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TRAVERSALLY GENERIC & VERSAL VECTOR FLOWS: SEMI-ALGEBRAIC MODELS OF TANGENCY TO THE BOUNDARY*

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Abstract. Let X be a compact smooth manifold with boundary. In this article, we study the spaces $\mathcal{V}^\dagger(X)$ and $\mathcal{V}^\ddagger(X)$ of so called boundary generic and traversally generic vector fields on X and the place they occupy in the space $\mathcal{V}(X)$ of all fields (see Theorems 3.4 and Theorem 3.5). The definitions of boundary generic and traversally generic vector fields v are inspired by some classical notions from the singularity theory of smooth Bordman maps [Bo]. Like in that theory (cf. [Morin]), we establish local versal algebraic models for the way a sheaf of v-trajectories interacts with the boundary ∂X. For fields from the space $\mathcal{V}^\ddagger(X)$, the finite list of such models depends only on $\dim(X)$; as a result, it is universal for all equidimensional manifolds. In specially adjusted coordinates, the boundary and the v-flow acquire descriptions in terms of universal deformations of real polynomials whose degrees do not exceed $2 \cdot \dim(X)$.

Key words. Vector fields, manifolds with boundary, singularity theory.

AMS subject classifications. 57R25, 57R45, 37C10.

1. Introduction. This paper is the second in a series that researches the Morse Theory, gradient flows, concavity and complexity on smooth compact manifolds with boundary. In the context of $3D$-flows, some of its ideas can be traced back to [K]. The paper serves as an analytical foundation for the investigation of *boundary generic* (see Definition 2.1) and, so called, *traversally generic*[1] (see Definition 3.2) vector fields v on manifolds X with boundary. These analytical tools provide us with local semi-algebraic models for the ways in which typical nonsingular vector flows interact with the boundary ∂X. Here the word "local" refers to the vicinity of a given trajectory γ of the v-flow.

The main observation is that, for smooth fields, each intersection point $a \in \gamma \cap \partial X$ comes with a positive integral multiplicity $j(a)$ attached to it. This multiplicity $j(a)$ can be given a number of competing but equivalent definitions, one of which uses the *Morse stratification* (see Definition 2.1 and formula (2.1)), which has been studied in [K1]. Naively, one can think of $j(a)$ as a *multiplicity of tangency* between γ and ∂X. So, surprisingly, the smooth topology of the flow can distinguish between, say, degree 2 and degree 4 tangency!

Lemma 3.1 and Lemma 3.4 describe the models for ∂X and v in the vicinity of a point $a \in \gamma \cap \partial X$ and in the vicinity of a trajectory γ, respectively. It turns out that, for traversally generic fields, in special flow-adjusted coordinates $(u, \vec{x})$, the boundary is given by a real polynomial equation $P(u, \vec{x}) = 0$ of degree that depends on γ and does not exceed $2 \cdot \dim(X)$. The manifold X is given by the polynomial inequality $P(u, \vec{x}) \leq 0$. The polynomial $P(u, \vec{x})$ depends only on the ordered sequence of multiplicities $\{j(a)\}_{a \in \gamma \cap \partial X}$. So, in each dimension, there are only finitely many semi-algebraic models for the vicinity of v-trajectories γ in X.

We introduce a variety of spaces that correspond to different types of vector fields on X, the space $\mathcal{V}^\dagger(X)$ of generic with respect to the boundary fields and the space

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[1]For vector fields v that vanish on X, the v-trajectory space is pathological; in contrast, the trajectory spaces of a traversally generic fields are a compact CW-complexes.

$\mathcal{V}^\ddagger(X)$ of traversally generic fields are among them. Two theorems describe our main results: Theorem 3.4 claims that $\mathcal{V}^\dagger(X)$ is an open and dense in the space of all smooth fields $\mathcal{V}(X)$, and Theorem 3.5 claims that $\mathcal{V}^\ddagger(X)$ is open and dense in the space $\mathcal{V}_{\mathsf{trav}}(X)$ of all *traversing* (equivalently, all gradient-like non-vanishing) fields. Traversing fields have only trajectories that are homeomorphic to closed intervals or singletons.

2. Morin's local models: how nonsingular flows interact with boundary. Let v be a vector field on a smooth compact $(n+1)$-manifold X with boundary ∂X. To achieve some uniformity in our notations, let $\partial_0 X := X$ and $\partial_1 X := \partial X$.

The vector field v gives rise to a partition $\partial_1^+ X \cup \partial_1^- X$ of the boundary $\partial_1 X$ into two sets: the locus $\partial_1^+ X$, where the field is directed inward of X, and $\partial_1^- X$, where it is directed outwards. We assume that v, viewed as a section of the quotient line bundle $T(X)/T(\partial X)$ over ∂X, is transversal to its zero section. This assumption implies that both sets $\partial_1^\pm X$ are compact manifolds which share a common boundary $\partial_2 X := \partial(\partial_1^+ X) = \partial(\partial_1^- X)$. Evidently, $\partial_2 X$ is the locus where v is *tangent* to the boundary $\partial_1 X$.

Morse has noticed that, for a generic vector field v, the tangent locus $\partial_2 X$ inherits a similar structure in connection to $\partial_1^+ X$, as $\partial_1 X$ has in connection to X (see [Mo]). That is, v gives rise to a partition $\partial_2^+ X \cup \partial_2^- X$ of $\partial_2 X$ into two sets: the locus $\partial_2^+ X$, where the field is directed inward of $\partial_1^+ X$, and $\partial_2^- X$, where it is directed outward of $\partial_1^+ X$. Again, let us assume that v, viewed as a section of the quotient line bundle $T(\partial_1 X)/T(\partial_2 X)$ over $\partial_2 X$, is transversal to its zero section.

For generic fields, this structure replicates itself: the cuspidal locus $\partial_3 X$ is defined as the locus where v is tangent to $\partial_2 X$; $\partial_3 X$ is divided into two manifolds, $\partial_3^+ X$ and $\partial_3^- X$. In $\partial_3^+ X$, the field is directed inward of $\partial_2^+ X$, in $\partial_3^- X$, outward of $\partial_2^+ X$. We can repeat this construction until we reach the zero-dimensional stratum $\partial_{n+1} X = \partial_{n+1}^+ X \cup \partial_{n+1}^- X$.

Thus a generic vector field v on X gives rise to two stratifications:

$$\begin{gathered}\partial X := \partial_1 X \supset \partial_2 X \supset \cdots \supset \partial_{n+1} X,\\ X := \partial_0^+ X \supset \partial_1^+ X \supset \partial_2^+ X \supset \cdots \supset \partial_{n+1}^+ X,\end{gathered} \tag{2.1}$$

the first one by closed submanifolds, the second one—by compact ones. Here $\dim(\partial_j X) = \dim(\partial_j^+ X) = n+1-j$. For simplicity, the notations "$\partial_j^\pm X$" do not reflect the dependence of these strata on the vector field v. When the field varies, we use a more accurate notation "$\partial_j^\pm X(v)$".

These considerations motivate

Definition 2.1. *Let X be a compact smooth $(n+1)$-dimensional manifold with boundary $\partial X \neq \emptyset$, and v a smooth vector field on X.*

We say that v is boundary generic *if v produces a filtrations of X as in (2.1) whose strata $\{\partial_j^+ X \subset \partial_j X\}_{1 \le j \le n+1}$ are defined inductively in j as follows:*

- $\partial_0 X := \partial X$, $\partial_1 X := \partial X$ [2],
- *v, viewed as a section of the tangent bundle $T(X)$, is transversal to its zero section,*
- $v|_{\partial X} \neq 0$,
- *for each $k \in [1, j]$, the v-generated stratum $\partial_k X$ is a closed smooth submanifold of $\partial_{k-1} X$,*

[2] So $\partial_0 X$ and $\partial_1 X$—the base of induction—do not depend on v.

- *the field v, viewed as section of the quotient 1-bundle*

$$T_k^\nu := T(\partial_{k-1}X)/T(\partial_k X) \to \partial_k X,$$

is transversal to the zero section of T_k^ν for all $k \leq j$.
- *the stratum $\partial_{j+1}X$ is the zero set of the section $v \in T_j^\nu$* [3].
- *the stratum $\partial_{j+1}^+ X \subset \partial_{j+1}X$ is the locus where v points inside of $\partial_j^+ X$.* □

Let v be a boundary generic vector field on X so that $v \neq 0$ along the boundary ∂X. We can add an external collar to X and smoothly extend the field into a larger manifold $\hat{X}$ without introducing new singularities. Let $\hat{v}$ denote the extended field.

At each point $x \in \partial_1 X$, the $(-\hat{v})$-flow defines the germ of the projection $p_x : \hat{X} \to S_x$, where S_x is a local section of the $\hat{v}$-flow which is transversal to it. The projection is considered at each point of $\partial_1 X \subset \hat{X}$. When $\hat{v}$ is a gradient-like field for a function $\hat{f} : \hat{X} \to \mathbb{R}$, we can choose the germ of the hypersurface $\hat{f}^{-1}(f(x))$ for the role of S_x.

Let $\mathcal{V}_{\neq 0}(X)$ be the space of smooth vector fields $v \neq 0$, equipped with the C^∞-topology.

A theorem of Morin [Morin] describes all local models of $p_x : \partial_1 X \to S_x$ for a G_δ, or *residual*[4], set of fields in $\mathcal{V}_{\neq 0}(X)$.

Let us introduce and depict these models. For any integer $s \in [1, n+1]$, consider the polynomial

$$Q_s(u_1, u_2, \dots, u_{n-1}, \mathsf{u}_n) := \mathsf{u}_n^s + \sum_{i=0}^{s-2} u_i \mathsf{u}_n^i \tag{2.2}$$

and the map $\mu_s : \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by

$$\mu_s : (u_1, u_2, \dots, u_{n-1}, \mathsf{u}_n) \to (u_1, u_2, \dots, u_{n-1}, Q_s,\ \mathsf{u}_n). \tag{2.3}$$

Let $(y_1, \dots, y_n, y_{n+1})$ be coordinates in $\mathbb{R}^{n+1}$. The constant field $e_{n+1} := \partial_{y_{n+1}}$, will play the role of the nonsingular field $\hat{v}$ on $\hat{X}$.

Consider the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by the formula

$$\pi : (y_1, y_2, \dots, y_n, y_{n+1}) \to (y_1, y_2, \dots, y_n). \tag{2.4}$$

Then the composition $\pi \circ \mu_s$ is given by the formula

$$(u_1, u_2, \dots, u_{n-1}, \mathsf{u}_n) \to (u_1, u_2, \dots, u_{n-1},\ \mathsf{u}_n^s + \sum_{i=0}^{s-2} u_i \mathsf{u}_n^i). \tag{2.5}$$

Let us denote by λ the line distribution $\ker(D\pi)$ tangent to the fibers of the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$.

The following result [Morin] is of key importance for our goals.

Theorem 2.1 (Morin). *For a G_δ-set of 1-dimensional distributions l on $\hat{X}$ and any point $x \in \partial_1 X$, there is a neighborhood U of x in $\hat{X}$, a diffeomorphism $h : U \to \mathbb{R}^{n+1}$, and an integer $s \in [1, n+1]$ such that*

[3]Thus if all $\{\partial_k X\}_{\{k \leq j\}}$ are smooth manifolds, then by the transversality of the section v, so is $\partial_{j+1}X$.

[4]that is, a countable intersection of open and dense subsets in $\mathcal{V}_{\neq 0}(X)$

- $h(x) = 0 \in \mathbb{R}^{n+1}$,
- $h(\partial_1 X \cap U) = \mu_s(\mathbb{R}^n)$, *where* μ_s *is defined by formula (2.3),*
- *the distribution* $l|_U$ *is mapped by the differential* $Dh : TU_x \to T\mathbb{R}^{n+1}$ *to the distribution* λ. □

Note that Morin's theorem a priori allows for the four-fold ambiguity: (1) the distribution l can be given two possible orientations (i.e. the model field can be $\pm e_{n+1}$), and (2) the manifold X can occupy each of the two chambers in which $\mathbb{R}^{n+1}$ is locally divided by $\mu_s(\mathbb{R}^n)$.

We can describe these local models by replacing their $\vec{u}$-parametric form in formula (2.3) with equations in the new coordinates

$$(u, x_0, x_1 \dots, x_{n-1}) := (y_{n+1}, -y_n, y_1, \dots, y_{n-1}).$$

Let

$$P_s(u, x) := Q_s(y_1, \dots, y_{n-1}, y_{n+1}) - y_n = u^s + \sum_{i=0}^{s-2} x_i u^i, \tag{2.6}$$

so that $\partial_1 X$ is given by the equation $P_s(u, x_0, x_1, \dots, x_{n-1}) = 0$. For a fixed vector $x := (x_0, \dots, x_{n-1})$, P_s is a depressed polynomial in u.

Therefore, for each point $x \in \mathbb{R}^n$, the points of the boundary $\partial_1 X$ residing in the fiber $\pi^{-1}(x)$ of the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are the real-valued zeros of the polynomial $P_s(u, x)$.

With each $x \in \mathbb{R}^n$, we associate the real zero divisor $D_s(x)$ of the u-polynomial $P_s(u, x)$–the collection of distinct real zeros of the polynomial with the positive integral multiplicities attached to them (see Fig. 1). We will be particularly interested in the *ordered* sequence $\omega = \{\omega_i\}_i$ of multiplicities that is generated by the divisor $D_s(x)$.

Put $e_{n+1} := \frac{\partial}{\partial u}$. We abbreviate $\big(\frac{\partial}{\partial u}\big)^j P_s$ as $P_s^{(j)}$. Let $w := (u, x) \in \mathbb{R} \times \mathbb{R}^n \approx \mathbb{R}^{n+1}$.

For the local models as in formula (2.6), the Morse stratification $\{\partial_j^+ X := \partial_j^+ X(\partial_u)\}_j$ (see formula (2.1)) can be expressed in terms of the divisor $D_s(x)$ and labeled by the ordered list of multiplicities $\omega(x)$.

Theorem 2.2. *For a* G_δ*-set of vector fields* v *on* X, $v|_{\partial_1 X} \neq 0$, *and for each point* $a \in \partial_1 X$, *there exists an integer* $s \in [1, n+1]$ *such that the models for* v *and the* v*-induced strata* $\{\partial_j^+ X\}_{1 \leq j \leq n+1}$, *in the vicinity of* a, *are given by one of the four real semi-algebraic sets:*

(1) $X = \{w \in \mathbb{R}^{n+1} |\ P_s(w) \geq 0\}$ *and* $v = e_{n+1}$;
- $\partial_j X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j\}$
- $\partial_j^+ X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j, \text{ and } P_s^{(j)}(w) \geq 0\}$

(2) $X = \{w \in \mathbb{R}^{n+1} |\ P_s(w) \leq 0\}$ *and* $v = e_{n+1}$;
- $\partial_j X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j\}$
- $\partial_j^+ X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j, \text{ and } P_s^{(j)}(w) \leq 0\}$

(3) $X = \{w \in \mathbb{R}^{n+1} |\ P_s(w) \geq 0\}$ *and* $v = -e_{n+1}$;
- $\partial_j X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j\}$
- $\partial_j^+ X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j, \text{ and } (-1)^j P_s^{(j)}(w) \geq 0\}$

(4) $X = \{w \in \mathbb{R}^{n+1} |\ P_s(w) \leq 0\}$ *and* $v = -e_{n+1}$;
- $\partial_j X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j\}$
- $\partial_j^+ X = \{w \in \mathbb{R}^{n+1} |\ P_s^{(i)}(w) = 0 \text{ for all } i < j, \text{ and } (-1)^j P_s^{(j)}(w) \leq 0\}$.

Proof. Our argument is based on Theorem 2.1. By the second bullet in that theorem, the boundary $\partial_1 X$ is given locally by the equation $P_s(w) = 0$.

By formula (2.6), the gradient

$$\nabla P_s = (P_s^{(1)}(u, x),\ 1, u, u^2,\ \ldots,\ u^{s-2},\ 0,\ \ldots\ 0). \tag{2.7}$$

The orthogonality of e_{n+1} and $\nu_1 := \nabla P_s$ defines the locus $\partial_2 X$. Hence, the orthogonality is equivalent to the constraint $P_s^{(1)}(x, u) = 0$, and $\partial_2 X$ is determined by two equations: $P_s = 0$ and $P_s^{(1)} = 0$. Formulas (2.6) and (2.7) imply that the vector $\nu_2 := \nabla[P_s^{(1)}] = \nu_1^{(1)}$ has $P_s^{(2)}$ for its first coordinate. Since $\partial_3 X$ is the locus where e_{n+1} is tangent to $\partial_2 X$ and ν_2 is orthogonal the hypersurface $\{P_s^{(1)} = 0\} \supset \partial_2 X$, the vectors ν_2 and e_{n+1} must be orthogonal along $\partial_3 X$. This leads to the equation $P_s^{(2)} = 0$.

Using this type of argument repeatedly for the linear independent vector fields $\{\nu_j := \nu_1^{(j)}\}_j$, proves the first bullets in claims (1) and (2) of the theorem.

When X is defined by the inequality $P_s \geq 0$, e_{n+1} points inside X at $w = (u, x) \in \partial_1 X$ if and only if the dot product $e_{n+1} \cdot \nu_1(w) \geq 0$. That is, $\partial_1^+ X$ is defined by $P_s^{(1)} \geq 0$ together with $P_s = 0$. On the other hand, when X is defined by the inequality $P_s \leq 0$, e_{n+1} points inside X at $w \in \partial_1 X$ if and only if $P_s^{(1)}(w) \leq 0$ and $P_s(w) = 0$.

Note that

$$\frac{\partial}{\partial u}(\nu_{j-1} \cdot e_{n+1}) = \frac{\partial}{\partial u}\nu_{j-1} \cdot e_{n+1} + \nu_{j-1} \cdot \frac{\partial}{\partial u} e_{n+1} = \nu_j \cdot e_{n+1}. \tag{2.8}$$

Along $\partial_2 X$, in view of formula (2.8), the property $\nu_2 \cdot e_{n+1} \geq 0$ is equivalent to the inequality $\frac{\partial}{\partial u}(\nu_1 \cdot e_{n+1}) \geq 0$. When, along $\partial_2 X$, the Lie derivative

$$\mathcal{L}_{e_{n+1}}(\nu_1 \cdot e_{n+1}) := \frac{\partial}{\partial u}(\nu_1 \cdot e_{n+1}) = \nu_2 \cdot e_{n+1}$$

is nonnegative, the field e_{n+1} points inside $\partial_1^+ X$; otherwise, it points inside $\partial_1^- X$. Therefore, $\partial_2^+ X$ is depicted by the inequality $P_s^{(2)} = \nu_2 \cdot e_{n+1} \geq 0$, coupled with the pair of equalities $P_s = 0, P_s^{(1)} = 0$. The general case of

$$\partial_j^+ X = \{P_s^{(i)} = 0\}_{i<j} \cap \{P_s^{(j)} \geq 0\}$$

is analogous.

The argument in case (2) of the theorem is very similar to case (1).

We notice that flipping the direction of the field $v = e_{n+1}$ does not change the polarity of the strata with even j's and reverses the polarity of the strata with odd ones.

Finally, we combine this description of stratifications $\big\{\partial_j^+\{P_s \geq 0\}\big\}_j$, $\big\{\partial_j^+\{P_s \leq 0\}\big\}_j$ with Morin's Theorem 2.1 to get the desired local models for a G_δ-set of fields in $\mathcal{V}(X)$. □

Remark 2.1. Note that Theorem 2.2 does not describe local models for generic Morse data (f, v) in the vicinity of a typical point $a \in \partial_1 X$, just $4(n+1)$ local models for generic nonsingular fields v; in other words, $\pm e_{n+1}$ mimics v, but the coordinate function $\pm u$ in general does not represent $f : X \to \mathbb{R}$. □

Corollary 2.1. *For the vector fields v that have Morin normal forms as in formula (2.5) (as in Theorem 2.1), all the strata $\partial_j X$ are manifolds, and the embeddings $\partial_j X \subset \partial_{j-1} X$ are regular. Moreover, these fields are boundary generic in the sense of Definition 2.1.*

In the vicinity of each $a \in \partial_1 X$ of the s-type, we get $\partial_{s+1} X = \emptyset$, so that the s-type fields are both boundary $(s+1)$-convex and $(s+1)$-concave in the sense of [K1].

Proof. According to Theorem 2.2, $\partial_j X$ is given by the system of equations

$$\{P_s = 0,\, P_s^{(1)} = 0,\, \dots\, ,\, P_s^{(j-1)} = 0\}.$$

It follows from formulas (2.7) and (2.8) that the gradient vector fields $\nu_1, \dots, \nu_j$, where $\nu_k = \nabla(P_s^{(k-1)})$, are linearly independent. Therefore, $\partial_j X$ is a submanifold of $\partial_1 X$. Moreover, since ν_j is a section of the normal line bundle $T(X_{j-1})/T(X_j)$ which is transversal to its zero section and since $e_{n+1} = \sum_k (e_{n+1} \cdot \nu_k)\nu_k$, the field e_{n+1} is boundary generic in the sense of Definition 2.1.

