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A CRITERION FOR A FINITE UNION OF INTERVALS TO BE A SELF-SIMILAR SET SATISFYING THE OPEN SET CONDITION*

ZHI-YING WEN† AND XUAN ZHAO‡

Abstract. Let $k_1, k_2, \ldots, k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ be positive numbers. Let $K(k_1, k_2, \ldots, k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ be the union of m closed intervals of lengths $k_1, k_2, \ldots, k_m$ and gap lengths $\lambda_1, \lambda_2, \cdots, \lambda_{m-1}$. In this paper, we will give a characterization over $k_1, k_2, \ldots, k_m$ and $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ such that $K(k_1, k_2, \ldots, k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ is a self-similar set satisfying the open set condition.

Key words. Self-similar sets, open set condition, multiple word, non-negative matrices, common eigenvector.

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1. Introduction. Self-similar sets (SSS) form the most important class of Fractals, which have been studied extensively and deeply. One of the basic problems is to construct new self-similar sets from known ones. For example, closed intervals are the simplest SSS, and a typical question is whether the union of two (or more generally a finite number of) disjoint closed intervals is still a SSS. In 2007, De-Jun FENG, Su HUA and Yuan JI [Fen] considered a special case of this problem, and characterized the self-similarity of the union of two disjoint unit intervals satisfying the open set condition (OSC) by the length of the gap between these two intervals.

In this work we shall generalize the above result to the general case. More precisely, we shall consider the union of a finite number of disjoint closed intervals with possibly different lengths, and characterize the similarity of the union satisfying OSC, by the lengths of the intervals and the gaps between consecutive intervals. Our results contain those of [Fen] as special cases.

1.1. SSOSC. Let $\{S_j(x) = c_j x + d_j\}_{j=1}^m$ be an iterated function system (IFS) on $\mathbb{R}$ such that $|c_j| < 1$ for all $1 \leq j \leq m$. Then there is a unique non-empty compact set $K \subset \mathbb{R}$ such that [Hut]

$$K = \bigcup_{j=1}^{m} S_j(K).$$

The set K is called the *self-similar set* generated by the IFS $\{S_j\}_{j=1}^m$.

If the IFS $\{S_j\}_{j=1}^m$ also satisfies the *open set condition* (OSC), i.e., there exists a non-empty bounded open set $U \subset \mathbb{R}$ such that

$$\bigcup_{j=1}^{m} S_j(U) \subset U$$

with the union disjoint, we say that K is a *self-similar set fulfilling the open set condition* (SSOSC).

For the further properties of OSC and SSOSC, we refer to [Sch] and [Fal].

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†Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, China (wenzy@mail.tsinghua.edu.cn).
‡The correspondent author. National Education Examinations Authority, 100084 Beijing, China (zhaox@mail.neea.edu.cn).

1.2. Multiple word. We begin with some definitions and notation. Let $n \geq 2$ be an integer, $\mathcal{A}(n) = \{1, 2, \cdots, n\}$ be an alphabet, $W_m := \{w = w_1 \cdots w_m | \; w_i \in \mathcal{A}(n)\}$ be the set of words of length of m over $\mathcal{A}(n)$, and $W^* := \bigcup_{m \geq 0} W_m$ the set of words of finite length over $\mathcal{A}(n)$, with convention $W_0 := \{\emptyset\}$. A word $\omega \in W_{2n}$ is called *double word over* $\mathcal{A}(n)$ if for any $1 \leq i \leq n$, the letter i appears exactly twice in ω, and ii does not appear in ω as a subword. For example, the word 123132 is a double word over $\mathcal{A}(3)$, and 121233 is not a double word. More generally, for any integers $m, n \geq 2$, a word $\omega \in W_{mn}$ is called a *m-multiple word over* $\mathcal{A}(n)$, if for any integer $1 \leq i \leq n$, the letter i appears exactly m times in ω, and ii does not appear in ω. For example, the word 123132321 is a 3-multiple word over $\mathcal{A}(3)$.

Let ω be a double word. For all letters i in ω, recall that i appears in ω twice exactly, and we replace it respectively by i_1 and i_2 (or by i_2 and i_1), the new word obtained in this way is called an *ordered double word over* $\mathcal{A}(n)$. For example, let $\omega = 123132$ be a double word over $\mathcal{A}(3)$, then $1_1 2_1 3_1 1_2 3_2 2_2$ and $1_1 2_2 3_1 1_2 3_2 2_1$ are two ordered double words induced by ω. Hence a double word can induce several different ordered double words. Similarly, if ω is a m-multiple word over $\mathcal{A}(n)$, then for all letters $1 \leq i \leq n$, each i appears m times in ω, we replace it respectively by the letters $i_1, i_2, \ldots, i_m$ according the orders $i_1, i_2, \ldots, i_m$ or $i_m, i_{m-1}, \ldots, i_1$, and we get a new word called an *ordered m-multiple word over* $\mathcal{A}(n)$. For example, let $\omega = 123132321$ be a 3-multiple word, then $1_1 2_3 3_1 1_2 3_2 2_2 3_3 2_1 1_3$ is an ordered 3-multiple word induced by ω. In the sequel we denote simply the ordered multiple words by γ.

Let ω be a m-multiple word over $\mathcal{A}(n)$. We call it ω *reducible* if there exist a strict sub-word ω' of ω and a sub-alphabet $\{i_1, \cdots, i_k\}$ $(k < n)$ of $\mathcal{A}(n)$ such that ω' is a m-multiple word over the alphabet $\{i_1, \cdots, i_k\}$. If ω is not reducible, we say that ω is *irreducible*. For instance, the word 123231 is reducible because it has a strict sub-word 2323, and 123132321 is irreducible. An ordered multiple word is called irreducible (*res.* reducible) if the corresponding multiple word is irreducible (*res.* reducible). For example, the word $1_1 2_3 3_1 1_2 3_2 2_2 3_3 2_1 1_3$ is an irreducible ordered 3-multiple word, for its corresponding 3-multiple word 123132321 is irreducible.

