

GENERALIZED PSEUDO-COEFFICIENTS OF DISCRETE SERIES OF p -ADIC GROUPS*

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Abstract. Let G be a connected reductive group over a p -adic field F of characteristic 0 and let M be an F -Levi subgroup of G . Given a discrete series representation σ of $M(F)$, we prove that there exists a locally constant and compactly supported function on $M(F)$, which generalizes a pseudo-coefficient of σ . This function satisfies similar properties to the pseudo-coefficient, and its lifting to $G(F)$ is applied to the Plancherel formula.

Key words. pseudo-coefficient, discrete series, p -adic group, trace Paley-Wiener theorem, orbital integral, character function, Plancherel formula.

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1. Introduction. Given a connected *semi-simple* group G over a p -adic field F of characteristic 0 and a discrete series representation σ of $G(F)$ (that is, σ is an irreducible, admissible, unitary representation whose matrix coefficients are square-integrable modulo the center of $G(F)$), it is well-known ([Kaz86], [SS97]) that there exists a pseudo-coefficient ϕ_σ for σ , which lies in the Hecke algebra $C_c^\infty(G(F))$ of locally constant and compactly supported functions on $G(F)$, satisfying the property that, for any tempered representation π of $G(F)$,

$$\text{trace}(\pi)(\phi_\sigma) = \begin{cases} 1, & \text{if } \pi \simeq \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

Historically, Harish-Chandra already knew that the Howe conjecture implied the existence (see [Clo91, p.102]). Another proof of the existence relies on the trace Paley-Wiener theorem ([BDK86]). As an explicit example of such a pseudo-coefficient, ϕ_{St} with the Steinberg representation St of $G(F)$ forms an Euler-Poincaré function in [Kot88], and this result is generalized to any discrete series representation of $G(F)$ in [SS97].

The pseudo-coefficient ϕ_σ satisfies interesting properties. The orbital integral of ϕ_σ at a regular semi-simple element γ of $G(F)$ vanishes unless γ is elliptic ([Kaz86], [SS97]). This property, so-called *cuspidality*, is so crucial as to get various trace formulas (see [Har11] for example). It is also applied to endoscopic transfers ([AC89], [HS11]) and the limit multiplicities of discrete series representations in spaces of automorphic forms ([Clo86], [FLM14]). The orbital integral is also related to the Harish-Chandra character function of σ ([Kaz86], [SS97]).

All work on pseudo-coefficients under the condition ‘semi-simplicity’ can be extended to the case that $G(F)$ has compact center. This is possible because the group $\Psi(G)$ of unramified characters on $G(F)$ is finite (see Remark 2.4). One then chooses a regular function ξ on the $\Psi(G)$ -orbit of σ such that $\xi(z)$ vanishes unless $z = \sigma$. This ξ thus gives the pseudo-coefficient ϕ_σ by means of the trace Paley-Wiener theorem ([BDK86]).

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In the general case that the group $G(F)$ has non-compact center, however, $\Psi(G)$ is no longer finite. It follows that there exists no such ξ as above. For such a group, instead, one can find a function ϕ_σ^ω in the Hecke algebra $C_c^\infty(G(F), \omega^{-1})$ of locally constant functions with compact support *modulo the center* and transforming under the center by ω^{-1} , where ω is the central character of σ . The function ϕ_σ^ω satisfies the property that, for any tempered representation π of $G(F)$ whose central character is ω ,

$$\text{trace}(\pi)(\phi_\sigma^\omega) = \begin{cases} 1, & \text{if } \pi \simeq \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

We also say such a function ϕ_σ^ω is a pseudo-coefficient of σ (see [Clo91], [DKV84, A.4]).

The purpose of this paper is to study a locally constant and compactly supported function on $G(F)$ with non-compact center, which generalizes both pseudo-coefficients $\phi_\sigma \in C_c^\infty(G(F))$ for the case with compact center and $\phi_\sigma^\omega \in C_c^\infty(G(F), \omega^{-1})$ for the case with non-compact center. We work with an arbitrary F -Levi subgroup M (possibly, $M = G$) of a connected reductive group G over F . Our generalized pseudo-coefficient is determined by a discrete series representation of $M(F)$ and a regular function on the $\Psi(G)$ -orbit of σ . It also satisfies the cuspidal property. Furthermore, its projection to $C_c^\infty(M(F), \omega^{-1})$ is related to the pseudo-coefficient ϕ_σ^ω and the Harish-Chandra character function of σ . As an application, we lift it to a function in $C_c^\infty(G(F))$ and simplify the Plancherel formula.

We explain our results more precisely. Let σ be a discrete series representation of $M(F)$ and let ξ be a regular function on the $\Psi(M)$ -orbit of σ (denoted by $\Omega(\sigma)$). Using the trace Paley-Wiener theorem ([BDK86]), we prove that there exists a locally constant and compactly supported function $\phi_{\sigma,\xi}$ on $M(F)$ such that, for all tempered representations π of $(M(F))$,

$$\text{trace}(\pi)(\phi_{\sigma,\xi}) = \begin{cases} \xi(\pi), & \text{if } \pi \simeq \sigma \otimes \psi \text{ for some } \psi \in \Psi(M), \\ 0, & \text{otherwise.} \end{cases}$$

We call $\phi_{\sigma,\xi}$ a *generalized pseudo-coefficient for σ and ξ* . We remark that the regular function $\xi(z)$ cannot be chosen to be 0 for $z \neq \sigma$, unless $M = G$ and $G(F)$ has compact center. The reason is that $\Omega(\sigma)$ is isomorphic to the quotient of $(\mathbb{C}^\times)^d$ by a finite subgroup, the stabilizer $\text{Stab}_\Psi(\sigma)$ of σ in $\Psi(M)$, where d is the dimension of the split component in the center Z_M of M (see Sections 2.3 and 2.4 for details). This part is different from the case with compact center. Furthermore, the fact that $\phi_{\sigma,\xi}$ has compact support on $M(F)$ makes a distinction from $\phi_\sigma^\omega \in C_c^\infty(M(F), \omega^{-1})$.

Despite such differences the generalized pseudo-coefficient $\phi_{\sigma,\xi}$ satisfies several properties similar to those pseudo-coefficients ϕ_σ and ϕ_σ^ω . Given σ and ξ , the function $\phi_{\sigma,\xi}$ is uniquely determined modulo the \mathbb{C} -subspace that is spanned by the functions of the form $m \mapsto (h(m) - h(xm.x^{-1}))$, for all $h \in C_c^\infty(M(F))$ and $x \in M(F)$ (see Remark 3.7). The function also satisfies the cuspidal property (Proposition 3.9)

$$O_\gamma^{M(F)}(\phi_{\sigma,\xi}) = 0 = O_\gamma^{M(F)}(\phi_{\sigma,\xi}^\omega),$$

for any γ regular semi-simple but not elliptic in $M(F)$, where $\phi_{\sigma,\xi}^\omega(m)$ denotes the canonical projection $\int_{Z_M(F)} \omega(z)\phi_{\sigma,\xi}(zm)dz$ of $\phi_{\sigma,\xi}$ onto $C_c^\infty(M(F), \omega^{-1})$. Furthermore, by making a suitable choice of ξ so that $\xi(z)$ vanishes on a finite subset of $\Omega(\sigma)$ whose central character is ω , unless $z = \sigma$ (see Lemma 3.12 for the finite subset),