To validate the last claim, note that $P_s^{(s)} \neq 0$, which implies that $\partial_{s+1} X = \emptyset$. □

Remark 2.2. For local models as in Theorem 2.2, the germs of intersections of the strata $\partial_j X$, $j \in [1, n+1]$, with the hypersurfaces of $\{u = c\}$ are affine subspaces of $\mathbb{R}^{n+1}$, while the germs of $\partial_j^+ X \cap \{u = c\}$ are affine half-spaces. Indeed, for each fixed value of u and any j, the equations $\{P_s^{(i)} = 0\}_{i<j}$ impose *linear* constraints on the rest of the variables; similarly, $\pm P_s^{(j)} \geq 0$ is a linear inequality in $x_0, \dots, x_{n-1}$. Therefore, the equations define a *ruled* real variety, and the inequality picks a semi-algebraic subvariety, ruled by the half-spaces. □

Example 2.1 For a 4-dimensional X, the eight local models

$$u : \{\pm P_s \geq 0\} \to \mathbb{R},$$

where $s \in [1, 4]$ and $v = e_4$[5], are given by the four polynomials:

- $P_1 = u$,
- $P_2 = u^2 + x_0$,
- $P_3 = u^3 + x_1 u + x_0$,
- $P_4 = u^4 + x_2 u^2 + x_1 u + x_0$.

Let us consider the fibers $\pi^{-1}(x) \cap X$ of the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ in the vicinity of the origin for the P_4-model. Since the coefficients of $P_4(\sim, u)$ are real, the divisor $D(P_4) \subset \mathbb{R}$ is: (i) either real of degree four, or (ii) real of degree two or (iii) real of degree zero. Note that, in case (i), the sum of all real roots, taken with their multiplicities, is zero.

The P_4-model is described by the diagrams a—k and a'—k' in Fig. 1. In the figure, we do not stress the balanced nature of the divisors $\{P_4 = 0\}$. The shaded portions of the number lines in the figure belong to the 4-fold X; in fact, one can think of X as being disjoint union of these shaded portions. Note the polarities $\{+, -\}$ attached to each root of P_4: they reflect the polarities in the Morse stratification $\partial_j^{\pm} X$. By the Viète Formula, each divisor in diagrams a—k and a'—k' determines a unique point $x = (x_0, x_1, x_2)$ over which it resides. □

[5] To save space, we do not list the other eight cases with $v = -e_4$.

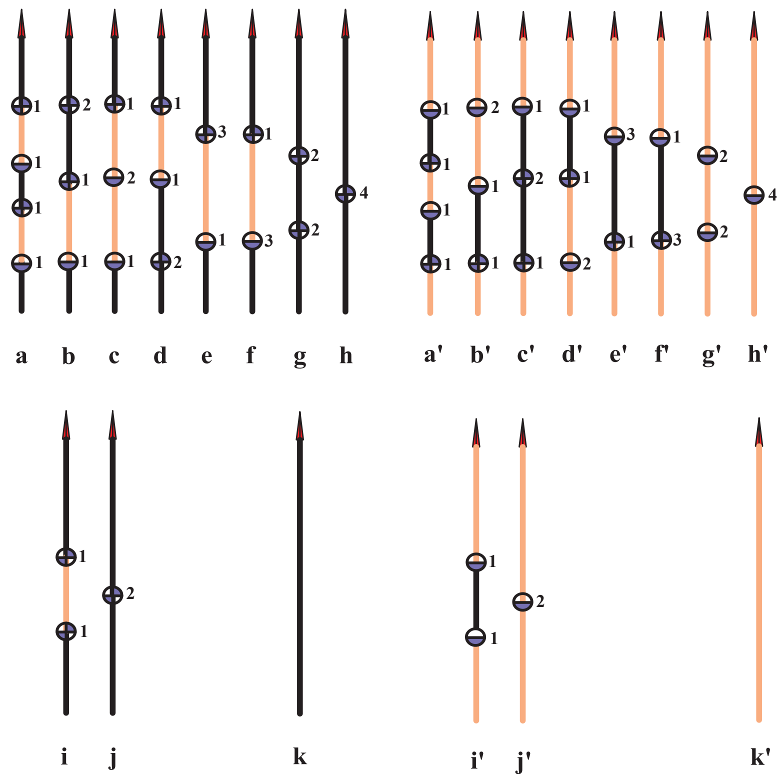

Fig. 1. *The patterns of solutions for $P_4 \geq 0$ (on the left) and for $P_4 \leq 0$ (on the right). The numbers 1, 2, 3, 4 indicate the multiplicities of the roots. Diagrams i, j, i', j' correspond to the case of one pair, and diagrams k, k' to the case of two pairs of complex conjugate roots.*

3. Traversally generic and versal fields. Guided by the geometry of the local models from Theorem 2.2, which describes the ways in which vector fields on X interact with its boundary $\partial_1 X$, we embark on an investigation of the "semi-local" dynamics of generic gradient-like *nonsingular* v-flows.

Here and on, we assume that each v-trajectory does not "end" at a point where it is tangent to the boundary; when possible, it "extends further" in the interior of X. Also, for technical reasons, we do regard a singleton x, the flow "curve" through $x \in \partial_2^- X$, as a trajectory.

For boundary generic (see Definition 2.1) vector fields fields v, elements of the space $\mathcal{V}^\dagger(X)$, we associate an ordered sequence of multiplicities with each trajectory γ. For any $v \in \mathcal{V}^\dagger(X)$, the intersection $\{\alpha_i\} := \gamma \cap \partial_1 X$ is automatically a finite set. We call a field v *traversing* (see Definition 4.6 from [K1]) if its trajectories are homeomorphic either to closed intervals, or to singletons. (This property of v is equivalent to $v \neq 0$ and being of the gradient type.) For traversing fields, the points $\{a_i\}$ of the intersection $\gamma \cap \partial_1 X$ are ordered by the field-oriented trajectory γ, and the index i reflects this ordering.

Definition 3.1. *Let $v \in \mathcal{V}^\dagger(X)$ be a generic field. Then every v-trajectory γ intersects the boundary $\partial_1 X$ at a finite number of points $\{a_i\}$. Each point a_i belongs to a unique pure stratum $\partial_{j_i} X^\circ$.*

The multiplicity $m(\gamma)$ *of* γ *is defined by the formula*

$$m(\gamma) = \sum_i j_i. \tag{3.1}$$

The reduced multiplicity $m'(\gamma)$ *of* γ *is defined by the formula*

$$m'(\gamma) = \sum_i (j_i - 1), \tag{3.2}$$

and the virtual multiplicity $\mu(\gamma)$ *of* γ *is defined by*

$$\mu(\gamma) = \sum_i \left\lceil \frac{j_i}{2} \right\rceil, \tag{3.3}$$

where $\lceil \sim \rceil$ *denotes the integral part function.* □

For an open and dense subspace $\mathcal{V}^\ddagger(X)$ of $\mathcal{V}^\dagger(X)$, one can interpret $\mu(\gamma)$ as the maximal number of tangency points that any trajectory γ' in the vicinity of γ may have (see Theorem 3.4).

When v is nonsingular on $\partial_1 X$, we can extend it into a larger manifold $\hat{X}$ so that $\hat{X}$ properly contains X and the extension $\hat{v}$ remains nonsingular in in the vicinity of $\partial_1 X \subset \hat{X}$. Throughout this text, we treat the pair $(\hat{X}, \hat{v})$ as a germ which extends (X, v).

When $v \in \mathcal{V}^\dagger(X)$, each pure stratum $\partial_j X^\circ := (\partial_j X)^\circ$ is an open manifold.

Consider the collection of tangent spaces $\{T_{a_i}(\partial_{j_i} X^\circ)\}_i$ to the pure strata $\{\partial_{j_i} X^\circ\}_i$ that have a non-empty intersection with a given trajectory γ. By Theorem 2.2, each space $T_{a_i}(\partial_{j_i} X^\circ)$ is transversal to the curve γ.

Let S be a local section of the $\hat{v}$-flow at some point $a_\star \in \gamma$ and let $\mathsf{T}_\star$ be the space tangent to S at $a_\star$. Each space $T_{a_i}(\partial_j X^\circ)$, with the help of the $\hat{v}$-flow, determines a vector subspace $\mathsf{T}_i = \mathsf{T}_i(\gamma)$ in $\mathsf{T}_\star$. It is the image of the tangent space $T_{a_i}(\partial_j X^\circ)$ under the composition of two maps: (1) the differential of the flow-generated diffeomorphism that maps a_i to $a_\star$, and (2) the linear projection $T_{a_\star}(X) \to \mathsf{T}_\star$ whose kernel is generated by $v(a_\star)$.

The configuration $\{\mathsf{T}_i\}$ of *affine* subspaces $\mathsf{T}_i \subset \mathsf{T}_\star$ is called *generic* (or *stable*) when all the multiple intersections of spaces from the configuration have the least possible dimensions consistent with the dimensions of $\{\mathsf{T}_i\}$. In other words,

$$\mathrm{codim}(\bigcap_s \mathsf{T}_{i_s}, \mathsf{T}_\star) = \sum_s \mathrm{codim}(\mathsf{T}_{i_s}, \mathsf{T}_\star)$$

for any subcollection $\{\mathsf{T}_{i_s}\}$ of spaces from the list $\{\mathsf{T}_i\}$.

Consider the case when $\{\mathsf{T}_i\}$ are *vector* subspaces of $\mathsf{T}_\star$. If we interpret each T_i as the kernel of a linear epimorphism $\Phi_i : \mathsf{T}_\star \to \mathbb{R}^{n_i}$, then the property of $\{\mathsf{T}_i\}$ being generic can be reformulated as the property of the direct product map $\prod_i \Phi_i : \mathsf{T}_\star \to \prod_i \mathbb{R}^{n_i}$ being an epimorphism. In particular, for a generic configuration of affine subspaces, if a point belongs to several T_i's, then the sum of their codimensions n_i does not exceed the dimension of the ambient space $\mathsf{T}_\star$.

Example 3.1 The configuration of a line and a plane in $\mathbb{R}^3$, the line being transversal to the plane, is generic; and so is the configuration of two planes in $\mathbb{R}^3$

which share only a line. On the other hand, two lines in $\mathbb{R}^3$ which share a point do not form a generic configuration. Also, any three lines in $\mathbb{R}^2$ which share a point do not form a generic configuration. □

The definition below resembles and is inspired by the "Condition NC" imposed on, so called, *Boardman maps* between smooth manifolds (see [GG], page 157, for the relevant definitions). In fact, for traversing generic fields v, the v-flow delivers germs of Boardman maps $p(v,\gamma) : \partial_1 X \to \mathbb{R}^n$, available in the vicinity of every trajectory γ.

Definition 3.2. *We say that a traversing field v on X is* traversally generic *if:*

- *the field is boundary generic in the sense of Definition 2.1,*
- *for each v-trajectory $\gamma \subset X$ (not a singleton), the collection of subspaces $\{\mathsf{T}_i(\gamma)\}_i$ is generic in $\mathsf{T}_\star$: that is, the obvious quotient map $\mathsf{T}_\star \to \prod_i \big(\mathsf{T}_\star/\mathsf{T}_i(\gamma)\big)$ is surjective.*

We denote by $\mathcal{V}^\ddagger(X)$ the space of all traversally generic fields on X. □

Remark 3.1. In particular, the second bullet of the definition implies the inequality

$$\sum_i \operatorname{codim}(\mathsf{T}_i(\gamma), \mathsf{T}_\star) \leq \dim(\mathsf{T}_\star) = n.$$

In other words, for traversally generic fields, the reduced multiplicity of each trajectory γ satisfies the inequality

$$m'(\gamma) = \sum_i (j_i - 1) \leq n. \tag{3.4}$$

Evidently the property of the configuration $\{\mathsf{T}_i(\gamma)\}_i$ being generic in $\mathsf{T}_\star$ does not depend on the choice of the point $a_\star \in \gamma$ and the smooth transversal flow section S at $a_\star$. □

Remark 3.2. Note that, if a smooth submersion $F : (\tilde{X}, \partial\tilde{X}) \to (X, \partial X)$ is a finite covering of $(n+1)$-manifolds, then the pull-back $\tilde{v}$ of a traversally generic field v on X is a traversally generic field on $\tilde{X}$. Indeed, each v-trajectory γ is a segment or a singleton. Therefore $F^{-1}(\gamma)$ is a disjoint union of arcs or points. These are trajectories of the pull-back field $\tilde{v}$ on $\tilde{X}$. Each of the lifted trajectories has a $\tilde{v}$-adjusted neighborhood $U_{\tilde{\gamma}}$ that projects by the local diffeomorphism F onto a v-adjusted neighborhood U_γ. That field-respecting diffeomorphism $F : U_{\tilde{\gamma}} \to U_\gamma$ maps $\partial\tilde{X} \cap U_{\tilde{\gamma}}$ to $\partial X \cap U_\gamma$, so that all the structures participating in Definition 3.2 are respected by F. □

Let us review the list of notations for various kinds of vector fields on X. Recall that we have introduced the following nested collection of spaces:

$$\mathcal{V}^\ddagger(X) \subset \mathcal{V}^\dagger(X) \subset \mathcal{V}(X)$$

based on traversally generic, boundary generic, and arbitrary smooth vector fields on X, respectively. We denote by $\mathcal{V}_{\neq 0}(X)$ the space of non-vanishing fields on X. We are also considering the space $\mathcal{V}_{\mathsf{grad}}(X)$ of gradient-like fields. Finally, $\mathcal{V}_{\mathsf{trav}}(X)$ denotes the space of traversing fields (see Definition 4.6 in [K1]).

In view of Corollary 4.1 from [K1], the following inclusions follow from the definitions:

$$\mathcal{V}_{\mathsf{grad}}(X) \cap \mathcal{V}_{\neq 0}(X) = \mathcal{V}_{\mathsf{trav}}(X) \subset \mathcal{V}_{\neq 0}(X), \tag{3.5}$$

$$\mathcal{V}^{\ddagger}(X) \subset \mathcal{V}_{\mathsf{trav}}(X) \cap \mathcal{V}^{\dagger}(X). \tag{3.6}$$

Recall also that, by the Phillips Theorem B [Ph], for a fixed Riemannian metric g on X, the gradient map

$$\nabla_g : Sub(X, \mathbb{R}) \to \mathcal{V}_{\mathsf{trav}}(X) \subset \mathcal{V}_{\neq 0}(X),$$

where $Sub(X, \mathbb{R})$ denotes the space of submersions $f : X \to \mathbb{R}$, is a *weak homotopy equivalence* between the spaces $Sub(X, \mathbb{R})$ and $\mathcal{V}_{\neq 0}(X)$.

One might speculate that $\nabla_g : Sub(X, \mathbb{R}) \to \mathcal{V}_{\mathsf{trav}}(X)$ is a weak homotopy equivalence as well. We can prove (see Corollary 4.1 [K2]) that if two gradient-like fields can be connected by a path in the space of all non-vanishing fields, then they can be connected by a path in the space of all non-vanishing gradient-like fields as well. Therefore, at least the map

$$(\nabla_g)_* : \pi_0(Sub(X, \mathbb{R})) \to \pi_0(\mathcal{V}_{\mathsf{trav}}(X))$$

is bijective.

It turns out that, for generic fields $v \in \mathcal{V}^{\dagger}(X)$, the way in which each trajectory γ intersects with the Morse strata $\{\partial_k X^{\circ}\}_k$ reflects "*the order of tangency*" between γ and $\partial_1 X$. The fundamental lemma below reflects this fact.

We embed the pair (X, v) into a pair $(\hat{X}, \hat{v})$ in a way that has been described previously.

Lemma 3.1. *Assume that $v \in \mathcal{V}^{\dagger}(X)$. Denote by γ_a the $\hat{v}$-trajectory through a point $a \in X$. Let $z : \hat{X} \to \mathbb{R}$ be a smooth function in the vicinity of $\partial_1 X \subset \hat{X}$ such that:*

(1) 0 *is a regular value of* z,
(2) $z^{-1}(0) = \partial_1 X$, *and*
(3) $z^{-1}((-\infty, 0]) = X$.

Then the following properties hold:

- *For each point $a \in \partial_k X^{\circ}$, the restricted function $z|_{\gamma_a}$ has zero of multiplicity k at the point a.*
- *In the vicinity of a in $\hat{X}$, there exists a coordinate system (u, x), where $u \in \mathbb{R}$ and $x \in \mathbb{R}^n$, so that:*
  1) each v-trajectory γ is defined by an equation $\{x = \vec{const}\}$,
  2) the boundary $\partial_1 X$ is defined by the equation

$$u^k + \sum_{j=0}^{k-2} x_j\, u^j = 0 \tag{3.7}$$

- *In the vicinity of each $a \in \partial_k X^{\circ}$, there exists a set $J(a) \subset \{0, 1, 2, , \dots n+1\}$ such that each v-trajectory γ hits only some strata $\{\partial_j X^{\circ}\}_{j \in J(a)}$ in such a way that*

$$\sum_{j \in J(a)} j \leq k \ \ and \ \sum_{j \in J(a)} j \equiv k \ (2).$$

Proof. Consider the space $\mathbb{R}^1 \times \mathbb{R}^n$ with coordinates $(z, x) := (z, x_1, \dots x_n)$ and the standard Euclidean scalar product $\langle \sim, \sim \rangle$. Let the sets $M_0 := \{z \geq 0\}$ and $M_1 := \{z = 0\}$ represent the germ of the pair $(X, \partial_1 X)$ in the vicinity of a typical point in $\partial_1 X$.

Consider a smooth vector field $v(z, x) = \big(u(z, x), w(z, x)\big)$ on $\mathbb{R}^1 \times \mathbb{R}^n$, where the component $u(z, x)$ is parallel to $\mathbb{R}^1$, and $w(z, x)$ to $\mathbb{R}^n$.

In the argument to follow, we view the coordinates (z, x) as "more permanent ingredients", while analyzing the restrictions on $u(z, x), w(z, x)$ imposed by the desired property of the field $v(x, z)$ being boundary generic. In fact, we can allow for some changes in the coordinates (z, x) as well, as long as the locus $\{z = 0\}$ (which models the boundary $\partial_1 X$) remains fixed.

The field v is tangent to $M_1 \subset \mathbb{R}^1 \times \mathbb{R}^n$ along the locus

$$M_2 := \{z = 0,\ u(z, x) = 0\}$$

which models $\partial_2 X := \partial_2 X(v)$. To reflect the generic nature of v, we require that $u(0, x)$, viewed as a section of the obvious projection $\mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, will be transversal to M_1 along M_2. In other words, the gradient $\nabla_x u$ of the function $u(0, x)$ with respect to the coordinates x should not vanish along the manifold M_2. The locus M_3, where v is tangent to M_2, is given by two equations $\{z = 0\}$ and $\{\langle \nabla_x u, w \rangle = 0\}$, while the transversality of v to M_3 can be expressed as the condition of linear independency of the two fields, $\nabla_x u$ and $\nabla_x \langle \nabla_x u, w \rangle$. The set

$$M_1^+ := \{x |\ z = 0,\ u \geq 0\} \subset M_1$$

mimics the locus $\partial_1^+ X$, while the set

$$M_2^+ := \{x |\ z = 0,\ u = 0,\ \langle \nabla_x u, w \rangle \geq 0\} \subset M_2$$

mimics $\partial_2^+ X$.

In order to capture the emerging pattern, for each pair of smooth maps $u : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}$ and $w : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, we introduce a sequence of new functions $\psi_k : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$ in the variables z, x by the recursive formula:

$$\begin{aligned} &\psi_1(z, x) := u(z, x), \\ &\psi_k(z, x) := \langle \nabla_x \psi_{k-1}(z, x),\ w(z, x) \rangle. \end{aligned} \tag{3.8}$$

For $k > 0$, locus M_k, a model of $\partial_k X$, is given by the equations

$$\{z = 0,\ \psi_1(z, x) = 0,\ \dots,\ \psi_{k-1}(z, x) = 0\}, \tag{3.9}$$

together with the requirement that on the solution set $\{(0, x)\}$ of equation (3.9) the vector fields

$$\Big\{\nabla_x \psi_1(0, x),\ \dots,\ \nabla_x \psi_{k-1}(0, x)\Big\} \tag{3.10}$$

are linearly independent (hence, M_k is a manifold). Thus the linear independence of the the gradient fields in (3.10), where $2 \leq k \leq n + 1$, is equivalent to v *locally* belonging to the space $\mathcal{V}^\dagger(X)$.

The submanifold $M_k^+ \subset M_k$ is defined by the additional inequality $\{\psi_k(z, x) \geq 0\}$: there the field v points inside of M_{k-1}^+.

Consider the following system of ordinary differential equations, determined by the field $v(z, x)$:

$$\begin{aligned} \frac{dz}{dt} &= u(z, x), \\ \frac{dx}{dt} &= w(z, x). \end{aligned} \tag{3.11}$$

Take a point $x \in M_k^\circ := M_k \setminus M_{k+1}$. Let $\gamma(t) = (z(t), x(t))$ be the solution of (3.11) such that $\gamma(0) = (x, 0)$. In order to prove the lemma, we need to verify that the function z, being restricted to $\gamma(t)$, has x as its zero of multiplicity k.