1.3. Incidence matrix. For all integers $m, n \geq 2$, we denote by Ω_n^m the set of all ordered m-multiple words $\mathcal{A}(n)$.

Given $q > 0$ and an ordered double word $\gamma \in \Omega_n^2$, by the definition of ordered double word, $\forall i \in \mathcal{A}(n)$, i_1 and i_2 appear once only in γ. The *incidence matrix* $A_\gamma^q = (a_{ij})_{n \times n}$ of γ is defined as follow:

$$a_{ij} = \begin{cases} 0 & \text{if } i = j\text{; or } i \neq j\text{, and } j_1, j_2 \text{ are not between } i_1 \text{ and } i_2, \\ 1 & \text{if } i \neq j \text{ and only } j_1 \text{ is between } i_1 \text{ and } i_2, \\ q & \text{if } i \neq j \text{ and only } j_2 \text{ is between } i_1 \text{ and } i_2, \\ 1+q & \text{if } i \neq j \text{ and both } j_1, j_2 \text{ are between } i_1 \text{ and } i_2. \end{cases}$$

For example, the incidence matrix of the ordered double word $1_1 2_2 3_1 1_2 3_2 2_1$ is

$$A_\gamma^q = \begin{bmatrix} 0 & q & 1 \\ q & 0 & q+1 \\ q & 0 & 0 \end{bmatrix}.$$

More generally, given m strictly positive numbers $k_1, k_2, \ldots, k_m$ and an ordered m-multiple word γ, we can define $m-1$ incidence matrices $A_{\gamma,1}^{(k_1,k_2,\ldots,k_m)}$, $A_{\gamma,2}^{(k_1,k_2,\ldots,k_m)}$, $\ldots$, $A_{\gamma,m-1}^{(k_1,k_2,\ldots,k_m)}$ as follows:

$$A_{\gamma,l}^{(k_1,k_2,\ldots,k_m)} = (a_{ij}^l)_{n \times n}, \; 1 \leq l \leq m-1,$$

where

$$a_{ij}^l = \begin{cases} 0 & \text{if } i = j; \text{ or } i \neq j, \text{ and none of } j_1, j_2, \ldots, j_m \text{ is} \\ & \text{between } i_l \text{ and } i_{l+1}, \\ k_s + k_{s+1} + \cdots + k_t & \text{if } i \neq j \text{ and only } j_s, j_{s+1}, \ldots, j_t \ (1 \leq s \leq t \leq \\ & m-1) \text{ are between } i_l \text{ and } i_{l+1}. \end{cases}$$

For example, for the ordered 3-multiple word $1_1 2_3 3_1 1_2 3_2 2_2 3_3 2_1 1_3$, its two incidence matrices are

$$A_{\gamma,1}^{(k_1,k_2,k_3)} = \begin{bmatrix} 0 & k_3 & k_1 \\ 0 & 0 & k_3 \\ k_2 & 0 & 0 \end{bmatrix}, \quad A_{\gamma,2}^{(k_1,k_2,k_3)} = \begin{bmatrix} 0 & k_1+k_2 & k_2+k_3 \\ k_2 & 0 & k_1+k_2 \\ 0 & k_2 & 0 \end{bmatrix}.$$

1.4. Main results. Firstly we consider the case that K is the union of two closed intervals, and the length ratio between two intervals is $q \in \mathbb{R}^+$. Let

$$K_\lambda^q = [0,1] \cup [1+\lambda, 1+\lambda+q] \subset \mathbb{R}, \text{ with } \lambda, q \in \mathbb{R}^+.$$

For a given $q \in \mathbb{R}^+$, we will determine the parameter λ for which K_λ^q is a SSOSC.

Let $n \geq 2$ be an integer, set

$$I_n^2 = \{\gamma \mid \gamma \text{ is an irreducible ordered double word over } \mathcal{A}(n)\},$$
$$\Omega_n^q = \{A_\gamma^q \mid A_\gamma^q \text{ is the incidence matrix of } \gamma, \gamma \in I_n^2\},$$
$$\Lambda_n^q = \{\rho(A) \mid A \in \Omega_n^q\}, \quad \Lambda^q = \bigcup_{n=2}^{+\infty} \Lambda_n^q,$$

where $\rho(A)$ denotes the spectral radius of the matrix A.

Now we state the first main result of this work:

Main Theorem 1.1. *K_λ^q is a SSOSC if and only if $\lambda \in \Lambda^q$.*

Next we consider the general case that K is the union of finitely many closed intervals. Let $I_1, \ldots, I_m (m \geq 2)$ be m closed intervals, arranged from left to right, of lengths $k_1, k_2, \ldots, k_m$. We denote the gap between I_j and I_{j+1} by λ_j $(1 \leq j \leq m-1)$. Put

$$K(k_1, k_2, \ldots, k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1}) := \bigcup_{i=1}^m I_i.$$

We will determine $k_1, k_2, \ldots, k_m$ and $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ such that $K(k_1, k_2, \ldots, k_m;$ $\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ is a SSOSC.

