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THE CONVEXITY RADIUS OF A RIEMANNIAN MANIFOLD*

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Abstract. The ratio of convexity radius over injectivity radius may be made arbitrarily small within the class of compact Riemannian manifolds of any fixed dimension at least two. This is proved using Gulliver's method of constructing manifolds with focal points but no conjugate points. The approach is suggested by a characterization of the convexity radius that resembles a classical result of Klingenberg about the injectivity radius.

Key words. Convexity radius, injectivity radius, focal points, conjugate points.

AMS subject classifications. Primary 53C20.

1. Introduction. A subset X of a Riemannian manifold M is **strongly convex** if any two points in X are joined by a unique minimal geodesic $\gamma : [0,1] \to M$ and each such geodesic maps entirely into X. It is well known that there exist functions $\mathrm{inj}, r : M \to (0,\infty]$ such that, for each $p \in M$,

$$\mathrm{inj}(p) = \max\big\{R > 0 \,\big|\, \exp_p|_{B(0,s)} \text{ is injective for all } 0 < s < R\big\}$$

and

$$r(p) = \max\big\{R > 0 \,\big|\, B(p,s) \text{ is strongly convex for all } 0 < s < R\big\},$$

where $B(0,s) \subset T_pM$ denotes the Euclidean ball of radius s around the origin. The number $\mathrm{inj}(p)$ is the **injectivity radius at** p, and $r(p)$ is the **convexity radius at** p. The **conjugate radius at** p is defined, as is customary, to be

$$r_c(p) = \min\big\{T > 0 \,\big|\, \exists \text{ a non-trivial normal Jacobi field } J \text{ along a unit-speed geodesic } \gamma \text{ with } \gamma(0) = p,\ J(0) = 0, \text{ and } J(T) = 0\big\},$$

and the **focal radius at** p is introduced here to be

$$r_f(p) = \min\big\{T > 0 \,\big|\, \exists \text{ a non-trivial normal Jacobi field } J \text{ along a unit-speed geodesic } \gamma \text{ with } \gamma(0) = p,\ J(0) = 0, \text{ and } \|J\|'(T) = 0\big\}.$$

Either of these is defined to be infinite if the corresponding Jacobi fields do not exist. Short arguments show that they are well defined and that $r_f(p) \le r_c(p)$, with equality if and only if both are infinite.

If $\gamma : [a,b] \to M$ is a geodesic connecting p to q, then p is **conjugate to** q **along** γ if there exists a non-trivial normal Jacobi field J along γ that vanishes at the endpoints. If $\sigma : I \to M$ is a geodesic and $\gamma : [a,b] \to M$ is a geodesic connecting p to $\sigma(s)$, where I is an interval and $s \in I$, then p is **focal to** σ **along** γ if there exists a non-trivial normal Jacobi field J along γ such that $J(a) = 0$ and $J(b) = \sigma'(s)$. Conjugate and focal points correspond to singularities of the restriction of the exponential map to T_qM and the normal bundle of $\sigma(I)$, respectively. Employing arguments similar

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to the proof of Proposition 4 in [10], one finds that the conjugate radius at p is the length of the shortest geodesic $\gamma : [a, b] \to M$ along which p is conjugate to $\gamma(b)$, while the focal radius at p is the length of the shortest geodesic $\gamma : [a, b] \to M$ along which p is focal to a non-constant geodesic normal to γ at $\gamma(b)$. Let $\mathrm{inj}(M) = \inf_{p \in M} \mathrm{inj}(p)$, and similarly define numbers $r(M)$, $r_c(M)$, and $r_f(M)$. It's widely known that, when M has sectional curvature bounded above, the first three are positive. The results in the third section of [4] imply the same about the focal radius.

When M is compact, it was shown by Berger [1] that $r(M) \le \frac{1}{2}\mathrm{inj}(M)$. Berger has also pointed out that there are no examples in the literature where this inequality is known to be strict [2]. It will be shown in this paper that $\inf \frac{r(M)}{\mathrm{inj}(M)} = 0$ over the class of compact manifolds of any fixed dimension at least two. The proof is suggested by alternative characterizations of the injectivity and convexity radiuses. Klingenberg [7] showed that $\mathrm{inj}(M) = \min\left\{r_c(M), \frac{1}{2}\ell_c(M)\right\}$, where $\ell_c(M)$ is the length of the shortest non-trivial closed geodesic in M. It will be shown here that $r(M) = \min\left\{r_f(M), \frac{1}{4}\ell_c(M)\right\}$. To the best of my knowledge, this equality does not appear elsewhere in the literature. Gulliver [6] introduced a method of constructing compact manifolds with focal points but no conjugate points. For such manifolds, $r_f(M) < \infty$ and $r_c(M) = \infty$. The main result follows by showing that Gulliver's method may be used to construct manifolds with $\frac{r_f(M)}{\ell_c(M)}$ arbitrarily small.