For any j, with the help of equations (3.8) and (3.11), we get

$$\frac{d\psi_j}{dt} = \frac{\partial \psi_j}{\partial z} \cdot \frac{dz}{dt} + \Big\langle \nabla_x \psi_j, \frac{dx}{dt} \Big\rangle = \frac{\partial \psi_j}{\partial z} \cdot \psi_1 + \psi_{j+1}.$$

Put $\xi_j := \dfrac{\partial \psi_j}{\partial z}$. Then the formula above can be rewritten as

$$\frac{d\psi_j}{dt} = \xi_j \cdot \psi_1 + \psi_{j+1}. \tag{3.12}$$

Next, we claim that the function $\dfrac{d^j z}{(dt)^j}$, being restricted to the integral curve γ, can be represented as $\psi_j + \sum_{i=1}^{j-1} \beta_i \cdot \psi_i$ for an appropriate choice of smooth functions $\beta_i = \beta_i(z, x)$. In other words, the functions $\dfrac{d^j z}{(dt)^j}|_\gamma$ and $\psi_j|_\gamma$ are congruent modulo the ideal of $C^\infty(\gamma, \mathbb{R})$ generated by $\{\psi_i|_\gamma\}_{i<j}$. In view of equations (3.9), this implies that $\dfrac{d^j z}{(dt)^j}(0) = 0$ for all $j < k$. Since $x \notin M_{k+1}$ translates as $\psi_k(x) \neq 0$, it follows that $\dfrac{d^k z}{(dt)^k}(0) \neq 0$. So $z|_\gamma$ indeed has a zero of multiplicity k at $\gamma(0) \in M_k^\circ$.

We proceed to prove the formula

$$\frac{d^j z}{(dt)^j}\Big|_\gamma = \big(\psi_j + \sum_{i=1}^{j-1} \beta_i \cdot \psi_i\big)\Big|_\gamma$$

by induction in j. The inductive step $j \Rightarrow j+1$ is carried out by differentiating with respect to t the identity conjectured for j and then by using formulas (3.12). The basis of induction is represented by the claim $\dfrac{dz}{(dt)}\Big|_\gamma = \psi_1|_\gamma := u|_\gamma$, the first equation in (3.11).

Finally, we notice that all these arguments do not depend on the choice of the auxiliary function z, subject to the properties (1)-(3) described in the lemma. This proves the claim in the first bullet of the lemma.

Next, we switch to new local coordinates (u, y), amenable to the v-flow (in contrast with the previous coordinates (z, x) which were adjusted to the boundary $\partial_1 X$). Consider a point $a \in M_k^\circ$ and a germ of a v-trajectory γ passing through a. Let S be a germ of a smooth transversal flow section at a, equipped with coordinates $y := (y_1, \dots, y_n)$, such that a is the origin. Let u be the coordinate obtained by integrating v so that S is the locus $\{u = 0\}$. The procedure for producing the coordinate u (under the name "$\int_S^x v$") has been described in the proof of Lemma 4.1 from [K1].

Now we view $z : \hat{X} \to \mathbb{R}$ as a smooth function of (u, y). We have shown that $z|_\gamma = z(u, 0)$ has a zero at $u = 0$ of multiplicity k. Using the Malgrange Preparation Theorem [Mal], there exist an invertible germ of a smooth function $Q(u, y)$ and a germ of the form

$$P(u, y) = u^k + \sum_{j=0}^{k-1} \phi_j(y)u^j,$$

where $\phi_j(0) = 0$, so that $z = P \cdot Q$. Due to this factorization, in the coordinates (u, y), the locus $\{z = 0\}$—the boundary $\partial_1 X$—is given by an equation $u^k + \sum_{j=0}^{k-1} \phi_j(y)u^j = 0$. We can improve the latter equation by modifying the coordinates (u, y) (by re-parametrizing the trajectories): just put $\tilde{u} = u + \frac{1}{k}\phi_{k-1}(y)$ and $\tilde{y} = y$. Then, in the new coordinates, the equation $z = 0$ is transformed into the equation

$$\tilde{u}^k + \sum_{j=0}^{k-2} \tilde{\phi}_j(y)\tilde{u}^j = 0,$$

a "surrogate" of the Morin canonical form (2.2).

We still need to bridge the gap between the surrogate and true Morin's canonical forms. With this goal in mind, consider the Jacobi matrix $D\Phi(y)$ formed by the partial derivatives $\big\{\frac{\partial}{\partial y_i}\tilde{\phi}_j(y)\big\}_{i,j}$. If $D\Phi(0)$ has the maximal possible rank $k-1$, then we can choose the functions $\{\tilde{\phi}_j(y)\}_{0 \leq j < k-2}$ for the role of the first new coordinates in the vicinity of the origin. They can be completed to a coordinate system comprising the function $\tilde{u}$ together with some functions $\{\tilde{y}_i\}_{k \leq i \leq n}$, drawn from the list of $\{y_i\}_{1 \leq i \leq n}$.

As before, each trajectory γ is produced by freezing all the coordinates but $\tilde{u}$. In the new coordinates $(\tilde{\phi}_0, \dots, \tilde{\phi}_{k-2}, \tilde{y}_k, \dots, \tilde{y}_n, \tilde{u})$, the germ of $\partial_1 X$ at a is given by the depressed polynomial equation

$$\tilde{u}^k + \sum_{j=0}^{k-2} \tilde{\phi}_j \tilde{u}^j = 0,$$

the true Morin canonical form. Therefore $\mathrm{rk}(D\Phi(0)) = k - 1$ implies the existence of the Morin canonical coordinates in the vicinity of a.

In fact (see Lemma 3.3), the requirement $\mathrm{rk}(D\Phi(0)) = s$ has an intrinsic meaning. In particular, it is independent on the factorization $z = P \cdot Q$.

Now consider a trajectory $\gamma := \{y = c\}$, where $c := (c_1, \dots, c_n)$ is a fixed point, in the vicinity of $a \in \partial_1 X \cap \gamma$ and the zeros of the function z, being restricted to γ. We claim that the two loci

$$\big\{z(u, c) = 0,\ \frac{\partial}{\partial u}z(u, c) = 0,\ \dots,\ \frac{\partial^j}{\partial u^j}z(u, c) = 0\big\},$$

$$\big\{P(u, c) = 0,\ \frac{\partial}{\partial u}P(u, c) = 0,\ \dots,\ \frac{\partial^j}{\partial u^j}P(u, c) = 0\big\} \tag{3.13}$$

coincide. The argument, inductive in j, employes the fact that $Q(u, c) \neq 0$.

Since we have established the validity of the first bullet in Lemma 3.1, we get that both systems of equations in (3.13) locally define the stratum $\partial_{j+1} X$. Moreover, as in the proof of Theorem 2.2 (in particular, see formula (2.7)), the fact that $v \in \mathcal{V}^\dagger(X)$, in the vicinity of $a \in \partial_k X^\circ$, can be translated as the property of linear independence for the gradients

$$\nabla z,\ \nabla \frac{\partial z}{\partial u},\ \dots,\ \nabla \frac{\partial^{k-2} z}{\partial u^{k-2}} \tag{3.14}$$

in the coordinates (u, z) at the origin $(0, 0)$[6]. Note that these are the gradients of functions in (3.13) that determine the stratum $\partial_k X$.

We claim that this linear independence is equivalent to the linear independence of the gradients

$$\nabla P,\ \nabla \frac{\partial P}{\partial u},\ \dots,\ \nabla \frac{\partial^{k-2} P}{\partial u^{k-2}}. \tag{3.15}$$

In order to validate this claim, we introduce the u-parameter curve $\alpha(u) := (1, u, \dots, u^{k-2})$. Note that the gradients in (3.15) admit the following representations:

$$\nabla P = \Big(\frac{\partial P}{\partial u},\ D\Phi \cdot \alpha\Big),\ \ \nabla \frac{\partial P}{\partial u} = \Big(\frac{\partial^2 P}{\partial u^2},\ D\Phi \cdot \frac{\partial \alpha}{\partial u}\Big),\ \ \dots\ ,$$

$$\nabla \frac{\partial^{k-2} P}{\partial u^{k-2}} = \Big(\frac{\partial^{k-1} P}{\partial u^{k-1}},\ D\Phi \cdot \frac{\partial^{k-2} \alpha}{\partial u^{k-2}}\Big),$$

where $\Big(\frac{\partial^j P}{\partial u^j},\ D\Phi\cdot \frac{\partial^{j-1}\alpha}{\partial u^{j-1}}\Big)$ is the vector whose first component is $\frac{\partial^j P}{\partial u^j}$, and $D\Phi\cdot\frac{\partial^{j-1}\alpha}{\partial u^{j-1}}$ denotes the multiplication of the matrix $D\Phi$ by the vector $\frac{\partial^{j-1}\alpha}{\partial u^{j-1}}$.

Since $\frac{\partial^j P}{\partial u^j}(0, 0) = 0$ for all $1 \le j \le k - 1$, we conclude that the dimension of the space spanned by vectors

$$\nabla P(0,0), \nabla \frac{\partial P}{\partial u}(0,0), \dots, \nabla \frac{\partial^{k-2} P}{\partial u^{k-2}}(0,0)$$

equals the dimension of the space spanned by vectors

$$D\Phi \cdot \alpha,\ D\Phi \cdot \frac{\partial \alpha}{\partial u},\ \dots\ ,\ D\Phi \cdot \frac{\partial^{k-2}\alpha}{\partial u^{k-2}},$$

being evaluated at $(0, 0)$. Since α and its u-derivatives are independent vectors, the dimension of the latter space is equal to the rank of $D\Phi(0)$. Now we employ Lemma 3.3 below to conclude that the dimension of the space spanned by vectors in formula (3.14) is the same as the dimension of the space spanned by vectors in (3.15), both dimensions being equal to $\mathrm{rk}(D\Phi(0))$.

If $v \in \mathcal{V}^\dagger(X)$, the vectors in formula (3.15) must be independent at the origin 0. Therefore, $\mathrm{rk}(D\Phi(0)) = k - 1$, which implies the existence of the Morin coordinates in the vicinity of $a \in \partial_k X^\circ$. This proves the claim in the second bullet of the lemma.

In view of formula (3.13), there exists $\epsilon > 0$, so that the sum of zero multiplicities of the smooth u-function $z(u, c)$ in the interval $-\epsilon < u < \epsilon$, is the same as the sum

[6] In the coordinates (u, y), this is an analogue of (3.10).

of zero multiplicities of the function $P(u,c)$ in the same interval; the later function is a polynomial of degree k in u. Therefore, in the vicinity of a, the sum of zero multiplicities of $z|_\gamma$ does not exceed k. In other words, if $a \in M_k^\circ$, then each trajectory γ, in the vicinity of a, hits only some strata $\{M_j^\circ\}$ so that $\sum j \leq k$.

Consider the polynomial family $P(u,x)$ in the formula (3.7). There is $\delta > 0$ such that $\|c\| < \delta$ implies that $|x_j(c)| < (\epsilon/\rho)^j$, where the universal constant ρ is introduced in the proof of Lemma 3.2 below. By that lemma, for such $\delta > 0$ all the real roots u of $P(u,x)$ are confined to the interval $(-\epsilon, \epsilon)$. Therefore, each trajectory $\gamma = \{x = c\}$, passing through the section $S := \{u = 0,\ \|c\| < \delta\}$, has *all* the roots of $P(u,c)$ concentrated in the interval $(-\epsilon, \epsilon)$.

Let D_γ be the zero divisor of $P(u,c)$. Thus, not only $\deg(D_\gamma) \leq k$, but also $\deg(D_\gamma) \equiv k\ (2)$. The third bullet of Lemma 3.1 has been validated. □

LEMMA 3.2. *Put $x := (x_0, \dots, x_{k-1})$. Let $P(u,x) = u^k + \sum_{j=0}^{k-1} x_j u^j$ be a x-parameter family of real monic u-polynomials. Then, there exists a universal constant $\rho := \rho(k) > 0$ with the following property: for any $\epsilon > 0$ and each polynomial $P(u,x)$ with the coefficients $\{|x_j| < (\epsilon/\rho)^j\}$, all the real roots u of $P(u,x)$ are located in the interval $(-\epsilon, \epsilon)$.*

Proof. Consider the complex Viète map $V : \mathbb{C}^k \to \mathbb{C}^k$ defined by elementary symmetric polynomials $\sigma_1, \dots, \sigma_k$ in the variables-roots $\alpha_1, \dots \alpha_k$.

For each $\beta > 0$, consider the set $K(\beta) := \{|\sigma_j| < \beta^j\}_{1 \leq j \leq k}$ in the complex space $\mathbb{C}^k$ with the coordinates $\alpha_1, \dots \alpha_k$. Because each polynomial σ_j is homogeneous of degree j, every real ray from the origin intersects with $K(\beta) \subset \mathbb{C}^k$ along a segment. Indeed, any ray that does not belong the coordinate hyperplanes, hits the hypersurface $\{\sigma_k = \beta^k\}$ at a singleton, any ray which belongs to the coordinate hyperplanes hits the hypersurfaces $\{\sigma_{k-1} = \beta^{k-1}\}$ at a singleton, and so on ... Note that $K(\beta')$ can be obtained from $K(\beta)$ by a conformal scaling with the scaling factor β'/β.

Denote by B the unit polydisk $\{|\alpha_j| \leq 1\}_j$ in the complex space $\mathbb{C}^k$ with the coordinates $\alpha_1, \dots \alpha_k$. For each point $\alpha \in \partial B$, consider the real ray $r(\alpha)$ through α, emanating from the origin, and the proportion $\rho(\alpha)$ between the lengths of segments $[0,\, r(\alpha) \cap \partial K(1)]$ and $[0,\, r(\alpha) \cap \partial B]$.

Put $\rho := \max_{\alpha \in \partial B}\ \rho(\alpha)$. Now, if $\alpha \in \partial K(\beta)$, then each $|\alpha_j| \leq \beta/\rho$.

For a given $\epsilon > 0$, take the coefficients $x_j = \sigma_j$ so that $|x_j| < (\epsilon/\rho)^j$. For such choice of coefficients, all the roots $\{\alpha_j\}$ of the polynomial

$$P(u,x) := u^k + \sum_{j=0}^{k-1} x_j u^j$$

are confined to the interval $(-\epsilon, \epsilon)$. □

Lemma 3.1, as well as some future arguments, relies on the technical lemma below.

LEMMA 3.3. *Let $\{\alpha_i\}$ be a finite set of distinct real numbers. Let $z(u,y)$ be a smooth function, where $u \in \mathbb{R}$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Assume that $z(u,y)$ is a product of two smooth functions,*

$$P(u,y) = \prod_i \left[(u - \alpha_i)^{k_i} + \sum_{l=0}^{k_i - 2} \phi_{il}(y)(u - \alpha_i)^l\right]$$

and $Q(u,y)$, where $Q(\alpha_i, 0) \neq 0$ and $\phi_{il}(0) = 0$. Put $m := \sum_i (k_i - 1) \leq n$.

For each i, consider the $(k_i - 1) \times n$-matrix

$$D\Phi_i := \Big(\frac{\partial \phi_{il}}{\partial y_m}(0)\Big).$$

Let $D\Phi$ be the $m \times n$-matrix obtained by stacking all the matrices $\{D\Phi_i\}_i$ one on top of the other.

Consider the matrix

$$M_i(z) := \Big(\frac{\partial^{l+1} z}{\partial u^l \partial y_m}(\alpha_i, 0)\Big),$$

where $0 \leq l \leq k_i - 2$ and $1 \leq m \leq n$. Denote by $M(z)$ the matrix formed by stacking all the matrices $\{M_i(z)\}_i$ one on top of the other.

Then the rank of the matrix $D\Phi$ equals to the rank of the matrix $M(z)$. As a result, the restriction $\mathrm{rk}(D\Phi) \leq s$, where $s \leq n$, is given by algebraic constraints imposed on the coefficients of the degree k_i Taylor polynomials of the function $z(u, y)$ at $(\alpha_i, 0)$. In other words, the constraint $\mathrm{rk}(D\Phi) \leq s$ defines a closed affine subvariety in the space

$$\prod_i \mathrm{Jet}^{k_i}\big((\mathbb{R} \times \mathbb{R}^n, \alpha_i \times 0); (\mathbb{R}, 0)\big),$$

the product of k_i-jet spaces.

Proof. To save the pain of complex multi-indexing, we will prove one special case of the lemma, when the polynomial $P(u, y)$ consists of a single multiplier

$$(u - \alpha_1)^{k_1} + \sum_{l=0}^{k_1 - 2} \phi_{1l}(y)(u - \alpha_1)^l.$$

By shifting all the relevant functions by α_1, we can replace the variable $u - \alpha_1$ with the variable u. Also put $k := k_1$.

The argument is an induction $k \Rightarrow k + 1$ by the degree k of the polynomial

$$P(u, y) = u^k + \sum_{l=0}^{k-2} \phi_{k,l}(y) u^l,$$

with $k = 2$ being the base of induction[7].

Consider the polynomial

$$P_{k+1}(u, y) := u^{k+1} + \sum_{l=0}^{k-1} \phi_{k+1,l}(y) u^l$$

with some smooth functional coefficients, where $\phi_{k+1,l}(0) = 0$.

Let $z_{k+1} := P_{k+1} \cdot Q_{k+1}$ with $Q_{k+1}(0, 0) \neq 0$. Then

$$z_{k+1} = \Big[u\big(u^k + \sum_{l=1}^{k-1} \phi_{k+1,l}(y) u^{l-1}\big) + \phi_{k+1,0}(y)\Big] Q_{k+1}(u, y).$$

[7]Note that here the coefficient $\phi_{k,l}$ is not the coefficient ϕ_{kl}, present in the formulation of Lemma 3.3!

Define

$$P_k := u^k + \sum_{l=1}^{k-1} \phi_{k+1,l}(y)u^{l-1}, \;\; \phi := \phi_{k+1,0}, \;\; Q := Q_{k+1},$$

and $z_k := P_k \cdot Q$.

In the new notations,

$$z_{k+1} = u \cdot z_k + \phi \cdot Q.$$

Then

$$\frac{\partial z_{k+1}}{\partial y}(u,0) = u \cdot \frac{\partial z_k}{\partial y}(u,0) + \frac{\partial \phi}{\partial y}(0) \cdot Q(u,0),$$

where $\frac{\partial}{\partial y} := (\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n})$. Applying the operators $\frac{\partial^j}{\partial u^j}$ to the previous identity, for $j \leq k$, we get

$$\frac{\partial^{j+1} z_{k+1}}{\partial u^j \partial y}(u,0) = j\frac{\partial^j z_k}{\partial u^{j-1}\partial y}(u,0) + u\frac{\partial^{j+1} z_k}{\partial u^j \partial y}(u,0) + \frac{\partial \phi}{\partial y}(0) \cdot \frac{\partial^j Q}{(\partial u)^j}(u,0).$$

The substitution $u = 0$ leads to the equations

$$\frac{\partial z_{k+1}}{\partial y}(0,0) = \frac{\partial \phi}{\partial y}(0) \cdot Q(0,0),$$
$$\frac{\partial^{j+1} z_{k+1}}{\partial u^j \partial y}(0,0) = j\frac{\partial^j z_k}{\partial u^{j-1}\partial y}(0,0) + \frac{\partial \phi}{\partial y}(0) \cdot \frac{\partial^j Q}{\partial u^j}(0,0). \tag{3.16}$$

By inductive assumption, the rank of the $(k-1) \times n$-matrix $M(z_k)$ whose rows are the vectors

$$\{\frac{\partial z_k}{\partial y}(0,0), \; \frac{\partial^2 z_k}{\partial u \partial y}(0,0), \; \dots, \; \frac{\partial^{k-2} z_k}{\partial u^{k-2}\partial y}(0,0)\}$$

is equal to the rank of the Jacobi matrix $N(P_k)$ whose rows are the vectors

$$\{\frac{\partial \phi_{k+1,1}}{\partial y}(0), \; \frac{\partial \phi_{k+1,2}}{\partial y}(0), \; \dots, \; \frac{\partial \phi_{k+1,k-1}}{\partial y}(0)\}.$$

Let $\hat{M}(z_k)$ be the $(k \times n)$-matrix obtained from $M(z_k)$ by adding the first row of zeros. Consider the column-vector

$$q := \{Q(0,0), \; \frac{\partial Q}{\partial u}(0,0), \; \dots, \; \frac{\partial^{k-1} Q}{\partial u^{k-1}}(0,0)\}.$$

Then

$$M(z_{k+1}) = \hat{M}(z_k) + q \times \frac{\partial \phi}{\partial y}(0),$$

where "$\times$" stands for the matrix multiplication.

Remembering that $Q(0,0) \neq 0$ and that $\phi := \phi_{k+1,0}$, in view of the equations above (or equations (3.16)), we get that the space spanned by the vectors

$$\frac{\partial z_{k+1}}{\partial y}(0,0), \; \text{and} \; \Big\{\frac{\partial^{j+1} z_{k+1}}{\partial u^j \partial y}(0,0)\Big\}_{1 \leq j \leq k-1}$$

coincides with the space spanned by the vectors

$$\frac{\partial \phi}{\partial y}(0),\ \frac{\partial z_k}{\partial y}(0,0),\ \text{and}\ \Big\{\frac{\partial^{j+1} z_k}{\partial u^j \partial y}(0,0)\Big\}_{1\leq j\leq k-2}.$$

Therefore, $\mathrm{rk}(M(z_{k+1})) = \mathrm{rk}(N(P_{k+1}))$.