Let $A_1, A_2, \ldots, A_n$ be n non-negative matrices. We call them *linked* if they have a positive common eigenvector. Given $k_1, k_2, \ldots, k_m \in \mathbb{R}^+$, an ordered m-multiple word γ is called an *ordered m-multiple linked word*, if its $m-1$ incidence matrices $A_{\gamma,1}^{(k_1,k_2,\ldots,k_m)}$, $A_{\gamma,2}^{(k_1,k_2,\ldots,k_m)}$, $\ldots$, $A_{\gamma,m-1}^{(k_1,k_2,\ldots,k_m)}$ are linked. In the following we denote the incidence matrices $A_{\gamma,1}^{(k_1,k_2,\ldots,k_m)}$, $A_{\gamma,2}^{(k_1,k_2,\ldots,k_m)}$, $\ldots$, $A_{\gamma,m-1}^{(k_1,k_2,\ldots,k_m)}$ simply as $A_{\gamma,1}, A_{\gamma,2}, \ldots, A_{\gamma,m-1}$.

Let $n \geq 2, m \geq 3$ be integers, and

$$I_n^m = \{\gamma \mid \gamma \text{ is an irreducible ordered } m\text{-multiple word over } \mathcal{A}(n)\},$$
$$L_n^m = \{\gamma \mid \gamma \text{ is an irreducible ordered } m\text{-multiple linked word over } \mathcal{A}(n)\},$$
$$L^m = \bigcup_{n=2}^{+\infty} L_n^m.$$

Then we state the second main result:

Main Theorem 1.2. *With the notation above, then $K(k_1, k_2, \ldots, k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ is a SSOSC if and only if there exists $\gamma \in L^m$ such that $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}) = (\rho(A_{\gamma,1}), \rho(A_{\gamma,2}), \ldots \rho(A_{\gamma,m-1}))$.*

The paper is organized as follows: In Section 2, we recall some known facts on non-negative matrix; Section 3 is devoted to the proof of Theorem 1.1 and Theorem 1.2; in Section 4, we will give some remarks and questions.

2. Irreducible word and non-negative matrices.

2.1. Some known facts. In this part, we recall some propositions that will be used in the proofs of Theorem 1.1 and Theorem 1.2.

Proposition 2.1 ([Fen]). *Let K be the union of finitely many closed intervals. Then K is an SSOSC if and only if K can tile an interval by using finitely many of its affine copies, i.e., there exist finitely many affine maps $\{\psi_i\}_{i=1}^k$ such that $\bigcup_{i=1}^k \psi_i(K)$ is a non-empty interval and $\psi_i(int(K))$ are disjoint.*

In the following we denote by M_n the set of all $n \times n$ matrices.

The following classic facts on the non-negative matrix can be found [HJ]:

Proposition 2.2. *Let $A \in M_n$ be irreducible and non-negative. Then*

(i) $\rho(A) > 0$;
(ii) *$\rho(A)$ is an eigenvalue of A;*
(iii) *There is a positive vector x such that $Ax = \rho(A)x$.*

Proposition 2.3. *Let $A \in M_n$. If $A \geq 0$ and for any integer $i = 1, 2, \ldots, n$,*

$$\sum_{j=1}^n a_{ij} > 0,$$

then $\rho(A) > 0$. In particular, $\rho(A) > 0$ if $A > 0$ or if A is irreducible and non-negative.

Proposition 2.4. *Let $A \in M_n$ be non-negative. If A has a positive eigenvector, then the corresponding eigenvalue is $\rho(A)$; that is, if $Ax = \lambda x$ and $x > 0$ and $A \geq 0$, then $\lambda = \rho(A)$.*

Proposition 2.5. *Let $A = (a_{ij}) \in M_n$. Then A is irreducible if and only if A has the following property: for any $1 \leq i, j \leq n$ and $i \neq j$, there exist integers $1 \leq b_1, b_2, \ldots, b_k \leq n$ such that $a_{ib_1} > 0, a_{b_1 b_2} > 0, \ldots, a_{b_{k-1} b_k} > 0, a_{b_k j} > 0$.*

2.2. Irreducibility of double word and incidence matrix. Keep the definitions and notation in Section 1. Given $q > 0$ and an ordered double word γ over the alphabet $\mathcal{A}(n)$, we will define the relationship of position among different letters. Let $i, j \in \mathcal{A}(n)$ with $i \neq j$. If both i_1, i_2 are located between j_1 and j_2, we call the letter j *includes* the letter i; if only one of the letters i_1 and i_2 is located between j_1 and j_2, we call the letters i and j *intersect*; if neither i_1 nor i_2 is located between j_1 and j_2 and *vice versa* (i.e., neither j_1 nor j_2 is located between i_1 and i_2), we call the letters i and j are *disjoint*. From the above definitions, we can see that the relationship of position between any two different letters will be one of the three relations: inclusion, intersection, and disjoint.

LEMMA 2.6. *Given an ordered double word γ over the alphabet $\mathcal{A}(n)$ with incidence matrix A^q_γ. Let i and j be two different letters, then $a_{ij} > 0$ if and only if either the letter i includes the letter j, or the letters i and j intersect.*

By the definition of incidence matrix, we can see that Lemma 2.6 is equivalent to: $a_{ij} = 0$ if and only if either the letter j includes the letter i, or the letters i and j are disjoint.

If there are different letters $i, j, b_1, b_2 \ldots, b_k \in \mathcal{A}(n)$ $(k \leq n-2)$ such that all the pairs i and b_1, b_1 and b_2, …, b_{k-1} and b_k, b_k and j intersect in the word γ, then we say that the letters i and j are *connected*. By convention, any letter $i \in \mathcal{A}(n)$ is connected with itself. Two letters i and j is called equivalent and denoted by $i \sim j$ if i and j are connected. For a given ordered double word γ, the relation $\sim$ defines an equivalence relation on $\mathcal{A}(n)$, and the equivalence class containing the letter i is

$$\Gamma_i := \{1 \leq j \leq n \mid i \text{ and } j \text{ are connected}\}. \tag{2.1}$$

Assume that the letters i, j are not connected. If there exists $i_0 \in \Gamma_i$ and $j_0 \in \Gamma_j$ such that i_0 includes j_0, then we say that the class Γ_i *includes* the class Γ_j, and in this case, i_0 includes all letters in Γ_j. If for any $l \in \Gamma_i$ and any $l' \in \Gamma_j$, the letters l and l' are disjoint, then we say that the two classes Γ_i and Γ_j are *disjoint*. Any two classes are either disjoint, or one is included in the other. Therefore, the relation "inclusion" over the equivalence classes on $\mathcal{A}(n)$ is indeed a partial ordering.