Proposition 3.13 provides a relationship between the orbital integral of $\phi_{\sigma,\xi}^\omega$ and the Harish-Chandra character Θ_σ

$$O_\gamma^{M(F)}(\phi_{\sigma,\xi}^\omega) = \xi(\sigma)\overline{\Theta_\sigma(\gamma)},$$

for every elliptic regular semi-simple γ in $M(F)$. Here the bar $\overline{}$ means the complex conjugation. We note that the value $\xi(\sigma)$ above is determined by $h_\sigma^\omega(1)$ and the formal degree $d_M(\sigma)$, where h_σ^ω is a matrix coefficient of σ (see Remark 3.16). By taking the regular function ξ with $\xi(\sigma) = 1$, that is, $h_\sigma^\omega(1) = d_M(\sigma)$, we get a simple relationship between $\phi_{\sigma,\xi}^\omega$ and a pseudo-coefficient ϕ_σ^ω of σ (see Proposition 3.18). Applying $\phi_{\sigma,\xi}^\omega$ to the Plancherel formula, we also get

$$\phi_{\sigma,\xi}(1) = d_M(\sigma) \int_{\psi \in \Psi_u(M)/\text{Stab}_\Psi(\sigma)} \xi(\sigma \otimes \psi) d\psi,$$

where $\Psi_u(M)$ is the subgroup of $\Psi(M)$ consisting of unitary characters (Proposition 3.24).

Using the cuspidal property and parabolic descent, we lift the generalized pseudo-coefficient $\phi_{\sigma,\xi} \in C_c^\infty(M(F))$ to $f_{\sigma,\xi} \in C_c^\infty(G(F))$. They are related by the formula (Theorem 3.22)

$$|D_{G/M}(\gamma)|^{1/2} O_\gamma^{G(F)}(f_{\sigma,\xi}) = O_\gamma^{M(F)}(\overline{f_{\sigma,\xi}}^{(P)}) = O_\gamma^{M(F)}(|W(\theta, \theta)| \cdot \phi_{\sigma,\xi}),$$

for every elliptic regular semi-simple γ in $M(F)$, where $D_{G/M}(\gamma) = \det(1 - ad(\gamma))|_{\text{Lie}(G)/\text{Lie}(M)}$. Here we impose the condition that ξ be a $W(\theta, \theta)$ -invariant regular function on $\Omega(\sigma)$. We refer to Section 3.3 for unexplained notations. As an application, this lifting $f_{\sigma,\xi}$ turns out to simplify the Plancherel formula as in Proposition 3.26.

This paper is organized as follows. In Section 2, we recall background material and review basic facts on the Plancherel formula, unramified characters, and regular functions on $\Psi(M)$ and $\Omega(\sigma)$. In Section 3, we prove the existence of a generalized pseudo-coefficient and study its properties: cuspidality, a relation to the Harish-Chandra character function as well as ‘usual’ pseudo-coefficients. We also lift the generalized pseudo-coefficient in $C_c^\infty(M(F))$ to a function in $C_c^\infty(G(F))$ and apply the generalized pseudo-coefficient and the lift to the Plancherel formula.

2. Preliminaries. This section is devoted to basic notions, terminologies and known results. We mainly refer to [HR10], [HC73], [Kot86], [Sha90], [Shi10], and [Wal03].

2.1. Notation and convention. Let F be a p -adic field of characteristic 0, that is, a finite extension of \mathbb{Q}_p , where p is a prime number. Let \overline{F} be an algebraic closure of F . Denote by \mathcal{O}_F the ring of integers of F and by \mathfrak{p} the maximal ideal in \mathcal{O}_F . Let q denote the cardinality of the residue field $\mathcal{O}_F/\mathfrak{p}$ and $|\cdot|_F$ the normalized absolute value on F .

Let G be a connected reductive algebraic group over F . Write $G(F)$ for the group of F -points of G . For a semi-simple element $\gamma \in G(F)$, write G_γ for the centralizer in G of γ and G_γ° for its identity component. We say γ is regular if G_γ° is a maximal torus in G . Let $G(F)^{\text{reg}}$ denote the set of regular semi-simple elements in $G(F)$. We say γ is elliptic if the quotient G_γ°/Z_G° is anisotropic over F . Let $G(F)^{\text{ell}}$ denote the set of elliptic regular semi-simple elements in $G(F)$.

We denote by $C_c^\infty(G(F))$ the Hecke algebra of locally constant functions with compact support. Let $\text{Irr}(G(F))$ (resp., $\text{Irr}_u(G(F))$) be the set of equivalence classes of irreducible admissible (resp., unitary) representations of $G(F)$. By abuse of notation, we identify an equivalence class with its representative. Let $\Pi_{\text{temp}}(G(F))$ and $\Pi_{\text{disc}}(G(F))$ be the subsets of $\text{Irr}(G(F))$ consisting of irreducible tempered representations and discrete series representations of $G(F)$, respectively. Here, we say an admissible representation is a discrete series representation if it is in $\text{Irr}_u(G(F))$ and has non-zero matrix coefficients which are square-integrable modulo the center of $G(F)$. Since it is irreducible, if one non-zero matrix coefficient is square-integrable, then so are all the others.

Fix a minimal F -parabolic subgroup P_0 of G with its Levi decomposition M_0N_0 , where M_0 is a minimal F -Levi subgroup and N_0 is the unipotent radical of P_0 . Let A_0 be the split component of M_0 (or P_0), that is, the maximal F -split torus in the center of M_0 . Denote by $W_G = W(G, A_0) := N_G(A_0)/Z_G(A_0)$ the Weyl group of A_0 in G , where $N_G(A_0)$ and $Z_G(A_0)$ are the normalizer and the centralizer of A_0 in G , respectively. We denote by Φ the set of roots of G with respect to A_0 . Let Φ^+ be the set of positive roots. Note that the choice of P_0 determines the set Φ^+ of positive roots. Let $\Delta \subset \Phi^+$ denote the set of simple roots.

Let P be a standard F -parabolic subgroup (that is, $P \supseteq P_0$) with its Levi factor $M = M_\theta \supseteq M_0$ generated by a subset $\theta \subseteq \Delta$ and its unipotent radical $N \subseteq N_0$. Let A_M be the split component of M . Then we write Φ_θ for the subset of Φ consisting of \mathbb{Z} -linear combinations of the roots in θ , and Φ_θ^+ for $\Phi_\theta \cap \Phi^+$. Denote by $W_M = W(G, A_M) := N_G(A_M)/Z_G(A_M)$ the Weyl group of A_M in G . We denote by $P^- = MN^-$ the parabolic subgroup opposite to P , where N^- is opposite to N . For a standard F -Levi subgroup $L = L_\vartheta$, we set $W(\vartheta, \theta) := \{w \in W_G : w(\vartheta) = \theta\}$.