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2. Geometric radiuses. When M is complete, each $v \in TM$ determines a geodesic $\gamma_v : (-\infty, \infty) \to M$ by the rule $\gamma_v(t) = \exp(tv)$. For each $p \in M$, the **cut locus at** p is the set

$$\mathrm{cut}(p) = \left\{v \in T_pM \,\middle|\, \gamma_v|_{[0,T]} \text{ is minimal if and only if } T \le 1\right\},$$

and the **conjugate locus at** p is

$$\mathrm{conj}(p) = \left\{v \in T_pM \,\middle|\, \exp_p : T_pM \to M \text{ is singular at } v\right\}.$$

A **geodesic loop** is a geodesic $\gamma : [a, b] \to M$ such that $\gamma(a) = \gamma(b)$, while a **closed geodesic** additionally satisfies $\gamma'(a) = \gamma'(b)$. For each $p \in M$, denote by $\ell(p)$ the length of the shortest non-trivial geodesic loop based at p. Let $\ell(M) = \inf_{p \in M} \ell(p)$, and recall from the introduction that, for compact M, $\ell_c(M)$ equals the length of the shortest non-trivial closed geodesic in M. According to a celebrated theorem of Fet–Lyusternik [5], $0 < \ell_c(M) < \infty$. A general relationship between inj and r_c is described by the following classical result of Klingenberg [7].

Theorem 2.1 (Klingenberg). *Let M be a complete Riemannian manifold. If $p \in M$ and $v \in \mathrm{cut}(p)$ has length $\mathrm{inj}(p)$, then one of the following holds:*

(i) $v \in \mathrm{conj}(p)$; or

(ii) $\gamma_v|_{[0,2]}$ is a geodesic loop.

Consequently, $\mathrm{inj}(p) = \min\left\{r_c(p), \frac{1}{2}\ell(p)\right\}$.

Klingenberg used this to characterize $\mathrm{inj}(M)$.

Corollary 2.2 (Klingenberg). *The injectivity radius of a complete Riemannian manifold M is given by* $\operatorname{inj}(M) = \min\{r_c(M), \frac{1}{2}\ell(M)\}$. *When M is compact, it is also given by* $\operatorname{inj}(M) = \min\{r_c(M), \frac{1}{2}\ell_c(M)\}$.

It's not clear that a pointwise result like that in Theorem 2.1 holds for the convexity radius, but global equalities like those in Corollary 2.2 will be proved.

The following lemma is an application of the second variation formula. Note that a C^2 function $f : M \to \mathbb{R}$ is **strictly convex** if its Hessian $\nabla^2 f$ is positive definite. This is equivalent to the condition that, for any non-constant geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$, $(f \circ \gamma)''(0) > 0$.

Lemma 2.3. *Let M be a Riemannian manifold and $p \in M$. If $R \le r_f(p)$ and $\exp_p$ is defined and injective on $B(0,R) \subset T_pM$, then $d^2(p,\cdot) : B(p,R) \to [0,R^2)$ is strictly convex.*

It will be useful to know that the convexity radius is globally bounded above by the focal radius. This may be proved using the following consequence of the Morse index theorem [9]: If γ and σ are unit-speed geodesics, $\gamma(0) = p$, $\gamma(T) = \sigma(0)$, $T = d\big(p, \sigma(0)\big) < \operatorname{inj}(p)$, and p is focal to σ along $\gamma|_{[0,T]}$, then, for sufficiently small s and ε satisfying $0 < \varepsilon < s$, the ball $B\big(\gamma(-s), T + s - \varepsilon\big)$ is not strongly convex. This implies an inequality relating the focal and convexity radiuses near each point.

Lemma 2.4. *Let M be a complete Riemannian manifold. Then, for each $p \in M$, $\liminf_{x \to p} r(x) \le r_f(p)$.*

The global inequality $r(M) \le r_f(M)$ follows immediately. One may also prove a global inequality relating the conjugate and focal radiuses.