From the discussions above, we can obtain immediately

PROPOSITION 2.7. *An ordered double word γ is irreducible if and only if the equivalence relation induced by γ has only one equivalence class.*

Proof. Given an ordered double word γ over $\mathcal{A}(n)$, choose a minimal element from the equivalence classes by the partial ordered relation. Since the inclusion is a partial ordering and the cardinality of the equivalence classes is finite, minimal element always exists. Suppose that Γ_i is such a minimal element, then for any $\Gamma_j \neq \Gamma_i$, we get either Γ_j includes Γ_i, or Γ_j and Γ_i are disjoint, which means the letters in Γ_i appear in the word γ forms a sub-word of γ. By the definition of irreducibility of the double word, γ is irreducible if and only if the equivalence classes induced by γ has only one equivalence class. □

The following proposition establishes the relationship between the irreducibility of an ordered double word and the irreducibility of its incidence matrix.

PROPOSITION 2.8. *An ordered double word γ is irreducible if and only if its incidence matrix is irreducible.*

Proof. If the ordered double word γ over $\mathcal{A}(n)$ is irreducible, then by Proposition 2.7, any two letters i and j are connected, that is, there exist letters $b_1, b_2, \ldots, b_k$ such that all pairs i and b_1, b_1 and b_2, ..., b_k and j intersect. By Lemma 2.6, $a_{ib_1} > 0$, $a_{b_1b_2} > 0$, ..., $a_{b_kj} > 0$. Notice that a_{ib_1}, $a_{b_1b_2}$, ..., a_{b_kj} are coefficients of the incidence matrix A^q_γ, by Proposition 2.5, A^q_γ is irreducible.

Conversely, suppose the matrix A^q_γ is irreducible. By contradiction, we assume that γ is reducible. By Proposition 2.8, there are at least two equivalence classes. Let Γ_i be a minimal element (by partial ordering on the equivalence classes), then the complement Γ^c_i is not empty. Taking $l \in \Gamma_i$ and $l' \in \Gamma^c_i$ arbitrarily, we will prove $a_{ll'} = 0$. Notice firstly that the letter l cannot include the letter l', otherwise by the definition, the class Γ_i will include the class $\Gamma_{l'}$; secondly the letters i and l' cannot intersect, otherwise l and l' will be connected, then $l' \in \Gamma_i$ by (2.1). Therefore we have either l' includes l, or l and l' are disjoint, and by Lemma 2.6, we get $a_{ll'} = 0$. By the way, from the irreducibility of matrix A^q_γ and Proposition 2.5, there exist letters $b_1, b_2, \ldots, b_k \in \mathcal{A}(n)$ such that $a_{lb_1} > 0$, $a_{b_1b_2} > 0$, ..., $a_{b_kl'} > 0$. From the discussion above, $a_{lb_1} > 0$ will imply $b_1 \in \Gamma_i$, and in turn $b_2 \in \Gamma_i$, ..., and finally we get $l' \in \Gamma_i$ which contradicts the hypothesis $l' \in \Gamma^c_i$. □

Remark 2.9. In the case of multiple word, we give two remarks about the incidence matrices in the following:

(i) Notice that the conclusions of Propositions 2.7 and 2.8 may be false when $m \geq 3$, that is, if γ is an irreducible ordered m-multiple word over $\mathcal{A}(n)$, we cannot conclude that all $m-1$ incidence matrices of γ must be irreducible. Here is an example. Let $\gamma = 1_1 2_1 1_2 2_2 1_3 3_3 2_3 4_1 3_2 4_2 3_1 4_3$, then γ is an irreducible ordered 3-multiple word over $\mathcal{A}(4)$, but the incidence matrix

$$A^{(k_1,k_2,k_3)}_{\gamma,1} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_2 \\ 0 & 0 & k_2 & 0 \end{bmatrix}$$

is reducible. In this case, the conditions of Proposition 2.2 do not hold.

(ii) Let γ be an irreducible ordered m-multiple word γ over $\mathcal{A}(n)$. By the definition of the multiple word in Section 1.2, the same letters appearing in γ cannot be adjacent, thus, for any $1 \leq l \leq m-1$, all incidence matrices of γ satisfy that $\sum_{j=1}^n a^l_{ij} > 0$ for all $i = 1, 2, \ldots, n$, that is, satisfy the condition of Proposition 2.3.

3. The proof of main theorems.

3.1. Proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1.

Theorem 1.1. *K^q_λ is a SSOSC if and only if $\lambda \in \Lambda^q$.*

Proof. If K^q_λ is a SSOSC, by Proposition 2.1, there exists a finite family of affine mappings $\{\psi_i(x) = c_i x + d_i\}_{i=1}^k$ such that $\bigcup_{i=1}^k \psi_i(K^q_\lambda)$ is a non-empty closed interval and for any $i \neq j$, $\psi_i(int(K^q_\lambda)) \cap \psi_j(int(K^q_\lambda)) = \varnothing$. To avoid trivial case, we suppose that $|c_i| > 0$ $(1 \leq i \leq k)$. Notice that K^q_λ may be generated by different families of affine mappings. Among these families, there exists a family which has the least number of affine mappings, which we denote by $\{\psi_i(x) = c_i x + d_i\}_{i=1}^m$. Recall that $K^q_\lambda = [0,1] \cup [1+\lambda, 1+\lambda+q]$. Let $K_1 = [0,1], K_2 = [1+\lambda, 1+\lambda+q]$, then