We let the reader to verify the validity of the lemma for $k = 2$. □

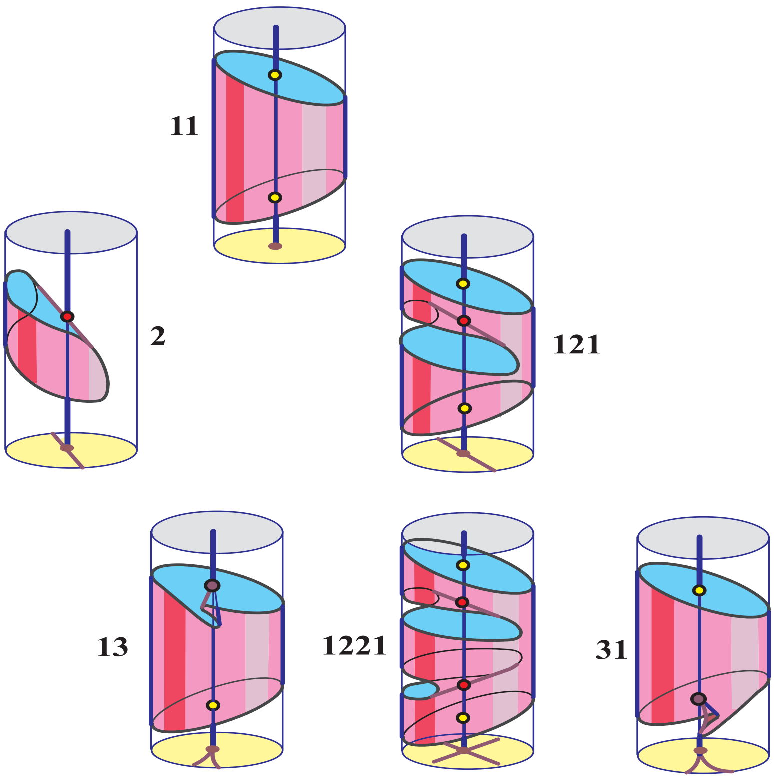

FIG. 2. *The six geometrical patters of traversally generic fields on 3-folds*

DEFINITION 3.3. *Let $\hat{v}$ be a nonvanishing vector field in $\hat{X}$. We say that a set $U \subset \hat{X}$ is $\hat{v}$-*adjusted*, if its intersection with every $\hat{v}$-trajectory $\hat{\gamma}$ is an interval.* □

DEFINITION 3.4. *Let v be a traversing and boundary generic vector field in X, and γ its trajectory. Consider the intersection $\gamma \cap \partial_1 X = \coprod_{1\leq i\leq p} a_i$, where each point $a_i \in \partial_{j_i} X(v)^\circ$. Pick a smooth transversal section S of the $\hat{v}$-flow at a point $b \in \gamma$.*

We say that v is traversally generic *at γ, if the images of the tangent spaces $\{T_{a_i}(\partial_{j_i} X(v))\}_i$ projected to $T_b(S)$ under the $\hat{v}$-flow are in general position*[8]. □

Fig. 2 shows all the local patterns of traversally generic fields for 3-folds X; the sequences of integers next to the figures label the corresponding tangency patterns of the core trajectory.

[8]This property does not depend on the choice of b and S.

The following proposition is a "semi-global" generalization of Morin's Theorem 3.1.

LEMMA 3.4. *Let $\hat{v}$ be a traversing field in a $\hat{v}$-adjusted neighborhood $U \subset \hat{X}$ of a given v-trajectory $\gamma \subset X$. Assume that v is traversally generic* at *γ.*

Then γ has a $\hat{v}$-adjusted neighborhood $V \subset U$ with a special system of coordinates

$$\big(u,\, \underbrace{x_{10}, \dots, x_{1j_1-2}},\, \dots\, , \underbrace{x_{i0}, \dots, x_{ij_i-2}},\, \dots\, , \underbrace{x_{p0}, \dots\, x_{pj_p-2}},\, \underbrace{y_1, \dots, y_{n-m'(\gamma)}}\big)$$

such that:

- *$\{u = const\}$ defines a transversal section of the $\hat{v}$-flow,*
- *each $\hat{v}$-trajectory in V is produced by fixing all the coordinates $\{x_{il}\}$ and $\{y_k\}$,*
- *there is an $\epsilon > 0$ such that $V \cap \partial_1 X \subset \coprod_i V_i$, where*

$$V_i := u^{-1}((\alpha_i - \epsilon, \alpha_i + \epsilon)) \cap V,$$

  and $\alpha_i = u(a_i)$,
- *for each i, the intersection $V_i \cap \partial_1 X$ is given by the equation*

$$(u - \alpha_i)^{j_i} + \sum_{l=0}^{j_i - 2} x_{i,l}(u - \alpha_i)^l = 0. \tag{3.17}$$

Proof. Let $U \subset \hat{X}$ be a $\hat{v}$-adjusted neighborhood of a typical trajectory $\gamma \subset X$. With the help of the flow, we introduce a system of coordinates $(u, y) := (u, y_1, \dots y_n)$ in U so that:

- $S_i = \{u = \alpha_i\}$ is a transversal section of $\hat{v}$-flow, which contains a_i and is diffeomorphic to a closed disk D^n,
- the locus $\{y = 0\} \cap X$ is the trajectory γ, and
- each $\hat{v}$-trajectory in U is given by freezing some point y.

Consider a smooth function $z : \hat{X} :\to \mathbb{R}$, or rather its germ, in the vicinity of X, such that:

- 0 is a regular value for z,
- $z^{-1}(0) = \partial_1 X$,
- $z^{-1}((-\infty, 0]) = X$. (3.18)

In the vicinity of each point $a_i \in \gamma \cap \partial_1 X$, the coordinates $(u - \alpha_i, y) := (u - \alpha_i, y_1, \dots y_n)$ are available. As in the proof of Lemma 3.1, in some $\hat{v}$-adjusted neighborhood $W_i \subset U$ of each point a_i, the globally-defined function z as in (3.18) can be written as the product of two smooth functions $P_i(u - \alpha_i, y) \cdot Q_i(u - \alpha_i, y)$, where $Q_i(u - \alpha_i, y) \neq 0$, and the polynomial

$$P_i(u - \alpha_i, y) = (u - \alpha_i)^{j_i} + \sum_{l=0}^{j_i - 2} \phi_{i,l}(y)(u - \alpha_i)^l.$$

Its smooth functional coefficients $\{\phi_{i,l}(y)\}_l$ are such that $\phi_{i,l}(0) = 0$.

We can adjust the size of the W_i's so that, for each pair W_i and W_k, the set of trajectories passing through W_i and the set of trajectories passing through W_k coincide. In other words, we can find a $\hat{v}$-adjusted cylindrical neighborhood $W \subset U$ of γ, such that $W \cap W_i = W_i$ for all i.

Let $\Phi_i : \mathbb{R}^n \to \mathbb{R}^{j_i-1}$ be given by the functions $\{\phi_{i,l}(y)\}_l$. So we can assume that, in such $\hat{v}$-adjusted neighborhood W of γ, the locus $\{z = 0\}$ is given by one of the polynomial equations $\{P_i(u - \alpha_i, y) = 0\}_i$.

As in the proof of Lemma 3.1 and by Lemma 3.3, $v \in \mathcal{V}^\dagger(X)$ implies that $\mathrm{rk}(D\Phi_i(0)) = j_i - 1$.

Since the $\hat{v}$-flow is assumed to be traversally generic at γ, the flow-generated images $\{\mathsf{T}_i\}_i$ of all tangent spaces $\{T_{a_i}(\partial_{j_i}X(v))\}_i$ of the minimal strata $\{\partial_{j_i}X(v) \ni a_i\}_i$ must be in general position in the tangent space T of the section S at the point $a = \gamma \cap S$ (a resides below the lowest point $a_1 \in \gamma \cap \partial_1 X$, the order in the set $\gamma \cap \partial_1 X$ being induced by v). The space T_i is of the codimension $j_i - 1$ in T.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^{m'(\gamma)}$ be the direct product of the maps Φ_i, where $m'(\gamma) := \sum_i (j_i - 1)$. Then, by the definition of true genericity at γ, $\mathrm{rk}(D\Phi(0)) = m'(\gamma) \leq n$ (see Remark 3.1).

We claim that if $\mathrm{rk}(D\Phi(0)) = m'(\gamma)$, then the desired system of coordinates in a new $\hat{v}$-adjusted neighborhood $V \subset W$ of γ is available: indeed, just put $x_{il} = \phi_{i,l}(y)$ and keep certain $(n - m'(\gamma))$-tuple of the original coordinates $\{y_k\}$ unchanged. These coordinates $\{y_k\}$ are chosen so that, together with Φ, they define a map $\mathbb{R}^n \to \mathbb{R}^{m'(\gamma)} \times \mathbb{R}^{n-m'(\gamma)}$ of the maximal rank n at the origin. □

Remark 3.3. Lemma 3.4 implies that, in such special coordinates, the intersection $V \cap \partial_1 X$ is given by the single polynomial equation

$$P(u, x) := \prod_i \Big[(u - \alpha_i)^{j_i} + \sum_{l=0}^{j_i-2} x_{i,l}(u - \alpha_i)^l\Big] = 0 \tag{3.19}$$

of degree $m(\gamma) = \sum_i j_i$, and $V \cap X$ — by the inequality $P(u, x) \leq 0$. □

Lemma 3.5. *Let the smooth manifold $X \subset \mathbb{R} \times \mathbb{R}^d$ be given by the polynomial inequality $\{P(u, x) := u^d + \sum_{k=0}^{d-1} x_k u^k \leq 0\}$.*

Then the field ∂_u is traversally generic with respect to the boundary $\partial X = \{P(u, x) = 0\}$.

Proof. Let $\pi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be the obvious projection.

For every point $w := (u, x) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique integer $j(w) \geq 0$ such that

$$P^{(k)}(w) = 0 \ \text{ for all } \ 0 \leq k < j(w), \ \text{ and } \ P^{(j(w))}(w) \neq 0,$$

where $P^{(j)}$ denotes the j-th partial derivative of P with respect to u. Evidently, $j(w) \leq d$.

We denote by $\Sigma_{[j]} \subset \mathbb{R} \times \mathbb{R}^d$ the locus $\{w |\; j(w) \geq j\}$. It is given by the j equations

$$\{P(u, x) = 0,\; P^{(1)}(u, x) = 0,\; \dots,\; P^{(j-1)}(u, x) = 0.\} \tag{3.20}$$

Consider any t-parameter curve $\gamma(t) := (u(t), x(t)) \subset \Sigma_{[j]}$ and let us compute the restrictions imposed on its tangent vector $\dot{\gamma}(0)$ at the point $\gamma(0)$. Differentiating the equations (3.20) that define $\Sigma_{[j]}$ with respect to t, we get :

$$\begin{aligned} P^{(1)}\dot{u} + P_x \cdot \dot{x} &= 0, \\ P^{(2)}\dot{u} + P_x^{(1)} \cdot \dot{x} &= 0, \\ &\dots \\ P^{(j)}\dot{u} + P_x^{(j-1)} \cdot \dot{x} &= 0, \end{aligned} \tag{3.21}$$

where P_x denotes the row vector $(u^{d-1}, \dots, u, 1)$ and $\dot{x}$ the column vector $(\dot{x}_{d-1}, \dots, \dot{x}_1, \dot{x}_0)$. By combining (3.21) with the equations (3.20) of $\Sigma_{[j]}$, we get

$$\begin{aligned} P_x \cdot \dot{x} &= 0, \\ P_x^{(1)} \cdot \dot{x} &= 0, \\ &\dots \\ P_x^{(j-2)} \cdot \dot{x} &= 0, \\ P^{(j)}\dot{u} + P_x^{(j-1)} \cdot \dot{x} &= 0. \end{aligned} \tag{3.22}$$

These equations imply that for any $\dot{x}$, subject to the first $j-1$ equations in (3.22), there exists a unique value $\dot{u} = -(P_x^{(j-1)} \cdot \dot{x})/P^{(j)}$, unless $P^{(j)}(u,x) = 0$. The latter vanishing does not happen in the proper stratum $\Sigma^\circ_{[j]} := \Sigma_{[j]} \setminus \Sigma_{[j+1]}$.

Therefore, for each vector $(\dot{u}, \dot{x}) \in T_{(u,x)}\Sigma^\circ_{[j]}$, tangent to $\Sigma^\circ_{[j]}$ at a point (u,x), the projection $D\pi((\dot{u},\dot{x})) = \dot{x}$ belongs to a vector subspace $V_{[j]}(u,x) \subset T_x(\mathbb{R}^d) \approx \mathbb{R}^d$ defined by the first $j-1$ equations in (3.22). Moreover, every vector $\dot{x}$ at x that belongs to $V_{[j]}(u,x)$ has a unique preimage in $T_{(u,x)}\Sigma^\circ_{[j]}$, so that $D\pi : T_{(u,x)}\Sigma^\circ_{[j]} \to V_{[j]}(u,x)$ is a linear isomorphism.

Let us pick now a generic point $x \in \mathbb{R}^d$. Let L_x denote the line $\pi^{-1}(x) \subset \mathbb{R} \times \mathbb{R}^n$. Put $Z_x := L_x \cap \{P = 0\}$. The u-coordinate maps Z_x to the set of zeros $\{\alpha_i = \alpha_i(x)\}_i$ of the u-polynomial $P(u,x)$. We denote by $j_i = j(\alpha_i, x)$ the multiplicity of the root α_i.

Our goal is to show that π maps the tangent spaces $\{T_{(\alpha_i,x)}\Sigma_{[j_i]}\}_i$ to a general position configuration in $\mathbb{R}^d$. We have shown that the $D\pi$ maps bijectively each tangent space $T_{(\alpha_i,x)}\Sigma^\circ_{[j_i]}$ to the vector space $V(\alpha_i,x) := V_{[j_i]}(\alpha_i,x)$—the solution set of the homogeneous linear system $L(\alpha_i, j_i)$:

$$\{P_x(\alpha_i)\cdot\dot{x} = 0,\ P_x^{(1)}(\alpha_i)\cdot\dot{x} = 0,\ \dots,\ P^{(j_i-2)}(\alpha_i)\cdot\dot{x} = 0\} \tag{3.23}$$

in the variables $\dot{x} := (\dot{x}_0, \dots, \dot{x}_{d-1})$. This space $V(\alpha_i,x)$ is viewed as the subspace of $T_x\mathbb{R}^d$, where (α_i,x) satisfies (3.20) (with u being replaced with α_i).

We need to verify that the system $L(\{\alpha_i, j_i\}_i)$, formed by collecting all the systems $L(\alpha_i, j_i)$ for individual $V(\alpha_i,x)$'s, is of the maximal possible rank, so that the spaces $\{V(\alpha_i,x)\}_i$ are in general position.

Let $m := \sum_i (j_i - 1)$. The matrix of the system $L(\{\alpha_i, j_i\}_i)$ is a generalized Vandermonde $(m \times d)$-matrix. Its rank is m—the maximal possible.

Let us validate this fact. Consider the auxiliary u-polynomial

$$T(u) := P_x \cdot \dot{x} = \sum_{k=0}^{d-1} \dot{x}_k u^k.$$

By (3.23), the solutions $\dot{x}$ of $L(\{\alpha_i, j_i\}_i)$ correspond exactly to the u-polynomials $T(u)$ with coefficients $\dot{x}_0, \dots, \dot{x}_{d-1}$ and the roots $\{\alpha_i\}$ of multiplicities $\{j_i - 1\}$. In other words, $T(u)$ are the u-polynomials that are divisible by the real polynomial

$$S(u) := \prod_i (u - \alpha_i)^{j_i - 1}$$

of degree m.

The equations (3.22) represent the requirement that $R(u)$, the the remainder of this division, vanishes (a priori, $R(u)$ is a polynomial of degree $m-1$ at most). The quotient $T(u)/S(u)$ consists of polynomials of the form $Q(u,y) = \sum_{l=0}^{q} y_l\, u^l$, where $q = d-1-m$ and $\{y_l\}_l$ are some real coefficients. We can view them as free variables, since any choice of y_l's produces the polynomial $T(u,y) := S(u)\cdot Q(u,y)$, which gives rise to a solution $\dot{x}(y)$ of the system $L(\{\alpha_i, j_i\}_i)$. Moreover, different parameters y produce different polynomials $S(u)\cdot Q(u,y)$. Thus y gives a $(d-m)$-parametric representation of the solution space of the system $L(\{\alpha_i, j_i\}_i)$.

So (3.22) is solvable, and the rank of its matrix equals m. Therefore, the spaces $\{V(\alpha^\star, x^\star)\}$ form a general position configuration in $\mathbb{R}^d$. As a result, the field ∂_u is generic with respect to the boundary $\partial X = \{P(u,x) = 0\}$. □

LEMMA 3.6. *Let $P(u,x)$ be the polynomial in (3.19) of degree $d = \sum_i j_i$. Put $m := \sum_i (j_i - 1)$. Consider the smooth manifold $X \subset \mathbb{R}\times\mathbb{R}^m$, given by the polynomial inequality $\{P(u,x) \leq 0\}$.*

Then, in the vicinity of the line $L_0 := \{x = 0\} \subset \mathbb{R}\times\mathbb{R}^m$, the field ∂_u is traversally generic with respect to the boundary $\partial X = \{P(u,x) = 0\}$.

Proof. We follow the outline of the arguments from the previous lemma.

We denote by Z the hypersurface $\{P(u,x) = 0\} \subset \mathbb{R}\times\mathbb{R}^m$.

Let $x_{\{i\}} := (x_{i,0}, x_{i,1}, \ldots, x_{i,j_i-2})$. By definition, the polynomial $P(u,x)$ of degree $d = \sum_i j_i$ is the product of the factors

$$P_i(u, x_{\{i\}}) := (u-\alpha_i)^{j_i} + \sum_{l=0}^{j_i-2} x_{i,l}(u-\alpha_i)^l.$$

Consider the hypersurfaces $Z_i := \{P_i(u, x_{\{i\}}) = 0\}$. There exists an $\epsilon > 0$ such that the hypersurfaces $\{Z_i\}_i$ are disjointed in the cylinder

$$\Pi_\epsilon := \big\{\{\|x_{\{i\}}\| < \epsilon\}_i\big\} \subset \mathbb{R}\times\mathbb{R}^m$$

with the polydisk base $\big\{B_\epsilon := \{\|x_{\{i\}}\| < \epsilon\}_i\big\} \subset \mathbb{R}^m$. In other words, for such choice of $\epsilon > 0$,

$$Z \cap \Pi_\epsilon = \coprod_i (Z_i \cap \Pi_\epsilon).$$

Indeed, the claim follows from Lemma 3.2, since all the roots $\{\alpha_i\}$ of $P(u,0)$ are distinct (note that the intersection $L_0 \cap Z = \coprod_i (\alpha_i, 0))$ and the roots of nearby polynomials $P(u,x)$, $x \in B_\epsilon$, are grouped around the α_i's.

Therefore, for any $x^\star \in B_\epsilon$, the intersection of the line

$$L_{x^\star} := \{x = x^\star\} \subset \mathbb{R}\times\mathbb{R}^m$$

with Z is of the form $\coprod_i \Big(\coprod_{k\in A_i}((\beta_{ik}, x^\star))\Big)$, where $\{\beta_{ik}\}_k$ are the real roots of $P_i(u, x^\star)$.

Let us denote the multiplicity of the root $\beta^\star_{ik} = \beta_{ik}(x^\star)$ by $j^\star_{ik}$.

Since, in the vicinity of the point $(\alpha_i, 0)$, the equations $\{P(u,x) = 0\}$ and $\{P_i(u,x) = 0\}$ determine the same solution sets, we can reduce the study of the locus $\Sigma_{[j^\star_{ik}]} \subset Z$ to the study of a similar locus $\Sigma_{[j^\star_{ik}]} \subset Z_i$. As in (3.23), both loci are given by the equations:

$$\{P_i(u,x) = 0,\ P_i^{(1)}(u,x) = 0,\ \ldots,\ P_i^{(j^\star_{ik}-1)}(u,x) = 0\}.$$

Thus we have reduced our settings to the ones studied in Lemma 3.5. Let us adapt equations (3.20)-(3.22) to our present environment[9]. We denote by $\tilde{P}_i$ the α_i-shifted polynomial, so that $\tilde{P}_i(u-\alpha_i, x) := P_i(u, x)$. In particular, the vector $P_x := (u^{d-1}, \dots, u, 1)$ in (3.21) and (3.22) must be replaced by the vector

$$\xi_i(u) := (\tilde{P}_i)_{x_{\{i\}}} = ((u-\alpha_i)^{j_i-2}, \dots, (u-\alpha_i), 1).$$

We denote by $\{\xi_i^{(l)}(u)\}_l$ its multiple derivatives with respect to the variable u.

As in (3.22), we conclude that each vector $(\dot{u}, \dot{x})$, tangent to the locus $\Sigma_{[j^\star_{ik}]}$ at the point $(\beta_{ik}, x^\star)$ is determined by its $D\pi$-projection $\dot{x} \in T_{x^\star}(\mathbb{R}^m) \approx \mathbb{R}^m$.