Lemma 2.5. *If M is a complete Riemannian manifold, then $r_f(M) \le \frac{1}{2} r_c(M)$.*

Proof. Fix $\varepsilon > 0$, and let $p \in M$ be such that $r_c(p) < r_c(M) + \varepsilon$. Choose a unit-speed geodesic $\gamma : [0, r_c(p)] \to M$ satisfying $\gamma(0) = p$ and a non-trivial normal Jacobi field J along γ with $J(0) = 0$ and $J\big(r_c(p)\big) = 0$. Write $q = \gamma\big(r_c(p)\big)$. There must exist $0 < T < r_c(p)$ such that $\|J\|'(T) = 0$. If $T \le \frac{1}{2} r_c(p)$, then $r_f(p) \le \frac{1}{2} r_c(p)$. If $T \ge \frac{1}{2} r_c(p)$, then, by reversing the parameterizations of γ and J, one finds that $r_f(q) \le \frac{1}{2} r_c(p)$. In either case, $r_f(M) < \frac{1}{2}[r_c(M) + \varepsilon]$. ◻

It's now possible to prove global equalities for the convexity radius.

Theorem 2.6. *The convexity radius of a complete Riemannian manifold M is given by $r(M) = \min\{r_f(M), \frac{1}{4}\ell(M)\}$. When M is compact, it is also given by $r(M) = \min\{r_f(M), \frac{1}{4}\ell_c(M)\}$.*

Proof. Assume for the sake of contradiction that $r(M) > \frac{1}{4}\ell(M)$. Since $\ell(M) < \infty$, one may set $\varepsilon = \frac{4}{5}[r(M) - \frac{1}{4}\ell(M)] > 0$. Let $\gamma : [0,1] \to M$ be a non-trivial geodesic loop with $L(\gamma) < \ell(M) + \varepsilon$. Then $B\big(\gamma(\frac{1}{4}), \frac{1}{4}L(\gamma) + \varepsilon\big)$ and $B\big(\gamma(\frac{3}{4}), \frac{1}{4}L(\gamma) + \varepsilon\big)$ are strongly convex, from which it follows that each of $\gamma|_{[0,\frac{1}{2}]}$ and $-\gamma|_{[\frac{1}{2},1]}$ is the unique minimal geodesic connecting $\gamma(0)$ to $\gamma(\frac{1}{2})$. This contradiction, together with Lemma 2.4, implies that $r(M) \le \min\{r_f(M), \frac{1}{4}\ell(M)\}$.

Assume that there exists $p \in M$ such that $r(p) < \min\{r_f(M), \frac{1}{4}\ell(M)\}$. Let $\varepsilon_i \to 0$ be a decreasing sequence such that each $B\big(p, r(p) + \varepsilon_i\big)$ is not strongly convex. Then there exist $x_i, y_i \in B\big(p, r(p) + \varepsilon_i\big)$ and minimal geodesics $\gamma_i : [0,1] \to M$ from x_i to y_i such that $\gamma_i([0,1]) \not\subset B\big(p, r(p) + \varepsilon_i\big)$. Define constants $0 \le \delta_i < \varepsilon_i$ by

$$r(p) + \delta_i = \max\{d(p, x_i), d(p, y_i)\},$$

and fix $t_i \in (0,1)$ such that $d\big(p,\gamma_i(t_i)\big) \geq r(p)+\varepsilon_i$. Let (a_i,b_i) be the connected component of $\big\{t \in (0,1) \,\big|\, d\big(p,\gamma_i(t)\big) > r(p)+\delta_i\big\}$ containing t_i. Without loss of generality, replace x_i and y_i with $\gamma_i(a_i)$ and $\gamma_i(b_i)$, respectively, and γ_i with $\gamma_i|_{[a_i,b_i]}$, reparameterizing the latter so that $\gamma_i(0)=x_i$ and $\gamma_i(1)=y_i$. Since $L(\gamma_i) \leq 2[r(p)+\varepsilon_1]$ for all i, one may, by passing to a subsequence, suppose without loss of generality that $x_i \to x \in \partial B\big(p,r(p)\big)$, $y_i \to y \in \partial B\big(p,r(p)\big)$, and γ_i uniformly converges to a minimal geodesic $\gamma : [0,1] \to M$ from x to y.