$\psi_1(K_1), \psi_1(K_2), \ldots, \psi_m(K_1), \psi_m(K_2)$ tile a closed interval U. Arrange these $2m$ intervals from left to right, and label them by $\psi_{i_1}(K_{j_1}), \psi_{i_2}(K_{j_2}), \ldots, \psi_{i_{2m}}(K_{j_{2m}})$, where $i_1, i_2, \ldots, i_{2m} \in \{1, 2, \ldots, m\}$, and $j_1, j_2, \ldots, j_{2m} \in \{1, 2\}$, then $\gamma = i_{1_{j_1}} i_{2_{j_2}} \cdots i_{2m_{j_{2m}}}$ is an ordered double word over $\mathcal{A}(m)$. If γ is reducible, by the definition of reducibility, there exists a strict ordered double sub-word of γ. Without loss of generality, suppose that the sub-word is composed of the letters $1_1, 1_2, \ldots, l_1, l_2$, then $l < m$ and $\psi_1(K_1), \psi_1(K_2), \ldots, \psi_l(K_1), \psi_l(K_2)$ tile a closed interval $U' \subset U$. That is, the affine mappings family $\{\psi_i(x)\}_{i=1}^{l}$ also satisfies the condition of Proposition 2.1, which contradicts the hypothesis of minimal cardinality of the family $\{\psi_i(x)\}_{i=1}^{m}$. So γ is irreducible.

Notice that $\psi_1(K_\lambda^q), \psi_2(K_\lambda^q), \ldots, \psi_m(K_\lambda^q)$ tile a closed interval, of which the position of each interval $\psi_i(K_j)(1 \leq i \leq m, 1 \leq j \leq 2)$ in U is determined by the word γ.

By the discussions above, we have for any $1 \leq i \leq m$

$$|c_i|\lambda = \sum_{j=1}^{m} a_{ij}|c_j|.$$

Let $v = (|c_1|, |c_2|, \ldots, |c_m|)^T$. Then by the equality above and the definition of the incidence matrix, we get $A_\gamma^q v = \lambda v$. Since $v > 0$, by Proposition 2.4 $\lambda = \rho(A_\gamma^q) > 0$. Since γ is irreducible and by definition $A_\gamma^q \in \Omega_m^q$, so $\lambda \in \Lambda_m^q \subset \Lambda^q$.

Conversely, if $\lambda \in \Lambda^q$, there exists $n \in \mathbb{N}$ and an irreducible ordered double word γ over $\mathcal{A}(n)$ such that $\lambda = \rho(A_\gamma^q)$. Since γ is irreducible, by Proposition 2.8, we know that A_γ^q is irreducible. Then by Proposition 2.2, $\lambda > 0$ and there exists a positive vector $x = (x_1, x_2 \cdots x_n)^T > 0$ such that $A_\gamma^q x = \lambda x$. Let $\gamma = i_{1_{j_1}} i_{2_{j_2}} \cdots i_{2n_{j_{2n}}}$, where $i_1, i_2, \ldots, i_{2n} \in \{1, 2, \ldots, n\}$, $j_1, j_2, \ldots, j_{2n} \in \{1, 2\}$. Let $1 \leq k \leq 2n$. To each letter $i_{k_{j_k}}$, we associate with a closed interval $I_{i_k}^{j_k}$ in the following way: the length of the interval $I_{i_k}^{j_k}$ is $x_{i_k} q^{j_k - 1}$, and the left endpoint of $I_{i_k}^{j_k}$ coincides with the right endpoint of $I_{i_{k-1}}^{j_{k-1}}$(the interval $I_{i_1}^{j_1}$ is fixed arbitrarily). By this construction, these $2n$ closed intervals tile a closed interval U. Let $1 \leq l \leq n$. By the definition of the incidence matrix, the gap between I_l^1 and I_l^2 is $\sum_{m=1}^{n} a_{lm} x_m$. But $x = (x_1, x_2 \cdots x_n)^T$ is a eigenvector of A_γ^q relating to the eigenvalue λ, thus we have

$$\sum_{m=1}^{n} a_{lm} x_m = \lambda x_l.$$

Furthermore, the length of I_l^1 is x_l, I_l^2 is $q x_l$, so $I_l^1 \cup I_l^2$ is an affine copy of K_λ^q. That is, n affine copies of K_λ^q tile an interval U. Hence by Proposition 2.1, K_λ^q is a SSOSC. □

3.2. Proof of Theorem 1.2 and some complement. The main idea of the proof of Theorem 1.2 is analogous to that of Theorem 1.1, but the details are much more complicated.

THEOREM 1.2. *$K(k_1, k_2 \cdots k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ is a SSOSC if and only if $\exists \gamma \in L^m$ such that $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}) = (\rho(A_{\gamma,1}), \rho(A_{\gamma,2}), \ldots, \rho(A_{\gamma,m-1}))$.*

Proof. Let $K = K(k_1, k_2 \cdots k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1}) = \bigcup_{j=1}^{m} I_j$ be the union of m closed intervals $\{I_j\}_{j=1}^{m}$ of lengths $k_1, k_2, \ldots k_m$ and gap lengths $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$. If K is a SSOSC, by Proposition 2.1, there exists a finite family of affine mappings $\{\psi_i(x)\}_{i=1}^{k}$ such that $\bigcup_{i=1}^{k} \psi_i(K)$ is a non-empty closed interval, and $\forall i \neq j$,