Let $X^*(M)_F$ be the group of F -rational characters of M . We denote by $\mathfrak{a}_M := \text{Hom}(X^*(M)_F, \mathbb{R}) = \text{Hom}(X^*(A_M)_F, \mathbb{R})$ the real Lie algebra of A_M . Let $\mathfrak{a}_{M, \mathbb{C}}^* := \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$ be the complex dual of \mathfrak{a}_M , where $\mathfrak{a}_M^* := X^*(A_M)_F \otimes_{\mathbb{Z}} \mathbb{R} = X^*(M)_F \otimes_{\mathbb{Z}} \mathbb{R}$ denotes the dual of \mathfrak{a}_M . We define the homomorphism $H_M : M(F) \rightarrow \mathfrak{a}_M$ by

$$q^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_F$$

for all $\chi \in X^*(M)_F$ and $m \in M(F)$. Fix a maximal special compact (parahoric) subgroup K of $G(F)$. Then we have the Iwasawa decomposition $G(F) = P(F)K$. Note that one can extend H_M to $G(F)$ using the Iwasawa decomposition by $q^{\langle \chi, H_M(mnk) \rangle} = |\chi(m)|_F$.

Let $\delta_P : M(F) \rightarrow \mathbb{R}_{>0}^\times$ be the modulus character of P defined by $m \mapsto |\det(\text{ad}(m))|_{\text{Lie}(P)/\text{Lie}(M)}|_F$. Note that $\delta_P(m) = q^{\langle 2\rho_P, H_M(m) \rangle}$, where $\rho_P := \frac{1}{2} \sum_{\alpha \in \Phi^+ \setminus \Phi_\theta^+} \alpha$. Moreover, δ_P can be extended to $G(F)$ in the same way as H_M .

Now we define the normalized induced representation and the intertwining operator. For $\sigma \in \text{Irr}(M(F))$ and $\nu \in \mathfrak{a}_{M, \mathbb{C}}^*$, we denote by $I(\nu, \sigma)$ the normalized induced representation

$$I(\nu, \sigma) = \text{Ind}_{P(F)}^{G(F)}(\sigma \otimes q^{\langle \nu, H_M(\cdot) \rangle} \otimes \mathbb{1}).$$

The space $V(\nu, \sigma)$ of $I(\nu, \sigma)$ consists of locally constant functions f from $G(F)$ into the representation space of σ such that $f(mng) = \sigma(m)q^{\langle \nu + \rho_P, H_M(m) \rangle} f(g)$, for $m \in M(F)$, $n \in N(F)$ and $g \in G(F)$. The group $G(F)$ acts on $V(\nu, \sigma)$ by the regular right action. We often write $i_{G, M}(\nu, \sigma)$ for $I(\nu, \sigma)$ in order to specify groups.

2.2. Plancherel formula. We review the Plancherel formula. We continue with the notation of Section 2.1. Fix a representative $w \in G$ of $\tilde{w} \in W_G$ such that $\tilde{w}(\theta) \subseteq \Delta$. For simplicity we often omit \sim for the representative (see [Sha90, p.280]). Denote $N_{\tilde{w}} := N_0 \cap wN^-w^{-1}$, that is, the subgroup of N_0 generated by the root groups $U_\alpha, \alpha \in \Phi^+$ such that $w^{-1}\alpha \in N^-$. We fix a Haar measure dn on $N_{\tilde{w}}$. Given $f \in V(\nu, \sigma)$, for $g \in G(F)$, the standard intertwining operator is defined as

$$A(\nu, \sigma, \tilde{w})f(g) = \int_{N_{\tilde{w}}(F)} f(w^{-1}ng)dn.$$

Note that the integral converges absolutely if $Re\langle \nu, \alpha^\vee \rangle \gg 0, \forall \alpha \in \Delta \setminus \theta$, where α^\vee is the coroot attached to α . Moreover, it turns out that $A(\nu, \sigma, \tilde{w}) : I(\nu, \sigma) \rightarrow I(\tilde{w}(\nu), \tilde{w}(\sigma))$, where $\tilde{w}(\sigma)(m_w) := \sigma(w^{-1}m_w w)$ for $m_w \in M_{\tilde{w}(\theta)}(F)$ with $M_{\tilde{w}(\theta)} = wMw^{-1}$. We set

$$\gamma_{\tilde{w}}(G|M) = \int_{\tilde{N}_{\tilde{w}}(F)} q^{(2\rho_P, H_M(\tilde{n}))} d\tilde{n},$$

where $\tilde{N}_{\tilde{w}} := w^{-1}N_{\tilde{w}}w = N^- \cap w^{-1}N_0w$ and $d\tilde{n}$ is the normalized Haar measure on $\tilde{N}_{\tilde{w}}(F)$ so that the measure of $K_0 \cap \tilde{N}_{\tilde{w}}(F)$ is one. Here, K_0 is a maximal special parahoric subgroup of $G(F)$.

We define the Plancherel measure, due to [HC73, Theorem 20], as follows.

DEFINITION 2.1. For $\nu \in \mathfrak{a}_{M, \mathbb{C}}^*, \sigma \in \text{Irr}_u(M(F))$ and $\tilde{w}(\theta) \subseteq \Delta$, the Plancherel measure attached to ν, σ and \tilde{w} for $i_{G, M}(\nu, \sigma)$ is defined as a non-zero complex number $\mu_M(\nu, \sigma, w)$ such that

$$A(\nu, \sigma, \tilde{w})A(\tilde{w}(\nu), \tilde{w}(\sigma), \tilde{w}^{-1}) = \mu_M(\nu, \sigma, w)^{-1} \gamma_{\tilde{w}}(G|M)^2.$$

The Plancherel measure $\mu_M(\nu, \sigma, w)$ depends only on ν, σ and \tilde{w} . It is independent of the choices of any Haar measure and of any representative w of \tilde{w} ([Sha90, p.280]).

EXAMPLE 2.2. Suppose $F = \mathbb{Q}_p, G = SL_2, M \simeq GL_1$ (the diagonal matrix) and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Denote by χ a unitary unramified character of $M(F)$. For $s \in \mathbb{C}$, we identify s with $s\tilde{\alpha} \in \mathfrak{a}_{M, \mathbb{C}}^* \simeq \mathbb{C}$, where $\tilde{\alpha} = \frac{1}{2}(e_1 - e_2)$. Then the Plancherel measure $\mu_M(s, \chi, w)$ attached to s, χ and w is

$$\left(1 + \frac{1}{p}\right)^2 \cdot \frac{1 - \chi(p)p^{-s}}{1 - \chi(p)p^{-s-1}} \cdot \frac{1 - \chi(p)^{-1}p^s}{1 - \chi(p)^{-1}p^{s-1}},$$

which lies in the field $\mathbb{C}(p^{-s})$ of rational functions in p^{-s} .