Therefore, $D\pi$ maps each tangent space $T_{(\beta^\star_{ik}, x^\star)}(\Sigma^\circ_{[j^\star_{ik}]})$ bijectively to the linear subspace of $T_{x^\star}\mathbb{R}^m$, given by the $j^\star_{ik} - 1$ homogeneous equations

$$V(\beta^\star_{ik}) := \big\{\xi_i(\beta^\star_{ik}) \cdot \dot{x}_{\{i\}} = 0,\ \ \xi_i^{(1)}(\beta^\star_{ik}) \cdot \dot{x}_{\{i\}} = 0, \\ \dots\ ,\ \xi_i^{(j^\star_{ik}-2)}(\beta^\star_{ik}) \cdot \dot{x}_{\{i\}} = 0\big\} \tag{3.24}$$

with respect to the $j_i - 1$ variables $\dot{x}_{\{i\}}$. Note that if some $j^\star_{ik} = 1$, then the simple root $\beta^\star_{ik}$ does not contribute to the equations.

To conclude the proof, we need to show that these spaces $\{V(\beta^\star_{ik})\}_{i,k}$ are in general position in $\mathbb{R}^d$.

Again, the argument is a modification of a similar proof from Lemma 3.5. Consider the entire collection of spaces $\{V(\beta^\star_{ik})\}_{i,k}$ and the collection $\mathcal{L}(x^\star) = \{\mathcal{L}_{ik}(x^\star)\}_{i,k}$ of equations as in (3.24) with respect to the $m := \sum_i (j_i - 1)$ variables $\{\dot{x}_{\{i\}}\}_i$. The number of such equations is $m^\star := \sum_{i,k}(j^\star_{ik} - 1)$.

So we have to show that the rank of the matrix of the system $\mathcal{L}(x^\star)$ is $m^\star$, the maximal possible. Note that m is the reduced multiplicity of the ∂_u-trajectory through the point $(0, 0)$, while $m^\star \leq m$ is the reduced multiplicity of the ∂_u-trajectory through the nearby point $(0, x^\star)$.

The equations (3.24) are equivalent to the requirement that, for each $x^\star$, the u-polynomial $T_i(u, \dot{x}) := \xi_i(u) \cdot \dot{x}_{\{i\}}$ of degree $j_i - 2$ is divisible by the polynomial

$$S_i(u) := \prod_k (u - \beta^\star_{ik})^{j^\star_{ik}-1}$$

of degree $m^\star_i := \sum_k (j^\star_{ik} - 1)$. So the quotient $Q_i := T_i(u, \dot{x})/S_i(u)$ is a polynomial of the form

$$Q_i(u, y_{\{i\}}) := \sum_{l=0}^{j_i-2-m^\star_i} y_{il}\, u^l,$$

which depends on $(j_i - 1 - m^\star_i)$ parameters $\{y_{il}\}_l$.

On the other hand, for any choice of these parameters $\{y_{il}\}_l$, we can form the polynomial $S_i(u) \cdot Q_i(u, y_{\{i\}})$ and compute its Taylor polynomial $\sum_{l=0}^{j_i-2} a_{il}(u-\alpha_i)^l$ at the point α_i. Then $\{\dot{x}_{il} = a_{il}\}_l$ must satisfy $\mathcal{L}_i(x^\star)$.

Thus any solution of the system $\mathcal{L}(x^\star)$ is generated via this linear mechanism by choosing the $m - m^\star$ parameters variables $\{y_{\{i\}}\}_i$. Different choices of $\{y_{\{i\}}\}_i$ lead to different ordered sets of u-polynomials $\{S_i(u) \cdot Q_i(u, y_{\{i\}})\}_i$ and therefore to different

[9] Note the difference: in the present setting, the polynomial $\tilde{P}_i$ is depressed, i.e., the coefficient x_{i,j_i-1} of $(u-\alpha_i)^{j_i-1}$ is zero.

solutions $\{\dot{x}_{il}\}$ of $\mathcal{L}(x^\star)$. Therefore, the rank of the matrix of $\mathcal{L}(x^\star)$ is $m - m^\star$, the maximal possible. In turn, this implies that the $\{V(\beta^\star_{ik})\}_{i,k}$ are in general position in $\mathbb{R}^m$. □

Definition 3.5. *We say that a field $v \in \mathcal{V}_{\mathsf{trav}}(X)$ is* versal *if each v-trajectory γ has a $\hat{v}$-adjusted neighborhood $U \subset \hat{X}$, equipped with special coordinates*

$$(u,\ \underbrace{x_{10}, \dots, x_{1j_1-2}},\ \dots\ \underbrace{x_{i0}, \dots, x_{ij_i-2}},\ \dots\ \underbrace{x_{p0}, \dots\ x_{pj_p-2}},\ \underbrace{y_1, \dots, y_{n-m'(\gamma)}})$$

as in Lemma 3.4 in which $\partial_1 X$ is given by the equation (3.19)[10].

We denote by $\mathcal{V}_{\mathsf{vers}}(X) \subset \mathcal{V}_{\mathsf{trav}}(X)$ the subspace of all versal fields on the manifold X. □

Corollary 3.1. *If a field v is versal in a v-adjusted neighborhood of its trajectory γ, then there is another v-adjusted neighborhood $U \subset \hat{X}$ of γ so that v is traversally generic in U (with respect to $\partial_1 X \cap U$).*

Proof. By the definition of a versal field, in special coordinates with the core γ, the boundary $\partial_1 X$ is given by a polynomial equation $P = 0$ as in (3.19). By applying Lemma 3.6 to P, we conclude that v is traversally generic in a smaller v-adjusted neighborhood $U \subset \hat{X}$ of γ. □

The next proposition claims that if a field is traversally generic *at* a trajectory γ, then it is traversally generic in its vicinity.

Corollary 3.2. *Let v be a traversing field on X which is traversally generic* at *a trajectory γ (see Definition 3.4).*

Then there exists a $\hat{v}$-adjusted neighborhood U of γ, such that the field v is traversally generic in U with respect to $\partial_1 X \cap U$.

Proof. If v is traversally generic at γ (as the hypotheses of the corollary spell out), then by Lemma 3.4, v is versal in the vicinity of γ. By Corollary 3.1, the field is traversally generic in the vicinity of γ. □

Theorem 3.1. *A vector field on X is versal if and only if it is traversally generic; in other words, $\mathcal{V}_{\mathsf{vers}}(X) = \mathcal{V}^\ddagger(X)$.*

Proof. By Lemma 3.4, any traversally generic field is versal. By Corollary 3.2, any versal field is traversally generic in the vicinity of each trajectory γ, and therefore, is traversally generic globally. □

In this section, our ultimate goal is to improve upon the Morin Theorem 3.1 by showing that, for a compact smooth manifold X, the space $\mathcal{V}_{\mathsf{vers}}(X) = \mathcal{V}^\ddagger(X)$ of versal/traversally generic vector fields is actually *open* and *dense* in the space $\mathcal{V}_{\mathsf{trav}}(X)$ of all traversing fields (see Theorem 3.5 below). We would like to provide the reader with an argument that does not rely heavily on the jet magic of the singularity theory in general, and on the Boardman Stratification Theory [Bo] in particular. However, reluctantly, we are going to use few implications of that theory.

Let M, N be two smooth manifolds. For a given smooth map $\Phi : M \to N$, consider the locus

$$\Sigma^{(j)}(\Phi) := \Big\{x \in M |\ \dim\big(\ker(D_x\Phi)\big) \geq j\Big\}$$

[10]In particular, for any γ, we require that the reduced multiplicity $m'(\gamma) \leq \dim(X) - 1$.

where the rank of the differential $D\Phi$ drops by j at least.

If $\Sigma^{(j)}(\Phi)$ is a smooth submanifold, then we may consider the smooth map $\Phi| : \Sigma^{(j)}(\Phi) \to N$ and the locus

$$\Sigma^{(j,k)}(\Phi) := \Sigma^{(k)}\big(\Phi| : \Sigma^{(j)}(\Phi) \to N\big).$$

In a similar way, we may introduce the locus

$$\Sigma^{(j,k,l)}(\Phi) := \Sigma^{(l)}\big(\Phi| : \Sigma^{(j,k)}(\Phi) \to N\big).$$

This recipe can be recycled to generate the Boardman stratum $\Sigma^{\omega}(\Phi)$, where $\omega := (j_1, j_2, j_3, \dots)$.

Bordman Theorem [Bo] claims that the maps $\Phi \in C^{\infty}(M,N)$, for which all the loci $\{\Sigma^{\omega}(\Phi)\}_{\omega}$ are smooth manifolds, form a residual set in $C^{\infty}(M,N)$. When M is compact, this set is open and dense in $C^{\infty}(M,N)$.

We abriviate by $\Sigma_{[j]}(\Phi)$ the locus $\Sigma^{\omega}(\Phi)$ in M, where $\omega = (1,1,1,\dots)$ is of the length j.

Note that although $\Sigma_{[j]}(\Phi) \subset \Sigma^{(j)}(\Phi)$, the two loci are different in general.

Given a smooth map $\Phi : M \to N$, for each $x \in M$, we denote by $j_{\Phi}(x)$ the maximal integer j such that $x \in \Sigma_{[j]}(\Phi)$.

Definition 3.6. *Let M, N be smooth n-dimensional manifolds and let $\Phi : M \to N$ be a smooth map. We say that Φ is* traversally generic *if*

- *for each $1 \leq j \leq n$, the singular set $\Sigma_{[j]}(\Phi)$ is a regularly embedded submanifold of M of codimension j,*
- *the map $\Phi : \Sigma_{[j]}(\Phi) \setminus \Sigma_{[j+1]}(\Phi) \to N$ is an immersion,*
- *for each $y \in N$, the images of the maps*

$$\big\{\Phi : \Sigma_{[j_{\Phi}(x)]}(\Phi) \to N\big\}_{\{x \in \Phi^{-1}(y)|\ j_{\Phi}(x) \geq 1\}}$$

are in general position in the target manifold N.
Equivalently, for each $y \in N$, the images of the tangent spaces

$$\{T_x(\Sigma_{[j_{\Phi}(x)]}(\Phi))\}_{\{x \in \Phi^{-1}(y)|\ j_{\Phi}(x) \geq 1\}}$$

under the linear monomorphisms

$$\big\{D_x\Phi : T_x(\Sigma_{[j_{\Phi}(x)]}(\Phi)) \to T_y(N)\big\}_{\{x \in \Phi^{-1}(y)|\ j_{\Phi}(x) \geq 1\}}$$

are in general position in the tangent space $T_y(N)$.

Let us denote the space of traversally generic maps by the symbol $\mathcal{G}^{\ddagger}(M,N)$. □

The next theorem is a well-known but nontrivial implication of the *Boardman Maps Theory* (see [Bo], [GG]).

Theorem 3.2. *Let M be a n-dimensional compact manifold, and $K \subset M$ a closed set. Assume that $\Phi_0 : M \to \mathbb{R}^n$ is a smooth map which is traversally generic in the vicinity of K.*

Then the traversally generic maps $\Phi : M \to \mathbb{R}^n$ that coincide with the given map Φ_0 in the vicinity of K form an open and dense set $\mathcal{G}^{\ddagger}_{\Phi_0}(M, \mathbb{R}^n)$ in the space $C^{\infty}_{\Phi_0}(M, \mathbb{R}^n)$ of all smooth maps which coincide with Φ_0 in the vicinity of K.

Proof. When $K = \emptyset$, the theorem is a special case of Theorem 5.2 from [GG]. That theorem claims that the set of all *Boardman maps* (see [Bo] for the relevant definitions) satisfying the *normal crossing condition* (the "NC condition" on page 157 in [GG]) is residual (open and dense when X is compact) in $C^\infty(X, Y)$. By their very definition, the Boardman maps satisfy the first two bullets of Definition 3.6, while the third bullet is exactly the formulation of the normal crossing property.

When $K \neq \emptyset$, the openness of traversally generic maps in the space $C^\infty(M, \mathbb{R}^n)$ implies that traversally generic maps which coincide with Φ_0 in the vicinity of K are open in the subspace $C^\infty_{\Phi_0}(M, \mathbb{R}^n) \subset C^\infty(M, \mathbb{R}^n)$.

Tracing the proof of Theorem 5.2 from [GG], the density of $\mathcal{G}^\ddagger_{\Phi_0}(M, \mathbb{R}^n)$ in $C^\infty_{\Phi_0}(M, \mathbb{R}^n)$ is validated by the following general observations. If, for a map from $\Phi : M \to \mathbb{R}^n$, its N-jet $j^N(\Phi)$ is transversal to some variety $W \subset Jet^N(M, \mathbb{R}^n)$ (the transversality of $j^N(\Phi)$ to this variety is equivalent to the property of f being a Bordman map) at a compact set $K \subset M$, then Φ can be perturbed to a map Φ' such that $\Phi' = \Phi$ in the vicinity of K and $j^N(\Phi')$ is transversal to $W \subset Jet^N(M, \mathbb{R}^n)$ everywhere on M. Using Thom's Multijet Transversality Theorem (see [GG], Theorem 4.13), similar extension principle applies to maps that satisfy the normal crossing condition.

If, for a map $\Phi : M \to \mathbb{R}^n$, its N-jet $j^N(\Phi)$ is transversal to some real algebraic variety $W \subset Jet^N(M, \mathbb{R}^n)$ (the transversality of $j^N(\Phi)$ to this variety is equivalent to the property of f being a Bordman map) at a compact set $K \subset M$, then Φ can be perturbed to a map Φ' such that $\Phi' = \Phi$ in the vicinity of K and $j^N(\Phi')$ is transversal to $W \subset Jet^N(M, \mathbb{R}^n)$ everywhere on M. Using Thom's Multijet Transversality Theorem (see [GG], Theorem 4.13), a similar extension principle applies to maps that satisfy the normal crossing condition.

Let us clarify these claims. If $j^N(\Phi) : M \to Jet^N(M, \mathbb{R}^n)$ is transversal to the variety W on the compact K, then there is a compact neighborhood U of K, $K \subset \mathrm{int}(U)$ on which $j^N(\Phi)$ is transversal to W. We can choose U with a smooth boundary ∂U.

Take a compact smooth collar $C \subset U \setminus K$ of ∂U and consider a smooth partition of unity $\{a, b \in C^\infty(M, \mathbb{R})\}$ on M, subordinate to the cover $U, (M \setminus U) \cup C$ and such that $a = 1$ on $U \setminus C$. Then $b = 1$ on $M \setminus U$.

The maps Ψ, such that of $j^N(\Psi)$ is transversal to W on a given compact T in M, form an open and dense subset $\mathcal{U}^T \subset C^\infty(M, \mathbb{R}^n)$. Therefore $\mathcal{U}^U =_{\mathsf{def}} \mathcal{U}^U_\Phi$ is an open neighborhood of the map Φ.

Next, consider the open and dense subsets $\mathcal{U}^M, \mathcal{U}^U, \mathcal{U}^C$, and $\mathcal{U}^{M \setminus U^\circ}$ of $C^\infty(M, \mathbb{R}^n)$. Note that $\Phi \in \overline{\mathcal{U}^M} \cap \mathcal{U}^U_\Phi$.

Our goal is to construct a map $\Theta \in \mathcal{U}^M$ such that $\Theta = \Phi$ on $U \setminus C$.

Let

$$\mathcal{B}_\Phi =_{\mathsf{def}} \{\Psi \in C^\infty(M, \mathbb{R}^n) \big|\ a \cdot \Phi + b \cdot \Psi \in \mathcal{U}^C\}.$$

Since the operator $\Psi \to a \cdot \Phi + b \cdot \Psi$ is continuous in the C^∞-topology and $\mathcal{U}^C$ is an open set, $\mathcal{B}_\Phi$ is an open set. Note that $\Phi \in \mathcal{B}_\Phi$, so $\mathcal{B}_\Phi \neq \emptyset$.

Since $\mathcal{B}_\Phi \neq \emptyset$ is open and $\mathcal{U}^M$ is open and dense, we get $\mathcal{B}_\Phi \cap \mathcal{U}^M \neq \emptyset$. Therefore there exists a map $\Psi_\star \in \mathcal{B}_\Phi \cap \mathcal{U}^M$. Let us form the map

$$\Theta =_{\mathsf{def}} a \cdot \Phi + b \cdot \Psi_\star.$$

By its definition, $\Theta = \Phi$ on $U \setminus C$ and $\Theta \in \mathcal{U}^M$ (since, by the construction, $\Phi \in \mathcal{U}^{U \setminus C^\circ}$, $\Theta \in \mathcal{U}^C$ and $\Psi_\star \in \mathcal{U}^{M \setminus U^\circ}$). ◻

Let A, B be topological spaces. Recall that a map $\Psi : A \to B$ is called *quasi-open*, if the interior $\mathrm{int}(\Psi(U)) \neq \emptyset$ for any open set $U \subset A$.

The next three simple lemmas will prepare us for the pivotal Lemma 3.10 below.

LEMMA 3.7. *Let A and B be two topological spaces. Assume that there exists a pair of continuous maps, $\Psi : A \to B$ and $\Theta : B \to A$, such that $\Psi \circ \Theta = Id_B$. Then Ψ is a quasi-open map.*

Proof. By the continuity of Θ, for any open set $U \subset A$, the set $\Theta^{-1}(U)$ is open in B. The identity $\Psi \circ \Theta = Id_B$ implies that $\Theta^{-1}(U) \subset \Psi(U)$. The same identity implies that if $U \neq \emptyset$ then $\Theta^{-1}(U) \neq \emptyset$. So the Ψ-image of any non-empty open set contains a non-empty open set. □

LEMMA 3.8. *Let $\Psi : A \to B$ be a continuous quasi-open map. Let a subset $E \subset B$ be open and dense in B. Then $\Psi^{-1}(E)$ is open and dense in A.*

Proof. By the continuity of Ψ, the set $\Psi^{-1}(E)$ is open. Assume to the contrary that $\Psi^{-1}(E)$ is not dense in A. Then there exists an open set U such that $U \cap \Psi^{-1}(E) = \emptyset$. Consider the set $\Psi(U)$. Since the map Ψ is quasi-open, $\Psi(U)$ must contain a nonempty open subset V. Since E is dense, the open set $V \cap E \neq \emptyset$. Therefore there exists a point $a \in U$ such that $\Psi(a) \in V \cap E$. This implies that there is $a \in \Psi^{-1}(E) \cap U$, a contradiction with the assumption that $\Psi^{-1}(E) \cap U = \emptyset$. □

By combining the previous two lemmas, we get the following corollary.

LEMMA 3.9. *Let A and B be two topological spaces. Assume that there exists a pair of continuous maps, $\Psi : A \to B$ and $\Theta : B \to A$, such that $\Psi \circ \Theta = Id_B$. Let a subset $E \subset B$ be open and dense in B. Then $\Psi^{-1}(E)$ is open and dense in A.* □

The next lemma claims that each trajectory γ of a traversing boundary generic field v has a pair of special neighborhoods $V \subset U$ such that the field has an arbitrary C^∞-small perturbation which is supported in U and is traversally generic with respect to $\partial_1 X \cap V$. It is important to clarify the nature of the claim: *first* one chooses the right neighborhoods, and *then*, in these neighborhoods, *arbitrary small* perturbations of v with the desired properties are available.

As usually, we extend a given traversing field v on X to a germ of manifold $\hat{X}$ that properly contains X so that the extended field $\hat{v}$ is traversing in $\hat{X}$.

LEMMA 3.10. *Let v be a traversing and boundary generic field on X, and γ its trajectory. Assume that v is traversally generic in the vicinity of a closed $\hat{v}$-adjusted set $F \subset \hat{X}$.*

Then there is a triple $W \subset V \subset U$ of compact $\hat{v}$-adjusted neighborhoods of γ in $\hat{X}$ with the following properties:

- *there exists an arbitrary C^∞-small and U-supported perturbation $\hat{v}'$ of $\hat{v}$, such that $v' := \hat{v}'|_X$ still is traversing and boundary generic,*
- *$\hat{v}'$ is* traversally generic *with respect to $\partial_1 X \cap V$,*
- *for any $\hat{v}'$-trajectory γ' which intersects W, $\gamma' \cap \partial_1 X \subset \partial_1 X \cap V$,*
- *$\hat{v}' = \hat{v}$ in the vicinity of $F \cup (\hat{X} \setminus U)$ in $\hat{X}$.*

Proof. Let $\dim(X) = n + 1$. We denote by D^n_r the closed Euclidean n-ball of radius r.

For any traversing vector field $v \in \mathcal{V}^\dagger(X)$, each trajectory γ has regular $\hat{v}$-adjusted neighborhoods $W \subset V \subset U \subset \hat{X}$ such that $W \subset \mathrm{int}(V)$ and $V \subset \mathrm{int}(U)$. With the help of the $\hat{v}$-flow, U, V, and W are diffeomorphic to the cylinders $[0,1] \times D^n_1$, $[0,1] \times D^n_{0.5}$ and $[0,1] \times D^n_{0.25}$, respectively. Let G denote this diffeomorphism.

Let $f : U \to [0,1]$ denote the height function produced by G, so that $df(\hat{v}) > 0$. Let $S := f^{-1}(0)$ be a transversal section of the $\hat{v}$-flow in U.

Put $\partial U := G^{-1}(\partial D^n_1 \times [0,1])$ and $\delta U := G^{-1}(D^n_1 \times \partial[0,1])$.

Evidently, one can pick the tube U with the core γ so narrow that $\delta U \subset \hat{X} \setminus X$.