$\psi_i(int(K)) \cap \psi_j(int(K)) = \varnothing$. Let $\{\psi_i(x) = c_i x + d_i\}_{i=1}^n$ be a family which has the least number of affine mappings. Then $\{\psi_i(I_j)\}_{1\le i\le n, 1\le j\le m}$ tile a closed interval. Arrange them from left to right and label them by $\psi_{i_1}(I_{j_1}), \psi_{i_2}(I_{j_2}), \ldots, \psi_{i_{mn}}(I_{j_{mn}})$, where $i_1, i_2, \ldots, i_{mn} \in \{1, 2, \ldots, n\}$ and $j_1, j_2, \ldots, j_{mn} \in \{1, 2, \ldots, m\}$, then $\gamma = i_{1_{j_1}} i_{2_{j_2}} \cdots i_{mn_{j_{mn}}}$ is an ordered m-multiple word over $\mathcal{A}(n)$. Since $\{\psi_i(x)\}_{i=1}^n$ has the least number of affine mappings, it follows as in the proof of Theorem 1.1, that γ is irreducible, and hence $\gamma \in I_n^m$. Let $\{A_{\gamma,l}^{(k_1,k_2\cdots k_m)} = (a_{ij}^l)_{n\times n}\}_{l=1}^{m-1}$ be the incidence matrices of γ. Then $\forall 1 \le l \le m-1$, $\forall 1 \le i \le n$,

$$|c_i|\lambda_l = \sum_{j=1}^n a_{ij}^l |c_j|.$$

Let $v = (|c_1|, |c_2| \cdots |c_n|)^T$. Then $\forall 1 \le l \le m-1$, $A_{\gamma,l} v = \lambda_l v$. Since $v > 0$, by Proposition 2.4, $\forall 1 \le l \le m-1$, $\lambda_l = \rho(A_{\gamma,l}) > 0$. Since v is a positive common eigenvector of $\{A_{\gamma,l}\}_{l=1}^{m-1}$, $\{A_{\gamma,l}\}_{l=1}^{m-1}$ are linked, and hence $\gamma \in L_n^m \subset L^m$. We get $\gamma \in L^m$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}) = (\rho(A_{\gamma,1}), \rho(A_{\gamma,2}), \ldots, \rho(A_{\gamma,m-1}))$.

Conversely, assume $\exists \gamma \in L^m$ such that $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}) = (\rho(A_{\gamma,1}), \rho(A_{\gamma,2}), \ldots, \rho(A_{\gamma,m-1}))$. Then $\exists n \in \mathbb{N}$, such that $\gamma = i_{1_{j_1}} i_{2_{j_2}} \cdots i_{mn_{j_{mn}}} \in L^m$, where $i_1, i_2, \ldots, i_{mn} \in \{1, 2, \ldots, n\}$ and $j_1, j_2, \ldots, j_{mn} \in \{1, 2, \ldots, m\}$. From Remark 2.9, $\forall 1 \le l \le m-1$, $A_{\gamma,l}$ satisfies the condition of Proposition 2.3, and hence $\rho(A_{\gamma,l}) > 0$. Since γ is linked, $\exists b_1, b_2, \ldots, b_{m-1} > 0$ and vector $x = (x_1, x_2, \ldots, x_n)^T > 0$ such that $\forall 1 \le l \le m-1$, $A_{\gamma,l} x = b_l x$. From Proposition 2.4 we know $b_l = \rho(A_{\gamma,l}) = \lambda_l$. Given mn closed intervals $\{I_{i_h}^{j_h}\}_{1\le h\le mn}$, let $|I_{i_h}^{j_h}| = x_{i_h} k_{j_h}$, $\forall 1 \le h \le mn$. Arrange the mn intervals $I_{i_1}^{j_1}, I_{i_2}^{j_2}, \ldots, I_{i_{mn}}^{j_{mn}}$ from left to right so that they tile a closed interval U. By the definition of the incidence matrices $\{A_{\gamma,l}\}_{l=1}^{m-1}$, $\forall 1 \le l \le m-1$, $\forall 1 \le s \le n$, the gap between I_s^l and I_s^{l+1} is $\sum_{r=1}^n a_{sr}^l x_r$. Since $x = (x_1, x_2 \cdots x_n)^T > 0$ is a positive common eigenvector, we have $\sum_{r=1}^n a_{sr}^l x_r = \lambda_l x_s$. So $\forall 1 \le s \le n$, $\bigcup_{t=1}^m I_s^t$ is an affine copy of K. The n affine copies of K tile an interval U. By Proposition 2.1, $K = K(k_1, k_2, \ldots, k_m; \lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ is a SSOSC. □

Remark 3.1. As a reducible matrix can be decomposed into blocks of irreducible matrices (correspondingly, a reducible ordered double word can be decompose into irreducible ordered double sub-words), the irreducibility condition in Theorems 1.1 and 1.2 can be removed. In this case, Λ^q is unchanged. We keep the condition "irreducible" in this paper to simplify the proof.

Remark 3.2. Notice that for any integers $m \ge 3$ and $k_1, k_2, \ldots, k_m > 0$, the sets $K(k_1, k_2, \ldots, k_m; k_{m-1}, k_{m-2}, \ldots, k_1)$ and $K(k_m, k_{m-1}, \ldots, k_1; k_1, k_2, \ldots, k_{m-1})$ will tile a closed interval. Notice the second set is obtained from the first by translation and reflection. Therefore for any integers $m \ge 3$ and $k_1, k_2, \ldots, k_m > 0$, we have $L^m \ne \varnothing$.

Let $m, n \ge 2$ and $\gamma \in I_n^m$. Let $A_{\gamma,1}, A_{\gamma,2}, \ldots, A_{\gamma,m-1}$ be the incidence matrices of γ. The *spectral radius vector* of γ is defined as the vector

$$(\rho(A_{\gamma,1}), \rho(A_{\gamma,2}) \cdots \rho(A_{\gamma,m-1})).$$

Notice that different multiple words have different incidence matrices, but they may have the same spectral radius vector. The following remark discusses the relationship between the multiple words having the same spectral radius vector, and establishes an equivalence relation over I_n^m, which will clarify the structure of I_n^m. Some examples will show that I_n^m can be simplified to the more simple sets by the equivalence relation.