We now recall the Plancherel formula of Harish-Chandra. We refer to [HC84, Section 16] and [Wal03, Theorem VIII.1.1]. We normalize Haar measures as in [HC84, Section 5]. Fix a maximal special parahoric subgroup K_0 of $G(F)$, which is in a good relative position with M_0 (cf. [Sil79, Section 0.6]). For any closed subgroup $H(F)$ of $G(F)$ with a Haar measure dh on $H(F)$, we normalize dh so that the volume $\text{Vol}_{dh}(H(F) \cap K_0) = 1$. Then, Harish-Chandra's γ -factor $\gamma(G|M)$ and Harish-Chandra's c -factor $c(G|M)$ are defined as

$$\gamma(G|M) := \int_{N^-} q^{(2\rho_P, H_M(n^-))} dn^-,$$

$$c(G|M) := \gamma(G|M)^{-1} \prod_{\alpha \in \Phi(P, A_M)} \gamma(M_\alpha|M).$$

Set

$$a(G|M) := c(G|M)^{-2} \gamma(G|M)^{-1} |W_M|^{-1}.$$

For $\pi \in \Pi_{\text{disc}}(M(F))$, we denote by $d_M(\pi)$ the formal degree of π with respect to the Haar measure dm on $M(F)$ and by $\mu_M(\pi)$ the Plancherel measure $\mu_M(0, \pi, w)$ as in Definition 2.1. We write $d\pi$ for the measure at $\pi \in \Pi_{\text{disc}}(M(F))$ that is transferred from the normalized Haar measure $d\psi$ on the quotient $\Psi_u(M)/\text{Stab}_\Psi(\pi)$ (see Sections 2.3 and 2.4 for definitions) via the natural isomorphism

$$\Psi_u(M)/\text{Stab}_\Psi(\pi) \simeq \{\pi \otimes \psi : \psi \in \Psi_u(M)\}$$

(see [MT11, Section 5.4] and [HC84, Section 2]). The following proposition states the Plancherel formula.

THEOREM 2.3. (*Harish-Chandra, [HC84, Section 16]*) For any $f \in C_c^\infty(G(F))$,

$$f(1) = \sum_M a(G|M) \int_{\pi \in \Pi_{\text{disc}}(M(F))} d_M(\pi) \mu_M(\pi) \text{trace}(i_{G, M} \pi)(f) d\pi,$$

where the sum runs over all F -Levi subgroups M up to $G(F)$ -conjugacy. \square

2.3. Unramified characters. We introduce the group $\Psi(M)$ of unramified characters of $M(F)$. We continue with the notation in Sections 2.1 and 2.2. Let M be a standard F -Levi subgroup of G with split component A_M . We denote by Z_M the center of M . We write

$$M(F)^1 := \{m \in M(F) : |\chi(m)|_F = 1 \text{ for all characters } \chi \in X^*(M)_F\}.$$

We define the group of unramified characters $\Psi(M)$ on $M(F)$ as follows

$$\Psi(M) := \text{Hom}(M(F)/M(F)^1, \mathbb{C}^\times).$$

Denote by $\Psi_u(M)$ the subgroup of $\Psi(M)$ consisting of unitary unramified characters.

REMARK 2.4. We note the following facts (see [MT11, Section 3.3] and [Roc09, Section 1.4.1]).

- (i) $M(F)^1 = \ker(H_M)$.
- (ii) Every compact subgroup in $M(F)$ is contained in $M(F)^1$. In particular, $U(F)$ is contained in $L(F)^1$ for all standard F -parabolic subgroups Q of M with Levi decomposition $Q = LU$.
- (iii) $M(F)^1$ is an open, closed, normal and unimodular subgroup in $M(F)$.
- (iv) The quotient $M(F)/M(F)^1$ is isomorphic to \mathbb{Z}^d , where d is the dimension of A_M . In particular, if the center of M is anisotropic over F , then we have $M(F) = M(F)^1$.
- (v) The index $[M(F) : Z_M(F)M(F)^1]$ is finite. Since the quotient $Z_M(F)/(Z_M(F) \cap M(F)^1)$ is a full rank sublattice of $M(F)/M(F)^1$, the index $[M(F)/M(F)^1 : Z_M(F)/(Z_M(F) \cap M(F)^1)]$ is also finite.

REMARK 2.5. We fix a \mathbb{Z} -basis (m_1, m_2, \dots, m_d) for $M(F)/M(F)^1 \simeq \mathbb{Z}^d$. Then the map

$$\psi \mapsto (\psi(m_1), \psi(m_2), \dots, \psi(m_d))$$

yields the following isomorphisms

$$\Psi(M) \simeq (\mathbb{C}^\times)^d \text{ and } \Psi_u(M) \simeq (S^1)^d,$$

where S^1 is the unit circle in \mathbb{C}^\times (see [Roc09]).

EXAMPLE 2.6. Suppose $G = M = GL_m$. One has that $X^*(G)_F = \{\det^n : n \in \mathbb{Z}\} \simeq \mathbb{Z}$. We then have a short exact sequence

$$1 \longrightarrow G^1(F) \longrightarrow G(F) \xrightarrow{|\det|_F} q^{\mathbb{Z}} \longrightarrow 1.$$

We get $G(F)/G^1(F) \simeq \text{Image}(|\det|_F) = \{q^k : k \in \mathbb{Z}\} \simeq \mathbb{Z}$. Thus we have $\Psi(G) \simeq \text{Hom}(\mathbb{Z}, \mathbb{C}^\times) \simeq \mathbb{C}^\times$ and $\Psi_u(G) \simeq \text{Hom}(\mathbb{Z}, S^1) \simeq S^1$. In particular, we note that

$$\Psi(G) = \{g \mapsto |\det(g)|_F^s : s \in \mathbb{C}\} \simeq \mathfrak{a}_{G, \mathbb{C}}^* / \mathcal{L}_G,$$

$$\Psi_u(G) = \{g \mapsto |\det(g)|_F^s : s \in \sqrt{-1}\mathbb{R}\} \simeq \sqrt{-1}\mathfrak{a}_G^* / \mathcal{L}_G.$$

Here $\mathcal{L}_G := \frac{2\pi\sqrt{-1}}{\log(q)} X^*(G)_F$ is a lattice in $\sqrt{-1}\mathfrak{a}_G^*$.

REMARK 2.7 (Epimorphism from $\mathfrak{a}_{M, \mathbb{C}}^*$ onto $\Psi(M)$). There is a natural homomorphism $\nu \mapsto \psi_\nu$ from the \mathbb{C} -vector space $\mathfrak{a}_{M, \mathbb{C}}^*$ to the group $\Psi(M)$ of unramified characters defined by

$$\psi_\nu(m) := q^{\langle \nu, H_M(m) \rangle}.$$

We thus have the following properties (see [MT11, Section 3.3]).

- (a) $\psi_\nu \in \Psi_u(M)$ (that is, ψ_ν is unitary) if and only if $\nu \in \sqrt{-1}\mathfrak{a}_M^*$.
- (b) ψ_ν is trivial if and only if $\nu \in \frac{2\pi\sqrt{-1}}{\log(q)} X^*(M)_F \subset \sqrt{-1}\mathfrak{a}_M^*$. Notice that $\mathcal{L}_M := \frac{2\pi\sqrt{-1}}{\log(q)} X^*(M)_F$ is a lattice in $\sqrt{-1}\mathfrak{a}_M^*$.
- (c) The homomorphism $\nu \mapsto \psi_\nu$ is surjective. Hence, we have natural isomorphisms

$$\mathfrak{a}_{M, \mathbb{C}}^* / \mathcal{L}_M \simeq \Psi(M) \text{ and } \sqrt{-1}\mathfrak{a}_M^* / \mathcal{L}_M \simeq \Psi_u(M).$$

REMARK 2.8. Let $\nu \in \mathfrak{a}_{M, \mathbb{C}}^*$, $\sigma \in \text{Irr}_u(M(F))$ and $\tilde{w} \in W_G$ be given as in Definition 2.1. We observe that if $\nu \in \mathcal{L}_M$, then we have $\mu_M(\nu, \sigma, w) = \mu_M(0, \sigma, w)$. Hence, given $\psi \in \Psi(M)$, we write $\mu_M(\sigma \otimes \psi)$ for $\mu_M(\nu, \sigma, w)$, where $\nu \in \mathfrak{a}_{M, \mathbb{C}}^*$ is obtained by letting $\psi_\nu = \psi$ in the above natural isomorphism $\nu \mapsto \psi_\nu$ from $\mathfrak{a}_{M, \mathbb{C}}^* / \mathcal{L}_M$ onto $\Psi(M)$.

