permutation.*

Proof. Since there is a surjection $C(\Omega_{\mathbb{Z}\pi}) \to C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ (see Definition 2.12), it follows that $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) = 0$ by Lemma 6.7.

Because of Theorem 2.14, $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$. Thus $T^g(\pi) = 0$. The remaining proof is similar to that of Proposition 6.6. $\square$

The same argument of the above proposition may be applied to the semi-dihedral groups and the modular groups also. Let $n = 2^t$ where $t \geq 3$, define $SD_{2n} = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+(n/2)} \rangle$ (the semi-dihedral group of order $2n$), and define $M_{2n} = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+(n/2)} \rangle$ (the modular group of order $2n$). If π is a 2-group of order ≥ 16 and contains a cyclic normal subgroup of index 2, then π is isomorphic to the dihedral group, the semi-dihedral group, the generalized quaternion group or the modular group [Su, page 107].

The proof of the following proposition is similar to that of Theorem 6.4, and is omitted.

PROPOSITION 6.9. *Let $n = 2^t$ where $t \geq 3$.*
(1) If $\pi = SD_{2n}$, then $C(\Omega_{\mathbb{Z}\pi}) \simeq (\oplus_{0 \leq s \leq t-1} C(\mathbb{Z}[\zeta_{2^s} + \zeta_{2^s}^{-1}])) \oplus C(\mathbb{Z}[\zeta_n - \zeta_n^{-1}])$.
(2) If $\pi = M_{2n}$, then $C(\Omega_{\mathbb{Z}\pi}) \simeq (\oplus_{0 \leq s \leq t-1} C(\mathbb{Z}[\zeta_{2^s}])^{(2)}) \oplus C(\mathbb{Z}[\zeta_{n/2}])$.

PROPOSITION 6.10. *If $\pi \simeq M_{16}, M_{32}$ or M_{64}, then $C(\Omega_{\mathbb{Z}\pi}) = 0 = T^g(\pi)$. It follows that an invertible π-lattice is always stably permutation.*

Proof. By Proposition 6.9, $C(\Omega_{\mathbb{Z}\pi}) = 0$ because of [Wa, page 205, Theorem 11.1]. Thus $T^g(\pi) = 0$. $\square$

REMARK. The situation of the group SD_{2n} is left open because we don't know the class number of $\mathbb{Q}(\zeta_n - \zeta_n^{-1})$ when $n \geq 16$.

REFERENCES

[AG]   M. AUSLANDER AND O. GOLDMAN, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc., 97 (1960), pp. 367–409.

[CTS]   J.-L. COLLIOT-THÉLÈNE AND J.-J. SANSUC, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4), 10 (1977), pp. 175–229.

[CR1]   C. W. CURTIS AND I. REINER, *Methods of representation theory vol. 1*, Wiley-Interscience Publ., New York, 1981.

[CR2]   C. W. CURTIS AND I. REINER, *Methods of representation theory vol. 2*, Wiley-Interscience Publ., New York, 1987.

[EM1]   S. ENDO AND T. MIYATA, *Invariants of finite abelian groups*, Journal Math. Soc. Japan, 25 (1973), pp. 7–26.

[EM2]   S. ENDO AND T. MIYATA, *Quasi-permutation modules over finite groups*, Journal Math. Soc. Japan, 25 (1973), pp. 397–421; II, ibid. 26 (1974), pp. 698–713.

[EM3]   S. ENDO AND T. MIYATA, *On a classification of the function fields of algebraic tori*, Nagoya Math. J., 56 (1975), pp. 85–104; Corrigenda, ibid. 79 (1980), pp. 187–190.

[EM4]   S. ENDO AND T. MIYATA, *Integral representations with trivial first cohomology groups*, Nagoya Math. J., 85 (1982), pp. 231–240.

[Ha]   H. HASSE, *Über die Klassenzahl abelscher Zahlkörper*, Akademie-Verlag, 1952, Berlin; Springer-Verlag reprinting, 1985, Berlin.

[HKY]   A. HOSHI, M. KANG, AND A. YAMASAKI, *Class numbers and algebraic tori*, arXiv: 1312.6738v2.

[Is]   I. M. ISAACS, *Finite group theory*, Amer. Math. Soc., Providence, 2008.

[Ja]   H. JACOBINSKI, *Genera and decompositions of lattices over orders*, Acta Math., 121 (1968), pp. 1–29.

[Le]   H. W. LENSTRA JR., *Rational functions invariant under a finite abelian group*, Invent. Math., 25 (1974), pp. 299–325.

[Lee]   M. P. LEE, *Integral representations of dihedral groups of order* $2p$, Trans. Amer. Math. Soc., 110 (1964), pp. 213–231.

[Mi]   J. C. MILLER, *Class numbers of totally real fields and applications to the Weber class number problem*, Acta Arith., 164 (2014), pp. 381–397.

[Ol]   R. OLIVER, *G-actions on disks and permutation representations*, J. Algebra, 50 (1978), pp. 44–62.

[On]   T. ONO, *Arithmetic of algebraic tori*, Ann. Math., 74 (1961), pp. 101–139.

[Re]   I. REINER, *Maximal orders*, Academic Press, 1975; Clarendon Press reprinting, 2003, Oxford.

[Ro]   M. ROSEN, *Representations of twisted group rings*, Ph. D. Dissertation, Princeton University, 1963.

[Sa]   D. J. SALTMAN, *Retract rational fields and cyclic Galois extensions*, Israel J. Math., 47 (1984), pp. 165–215.

[Su]   M. SUZUKI, *Group theory II*, Grund. math. Wiss. Vol. 248, Springer-Verlag, Berlin, 1986.

[Sw1]   R. G. SWAN, *Induced representations and projective modules*, Ann. Math., 71 (1960), pp. 552–578.

[Sw2]   R. G. SWAN, *K-theory of finite groups and orders*, Springer LNM vol. 149, Springer-Verlag, 1970, New York.

[Sw3]   R. G. SWAN, *Projective modules over binary polyhedral groups*, J. Reine Angew. Math., 342 (1983), pp. 66–172.

[Sw4]   R. G. SWAN, *Noether's problem in Galois theory*, in "Emmy Noether in Bryn Mawr", edited by B. Srinivasan and J. Sally, Springer-Verlag, Berlin, 1983, pp. 21–40.

[Sw5]   R. G. SWAN, *The flabby class group of a finite cyclic group*, in "Proceedings of the 4th International Congress of Chinese Mathematicians vol. 1", edited by L. Ji, K. Liu, L. Yang and S.-T. Yau, Higher Education Press and International Press, Beijing and Somerville, MA, 2008, pp. 241–251.

[Vo]   V. E. VOSKRESENSKII, *Algebraic groups and their birational invariants*, Translations of Math. Monographs, vol. 179, Amer. Math. Soc., 1998, Providence, Rhode Island.

[Wa]   L. C. WASHINGTON, *Introduction to cyclotomic fields*, Second edition, GTM vol. 83, Springer-Verlag, 1997, New York.

[Za]   H. J. ZASSENHAUS, *The theory of groups*, 2nd edition, Chelsea Publ. Company, New York, 1958.