Consider two sets:

$$\partial^F U := \partial U \cup (F \cap U) \text{ and } \partial^F S := \partial S \cup (F \cap S).$$

Let $\mathcal{V}_f(U, \partial^F U)$ be the space of all smooth vector fields w in U such that:

- the field $w|_{\partial^F U}$ is proportional to the given field $\hat{v}|_{\partial^F U}$[11],
- $df(w) > 0$ in $U \setminus \delta U$,
- $w|_{\delta U} = 0$. (3.25)

Such fields w generate 1-parameter flows $\Phi_w : U \times \mathbb{R} \to U$ for which $F \cap U$ is invariant.

We may assume that $\hat{v} \in \mathcal{V}_f(U, \partial^F U)$ without modifying v in X. The $\hat{v}$-flow $\Phi_{\hat{v}}$ defines the obvious submersion $p_{\hat{v}} : U \to S$.

In fact, using the product structure in U, there exists a Riemannian metric g on U such that the gradient $\nabla_g(f) = \hat{v}$. (In such a metric, the submersion $p_{\hat{v}} : U \to S$ is a harmonic map.)

We denote by $\mathrm{Sub}\big((U, \partial^F U), (S, \partial^F S)\big)$ the space of submersions $p : U \to S$ whose restriction to $\partial^F U$ is equal to the given map $p_{\hat{v}}$. Since each $w \in \mathcal{V}_f(U, \partial^F U)$, with the help of the flow, defines a submersion p_w whose restriction to $\partial^F U$ is prescribed and equals to $p_{\hat{v}}$, we get a continuous map

$$J : \mathcal{V}_f(U, \partial^F U) \to \mathrm{Sub}\big((U, \partial^F U), (S, \partial^F S)\big). \tag{3.26}$$

This map J is surjective. Indeed, the fibers of any submersion $p \in \mathrm{Sub}\big((U, \partial^F U), (S, \partial^F S)\big)$ form an oriented 1-dimensional foliation $\mathcal{F}(p)$. Using the product metric g on U, we form a unit vector field $w(p)$, tangent to the fibers of $\mathcal{F}(p)$. By multiplying $w(p)$ by an appropriate universal smooth function $\phi : U \to \mathbb{R}_+$, such that $\phi > 0$ in $U \setminus \delta U$ and $\phi|_{\delta U} = 0$, we produce a vector field $\tilde{w}(p) \in \mathcal{V}_f(U, \partial^F U)$. Evidently, $p_{\tilde{w}(p)} = p$.

This construction $p \Rightarrow \mathcal{F}(p) \Rightarrow \phi \cdot w(p)$ defines a continuous map

$$K : \mathrm{Sub}\big((U, \partial^F U), (S, \partial^F S)\big) \to \mathcal{V}_f(U, \partial^F U)$$

such that $J \circ K = Id$. Therefore, by Lemmas 3.7-3.8, J is a quasi-open map.

For any $p \in \mathrm{Sub}\big((U, \partial^F U), (S, \partial^F S)\big)$, the restriction of p to $\partial_1 X \cap U$ produces a continuous restriction operator

$$\Psi : \mathrm{Sub}\big((U, \partial^F U),\ (S, \partial^F S)\big) \to C^\infty\big((\partial_1 X \cap U, \partial_1 X \cap \partial^F U),\ (S, \partial^F S)\big). \tag{3.27}$$

Let us show that Ψ is a *quasi-open* map. To validate this claim, guided by Lemma 3.9, we will construct a continuous extension operator

$$\Theta : C^\infty\big((\partial_1 X \cap U, \partial_1 X \cap \partial^F U),\ (S, \partial^F S)\big) \to C^\infty\big((U, \partial^F U),\ (S, \partial^F S)\big) \tag{3.28}$$

[11] and thus tangent to $\partial^F U$.

whose composition with the restriction operator

$$\Psi : C^\infty\big((U, \partial^F U),\ (S, \partial^F S)\big) \to C^\infty\big((\partial_1 X \cap U, \partial_1 X \cap \partial^F U),\, (S, \partial^F S)\big)$$

is the identity. The construction of operator Θ will depend on the choice of a map $p : (U, \partial^F U) \to (S, \partial^F S)$.

Consider a regular neighborhood $\mathcal{N}(\partial_1 X)$ of the submanifold $\partial_1 X \cap U$ in U. The neighborhood fibers over its core $\partial_1 X \cap U$ with the fiber being a segment. Formula (3.7) in Lemma 3.1 implies that the hypersurfaces ∂U and $\partial_1 X$ are transversal. Hence, we can choose a product structure $\pi : \mathcal{N}(\partial_1 X) \approx (\partial_1 X \cap U) \times [-1, 1]$ so that the intersection $\partial_1 X \cap U = \pi^{-1}\big((\partial_1 X \cap U) \times \{0\}\big)$ and $\mathcal{N}(\partial_1 X) \cap \partial U$ is entirely built of π-fibers.

For any smooth function $h : \partial_1 X \cap U \to \mathbb{R}$, we aim to construct its canonical smooth extension $\hat{H} : \mathcal{N}(\partial_1 X) \to \mathbb{R}$. With this goal in mind, consider a 1-parameter family $\{\phi_a : [-1, 1] \to \mathbb{R}\}_a$ of smooth bell-shaped functions such that $\phi_a(-1) = 0 = \phi_a(1)$ and $\phi_a := a \cdot \phi_0$ (the parameter a being the height $\phi_a(0) = a$ of the bell). Thus $\phi_{a+b} = \phi_a + \phi_b$. Note that, for $a > 0$, the function $\phi_a > 0$ in $(0, 1)$; for $a < 0$, the function $\phi_a < 0$ in $(0, 1)$; for $a = 0$, $\phi_0 = 0$.

For a given $h : \partial_1 X \cap U \to \mathbb{R}$, define a function $\tilde{H} : (\partial_1 X \cap U) \times [0, 1] \to \mathbb{R}$ by the formula $\tilde{H}(x, t) := \phi_{h(x)}(t)$. Then put $\hat{H} := \pi \circ \tilde{H}$. This function $\hat{H} : \mathcal{N}(\partial_1 X) \to \mathbb{R}$ extends to a smooth function $H : U \to \mathbb{R}$ by letting H vanish on the complementary set $U \setminus \mathcal{N}(\partial_1 X)$.

Let

$$\mathcal{E} : C^\infty(\partial_1 X \cap U, \mathbb{R}) \to C^\infty(U, \mathbb{R})$$

be the continuous extension operator, defined by the formula $\mathcal{E}(h) := H$ for all $h \in C^\infty(\partial_1 X \cap U, \mathbb{R})$. Since $\phi_a := a \cdot \phi_0$, the operator $\mathcal{E}$ is linear. Evidently, the operator $\mathcal{E}$ gives rise to a continuous linear operator

$$\mathcal{E}_n : C^\infty(\partial_1 X \cap U, \mathbb{R}^n) \to C^\infty(U, \mathbb{R}^n).$$

This operator $\mathcal{E}_n$ will be instrumental in the construction of extension operator Θ from (3.28). In a sense, $\mathcal{E}_n$ can be viewed as "the variation operator" for the operator Θ.

For a given submersion $p \in \mathrm{Sub}\big((U, \partial^F U),\, (S, \partial^F S)\big)$, consider its restriction $\Psi(p) \in C^\infty\big((\partial_1 X \cap U, \partial_1 X \cap \partial^F U),\, (S, \partial^F S)\big)$.

Note that, for a space Y, the difference $g_1 - g_0$ between two maps, $g_0 : Y \to \mathbb{R}^n$ and $g_1 : Y \to \mathbb{R}^n$, makes sense as a map from Y to $\mathbb{R}^n$.

Since the flow section $S \subset \mathbb{R}^n$, for any two maps

$$h, h' \in C^\infty\big((\partial_1 X \cap U, \partial_1 X \cap \partial^F U),\, (S, \partial^F S)\big),$$

the difference $h' - h : (\partial_1 X \cap U, \partial_1 X \cap \partial^F U) \to (\mathbb{R}^n, 0)$ is well-defined. Take $h := \Psi(p)$ and consider the map

$$p + \mathcal{E}_n(h' - \Psi(p)) : (U, \partial^F U) \to (\mathbb{R}^n, \partial^F S).$$

Note that the restriction of this map to $\partial^F U$ indeed takes $\partial^F U$ to $\partial^F S$: by definition, the restrictions of $\Psi(p)$ and h' to $\partial_1 X \cap \partial^F U$ coincide, and p takes $\partial_1 X \cap \partial^F U$ to $\partial^F S$.

Finally, we define the operator $\Theta := \Theta(p)$ in (3.28) by the formula

$$\Theta(h') := p + \mathcal{E}_n(h' - \Psi(p)).$$

Due to the properties of $\{\phi_a\}_a$, $\mathcal{E}_n(0) = 0$. Thus, $\Theta(\Psi(p)) = p$. By the linearity of the operator $\mathcal{E}_n$ and using that $\Psi \circ \mathcal{E}_n = Id$, we get

$$\begin{aligned}\Psi(\Theta(h')) := \Psi\big(p + \mathcal{E}_n(h' - \Psi(p))\big) &= \Psi(\mathcal{E}_n(h')) + \Psi(p - \mathcal{E}_n(\Psi(p))) \\ &= h' + \Psi(p) - \Psi(p) = h'.\end{aligned}$$

Since $\Psi \circ \Theta = Id$, by Lemma 3.7, the restriction operator Ψ is a quasi-open map.

Consider the subset

$$\mathcal{G}^{\ddagger}_V \subset C^{\infty}\big((\partial_1 X \cap U, \partial_1 X \cap \partial^F U),\ (S, \partial^F S)\big)$$

such that, for any $h \in \mathcal{G}^{\ddagger}_V$, the restriction $h : V \cap \partial_1 X \to S$ is a traversally generic map in the sense of Definition 3.6. By Theorem 3.4, $\mathcal{G}^{\ddagger}_V$ is an open and dense subset. Since Ψ is a quasi-open map, by Lemmas 3.7-3.9, $\Psi^{-1}(\mathcal{G}^{\ddagger}_V)$ is open and dense in the space $C^{\infty}\big((U, \partial^F U),\ (S, \partial^F S)\big)$. Since the space of submersions $\mathrm{Sub}\big((U, \partial^F U),\ (S, \partial^F S)\big)$ is open in the space $C^{\infty}\big((U, \partial^F U),\ (S, \partial^F S)\big)$, we conclude that the set

$$\mathcal{E}^{\ddagger}_V := \Psi^{-1}(\mathcal{G}^{\ddagger}_V) \cap \mathrm{Sub}\big((U, \partial^F U),\ (S, \partial^F S)\big)$$

is open and dense in the space $\mathrm{Sub}\big((U, \partial^F U),\ (S, \partial^F S)\big)$.

Let us revisit the map J from (3.26). Recall that J is a quasi-open map. Therefore, $J^{-1}(\mathcal{E}^{\ddagger}_V)$ is open and dense in the space $\mathcal{V}_f(U, \partial^F U)$.

Thus we have shown that the fields that are traversally generic with respect $\partial_1 X \cap V$ form an open and dense set in the space $\mathcal{V}_f(U, \partial^F U)$. Let us spell out what this claim means: for fields $\hat{v}' \in J^{-1}(\mathcal{E}^{\ddagger}_V)$, each $\hat{v}'$-trajectory γ intersects with $\partial_1 X \cap V$ along a finite set of points $\{a_i \in \partial_{j_i} X(\hat{v}')^{\circ}\}_i$ in such a way that the differential $Dp_{\hat{v}'}$ places the tangent spaces $\{T_{a_i}\big(\partial_{j_i} X(\hat{v}')^{\circ}\big)\}_i$ in general position in $T(S)$. This behavior still leaves out an option: some $\hat{v}'$-trajectory, passing through a point of $\partial_1 X \cap V$, may hit $\partial_1 X \cap U$ at a point $b \in \partial_k X(\hat{v}')^{\circ}$ such that $b \notin V$. The $Dp_{\hat{v}'}$-image of the tangent space $T_b(\partial_k X(\hat{v}')^{\circ})$ may not be in general position with respect to $Dp_{\hat{v}'}$-images of the spaces $\{T_{a_i}\big(\partial_{j_i} X(\hat{v}')^{\circ}\big)\}_i$.

To control better the flows in the vicinity of W, consider the fields $w \in \mathcal{V}_f(U, \partial^F U)$, having the following "property A": if a w-trajectory γ' has a nonempty intersection with the set W (recall that $W \subset V \subset U$), then the intersection $\gamma' \cap \partial_1 X \subset \partial_1 X \cap \mathrm{int}(V)$. Let us denote by $\mathcal{A}_{V,W}$ the set of fields w possessing the property A.

Since W is compact and properly contained in the interior of V, the set $\mathcal{A}_{V,W} \subset \mathcal{V}_f(U, \partial^F U)$ is open due to the smooth dependence of the solutions of ODE's on the initial data and coefficients (non-vanishing vector fields).

Therefore $J^{-1}(\mathcal{E}^{\ddagger}_V) \cap \mathcal{A}_{V,W}$ is open in $\mathcal{V}_f(U, \partial^F U)$ and dense in $\mathcal{A}_{V,W}$. Evidently, the original field $\hat{v} \in \mathcal{A}_{V,W}$. So $\hat{v}$ admits an arbitrary C^{∞}-small perturbation $\hat{v}'$ which belongs to $J^{-1}(\mathcal{E}^{\ddagger}_V) \cap \mathcal{A}_{V,W}$. By the definitions of all relevant spaces, $\hat{v}'$ possesses the properties listed in the lemma. □

Let $v \in \mathcal{V}^{\dagger}(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$ and let $z : \hat{X} \to \mathbb{R}$ be as in (3.18). Then along each trajectory γ, in the appropriate coordinates (u, y), the locus $\partial_1 X$ can be described as the zero set of the function

$$z(u,y) = \prod_{a_i \in \gamma \cap \partial_1 X} \Big[(u - u(a_i))^{j_i} + \sum_{l=0}^{j_i - 2} \phi_{i,l}(y)(u - u(a_i))^l\Big], \tag{3.29}$$

where j_i denotes the multiplicity of $z|_\gamma$ at a_i, and $\phi_{i,l}(0) = 0$.

We have seen the crucial role played in the previous arguments by the Jacobi $m'(\gamma) \times n$ matrix $D\Phi_\gamma$ whose rows are the vectors $\{\nabla_y \phi_{i,l}(0)\}_{i,l}$. Based on Lemma 3.3, we can give still another interpretation to the condition $\mathrm{rk}(D\Phi_\gamma) = s$.

As before, let $\gamma \cap \partial_1 X = \{a_i\}$, where $\{a_i\}$ are ordered by v. For each pair $(a_i, a_{i'})$, in the vicinity of γ, consider the germ of the $\hat{v}$-flow generated diffeomorphism $\Psi_{i,i'}(\gamma) : \hat{X} \to \hat{X}$ that takes $a_{i'}$ to a_i. In fact, the flow sections $S_i, S_{i'}$ at a_i and $a_{i'}$ can be chosen so that $\Psi_{i,i'}(\gamma)(S_{i'}) = S_i$.

With the function $z : \hat{X} \to \mathbb{R}$ as in (3.18) in place, at each $a_i \in \partial_{j_i} X^\circ$, we consider the 1-form dz and its successive v-directed Lie derivatives

$$\mathcal{L}_v(dz),\ \mathcal{L}_v^2(dz),\ \dots,\ \mathcal{L}_v^{j_i-2}(dz),$$

viewed as elements of the cotangent space $T^*_{a_i}(X)$. Note that points $a_i \in \partial_1 X^\circ$ do not contribute to this list (there are two such points at most). By Theorem 3.3 below, $\mathrm{rk}(D\Phi_\gamma)$ is the dimension of the space spanned by the 1-forms

$$\Big\{\Psi^*_{i,1}(\gamma)\big(dz|_{T_{a_i}S_i}\big),\ \Psi^*_{i,1}(\gamma)\big(\mathcal{L}_v(dz)|_{T_{a_i}S_i}\big),\ \dots \\ \dots,\ \Psi^*_{i,1}(\gamma)\big(\mathcal{L}_v^{j_i-2}(dz)|_{T_{a_i}S_i}\big)\Big\}_i \tag{3.30}$$

in the cotangent space $T^*_{a_1}(S_1)$ of the section S_1.

Theorem 3.3 expands the scope of this observation and incorporates the main claim from Theorem 3.1.

THEOREM 3.3. *For a boundary generic traversing field $v \in \mathcal{V}^\dagger(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$ the following properties are equivalent:*

- *v is traversally generic in the sense of Definition 3.2,*
- *v is versal in the sense of Definition 3.5,*
- *for each v-trajectory γ, $\mathrm{rk}(D\Phi_\gamma) = m'(\gamma)$,[12] where the Jacobi matrix $D\Phi_\gamma$ is produced as in Lemma 3.3 from the representation (3.19),*
- *for each v-trajectory γ, the dimension of the space spanned by the 1-forms in (3.30) is $m'(\gamma)$.*

Proof. By Theorem 3.1, if v is versal, then it is traversally generic and vice versa. So the first two bullets in the theorem are equivalent.

By the argument in Lemma 3.4, the property of a traversing $v \in \mathcal{V}^\dagger(X)$ being traversally generic implies that the boundary $\partial_1 X$, in the vicinity of each v-trajectory γ and in $\hat{v}$-adjusted coordinates $(u, y) \in \mathbb{R} \times \mathbb{R}^n$, is given by an equation

$$\prod_{a_i \in \gamma \cap \partial_1 X} \big[(u - u(a_i))^{j_i} + \sum_{l=0}^{j_i-2} \phi_{i,l}(y)(u - u(a_i))^l\big] = 0$$

[12] See formula (3.2).

such that the map $\Phi_\gamma : \mathbb{R}^n \to \mathbb{R}^{m'(\gamma)}$, produced by the functions $\{\phi_{i,l}(y)\}_{i,l}$, has the Jacobi matrix $D\Phi_\gamma(0)$ of the maximal rank $m'(\gamma)$ at the origin $0 \in \mathbb{R}^n$. Further, reasoning in Lemma 3.4 has established that $\mathrm{rk}(D\Phi_\gamma(0)) = m'(\gamma)$ implies the existence of better coordinates $(u, x, \tilde{y})$ in the vicinity of γ, the coordinates in which the boundary $\partial_1 X$ is given by a simpler $\tilde{y}$-independent equation

$$\prod_{a_i \in \gamma \cap \partial_1 X} \left[(u - u(a_i))^{j_i} + \sum_{l=0}^{j_i - 2} x_{i,l}(u - u(a_i))^l\right] = 0.$$

This is exactly the property of v being versal. On the other hand, by the argument as in Lemma 3.6, the versality of a vector field field implies that $\mathrm{rk}(D\Phi_\gamma(0)) = m'(\gamma)$

Thus the equivalence of the properties in the first three bullets has been proven.

Finally, the equivalence of the last bullet with the rest is basically implied by Lemma 3.3. Here is a computation that establishes the last equivalence.

Let $v \in \mathcal{V}^\dagger(X)$. Let us choose special coordinates $(u, \vec{x}) \in \mathbb{R} \times \mathbb{R}^n$ in the vicinity of a typical γ so that the "versal" boundary equation is given by (3.19). Consider a typical multiplier P_i in (3.19). Then

$$\frac{\partial P_i}{\partial \vec{x}} = \sum_{l=0}^{j_i - 2} \big(u - u(a_i)\big)^l \, \frac{\partial x_{i,l}}{\partial \vec{x}}.$$

Thus

$$dP_i = j_i\big(u - u(a_i)\big)^{j_i - 1} du + \Big\langle \frac{\partial P_i}{\partial \vec{x}}, d\vec{x} \Big\rangle$$

and its restriction to the section $S_i := \{u = u(a_i)\}$ is given by $\big\langle \frac{\partial P_i}{\partial \vec{x}}, d\vec{x} \big\rangle$. Therefore

$$\mathcal{L}^s_{\partial_u}(dP_i)|_{S_i} = \sum_{k=1}^{n} \frac{\partial^s}{\partial u^s} \frac{\partial P_i}{\partial x_k} \, dx_k.$$

As a result, $\mathrm{rk}(D\Phi_i(\gamma))$ is equal to the dimension of the space spanned by

$$\{\mathcal{L}^s_{\partial_u}(dP_i)|_{S_i} \in T^*_{a_i}(S_i)\}_{0 \leq s \leq j_i - 2}.$$

Similarly, $\mathrm{rk}(D\Phi(\gamma))$ equals the dimension of the space spanned by

$$\{\Psi^*_{a_i, a_1}(\mathcal{L}^s_{\partial_u}(dP_i)|_{a_i}) \in T^*_{a_1}(X)\}_{i,\, 0 \leq s \leq j_i - 2}.$$

By Lemma 3.3, the latter space coincides with the space spanned by

$$\{\Psi^*_{a_i, a_1}(\mathcal{L}^s_{\partial_u}(dz)|_{a_i}) \in T^*_{a_1}(X)\}_{i,\, 0 \leq s \leq j_i - 2},$$

where

$$z(u, \vec{x}) = Q(u, \vec{x}) \times \prod_i P_i(u, \vec{x}))$$

with $Q(u, 0) \neq 0$.