2.4. Regular functions on $\Psi(M)$ and $\Omega(\sigma)$. We discuss a regular function (resp., a rational function) on $\Psi(M)$, based on the notions and facts in [Wal03, Sections III - V]. We continue with the notation as in Sections 2.2 and 2.3. We fix a \mathbb{Z} -basis (m_1, m_2, \dots, m_d) for $M(F)/M(F)^1 \simeq \mathbb{Z}^d$. Then $\Psi(M)$ is an abelian complex algebraic group that is isomorphic to $(\mathbb{C}^\times)^d$ via

$$\lambda(\psi) = (\psi(m_1), \psi(m_2), \dots, \psi(m_d)).$$

Here d is the dimension of A_M . Note that $\mathbb{C}[z_1, z_1^{-1}, \dots, z_d, z_d^{-1}]$ is the set of all regular functions on $(\mathbb{C}^\times)^d$.

DEFINITION 2.9. A function $\xi : \Psi(M) \rightarrow \mathbb{C}$ is said to be a *regular function* (resp., a *rational function*) on $\Psi(M)$ if there exists a regular (resp., rational) function $p(z)$ on $(\mathbb{C}^\times)^d$ such that $\xi(\psi) = p(\lambda(\psi))$ for all $\psi \in \Psi(M)$. Here $z = (z_1, \dots, z_d)$.

REMARK 2.10. ([Wal03, Section IV.3 and Lemma V.2.1] and [HC73, Theorem 20]) Let $\sigma \in \text{Irr}(M(F))$ be given. The Plancherel measure $\mu_M(\sigma \otimes \psi)$ (see Remark 2.8) is a rational function on $\Psi(M)$. Moreover, it is a non-negative and regular function on $\psi \in \Psi_u$.

REMARK 2.11. There is a natural action of W_M on the set of regular functions on $\Psi(M)$ as follows. Given a regular function ξ on $\Psi(M)$, there exists a regular function $p(z)$ on $(\mathbb{C}^\times)^d$ such that $\xi(\psi) = p(\lambda(\psi))$ for all $\psi \in \Psi(M)$. We define

$${}^w\xi(\psi) = \xi({}^w\psi).$$

Here, the action of the Weyl group W_M on $\Psi(M)$ is defined by ${}^w\psi(m) = \psi(w^{-1}mw)$, for $w \in W_M$, $\psi \in \Psi(M)$, and $m \in M(F)$. It then turns out that ${}^w\xi$ is again a regular function on $\Psi(M)$, since ${}^w\xi(\psi) = {}^wp(\lambda(\psi))$, where ${}^wp(z) := p(\lambda({}^w(\lambda^{-1}(z))))$ is a regular function on $(\mathbb{C}^\times)^d$.

Next, we introduce a regular function (resp., a rational function) on the $\Psi(M)$ -orbit

$$\Omega(\sigma) := \{\sigma \otimes \psi : \psi \in \Psi(M)\}$$

of $\sigma \in \text{Irr}(M(F))$. Set

$$\text{Stab}_\Psi(\sigma) := \{\psi \in \Psi(M) : \sigma \otimes \psi \simeq \sigma\}$$

to be the stabilizer of σ in $\Psi(M)$.

REMARK 2.12. We note that $\text{Stab}_\Psi(\sigma)$ is a finite subgroup in $\Psi_u(M)$ (see [Roc09, Section 1.4.1]) and thus a finite subgroup in $(S^1)^d$, where d is the integer with $\Psi_u(M) \simeq (\mathbb{C}^\times)^d$. There is also an isomorphism $\Omega(\sigma) \simeq \Psi(M)/\text{Stab}_\Psi(\sigma)$, whence $\Omega(\sigma)$ is naturally equipped with a complex (quotient) variety structure.

DEFINITION 2.13. A function $\xi : \Omega(\sigma) \rightarrow \mathbb{C}$ is said to be a *regular function* (resp., a *rational function*) on $\Omega(\sigma)$ if the function $\psi \mapsto \xi(\sigma \otimes \psi)$ is a regular function (resp., a rational function) on $\Psi(M)$.

REMARK 2.14. For a given subgroup $H \subset \Psi(M)$, a regular (resp., rational) function ξ on $\Psi(M)$ is said to be *invariant under H* if $\xi(\psi) = \xi(\psi')$ for all $\psi, \psi' \in H$.

We consider an isomorphism $\Omega(\sigma) \simeq \Psi(M)/\text{Stab}_\Psi(\sigma)$. Then, given a regular (resp., rational) function ξ_σ on $\Psi(M)$ which is invariant under $\text{Stab}_\Psi(\sigma)$, we set

$$\xi(z) := \xi_\sigma(\psi)$$

for $z = \sigma \otimes \psi \in \Omega(\sigma)$. It then turns out that ξ is a regular function (resp., a rational function) on $\Omega(\sigma)$.

Here are some examples of regular functions on $\Psi(M)$ and $\Omega(\sigma)$.

EXAMPLE 2.15. Let $\sigma \in \text{Irr}_u(M(F))$ be given. We denote by m the cardinality of the finite group $\text{Stab}_\Psi(\sigma)$. For any polynomial $p(z) \in \mathbb{C}[z_1^m, z_1^{-m}, \dots, z_d^m, z_d^{-m}]$, we set $\xi_\sigma(\psi) = p(\lambda(\psi))$ for all $\psi \in \Psi(M)$. Then ξ_σ is a regular function on $\Psi(M)$ that is invariant under $\text{Stab}_\Psi(\sigma)$, since $\chi^m = 1$ for any $\chi \in \text{Stab}_\Psi(\sigma)$. Hence, a function

$$\xi(z) = \xi(\sigma \otimes \psi) := \xi_\sigma(\psi)$$

on $\Omega(\sigma)$ is regular.

REMARK 2.16. Let ξ_σ be a regular function on $\Psi(M)$. For any $w \in W_M$, Remark 2.11 implies that ${}^w\xi_\sigma$ is again a regular function on $\Psi(M)$. Define

$$\xi(z) := \frac{1}{|W_M|} \sum_{w \in W_M} {}^w\xi_\sigma(z). \tag{2.1}$$

Then $\xi(z)$ turns out to be a W_M -invariant regular function on $\Psi(M)$, that is, ${}^w\xi = \xi$ for any $w \in W_M$. Therefore, there is always a W_M -invariant regular function on $\Psi(M)$. In fact, this holds for any subgroup of W_M by replacing W_M in (2.1) with the subgroup.