Assume that $x=y$, and choose $\delta>0$ such that

$$r(p)+3\delta < \min\left\{r_f(M), \frac{1}{4}\ell(M)\right\} \leq \frac{1}{2}\min\left\{r_c(M), \frac{1}{2}\ell(M)\right\} = \frac{1}{2}\mathrm{inj}(M).$$

Let i be large enough that $x_i, y_i \in B(x,\delta)$. Then $L(\gamma_i)=d(x_i,y_i)<2\delta$, so $\gamma_i([0,1]) \subset B\big(p,r(p)+3\delta\big)$. Write $R=\min\{r_f(p),\mathrm{inj}(p)\}$. Since

$$d(p,x_i)=d(p,y_i)=r(p)+\delta_i<r(p)+3\delta<R$$

and γ_i is not constant, it follows from Lemma 2.3 that $d\big(p,\gamma_i(t)\big)<r(p)+\delta_i$ for all $t \in (0,1)$. This is a contradiction. So $x \neq y$.

Since $d(x,y) \leq 2r(p) < \mathrm{inj}(M)$, γ is the unique minimal geodesic connecting x to y. Let $w_i, z_i \in B\big(p,r(p)\big)$ be sequences such that $w_i \to x$ and $z_i \to y$. Then there exist unique minimal geodesics $\sigma_i : [0,1] \to M$ from w_i to z_i, which satisfy $\sigma_i([0,1]) \subset B\big(p,r(p)\big)$. Since $\sigma_i \to \gamma$, one finds that $\gamma([0,1]) \subset B(p,R)$. Because γ is not constant, Lemma 2.3 implies that $d\big(p,\gamma(t)\big)<r(p)$ for all $t \in (0,1)$. However, by construction, $d\big(p,\gamma(t)\big) \geq r(p)$ for all $t \in [0,1]$. It follows from this contradiction that $r(M)=\min\big\{r_f(M),\frac{1}{4}\ell(M)\big\}$.

In the case that M is compact, $\ell_c(M) \leq \ell(M)$, so $r(M) \leq \min\big\{r_f(M),\frac{1}{4}\ell_c(M)\big\}$. Since $\mathrm{inj}(M) = \min\big\{r_c(M),\frac{1}{2}\ell_c(M)\big\}$, the argument in the preceding three paragraphs shows, essentially without modification, that $r(M)=\min\big\{r_f(M),\frac{1}{4}\ell_c(M)\big\}$. □

3. Construction of compact manifolds with $\frac{r(M)}{\mathrm{inj}(M)}$ arbitrarily small. According to the characterizations of the injectivity and convexity radiuses in the preceding section, $r(M)=\frac{1}{2}\mathrm{inj}(M)$ whenever $r_f(M)=\frac{1}{2}r_c(M)$. Gulliver's examples of compact manifolds with focal points but no conjugate points show that this latter equality may fail to hold [6].

THEOREM 3.1 (Gulliver). *Let (M,g) be a compact Riemannian manifold with constant sectional curvature -1. Suppose $p \in M$ satisfies $\mathrm{inj}(p) \geq 1.7$. Then there exists a Riemannian metric h on M that agrees with g except on a g-ball $B_R=B(p,R)$ of radius $R=1.7$ and that satisfies the following:*

(i) $r_c(M,h)=\infty$; and

(ii) $r_f(B_R,h|_{B_R})<\infty$.

The Riemannian manifold $(B_R,h|_{B_R})$ depends only on the dimension of M.

Gulliver's construction is to write B_R as the union of a g-ball B_r and an annulus $B_R \setminus B_r$, change the metric on B_r to have constant curvature $(0.55)^2$, where B_r is large enough that it contains focal points but no conjugate points, and interpolate between the metrics on B_r and $M \setminus B_R$ through a radially symmetric metric on B_R. Provided $\mathrm{inj}(p) \geq 1.7$, this can be done without introducing conjugate points.

It will be useful to know that the fundamental group of a connected hyperbolic manifold is **residually finite**, which means that, for any non-trivial $[\gamma] \in \pi_1(M,p)$, there is a normal subgroup G of $\pi_1(M,p)$ of finite index such that $[\gamma] \notin G$. This is a special case of the following theorem of Mal'cev [8], also sometimes attributed to Selberg [12]. Note that a group is **linear** if it is isomorphic to a subgroup of the matrix group $\mathrm{GL}(F,n)$ for some field F.