For any field $v \in \mathcal{V}^\dagger(X)$, this proves that the properties described in the last two bullets are equivalent. $\square$

It is possible to globalize the local construction, defined by formulas (3.8) - (3.9).

Let X be a $(n+1)$-dimensional smooth compact manifold. Let $z : \hat{X} \to \mathbb{R}$ be a smooth function as in Lemma 3.1. Consider the sequence of already familiar functions:

$$\psi_0 := z,\ \psi_1 := \mathcal{L}_v(\psi_0),\ \psi_2 := \mathcal{L}_v(\psi_1),\ \dots,\ \psi_n := \mathcal{L}_v(\psi_{n-1}). \tag{3.31}$$

They gives rise to the smooth maps

$$\begin{aligned} \Psi(v,z) &:= (\psi_0,\dots,\psi_n) : X \to \mathbb{R}^{n+1}, \\ \Psi^{\partial}(v,z) &:= (\psi_1,\dots,\psi_n) : \partial_1 X \to \mathbb{R}^{n}. \end{aligned} \tag{3.32}$$

As in (3.8) - (3.10), the locus $\partial_1 X$ is defined by the equation $\{\psi_0 = 0\}$, the locus $\partial_2 X$ by the equations $\{\psi_0 = 0, \psi_1 = 0\}$, and so on. The locus $\partial_j X$ is defined by the equations $\{\psi_0 = 0, \psi_1 = 0, \dots, \psi_{j-1} = 0\}$. We notice that $\partial_j^+ X$ is characterized by the additional inequality $\psi_j \geq 0$. Recall that, unlike the maps in (3.32), these loci do not depend on the choice of the auxiliary function $z : \hat{X} \to \mathbb{R}$.

For a $(n+1)$-dimensional X, the formulas (3.32) give rise to the continuous maps

$$\mathbf{\Psi}_z : \mathcal{V}(X) \to C^\infty(X,\, \mathbb{R}^{n+1}),$$

$$\mathbf{\Psi}_z^{\partial} : \mathcal{V}(X) \to C^\infty(\partial_1 X,\, \mathbb{R}^{n}).$$

Let $\psi_0, \psi_1, \dots \psi_{l-1}$ be the standard coordinates in $\mathbb{R}^l$. Consider the complete flag

$$\mathsf{F}^l = \{\mathbb{R}^l := F_0 \supset F_1 \supset F_2 \dots \supset F_l := \{0\}\},$$

where $F_j \subset \mathbb{R}^l$ is defined by the equations

$$\{\psi_0 = 0,\ \dots,\ \psi_{j-1} = 0\}.$$

Each space F_j is divided by F_{j+1} into two halves: F_j^+ and F_j^-; the half-space F_j^+ is characterized by the inequality $\psi_j \geq 0$.

Let $\mathsf{Diff}_+^{\mathsf{F}}(\mathbb{R}^l)$ denote the group of smooth diffeomorphisms of $\mathbb{R}^l$ that preserve all the half-spaces $\{F_j^{\pm}\}$.

Definition 3.7. *Let M^k be a smooth compact k-manifold. We say that a map $\Psi : M^k \to \mathbb{R}^l$ is* F-stable *if, for an open neighborhood $\mathcal{O}$ of Ψ in $C^\infty(M^k, \mathbb{R}^l)$ and each $\Psi' \in \mathcal{O}$, there exists a smooth diffeomorphism $\chi : M^k \to M^k$ and a diffeomorphism $\phi \in \mathsf{Diff}_+^{\mathsf{F}}(\mathbb{R}^l)$ such that*

$$\phi \circ \Psi' = \Psi \circ \chi.$$

□

Remark 3.4. Let $\mathbf{\Psi}_z : \mathcal{V}^\dagger(X) \to C^\infty(X,\, \mathbb{R}^{n+1})$ be the map which takes each boundary generic field v to the map $\Psi(v,z)$ from (3.32).

Evidently, if the map $\Psi(v,z) : \partial_1 X \to \mathbb{R}^n$ is F^{n+1}-stable, then for each $v' \in \mathbf{\Psi}_z^{-1}(\mathcal{O})$, the appropriate χ from Definition 3.7 will map each stratum $\partial_j^{\pm} X(v')$ to the stratum $\partial_j^{\pm} X(v)$.

By definition, the F^{n+1}-stable maps form an open set in $C^\infty(X, \mathbb{R}^{n+1})$. However, in general, they do not form a dense subset. Recall that even the common stable maps

$f : Y \to Z$, where $\dim(Y) = \dim(Z) = n+1$, are dense in the space of all smooth maps $C^\infty(Y, Z)$ only for $n < 8$ (see [GG], page 163.)!

Nevertheless, in Theorem 3.4 below, we will establish the local stability of stratifications $\{\partial_j^{\pm} X(v)\}_j$ in the vicinity of *any* $v \in \mathcal{V}^\dagger(X)$—a much weaker property than the F^{n+1}-stability of the map $\Psi(v, z)$. ◻

By the arguments from Lemma 3.1, for any $v \in \mathcal{V}^\dagger(X)$, the map $\Psi(v, z) : X \to \mathbb{R}^{n+1}$ is transversal to each space F_j from the flag

$$\mathsf{F}^{n+1} := \{\mathbb{R}^{n+1} \supset F_1 \supset F_2 \supset \dots \supset F_{n+1} = \{0\}\},$$

and $\Psi(v, z)^{-1}(F_1) = \partial_1 X$. We describe this transversality by saying that Ψ is "transversal to the flag F^{n+1}".

Question 3.1. Which maps $\Theta : X \to \mathbb{R}^{n+1}$, transversal to the flag F^{n+1}, have the form $\Psi(v, z)$ for some function z as in Lemma 3.1 and $v \in \mathcal{V}^\dagger(X)$? For some $v \in \mathcal{V}^\dagger(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$? ◻

The answer to this question eludes us. However, if we extend the function list in (3.32) by introducing an additional function $\psi_{n+1} := \mathcal{L}_v(\psi_n)$, we will get an extension $\hat{\Psi}(z, v) : X \to \mathbb{R}^{n+2}$ of the map $\Psi(z, v) : X \to \mathbb{R}^{n+1}$.

According to the lemma below, for a given "extended" map $\hat{\Theta} : X \to \mathbb{R}^{n+2}$, the field v such that $\hat{\Theta} = \hat{\Psi}(z, v)$ is often unique.

Lemma 3.11. *Let X be a compact $(n+1)$-dimensional manifold. For a smooth map $\hat{\Theta} : X \to \mathbb{R}^{n+2}$, consider the composition $\Theta := \pi \circ \hat{\Theta}$, where $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ is the projection $(\psi_0, \dots, \psi_{n+1}) \to (\psi_0, \dots, \psi_n)$.*

Assume that Θ is transversal to the flag F^{n+1} and that $\mathrm{rk}(D\Theta) = n+1$ on a dense subset A of X. Then there is at most one boundary generic field v on X such that $\Theta = \Psi(z, v)$ and $\hat{\Theta} = \hat{\Psi}(z, v)$.

Proof. We denote by θ_j the j-th component of the given map $\hat{\Theta}$. Let g be a Riemannian metric on X, such that $\nabla_g(\theta_0) = \nu$, the unitary inward normal to $\partial_1 X$. Such metric g exists since Θ is transversal to F_1.

If $\Theta = \Psi(z, v)$, then $\theta_0 := z$, and v must satisfy the equations:

$$\theta_1 = \langle \nabla_g \theta_0, v \rangle,\ \theta_2 = \langle \nabla_g \theta_1, v \rangle,\ \dots, \theta_n = \langle \nabla_g \theta_{n-1}, v \rangle.$$

We impose an additional relation

$$\theta_{n+1} = \langle \nabla_g \theta_n, v \rangle$$

which reflects the assumption that Θ extends to $\hat{\Theta} := \hat{\Psi}(z, v)$.

Since the fields $\nabla_g \theta_0, \nabla_g \theta_2, \dots \nabla_g \theta_n$ are assumed to be independent on the dense set $A \subset X$, the field v is uniquely determined there by its scalar products with the fields $\{\nabla_g \theta_j\}_{0 \le j \le n}$. Since A is dense, by continuity, there is at most a single field v on X, such that $\hat{\Theta} = \hat{\Psi}(z, v)$. ◻

Recall that, for a boundary generic field v on X and its trajectory γ, each point $a \in \gamma \cap \partial_1 X$ acquires some multiplicity $j(a)$. If the set $\gamma \cap \partial_1 X$ is finite and γ is not a closed trajectory (for instance, if $v \in \mathcal{V}_{\mathsf{trav}}(X) \cap \mathcal{V}^\dagger(X)$), then the points of $\gamma \cap \partial_1 X$ are ordered. So we get an ordered sequence of points $a_i \in \gamma \cap \partial_1 X$, together with their multiplicities $j(a_i)$. We call such weighted sequence D_γ of points on γ the *divisor of* γ. Its degree is the multiplicity $m(\gamma) := \sum_{a \in \gamma \cap \partial_1 X} j(a)$ of γ (see Definition 3.1).

Theorem 3.4. *Let X be a compact smooth $(n+1)$-manifold with boundary.*

- *The space $\mathcal{V}^\dagger(X)$ of boundary generic fields*[13] *is open and dense in the space $\mathcal{V}(X)$ of all vector fields on X.*
- *Up to the natural action by diffeomorphisms of X, the Morse stratification $\{\partial_j X(v)\}_j$ is locally constant within each path-connected component of $\mathcal{V}^\dagger(X)$.*
- *For any field $v \in \mathcal{V}^\dagger(X)$, there is a neighborhood E of $\partial_1 X$ in $\hat{X}$, such that the portion $\gamma \cap E$ of every $\hat{v}$-trajectory γ has the E-localized multiplicity $m(\gamma \cap E) \leq n+1$. Moreover, for each point $a \in \partial_k X^\circ(v)$, $m(\gamma \cap E) \leq k$ for all γ's in the vicinity of the point a.*
- *Each point $a \in \partial_k X(v)^\circ$ has a $\hat{v}$-adjusted neighborhood U such that any divisor D in $\mathbb{R}$ with the properties* $\deg(D) \leq k$ *and* $\deg(D) \equiv k$ (2) *is realized, up to a diffeomorphism of $\mathbb{R}$, as the divisor $D_{\gamma \cap U}$ of the portion $\gamma \cap U$ for some $\hat{v}$-trajectory γ.*

Proof. We fix a Riemannian metric g on $\hat{X}$. Consider the functions $\{\psi_i\}$ from (3.31) and the map $\Psi = \Psi(v, z)$ from (3.32) that they generate. As in (3.7)-(3.9), the property of v being boundary-generic is equivalent to the requirement that the gradient vectors $\nabla\psi_0, \nabla\psi_1, \dots, \nabla\psi_{j-1}$ are linearly independent on the solution set of

$$\{\psi_0 = 0, \psi_1 = 0, \dots, \psi_{j-1} = 0\}$$

for all j.

In terms of the complete flags F^{n+1} (with $F_0 := \mathbb{R}^{n+1}$), a field $v \in \mathcal{V}^\dagger(X)$ if and only if, for each subspace $F_j \subset \mathbb{R}^{n+1}$, the rank of the differential $D\Psi := D\Psi(v, g)$, being restricted to the bundle, normal to the set $\partial_j X(v) = \Psi^{-1}(F_j)$ in X, is equal to j.

Let us denote by $\mathcal{V}_{\{|\partial \neq 0\}}(X)$ the space of fields that do not vanish on the boundary $\partial_1 X$. Evidently, any smooth section of the tangent bundle $T(X)$ can be approximated by a smooth section that does not vanish on $\partial_1 X$. Therefore it suffices to show that $\mathcal{V}^\dagger(X)$ is a dense (in the C^∞-topology) subset of $\mathcal{V}_{\{|\partial \neq 0\}}(X)$ in order to conclude that $\mathcal{V}^\dagger(X)$ is a dense subset of $\mathcal{V}(X)$.

Given any field v which does not vanish on $\partial_1 X$, we decompose it along $\partial_1 X$ into the normal component n_1 and the tangent component v_1. Then we extend the decomposition $v = v_1 \oplus n_1$ in a collar of $\partial_1 X$ in $\hat{X}$. It is possible to perturb n_1 to make sure that it defines a section of the normal bundle $\nu(\partial_1 X, \hat{X})$ that is transversal to its zero section. The perturbation can be smoothly extended into a collar of $\partial_1 X$ in $\hat{X}$, where it will be supported. Let us use the same notations for the perturbed field.[14] The component n_1 will be fixed in the further perturbations. Its zero locus is the manifold $\partial_2 X$. Next, we consider the orthogonal decomposition $v_1 = v_2 \oplus n_2$, where v_2 is tangent to $\partial_2 X$ and n_2 is a section of the normal bundle $\nu(\partial_2 X, \partial_1 X)$. Again, it is possible to extend this decomposition into a collar of $\partial_2 X$ in $\partial_1 X$. Then we perturb n_2 to make it transversal to $\partial_2 X$ and extend this perturbation first into a collar of $\partial_2 X$ in $\partial_1 X$ and then further into a collar of $\partial_1 X$ in $\hat{X}$. The zero locus of n_2 defines a submanifold $\partial_3 X \subset \partial_2 X$. Continuing this sequence of perturbations, we will produce a field from $\mathcal{V}^\dagger(X)$. Therefore such fields form a dense set in the space $\mathcal{V}_{\{|\partial \neq 0\}}(X)$, and thus, in the space of all fields $\mathcal{V}(X)$ (this fact can be derived from the Morin Theorem 2.1 as well).

To show that $\mathcal{V}^\dagger(X)$ is open, we use the local model employed in the proof of Lemma 3.1. Evidently, the linear independence of the gradient fields in (3.10) in the

[13] see Definition 2.1.

[14] In fact, the transversality of n_1 to $\partial_1 X$ defines an open set in the space $\mathcal{V}(\hat{X})$.

vicinity of the solution set of (3.9) is an open property imposed on the pair of functions $u(z, x), w(z, x)$ from that model. By Lemma 3.1, locally, this independence of fields is exactly the property of v to define transversal sections of the quotient 1-bundles $\{T(\partial_{j-1}X)/T(\partial_j X)\}_j$.

By covering a compact collar E of $\partial_1 X$ in $\hat{X}$ with a finite system of compact coordinate charts (z, x) as in the proof of Lemma 3.1, we conclude that if (3.9) and (3.10) are satisfied by a field v in each of the charts, then all sufficiently C^∞-small perturbations of v will satisfy similar conditions. Thus, $\mathcal{V}^\dagger(X)$ is open and dense in the space $\mathcal{V}_{\{|_\partial \neq 0\}}(X)$ which, in turn, is open and dense in the space $\mathcal{V}(X)$.

Now let us prove the claim in the second bullet of the theorem.

Let us fix a Riemannian metric g on X. Recall that the normal line bundle ν_j to $\partial_j X(v)$ in $\partial_{j-1}X(v)$ is trivial for all j. We assume that the fibers of ν_j are orthogonal to $\partial_j X(v)$ in the metric $g|_{\partial_{j-1}X(v)}$. For a sufficiently small $\epsilon > 0$, with the help of the exponential map, the bundle ν_j gives rise to a 1-disk bundle structure in a regular ϵ-neighborhood $V_j(\epsilon) \subset \partial_{j-1}X(v)$ of $\partial_j X(v)$. These bundle structures in $\{V_k(\epsilon_k)\}_{k \leq j}$ are instrumental in picking a particular j-disk bundle structure in a regular neighborhood $W_{j,\epsilon}$ of $\partial_j X(v)$ in $\hat{X}$, so that the fibers of the bundle $\xi_j : W_{j,\epsilon} \to \partial_j X(v)$ carry j global coordinates. By definition, a typical fiber of ξ_j over a point $a_j \in \partial_j X(v)$ consists of all points in X that can be reached by moving from a_j first in the direction of ν_j along the geodesic $\gamma_j \subset \partial_{j-1}X(v)$ for a distance l_{j-1}, then moving from a typical point $a_{j-1} \in \gamma_j$ in the direction of ν_{j-1} along the geodesic $\gamma_{j-1} \subset \partial_{j-2}X(v)$ for a distance l_{j-2}, then moving from a typical point $a_{j-2} \in \gamma_{j-1}$ in the direction of ν_{j-2} along the geodesic $\gamma_{j-2} \subset \partial_{j-3}X(v)$ for a distance l_{j-3}, and so on. Here $\sum_k l_{j-k}^2 \leq \epsilon^2$, where $\epsilon > 0$ being sufficiently small, and the geodesic curves $\gamma_k \subset \partial_{k-1}X(v)$ are taken in the metric $g|_{\partial_{k-1}X(v)}$.

Thanks to this structure of the bundle $\xi_j : W_{j,\epsilon} \to \partial_j X(v)$, the map ξ_j decomposes into a sequence of j surjections:

$$\xi_{j,1} : W_{j,\epsilon} \to W_{j,\epsilon} \cap \partial_1 X,$$

$$\xi_{j,2} : W_{j,\epsilon} \cap \partial_1 X \to W_{j,\epsilon} \cap \partial_2 X(v),$$

$$\xi_{j,3} : W_{j,\epsilon} \cap \partial_2 X(v) \to W_{j,\epsilon} \cap \partial_3 X(v),$$

and so on. The fibers of $\xi_{j,k}$ (of dimension one at most) foliate the space $W_{j,\epsilon} \cap \partial_{j-1}X(v)$.

Consider a complete coordinate flag

$$\mathsf{F}^n = \{\mathbb{R}^n := F_1 \supset F_2 \supset \cdots \supset F_{n+1} := \{0\}\}$$

in $\mathbb{R}^n$. We denote by $F_j^\perp$ be the affine j-space that is orthogonal to F_j in $\mathbb{R}^n$ at a point b. Let $\pi_j : \mathbb{R}^n \to F_j^\perp(0)$ be the orthogonal projection whose fiber is F_j.

Recall that, for $v \in \mathcal{V}^\dagger(X)$, the map $\Psi^\partial(v, z)$ in (3.32) is transversal to each $F_j \subset \mathbb{R}^n$, $j \in [2, n+1]$. Abusing the notations in (3.32), put $\Psi := \Psi^\partial(v, z)$. Therefore, for any boundary generic v, there exists an open ball $B_\epsilon^j := B_\epsilon^j(v) \subset F_j^\perp$ of radius $\epsilon := \epsilon(j)$, centered on the origin, such that the map

$$\pi_j \circ \Psi : U_{j,\epsilon} \to B_\epsilon^j,$$

where $U_{j,\epsilon} := (\pi_j \circ \Psi)^{-1}(B^j_\epsilon)$, is a surjection and thus a fibration. We choose $\epsilon > 0$ appropriate for all j's.

Next we pick the ϵ'-neighborhoods $\{W_{j,\epsilon'}\}_j$ as above and with $\epsilon' > 0$ so small that $\{W_{j,\epsilon'} \subset U_{j,\epsilon}\}_j$. Moreover, since the fibers of the bundle $\xi_j : W_{j,\epsilon'} \to \partial_j X(v)$ are transversal to its core $\partial_j X(v) = (\pi_j \circ \Psi)^{-1}(0)$ and the since the map $\pi_j \circ \Psi$ is transversal to $0 \in B^j_\epsilon$, there exists $\epsilon' > 0$ such that, for each j, all the fibers of the surjection $\pi_j \circ \Psi$ are transversal to the fibers of the closed disk bundle $\xi_j : W_{j,\epsilon} \to \partial_j X(v)$.

For any map $\Psi : \partial_1 X \to \mathbb{R}^n$, transversal to the complete flag F^n in $\mathbb{R}^n$, we fix the neighborhoods $W_{j,\epsilon'} \subset U_{j,\epsilon}$ of $\partial_j X(v)$ as above. Then there exists an open neighborhood $\mathcal{O}_j(\Psi) \subset C^\infty(\partial_1 X, \mathbb{R}^n)$ of the map Ψ, such that, for each map $\Psi' \in \mathcal{O}_j(\Psi)$, the following properties are valid:

- $(\Psi')^{-1}(F_j) \subset W_{j,\epsilon'}$,
- $(\Psi')^{-1}(F_j)$ is a smooth section of the trivial normal j-disk bundle $\xi_j : W_{j,\epsilon'} \to \partial_j X(v)$, transversal to its fibers. (3.33)

The existence of the neighborhood $\mathcal{O}_j(\Psi)$ follows routinely from the openness of transversal families of smooth maps on compact sets. For example, to insure that $(\Psi')^{-1}(F_j)$ is transversal to the fibers $\{D_a\}_{a \in \Psi^{-1}(F_j)}$ of the bundle ξ_j, we need to take Ψ' so close to Ψ that the space $D\Psi'(TD_x)$ is still complementary in $\mathbb{R}^n$ to the space F_j at the point $\Psi'(x)$ for all $x \in W_{j,\epsilon'} \supset (\Psi')^{-1}(F_j)$.

Next, we form the open set $\mathcal{O}(\Psi) := \cap_{j=1}^n \mathcal{O}_j(\Psi)$ in the vector space $C^\infty(\partial_1 X,\, \mathbb{R}^n)$, equipped with the Whitney topology.

Let

$$\boldsymbol{\Psi} : \mathcal{V}^\dagger(X) \to C^\infty(\partial_1 X, \mathbb{R}^n)$$

be the continuous map that takes a field $v' \in \mathcal{V}^\dagger(X)$ to the map $\Psi^\partial(v', z) \in C^\infty(\partial_1 X, \mathbb{R}^n)$.