3. Generalized pseudo-coefficients. We continue with the notation of Section 2. Let G be a connected reductive algebraic group over a p -adic field F of characteristic 0, and let M be a standard F -Levi subgroup (possibly, $M = G$) of G with split component A_M . In this section, for any $\sigma \in \Pi_{\text{disc}}(M(F))$ and regular function ξ on the $\Psi(M)$ -orbit $\Omega(\sigma)$ of σ , we obtain a function $\phi_{\sigma,\xi} \in C_c^\infty(M(F))$ satisfying a certain trace relation (Theorem 3.6). We then explore several properties of $\phi_{\sigma,\xi}$ and lift $\phi_{\sigma,\xi} \in C_c^\infty(M(F))$ to $f_{\sigma,\xi} \in C_c^\infty(G(F))$. Finally, we apply the Plancherel formula to $\phi_{\sigma,\xi}$ and $f_{\sigma,\xi}$.

Following the notation of [BDK86], we denote by $R(M)$ the Grothendieck group of smooth representations of $M(F)$ ($M(F)$ -module) of finite length. Note that $R(M)$ is a free abelian group with basis $\text{Irr}(M(F))$. For a standard F -parabolic subgroup $Q = LU$ of M , the normalized (twisted by $\delta_Q^{1/2}$) induced representation $i_{M,L}(\pi)$ from $\pi \in \text{Irr}(L(F))$ defines the morphism $i_{M,L} : R(L) \rightarrow R(M)$. Here δ_Q is the modulus character of $Q(F)$. A cuspidal pair is defined by (L, ρ) , where L is a standard F -Levi subgroup of M and $\rho \in \text{Irr}(L(F))$ is an irreducible supercuspidal $L(F)$ -module. We denote by $\Theta(M)$ the set of all cuspidal pairs up to $M(F)$ -conjugation. We say (L, ρ) and (L', ρ') are $M(F)$ -conjugate if there exists $m \in M(F)$ such that $L' = mLm^{-1}$ and $\rho' = m\rho m^{-1}$.

Given a cuspidal pair (L, ρ) , the set $\{(L, \psi\rho) \mid \psi \in \Psi(L)\}$ is called a connected component of $\Theta(M)$. For each $\pi \in \text{Irr}(M(F))$, there exists a cuspidal pair (L, ρ) unique up to conjugation by M such that π is a sub-quotient of $i_{M,L}(\rho)$. By sending π to (L, ρ) , we define a map $\text{inf.ch.} : \text{Irr}(M(F)) \rightarrow \Theta(M)$. The map inf.ch. is $\Psi(M)$ -equivariant, onto, and finite-to-one. Given a connected component Θ of $\Theta(M)$, the inverse image $\text{inf.ch.}^{-1}(\Theta)$ is called a connected component of $\text{Irr}(M(F))$.

3.1. Construction of a generalized pseudo-coefficient. We recall the trace Paley-Wiener theorem from [BDK86, Section 1.2] and [Clo86, Section 4.2].

DEFINITION 3.1. A linear functional (group homomorphism) $\lambda : R(M) \rightarrow \mathbb{C}$ is called *good* if

- (i) the function $\lambda : \text{Irr}(M(F)) \rightarrow \mathbb{C}$ is supported on a finite number of components;
- (ii) for any standard F -Levi subgroup L and $\pi \in R(M)$, the function $\psi \mapsto \lambda(i_{M,L}(\pi \otimes \psi))$ is a regular function on the complex variety $\Psi(L)$.

PROPOSITION 3.2 (Bernstein-Deligne-Kazhdan). *Let λ be as in Definition 3.1. λ is good if and only if there is a function $h \in C_c^\infty(M(F))$ such that for any $\pi \in R(M)$*

$$\lambda(\pi) = \text{trace}(\pi)(h).$$

REMARK 3.3. Instead of (i) in Definition 3.1, it is customary to use the condition that there exists an open compact subgroup $K_M \subset M(F)$ such that λ is non-zero only on the $M(F)$ -modules having nontrivial K_M -invariant vectors. We note that both are equivalent by Decomposition Theorem in [BDK86, Section 2.3].

DEFINITION 3.4. We say that the normalized induced representation $i_{M,L}(\pi)$ is a *standard module* if $\pi \simeq \tau \otimes q^{\langle \nu, H_L(\cdot) \rangle}$ for some $\tau \in \Pi_{\text{temp}}(L(F))$ and $\nu \in (\mathfrak{a}_L^*)^+ := \{ \nu \in \mathfrak{a}_L^* : \langle \alpha^\vee, \nu \rangle > 0, \forall \alpha \in \Delta_M \setminus \Delta_L \}$ (the positive Weyl chamber), where Δ_M and Δ_L are the sets of simple roots in M and L respectively (see [Clo86, p.279] where it is called a standard character).

PROPOSITION 3.5. ([Clo86, Proposition 2 p.279]) *The standard modules form a basis over \mathbb{Z} for the Grothendieck group $R(M)$.*

THEOREM 3.6 (Existence of generalized pseudo-coefficients). *For any $\sigma \in \Pi_{\text{disc}}(M(F))$ and any regular function ξ on $\Omega(\sigma)$, there exists a function $\phi_{\sigma,\xi} \in C_c^\infty(M(F))$ such that for all $\pi \in \Pi_{\text{temp}}(M(F))$,*

$$\text{trace}(\pi)(\phi_{\sigma,\xi}) = \begin{cases} \xi(\pi), & \text{if } \pi \simeq \sigma \otimes \psi \text{ for some } \psi \in \Psi(M), \\ 0, & \text{otherwise.} \end{cases}$$

We call the function $\phi_{\sigma,\xi} \in C_c^\infty(M(F))$ a *generalized pseudo-coefficient* for σ and ξ .

REMARK 3.7 (Uniqueness of generalized pseudo-coefficients). For given a regular function ξ on $\Omega(\sigma)$, the function $\phi_{\sigma,\xi}$ is determined *uniquely modulo the \mathbb{C} -subspace J_M* spanned by the functions of the form $m \mapsto (h(m) - h(xmx^{-1}))$ for all $h \in C_c^\infty(M(F))$ and $x \in M(F)$. This follows from the fact ([Kaz86, Theorem 0]) that

$$\phi_1 - \phi_2 \in J_M \iff \text{trace}(\pi)(\phi_1) = \text{trace}(\pi)(\phi_2), \quad \forall \pi \in \Pi_{\text{temp}}(M(F)).$$

REMARK 3.8. We note that, since $\pi \in \Pi_{\text{temp}}(M(F))$ is unitary, the unramified character ψ satisfying $\pi \simeq \sigma \otimes \psi$ must be unitary, that is, in $\Psi_u(M)$.

Proof of Theorem 3.6. We refer to Clozel’s proof of [Clo86, Proposition 1]. Given a regular function ξ on $\Omega(\sigma)$, we define a linear functional $\lambda : R(M) \rightarrow \mathbb{C}$ such that

$$\lambda(\pi) = \begin{cases} \xi(\pi), & \text{if } \pi \simeq \sigma \otimes \psi \text{ for some } \psi \in \Psi(M), \\ 0, & \text{if } \pi \text{ is a standard module but not in } \Omega(\sigma). \end{cases}$$

Since essentially square-integrable representations can never be realized by an induced representation from a proper F -Levi subgroup and by Proposition 3.5, the linear functional λ is well-defined.