THEOREM 3.2 (Mal'cev). *Every finitely generated linear group is residually finite.*

If M is compact and has a hyperbolic metric, then, for each $C > 0$, there exist only finitely many non-trivial closed geodesics $\{\gamma_1, \ldots, \gamma_k\}$ in M of length less than twice C (see Theorem 12.7.8 in [11]). For each corresponding $[\gamma_i] \in \pi_1(M, q_i)$, there exists a normal subgroup G_i of $\pi_1(M, q_i)$ of finite index such that $[\gamma_i] \notin G_i$. Each G_i is identified with a unique finite-index subgroup of $\pi_1(M,p)$ via conjugation by any path connecting p to q_i. Letting $G = \cap_{i=1}^k G_i$, one obtains a finite-index normal subgroup of $\pi_1(M,p)$ that does not contain, up to conjugation, any of the $[\gamma_i]$. Therefore, all non-trivial closed geodesics in the finite covering space $\tilde{M} = H^n/G$ have length at least twice C. Since $r_c(\tilde{M}) = \infty$, an application of Corollary 2.2 proves the following result, which is well known to hyperbolic geometers.

LEMMA 3.3. *For each $n \geq 2$ and $C > 0$, there exists a compact n-dimensional Riemannian manifold M that has constant sectional curvature -1 and satisfies $\mathrm{inj}(M) \geq C$.*

It may now be shown that Gulliver's construction can produce compact manifolds M of any dimension $n \geq 2$ with $\frac{r(M)}{\mathrm{inj}(M)}$ arbitrarily small.

THEOREM 3.4. *For each $n \geq 2$ and $\varepsilon > 0$, there exists a compact n-dimensional Riemannian manifold M with $\frac{r(M)}{\mathrm{inj}(M)} < \varepsilon$.*

Proof. Let $D < \infty$ denote the focal radius of the n-dimensional manifold in part (ii) of Theorem 3.1. According to Lemma 3.3, there exists a compact n-dimensional Riemannian manifold (M,g) that has constant sectional curvature -1 and satisfies $\mathrm{inj}(M,g) > \max\{2R, \frac{D}{\varepsilon} + R\}$, where $R = 1.7$. Apply Gulliver's construction to produce a metric h on M that agrees with g except on a g-ball $B_R = B(p,R)$, has no conjugate points, and satisfies $r_f(M,h) < D$. By Corollary 2.2 and Lemma 2.4, $\mathrm{inj}(M,h) = \frac{1}{2}\ell_c(M,h)$ and $r(M,h) \leq r_f(M,h) < D$.

Let $\gamma : [0,1] \to M$ be a non-trivial closed h-geodesic. If $\gamma([0,1]) \cap B_R = \emptyset$, then $L_h(\gamma) = L_g(\gamma) \geq 2\mathrm{inj}(M,g)$. If $\gamma([0,1]) \cap B_R \neq \emptyset$, then one may suppose without loss of generality that $\gamma(0) \in B_R$. Since (M,h) has no conjugate points, $[\gamma] \neq 0$, which implies the existence of $t_0 \in (0,1)$ such that $d_g(p, \gamma(t_0)) > R$. Let (a,b) be the connected component of $\{t \in (0,1) \,|\, d_g(p,\gamma(t)) > R\}$ containing t_0. Because $R < r(M,g)$, there exists a unique minimal geodesic $\sigma : [0,1] \to M$ of (M,g) connecting $\gamma(a)$ to $\gamma(b)$, which satisfies $\sigma([0,1]) \subset B_R$. Note that $L_g(\sigma) \leq 2R$. Since (M,g) has no conjugate points, the concatenation $\gamma|_{[a,b]} \cdot \sigma^{-1}$ is homotopically non-trivial, which implies that $L_g(\gamma|_{[a,b]} \cdot \sigma^{-1}) \geq 2\mathrm{inj}(M,g)$. Hence

$$L_h(\gamma) > L_g(\gamma|_{[a,b]}) \geq 2\mathrm{inj}(M,g) - 2R.$$

It follows that $\mathrm{inj}(M,h) \geq \mathrm{inj}(M,g) - R$ and, consequently, that

$$\frac{r(M,h)}{\mathrm{inj}(M,h)} < \frac{D}{\mathrm{inj}(M,g) - R} < \varepsilon.$$

□

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