Consider the open neighborhood of v:

$$\mathcal{U}(v) := \boldsymbol{\Psi}^{-1}\big(\mathcal{O}(\Psi(v))\big) \subset \mathcal{V}^\dagger(X).$$

We claim that, for any $v' \in \mathcal{U}(v)$, the Morse stratifications $\{\partial_j X(v)\}_j$ and $\{\partial_j X(v')\}_j$ can be transformed one into another by a diffeomorphism of X (actually, by an isotopy). Let us explain how to construct the matching diffeomorphism.

Put $\Psi' := \Psi^\partial(v', z)$ and $\Psi := \Psi^\partial(v, z)$. For any point

$$a \in \partial_{j+1} X(v) = (\Psi)^{-1}(F_j),$$

consider the unique point

$$b_j(a) := (\Psi')^{-1}(F_j) \cap \xi_j^{-1}(a)$$

where $\xi_j^{-1}(a)$ is the fiber over a of the disk bundle $\xi_j : W_{j,\epsilon'} \to \partial_{j+1} X(v)$. The two properties in (3.33) imply that the correspondence $\phi_j : b_j(a) \to a$ is a diffeomorphism which maps $(\Psi')^{-1}(F_j)$ to $\Psi^{-1}(F_j)$.

Consider the linear (in the fibers of the bundle ξ_j) diffeotopy

$$\big\{\phi_j(a, t) : \partial_j X(v') \to \partial_1 X\big\}_{t \in [0,1]}$$

that links each point $a \in \partial_j X(v)$ to the point $b_j(a) \in (\Psi')^{-1}(F_j) = \partial_j X(v')$, so that $\phi_j(\sim, 1)$ maps $\partial_j X(v')$ to $\partial_j X(v)$. It can be induced by an ambient isotopy (see [Thom])

$$\big\{\tilde{\Phi}_j(v, v') : \; (X, \partial_1 X) \times [0,1] \to (X, \partial_1 X)\big\}, \tag{3.34}$$

supported in the neighborhood $U_{j,\epsilon}$. That isotopy matches $\partial_j X(v)$ with $\partial_j X(v')$.

Now we will improve the previous construction by building a *single* ambient isotopy $\tilde{\Phi} := \tilde{\Phi}(v, v')$ that matches at once all the strata $\{\partial_j X(v)\}_j$ with the corresponding strata $\{\partial_j X(v')\}_j$. This will be achieved in stages, indexed by $j = 1, 2, 3, \dots$. In what follows, we assume that $v' \in \mathcal{U}(v)$ and take advantage of the special "foliated nature" of the j-disk bundles $\{\xi_j : W_{j,\epsilon'} \to \partial_j X(v)\}_j$ (recall that each ξ_j is foliated by the fibers of the line bundles $\{\xi_{j,k}\}_k$).

We start with a linear in the fibers of the bundle $\xi_2 : W_{2,\epsilon'} \to \partial_2 X(v)$ diffeotopy

$$\phi_1 : \partial_2 X(v') \times [0,1] \to \partial_1 X$$

that takes $\partial_2 X(v')$ to $\partial_2 X(v)$. It extends to an ambient isotopy $\Phi_1 : \partial_1 X \times [0,1] \to \partial_1 X$. In turn, Φ_1 extends to an isotopy $\tilde{\Phi}_1 : X \times [0,1] \to X$ which preserves $\partial_1 X$ invariant.

Then we construct a linear in the fibers of $\xi_3 : W_{3,\epsilon'} \to \partial_3 X(v)$ diffeotopy

$$\phi_2 : \Phi_1(\partial_3 X(v')) \times [0,1] \to \partial_2 X(v)$$

that takes $\Phi_1(\partial_3 X(v'))$ to $\partial_3 X(v)$. It extends to an ambient isotopy

$$\Phi_2 : \partial_2 X(v) \times [0,1] \to \partial_2 X(v).$$

In turn, by an argument as in [Thom], Φ_2 extends to an isotopy $\tilde{\Phi}_2 : X \times [0,1] \to X$ which preserves $\partial_1 X$ and $\partial_2 X(v)$ invariant.

Next we construct a linear in the fibers of $\xi_4 : W_{4,\epsilon'} \to \partial_4 X(v)$ diffeotopy of

$$(\Phi_2 \circ \Phi_1)(\partial_4 X(v'))$$

in $\partial_3 X(v)$ that takes $(\Phi_2 \circ \Phi_1)(\partial_4 X(v'))$ to $\partial_4 X(v)$. It extends to an ambient isotopy $\Phi_3 : \partial_3 X(v) \times [0,1] \to \partial_3 X(v)$. In turn, Φ_3 extends to an isotopy $\tilde{\Phi}_3 : X \times [0,1] \to X$ which preserves $\partial_1 X$, $\partial_2 X(v)$, and $\partial_3 X(v)$ invariant.

Continuing this way, we construct eventually an ambient isotopy $\tilde{\Phi}_n : X \times [0,1] \to X$ which preserves the strata $\partial_1 X, \partial_2 X(v), \dots, \partial_n X(v)$ invariant and matches $\partial_{n+1} X(v')$ with $\partial_{n+1} X(v)$.

Now, for any $v' \in \mathcal{U}(v)$, the composite diffeomorphism

$$\tilde{\Phi}_n(\sim, 1) \circ \cdots \circ \tilde{\Phi}_2(\sim, 1) \circ \tilde{\Phi}_1(\sim, 1) : X \to X$$

matches all the strata $\{\partial_j X(v')\}_j$ with the corresponding strata $\{\partial_j X(v)\}_j$.

Therefore, the smooth type of the stratification $\{\partial_j X(v)\}_j$ is locally stable as the function of $v \in \mathcal{V}^\dagger(X)$. Note that we do not claim that $\Psi := \Psi^\partial(v, z)$ is a F^n-stable map in the sense of Definition 3.7, a much stronger assertion!

In fact, for any path-connected component of $\mathcal{V}^\dagger(X)$, the smooth topological type of the Morse stratification remains constant. Indeed, if the points-fields v_0 and v_1 are connected by a continuous path $\gamma : [0,1] \to \mathcal{V}^\dagger(X)$, then each point $\gamma(t)$ produces an open neighborhood $\mathcal{U}(\gamma(t)) \subset \mathcal{V}^\dagger(X)$ as above. Using the compactness of image $\gamma([0,1])$, we can cover it by a finite number of open sets $\{\mathcal{U}(\gamma(t_i))\}_i$. By the previous arguments, any pair of Morse stratifications $\{\partial_j X(v)\}_j$ and $\{\partial_j X(v')\}_j$, where $v, v' \in \mathcal{U}(\gamma(t_i))$, can be transformed one into the other by a diffeomorphism of X. Therefore $\{\partial_j X(v_0)\}_j$ and $\{\partial_j X(v_1)\}_j$ can be transformed one into the other by a finite composition of locally available diffeomorphisms.

Now, let us validate the last two bullets of the theorem. Formula (3.7) implies that $m(\gamma \cap E) \leq n+1$ for all trajectories γ in E. Thus $m'(\gamma \cap E) \leq n$. By the same token, if $a \cap \partial_k X(v)^\circ$, then $m(\gamma \cap U_a) \leq k$ for all γ in a $\hat{v}$-adjusted tubular neighborhood U_a of a. Of course, this implies that $m'(\gamma \cap U_a) \leq k-1$.

The last bullet of the theorem follows from Lemma 3.1, in particular from the existence of special coordinates (u, x) in which formula (3.7) is valid. ☐

Finally, we have reached the summit of this paper.

Theorem 3.5. *Let X be a smooth compact $(n+1)$-dimensional manifold with boundary.*

- *The subspace $\mathcal{V}^\ddagger(X)$ of traversally generic fields is open and dense in the space $\mathcal{V}_{\mathsf{trav}}(X)$ of all traversing fields*[15].
- *If $v \in \mathcal{V}^\ddagger(X)$, then for every v-trajectory γ, we get $m'(\gamma) \leq n$ and $m(\gamma) \leq 2(n+1)$.*

Proof. First we would like to show that the space $\mathcal{V}^\ddagger(X)$ of traversally generic fields is open in the space $\mathcal{V}^\dagger(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$ of boundary generic traversing fields, and thus by Theorem 3.4, in the space of all traversing fields.

Let us start with a traversally generic field $v \in \mathcal{V}^\ddagger(X)$. By Theorem 3.4, the first bullet, there exists an open neighborhood $\mathcal{O}^\dagger(v) \subset \mathcal{V}(X)$ of v such that $\mathcal{O}^\dagger(v) \subset \mathcal{V}^\dagger(X)$.

We pick a finite set $\{S_\alpha\}$ of transversal sections of the $\hat{v}$-flow in the vicinity of $X \subset \hat{X}$, each section S_α being diffeomorphic to an open n-disk. We denote by T_α a closed n-disk which is properly contained in S_α. We pick the sections $\{S_\alpha \supset T_\alpha\}_\alpha$ so that each $\hat{v}$-trajectory that intersects with X hits at least one of the flow sections T_α in its interior. Let U_α be the union of $\hat{v}$-trajectories through S_α, and let V_α be the union of $\hat{v}$-trajectories through T_α (so that $V_a \subset U_\alpha$). Thus, $\{V_\alpha \cap X\}_\alpha$ is a cover of X.

We denote by $p_\alpha(\hat{v}) : U_\alpha \to S_\alpha$ the $\hat{v}$-directed projection, defined by the formula $x \to \hat{\gamma}_x \cap S_\alpha$, where $\hat{\gamma}_x$ is the $\hat{v}$-trajectory through x.

Since $\hat{v}$ is transversal to the closure of each S_α, there is an open neighborhood $\mathcal{O}^\star(v) \subset \mathcal{V}^\dagger(X)$ of v such that, for every field $v' \in \mathcal{O}^\star(v)$, each $\hat{v}'$-trajectory hits every section S_α transversally or misses it. Moreover, by the C^∞-continuous dependence of ODE's solutions on the initial values and on the non-vanishing vector field, we can assume that each $\hat{v}'$-trajectory through T_α is contained in the set U_α, and each $\hat{v}'$-trajectory through the hypersurface $\delta V_\alpha := \partial_1 X \cap V_\alpha$ hits S_α transversally at a singleton.

We form $\mathcal{O}^{\dagger\star}(v) := \mathcal{O}^\star(v) \cap \mathcal{O}^\dagger(v)$, an open neighborhood of v in $\mathcal{V}^\dagger(X)$.

Consider the $\hat{v}$-directed maps $\{p_\alpha(\hat{v}) : \delta V_\alpha \to S_\alpha\}_\alpha$. Since $v \in \mathcal{V}^\ddagger(X)$, each map $p_\alpha(\hat{v})$ is traversally generic in the sense of Definition 3.6 (where $M = \delta V_\alpha$ and $N = S_\alpha$). Examining Definition 3.6 and Definition 3.2, we see that the converse is true as well: if all $\{p_\alpha(\hat{v}) : \delta V_\alpha \to S_\alpha\}_a$ are traversally generic maps, then v is a traversally generic field.

[15]By definition, traversing fields do not vanish on X.

By Theorem 3.2, there is an open neighborhood $\mathcal{U}_\alpha$ of the map $p_\alpha(\hat{v})$ in $C^\infty(\delta V_\alpha, S_\alpha)$ such that each map $\Phi \in \mathcal{U}_\alpha$ is traversally generic.

Consider the map

$$\Xi : \mathcal{O}^{\dagger\star}(v) \to \prod_\alpha C^\infty(\delta V_\alpha, S_\alpha)$$

that takes each field $v' \in \mathcal{O}^{\dagger\star}(v)$ to the collection of maps $\{p_\alpha(\hat{v}')\}_\alpha$, defined by the $\hat{v}'$-flow. By the continuity of Ξ, the set $\mathcal{O}^\ddagger(v) := \Xi^{-1}(\prod_\alpha \mathcal{U}_\alpha)$ is open in $\mathcal{O}^{\dagger\star}(v)$ and thus in $\mathcal{V}^\dagger(X)$.

Note that $\{\mathrm{Int}(\delta V_\alpha)\}_\alpha$ form an open cover of $\partial_1 X$, so that each $\hat{v}'$-trajectory through X hits one set $\mathrm{Int}(\delta V_\alpha)$ at least. Since the property of a vector field v' being traversally generic can be faithfully expressed in "semi-local" terms of the vicinities of its $\hat{v}'$-trajectories, we conclude that any $v' \in \mathcal{O}^\ddagger(v)$ is traversally generic, in other words, that $\mathcal{O}^\ddagger(v)$, open in $\mathcal{V}^\dagger(X)$, is also open in $\mathcal{V}^\ddagger(X)$.

Now we would like to show that $\mathcal{V}^\ddagger(X)$ is dense in the space $\mathcal{V}^\dagger(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$. So we start with a boundary generic and traversing field v. By Lemma 4.1 [K1], for such a field, there exists a smooth function $f : X \to \mathbb{R}$ so that $df(v) > 0$ in X. Let us denote by $\mathcal{C}_f$ the open neighborhood of v in $\mathcal{V}^\dagger(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$ defined by the inequality $\{df(v') > 0 |\ v' \in \mathcal{V}^\dagger(X)\}$.

Each $\hat{v}$-trajectory γ has a nested triple $W \subset V \subset U$ of $\hat{v}$-adjusted neighborhoods in $\hat{X}$ with the properties described in the key Lemma 3.10. Since X is compact, we can choose a finite collection $\{W_i \subset V_i \subset U_i\}_i$ of such triples so that $\{\mathrm{int}(W_i)\}_i$ form a finite cover of $\partial_1 X$. Let us order the triples.

We denote by S_i a transversal section of the $\hat{v}$-flow in U_i. Let $T_i := S_i \cap V_i$, and $Q_i := S_i \cap W_i$.

Recall some old notations: for a given X-traversing field $\hat{w}$ in $\hat{X}$ and a subset $A \subset \hat{X}$, we denote by $\hat{X}(\hat{w}, A)$ the union of $\hat{w}$-trajectories that pass through A. Let $X(w, A) := \hat{X}(\hat{w}, A) \cap X$.

As we perturb the given boundary generic and traversing field $\hat{v}$, we will insist on all the perturbations $\hat{v}'$ being so small that:

1. $df(v') > 0$ (that is, $v' \in \mathcal{C}_f$),
2. $v' \in \mathcal{V}^\dagger(X)$,
3. $\hat{v}'$ is transversal to all the sections S_i,
4. the $\hat{v}'$-adjusted sets $\{\hat{X}(v', Q_i)\}_i$ cover X,
5. $\hat{X}(v', Q_i) \cap \partial_1 X \subset \hat{X}(v, T_i) \cap \partial_1 X$ for all i. (3.35)

We denote by $\hat{\mathcal{U}}^\bullet$ the set of such fields $\hat{v}'$ on $\hat{X}$. It depends on the choice of sections $\{S_i \supset T_i \supset Q_i\}_i$ and, via these sections, on the original field $\hat{v}$. Evidently, $\hat{v} \in \hat{\mathcal{U}}^\bullet$. In fact, $\mathcal{U}^\bullet$, formed by the restrictions to X of the fields from $\hat{\mathcal{U}}^\bullet$, is an open set in the space $\mathcal{V}^\dagger(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$. Indeed, the openness of sets of fields satisfying (1) and (3) is obvious, satisfying (2) follows from Theorem 3.4, and (4) and (5) follows from the smooth dependence of solutions of ODE's on initial data and on non-vanishing vector fields (on the "coefficients").

Let us pick an arbitrary open neighborhood $\mathcal{W}_{\hat{v}} \subset \mathcal{V}(\hat{X})$ of $\hat{v}$. Put

$$\mathcal{W}^\bullet_{\hat{v}} := \mathcal{W}_{\hat{v}} \cap \hat{\mathcal{U}}^\bullet \text{ and } \mathcal{W}^\bullet_v := \mathcal{W}_{\hat{v}} \cap \mathcal{U}^\bullet.$$

We intend to find a field $\hat{v}' \in \mathcal{W}^\bullet_{\hat{v}}$ that is traversally generic with respect to $\partial_1 X$. This will prove that traversally generic fields form a dense set in $\mathcal{V}^\dagger(X) \cap \mathcal{V}_{\mathsf{trav}}(X)$.

Let us order the $\hat{v}$-sections by forming a finite list: $(S_1, S_2, \dots, S_N)$. By an inductive argument in the number i of sections from the list, we will systematically "enlarge" a set of trajectories which are traversally generic with respect to a growing portion of $\partial_1 X$.

Here is how the induction step $i-1 \Rightarrow i$ works. Assume that we have managed to find a field $\hat{v}' \in \mathcal{W}^\bullet_{\hat{v}}$ such that it is traversally generic, when restricted to the closed $\hat{v}'$-adjusted set

$$F_{i-1}(\hat{v}') := \hat{X}(\hat{v}', \coprod_{1 \leq k < i} Q_k).$$

By key Lemma 3.10, there is a $\hat{X}(\hat{v}', S_i)$-supported perturbation $\hat{v}'' \in \mathcal{W}^\bullet_{\hat{v}}$ of $\hat{v}'$ such that: (1) $\hat{v}''$ is traversally generic in $\hat{X}(\hat{v}'', Q_i)$ and (2) $\hat{v}'' = \hat{v}'$, when restricted to $[\hat{X} \setminus \hat{X}(\hat{v}', S_i)] \cup F_{i-1}(\hat{v}')$. Such a field $\hat{v}''$ is traversally generic in $F_i(\hat{v}'')$.

Since the property (4) from (3.35) is enforced through the induction arguments, eventually (for $i = N$), we will construct a field $\hat{w} \in \mathcal{W}^\bullet_{\hat{v}}$ which is traversally generic everywhere in X.

Note that the case $i = 1$, the base of induction, is exactly the claim of Lemma 3.10.

It remains to prove the last bullet of the theorem. For any field $v \in \mathcal{V}^\ddagger(X)$, we have shown (see (3.4)) that $m'(\gamma) \leq n$ for all trajectories γ.

There are only two points of odd multiplicity in the set $\gamma \cap \partial_1 X$—the two ends of γ. Thus $m(\gamma)$ can be written in the form

$$(2t_0 + 1) + \sum_{i=1}^{q} 2s_i + (2t_1 + 1).$$

As a result,

$$m'(\gamma) = 2t_0 + \sum_{i=1}^{q} (2s_i - 1) + 2t_1 \leq n.$$

The latter inequality implies that $q \leq n$. Therefore,

$$m(\gamma) = m'(\gamma) + q + 2 \leq n + q + 2 \leq 2n + 2,$$

twice the dimension of X. □

Corollary 3.3. *For a given smooth nonsingular function $f : X \to \mathbb{R}$, the traversally generic f-gradient-like fields form an open and dense set in the space of all f-gradient-like fields.*

Proof. For a fixed nonsingular $f : X \to \mathbb{R}$, the set of f-gradient-like fields is defined by the inequality $df(v) > 0$, and therefore is open in the space of all fields $\mathcal{V}(X)$. With this remark in mind, the corollary follows from Theorem 3.5. □

The semi-local models of traversally generic flows that we have developed in this paper will form a foundation of our future investigations of the rich and universal combinatorics that governs such flows. These models will also enable us to study the topology of the trajectory spaces, generated by the traversally generic flows, an interesting class of CW-complexes that behave as surrogate manifolds [K3]. Finally, Theorems 3.4 and 3.5 insure that the traversally generic flows are typical among the traversing flows, thus justifying these future endeavors [K2].

REFERENCES

[Bo] J. M. Boardman, *Singularities of Differentiable Maps*, Publ. Math. I.H.E.S., 33 (1967), pp. 21–57.

[GG] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Graduate Texts in Mathematics 14, Springer-Verlag, New York Heidelberg Berlin, 1973.

[K] G. Katz, *Convexity of Morse Stratifications and Spines of 3-Manifolds*, JP Journal of Geometry and Topology, 9:1 (2009), pp. 1–119.

[K1] G. Katz, *Stratified Convexity & Concavity of Gradient Flows on Manifolds with Boundary*, Applied Mathematics, 5 (2014), pp. 2823–2848.

[K2] G. Katz, *Morse Theory, Gradient Flows, Concavity, and Complexity on Manifolds with Boundary*, to be published by World Scientific.

[K3] G. Katz, *The Stratified Spaces of Real Polynomials and Trajectory Spaces of Traversing Flows*, JP Journal of Geometry and Topology, 19:2 (2016), pp. 95–160.

[Mal] B. Malgrange, *The preparation theorem for differential functions*, Differential Analysis (Papers presented at the Bombay Colloquium, 1964), pp. 203–208.

[Morin] B. Morin, *Formes canoniques des singularities d'une application differentiable*, Comptes Rendus Acad. Sci., Paris 260 (1965), pp. 5662–5665, pp. 6503–6506.

[Mo] M. Morse, *Singular points of vector fields under general boundary conditions*, Amer. J. Math., 51 (1929), pp. 165–178.

[Ph] A. Phillips, *Submersions of open manifolds*, Topology, 6 (1967), pp. 171–206.

[Thom] R. Thom, *La classifications des immersions*, Sémin. Bourbaki, 157, pp. 1957–1958.