Now we show that λ is *good* (see Proposition 3.2). If the image $\text{inf.ch.}(\sigma)$ is (L, ρ) for some standard F -Levi subgroup L of M and a supercuspidal representation ρ of $L(F)$, then inf.ch. maps $\Omega(\sigma)$ into the connected component Θ containing (L, ρ) . Indeed, since the map inf.ch. is $\Psi(M)$ -equivariant, we have $\text{inf.ch.}(\Psi(M)\sigma) = (L, \Psi(M)\rho) \subset (L, \Psi(L)\rho) = \Theta$. It follows that the linear functional is supported only on Θ , so the condition (i) of Definition 3.1 is fulfilled.

For the condition (ii) of Definition 3.1, it suffices to show that λ vanishes on any induced representation $i_{M,L}(\tau)$ with proper standard F -Levi subgroup L of M and $\tau \in R(L)$. By Proposition 3.5, we may assume that τ is a standard module $i_{L,L_1}(\varsigma)$, where L_1 is a standard F -Levi subgroup of L and $\varsigma \simeq \varsigma_1 \otimes q^{(\nu_1, H_{L_1}(\cdot))}$ for some $\varsigma_1 \in \Pi_{\text{temp}}(L_1(F))$ and $\nu_1 \in (\mathfrak{a}_{L_1}^*)^+$. Then $i_{M,L}(\tau)$ is isomorphic to $i_{M,L_1}(\varsigma)$ which is a standard module but not an essentially square-integrable representation (since L_1 is proper), that is, $i_{M,L}(\tau) \notin \Omega(\sigma)$. It turns out that $\lambda(i_{M,L}(\tau)) = 0$ for any proper standard F -Levi subgroup L of M and any $\tau \in R(L)$. Hence, the condition (ii) also follows, and λ is *good*. Due to the trace Paley-Wiener theorem (Proposition 3.2), the proof is complete. \square

3.2. Properties of $\phi_{\sigma,\xi}$. We provide several properties of the generalized pseudo-coefficient $\phi_{\sigma,\xi}$ for σ and ξ defined in Theorem 3.6. Denote by ω a unitary character of $Z_M(F)$, where Z_M is the center of M . By $C_c^\infty(M(F), \omega^{-1})$ we denote the Hecke algebra of locally constant functions with compact support modulo the center and transforming under the center by ω^{-1} . Let $\Pi_{\text{temp}}(M(F), \omega)$ and $\Pi_{\text{disc}}(M(F), \omega)$ be the sets of equivalent classes of irreducible tempered representations and discrete series representations of $M(F)$, respectively, whose central character is ω .

Let $\sigma \in \Pi_{\text{disc}}(M(F), \omega)$ be given. The projection $\phi_{\sigma,\xi}^\omega$ of $\phi_{\sigma,\xi} \in C_c^\infty(M(F))$ onto $C_c^\infty(M(F), \omega^{-1})$ is defined by

$$\phi_{\sigma,\xi}^\omega(m) := \int_{Z_M(F)} \omega(z)\phi_{\sigma,\xi}(zm)dz, \tag{3.1}$$

where dz is a Haar measure on $Z_M(F)$.

Given $h \in C_c^\infty(M(F))$, the orbital integral $O_\gamma^{M(F)}(h)$ of h for $\gamma \in M(F)$ is defined by

$$O_\gamma^{M(F)}(h) := \int_{M_\gamma^\circ(F) \backslash M(F)} h(x^{-1}\gamma x)dx, \tag{3.2}$$

where dx is a quotient Haar measure on the quotient $M_\gamma^\circ(F) \backslash M(F)$. Given a character ω of $Z_M(F)$, we define $O_\gamma^{M(F)}(h^\omega)$ for $h^\omega \in C_c^\infty(M(F), \omega^{-1})$ in the same way.

We present the cuspidal properties of $\phi_{\sigma,\xi}$ and $\phi_{\sigma,\xi}^\omega$.

PROPOSITION 3.9 (Cuspidality of $\phi_{\sigma,\xi}$ and $\phi_{\sigma,\xi}^\omega$). *With the function $\phi_{\sigma,\xi} \in C_c^\infty(M(F))$ in Theorem 3.6 and its projection $\phi_{\sigma,\xi}^\omega$ defined in (3.1), we have*

$$O_\gamma^{M(F)}(\phi_{\sigma,\xi}) = 0 = O_\gamma^{M(F)}(\phi_{\sigma,\xi}^\omega)$$

for any $\gamma \in M(F)^{\text{reg}} \setminus M(F)^{\text{ell}}$.

Proof. We first notice from Definition (3.1) that

$$O_\gamma^{M(F)}(\phi_{\sigma,\xi}^\omega) = \int_{Z_M(F)} \omega(z) O_{z\gamma}^{M(F)}(\phi_{\sigma,\xi}) dz.$$

It is sufficient to show that for every $\gamma \in M(F)^{\text{reg}} \setminus M(F)^{\text{ell}}$,

$$O_\gamma^{M(F)}(\phi_{\sigma,\xi}) = 0.$$

Let $\gamma \in M(F)^{\text{reg}} \setminus M(F)^{\text{ell}}$ be given. Then there exists a proper standard F -Levi subgroup L of M containing γ . We denote by Q the standard F -parabolic subgroup in M with its Levi factor L . We consider the following parabolic descent $\bar{\phi}_{\sigma,\xi}^{(Q)}$ of $\phi_{\sigma,\xi}$ which is due to Harish-Chandra (cf. [vD72, p.236] and [Kot05, Section 13]). By [vD72, Lemma 9], we have

$$|D_{M/L}(\gamma)|^{1/2} O_\gamma^{M(F)}(\phi_{\sigma,\xi}) = O_\gamma^{L(F)}(\bar{\phi}_{\sigma,\xi}^{(Q)}), \tag{3.3}$$

where $D_{M/L}(\gamma) = \det(1 - ad(\gamma))|_{\text{Lie}(M)/\text{Lie}(L)}$. We claim that

$$O_\gamma^{L(F)}(\bar{\phi}_{\sigma,\xi}^{(Q)}) = 0$$

for all $\gamma \in L(F)$. It is well-known (cf. [vD72, p.237] and [Kot05, Section 13.7]) that

$$\text{trace}(\pi)(\bar{\phi}_{\sigma,\xi}^{(Q)}) = \text{trace}(i_{M,L}(\pi))(\phi_{\sigma,\xi})$$

for any $\pi \in \text{Irr}(L(F))$. Since L is proper and the orbit $\Omega(\sigma)$ consists only of essentially square-integrable representations, $i_{M,L}(\pi)$ cannot be in $\Omega(\sigma)$. Theorem 3.6 implies that

$$\text{trace}(i_{M,L}(\pi))(\phi_{\sigma,\xi}) = 0 \tag{3.4}$$

for any $\pi \in \text{Irr}(L(F))$. It follows that $\text{trace}(\pi)(\bar{\phi}_{\sigma,\xi}^{(Q)}) = 0$. Therefore, the claim is true from the fact ([Kaz86, Theorem 0]) that

$$\text{trace}(\pi)(\bar{\phi}_{\sigma,\xi}^{(Q)}) = 0 \quad \text{for all } \pi \in \Pi_{\text{temp}}(L(F))$$

if and only if

$$O_\gamma^{L(F)}(\bar{\phi}_{\sigma,\xi}^{(Q)}) = 0 \quad \text{for all } \gamma \in M(F)^{\text{reg}}.$$

Since $D_{M/L}$ is never zero for any element in $M(F)^{\text{reg}}$, the proof is complete. \square

REMARK 3.10. Proposition 3.9 is called the Selberg principle (cf. [HC70, Lemma 45]). Furthermore, it turns out that $\phi_{\sigma,\xi}$ is a cuspidal function in the sense of Arthur (see [Art93, p.130]).