REMARK 3.3. Let $n, m \geq 2$. We define permutation map and reflection map respectively over I_n^m as follow:

Let σ is a permutation over the alphabet $\mathcal{A}(n)$. For any $\gamma = i_{1_{j_1}} i_{2_{j_2}} \cdots i_{nm_{j_{nm}}} \in I_n^m$, define

$$P_\sigma(\gamma) := \sigma(i_1)_{j_1} \sigma(i_2)_{j_2} \cdots \sigma(i_{nm})_{j_{nm}} \in I_n^m.$$

Then P_σ is a permutation on I_n^m, called a permutation map on I_n^m.

For any $\gamma = i_{1_{j_1}} i_{2_{j_2}} \cdots i_{nm_{j_{nm}}} \in I_n^m$, define

$$R(\gamma) := i_{nm_{j_{nm}}} i_{nm-1_{j_{nm-1}}} \cdots i_{1_{j_1}} \in I_n^m.$$

Then R is also a permutation on I_n^m, called a reflection map on I_n^m.

Given $k_1, k_2, \ldots, k_m > 0$, taking any $1 \leq l \leq m-1$, and $\gamma \in I_n^m$, then for any permutation σ over the alphabet $\mathcal{A}(n)$, we see that the matrix $A_{\gamma,l}$ is similar to the matrix $A_{P_\sigma(\gamma),l}$ and $A_{\gamma,l} = A_{R(\gamma),l}$, so

$$\rho(A_{\gamma,l}) = \rho(A_{P_\sigma(\gamma),l}) = \rho(A_{R(\gamma),l}). \tag{3.1}$$

We also see that if the matrices $\{A_{\gamma,l}\}_{l=1}^{m-1}$ are linked, then the matrices $\{A_{P_\sigma(\gamma),l}\}_{l=1}^{m-1}$ and $\{A_{R(\gamma),l}\}_{l=1}^{m-1}$ are linked too. That is, for any $\gamma \in L_n^m$, $P_\sigma(\gamma), R(\gamma) \in L_n^m$, P_σ and R are also permutations on L_n^m.

Denote by Q_n^m the set of all maps on I_n^m that is the composition by finite permutation maps and reflection maps. Then Q_n^m defines an equivalence relation $\sim$ on I_n^m by the following way: let $\gamma, \gamma' \in I_n^m$, then $\gamma \sim \gamma'$ if and only if there is $f \in Q_n^m$ such that $f(\gamma) = \gamma'$. If $\gamma \sim \gamma'$, by (3.1) we have

$$(\rho(A_{\gamma,1}), \ldots, \rho(A_{\gamma,m-1})) = (\rho(A_{\gamma',1}), \ldots, \rho(A_{\gamma',m-1})). \tag{3.2}$$

Let $\overline{I}_n^m = I_n^m / \sim$ be the set of equivalence classes over I_n^m induced by the equivalence relation $\sim$. Then $\sim$ also induces an equivalence relation on L_n^m by (3.2), and we denote by $\overline{L}_n^m$ this set of equivalence classes over L_n^m.

In particular, for $m = 2$, Q_n^2 induces an equivalence relation on I_n^2. If $\gamma \sim \gamma'$, then for any $q > 0$, we have $\rho(A_\gamma^q) = \rho(A_{\gamma'}^q)$. Let

$$\overline{I}_n^2 = I_n^2 / \sim, \qquad \overline{\Omega}_n^q = \{A_\gamma^q | \gamma \in \overline{I}_n^2\},$$
$$\overline{\Lambda}_n^q = \{\rho(A) | A \in \overline{\Omega}_n^q\}, \qquad \overline{\Lambda}^q = \bigcup_{n=2}^{+\infty} \overline{\Lambda}_n^q.$$

So, to compute all $\lambda \in \Lambda_n^q$, we only need to compute all $\lambda \in \overline{\Lambda}_n^q$. Similarly, for computing the eigenvalues over all matrices, we can replace Λ^q, L^m by $\overline{\Lambda}^q, \overline{L}^m$ respectively in Theorem 1.1 and 1.2, which will simplify the computation.

Some examples are given below to illustrate the above discussions.

EXAMPLE 3.4. The set I_2^2 has 12 elements, and by the discussion above, we can simplify I_2^2 to the set $\overline{I}_2^2$, which only has 3 elements.

The table below shows two examples for $\gamma \in \overline{I}_2^2$:

$\gamma \in \overline{I}_2^2$	Characteristic polynomial of A_γ^q	$q = \frac{\sqrt{5}-1}{2}$	$q = \sqrt{2}$
$1_1 2_1 1_2 2_2$	$x^2 - q$	$\rho(A_\gamma^q) = 0.7862$	$\rho(A_\gamma^q) = 1.1892$
$1_1 2_2 1_2 2_1$	$x^2 - q^2$	$\rho(A_\gamma^q) = 0.618$	$\rho(A_\gamma^q) = 1.4142$
$1_2 2_1 1_1 2_2$	$x^2 - 1$	$\rho(A_\gamma^q) = 1$	$\rho(A_\gamma^q) = 1$

Example 3.5. The set I_3^2 has 192 elements, and the set $\overline{I}_3^2$ has only 16 elements.

$\gamma \in \overline{I}_3^2$	Characteristic polynomial of A_γ^q	$q = \frac{\sqrt{5}-1}{2}$	$q = \sqrt{2}$
$1_1 2_1 1_2 3_2 2_2 3_1$	$x^3 - (q^2+q)x$	$\rho(A_\gamma^q) = 1$	$\rho(A_\gamma^q) = 1.8478$
$1_1 2_1 1_2 3_1 2_2 3_2$	$x^3 - 2qx$	$\rho(A_\gamma^q) = 1.1118$	$\rho(A_\gamma^q) = 1.6818$
$1_1 2_2 1_2 3_1 2_1 3_2$	$x^3 - (q^2+1)x$	$\rho(A_\gamma^q) = 1.1756$	$\rho(A_\gamma^q) = 1.7321$
$1_2 2_1 1_1 3_1 2_2 3_2$	$x^3 - (q+1)x$	$\rho(A_\gamma^q) = 1.272$	$\rho(A_\gamma^q) = 1.5538$
$1_1 2_1 3_2 1_2 2_2 3_1$	$x^3 - (2q^2+q)x + (q^3+q^2)$	$\rho(A_\gamma^q) = 1.3557$	$\rho(A_\gamma^q) = 2.6855$
$1_1 2_1 3_1 1_2 2_2 3_2$	$x^3 - 3qx + (q^2+q)$	$\rho(A_\gamma^q) = 1.5774$	$\rho(A_\gamma^q) = 2.3824$
$1_1 2_2 3_1 1_2 2_1 3_2$	$x^3 - (q^2+q+1)x + (q^2+q)$	$\rho(A_\gamma^q) = 1.618$	$\rho(A_\gamma^q) = 2.4142$
$1_2 2_1 3_1 1_1 2_2 3_2$	$x^3 - (q+2)x + (q+1)$	$\rho(A_\gamma^q) = 1.8668$	$\rho(A_\gamma^q) = 2.1322$
$1_1 2_2 3_2 1_2 3_1 2_1$	$x^3 - 2q^2x + (q^3+q^2)$	$\rho(A_\gamma^q) = 1.1423$	$\rho(A_\gamma^q) = 2.4444$
$1_1 2_2 3_1 1_2 3_2 2_1$	$x^3 - (q^2+q)x + (q^3+q^2)$	$\rho(A_\gamma^q) = 1.2264$	$\rho(A_\gamma^q) = 2.3404$
$1_1 2_1 3_2 1_2 3_1 2_2$	$x^3 - (q^2+q)x + (q^2+q)$	$\rho(A_\gamma^q) = 1.3247$	$\rho(A_\gamma^q) = 2.2246$
$1_1 2_1 3_1 1_2 3_2 2_2$	$x^3 - 2qx + (q^2+q)$	$\rho(A_\gamma^q) = 1.3971$	$\rho(A_\gamma^q) = 2.1089$
$1_2 2_2 3_2 1_1 3_1 2_1$	$x^3 - 2qx + (q^2+q)$	$\rho(A_\gamma^q) = 1.3971$	$\rho(A_\gamma^q) = 2.1089$
$1_2 2_2 3_1 1_1 3_2 2_1$	$x^3 - (q+1)x + (q^2+q)$	$\rho(A_\gamma^q) = 1.5101$	$\rho(A_\gamma^q) = 2.0249$
$1_2 2_1 3_2 1_1 3_1 2_2$	$x^3 - (q+1)x + (q+1)$	$\rho(A_\gamma^q) = 1.618$	$\rho(A_\gamma^q) = 1.9167$
$1_2 2_1 3_1 1_1 3_2 2_2$	$x^3 - 2x + (q+1)$	$\rho(A_\gamma^q) = 1.7156$	$\rho(A_\gamma^q) = 1.8232$

4. Further remarks and open questions. Let $m \geq 2$, and define

$$I^m = \bigcup_{n=2}^{+\infty} I_n^m, \quad L^m = \bigcup_{n=2}^{+\infty} L_n^m.$$

Since for any $n \geq 2$, I_n^2 is a finite set, hence I^2 is a countable set. Consequently for any $q > 0$, Λ^q is also a countable set.

Question 1. We wish to know the structure of Λ^q (the set of all eigenvalues). Furthermore could we determine all limit points of Λ^q?

Question 2. Given $\lambda > 0$, could we talk about the structure of the set $\{q > 0 \mid \lambda \in \Lambda^q\}$?

Let $\gamma \in I_n^2$ and $q > 0$ (recall q is the ratio of the lengths of two intervals), and let $\lambda(\gamma, q) := \rho(A_\gamma^q)$, i.e., the spectral radius of the incidence matrix A_γ^q. Given $\gamma \in I_n^2$, as a function of q, $\lambda(\gamma, q)$ is continuous on $(0, +\infty)$.

Question 3. Given $\gamma \in I^2$, does the limit $\lim_{q\to+\infty} \lambda(\gamma, q)$ exist? If does, is the limit finite?

Question 4. Is I^2 finite for any $q > 0$?

For $m \geq 3$, considering the same question, that is, for any $k_1, k_2, \ldots, k_m > 0$, is L^m finite?

Question 5. Looking for $\lambda > 0$ such that $K(1,1,1;1,\lambda)$ is a SSOSC. The same question for $K(1,1,1;\lambda,\lambda)$.

We know that for any $\gamma \in I^3$, γ has two incidence matrices $A_{\gamma,1}, A_{\gamma,2}$, and if γ is linked, then two matrices have a positive common eigenvector x. By Proposition 2.4, we get

$$A_{\gamma,1}A_{\gamma,2}x = A_{\gamma,2}A_{\gamma,1}x = \rho(A_{\gamma,1})\rho(A_{\gamma,2})x,$$

therefore

$$\det(A_{\gamma,1}A_{\gamma,2} - A_{\gamma,2}A_{\gamma,1}) = 0.$$

We think this equation, associating with Theorem 1.2, could give more information about the above question.

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