We give a relationship between the orbital integral of $\phi_{\sigma,\xi}^\omega$ and the Harish-Chandra character Θ_σ (defined below) over the elliptic regular semi-simple set $M(F)^{\text{ell}}$.

DEFINITION 3.11. (Harish-Chandra, [HC81, p.96]) For any $\pi \in \text{Irr}(M(F))$, there exists a unique locally constant function on $M(F)^{\text{reg}}$ which is invariant under conjugation by $M(F)$ such that for all $h \in C_c^\infty(M(F))$,

$$\text{trace}(\pi)(h) = \int_{M(F)^{\text{reg}}} h(m) \Theta_\pi(m) dm,$$

with dm a Haar measure on $M(F)^{\text{reg}}$. We call Θ_π , the Harish-Chandra character of σ .

We note that the distribution character $\text{trace}(\pi)$ is dependent upon the choice of Haar measure dm , but the Harish-Chandra character Θ_π is not. Furthermore, Definition 3.11 stems from a theorem of Harish-Chandra, which proves the existence and uniqueness (see [HC99] for details).

We present the following lemma to state Proposition 3.13.

LEMMA 3.12. *For any $\sigma \in \Pi_{\text{disc}}(M(F), \omega)$, the set $\Omega(\sigma) \cap \Pi_{\text{disc}}(M(F), \omega)$ is finite.*

Proof. Let $\pi \in \Pi_{\text{disc}}(M(F), \omega)$ be given such that $\pi \simeq \sigma \otimes \psi$ for some $\psi \in \Psi(M)$. Since both central characters of σ and π are the same as ω , the restriction $\psi|_{Z_M(F)}$ of ψ to $Z_M(F)$ is trivial. The set $\Omega(\sigma) \cap \Pi_{\text{disc}}(M(F), \omega)$ is a subset of

$$\{\psi \in \Psi_u(M) : \psi|_{Z_M(F)} = \mathbb{1}\}. \tag{3.5}$$

Note that ψ is already trivial on $M(F)^1$. Since the index $[M(F) : Z_M(F)M(F)^1]$ is finite from Remark 2.4, there are only finitely many characters on $M(F)$ whose restriction to $Z_M(F)M(F)^1$ is trivial. Thus the set (3.5) is finite, which implies that the set $\Omega(\sigma) \cap \Pi_{\text{disc}}(M(F), \omega)$ is finite. \square

PROPOSITION 3.13 (Relationship between $\phi_{\sigma, \xi}$ and Θ_σ). *Let $\sigma \in \Pi_{\text{disc}}(M(F), \omega)$ be given, and let ξ be a regular function on $\Omega(\sigma)$. Suppose that $\xi(z)$ vanishes on $\Omega(\sigma) \cap \Pi_{\text{disc}}(M(F), \omega)$ unless $z \simeq \sigma$. Then we have*

$$O_\gamma^{M(F)}(\phi_{\sigma, \xi}^\omega) = \xi(\sigma)\overline{\Theta_\sigma(\gamma)} \tag{3.6}$$

for every $\gamma \in M(F)^{\text{ell}}$. Here the bar $\bar{}$ means the complex conjugation.

REMARK 3.14. The orbital integral in the left-hand side is computed with respect to the Haar measure dx in Definition (3.2). Changing dx to $c dx$ for some non-zero constant c , we then get $c^{-1} \phi_{\sigma, \xi}$ from Proposition 3.6 in place of $\phi_{\sigma, \xi}$. Thus the left-hand side is independent of choice of Haar measure, as the right-hand side should be. Note that the right-hand side depends only on ξ and σ .

Proof of Proposition 3.13. For any matrix coefficient h_σ^ω of σ , from the proof of [Clo91, Proposition 6], there exists $\widetilde{h}_\sigma^\omega \in C_c^\infty(M(F), \omega^{-1})$ such that

$$O_\gamma^{M(F)}(\widetilde{h}_\sigma^\omega) = O_\gamma^{M(F)}(h_\sigma^\omega), \quad \forall \gamma \in M(F). \tag{3.7}$$

We recall the Weyl integration formula (cf. [DKV84, A.3.f] and [Kot05, Section 7]). For any function $h^\omega \in C_c^\infty(M(F), \omega^{-1})$ and any $\pi \in \Pi_{\text{temp}}(M(F), \omega)$,

$$\text{trace}(\pi)(h^\omega) = \sum_T |W_T|^{-1} \int_{Z_M(F) \backslash T(F)^{\text{reg}}} |D_M(t)| O_\gamma^{M(F)}(h^\omega) \Theta_\pi(t) dt,$$

where the sum is taken over $M(F)$ -conjugacy classes of maximal F -tori T in M , $W_T := N_M(T)/Z_M(T)$, $T(F)^{\text{reg}} := M(F)^{\text{reg}} \cap T(F)$, $D_M(t) := \det(1 - \text{ad}(t))|_{\text{Lie}(M)/\text{Lie}(T)}$, and dt is a Haar measure on $Z_M(F) \backslash T(F)^{\text{reg}}$. From (3.7) and the Weyl integration formula, we deduce that

$$\text{trace}(\pi)(\widetilde{h}_\sigma^\omega) = \text{trace}(\pi)(h_\sigma^\omega) \tag{3.8}$$

for any $\pi \in \Pi_{\text{temp}}(M(F), \omega)$. Moreover, it is well-known [DKV84, Proposition A.3.g] that

$$\text{trace}(\pi)(h_\sigma^\omega) = \begin{cases} \frac{h_\sigma^\omega(1)}{d_M(\sigma)}, & \text{if } \pi \simeq \sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Here $d_M(\sigma)$ is the formal degree. We also recall from [DKV84, Proposition A.3.e.2)] that for all $\gamma \in M(F)^{\text{reg}}$, we have

$$O_\gamma^{M(F)}(h_\sigma^\omega) = \begin{cases} \frac{h_\sigma^\omega(1)}{d_M(\sigma)} \overline{\Theta_\sigma(\gamma)}, & \text{if } \gamma \in M(F)^{\text{ell}}, \\ 0, & \text{if } \gamma \in M(F)^{\text{reg}} \setminus M(F)^{\text{ell}}. \end{cases} \tag{3.10}$$

For any given regular function ξ on $\Omega(\sigma)$, we choose a matrix coefficient h_σ^ω of σ such that

$$h_\sigma^\omega(1) = \xi(\sigma)d_M(\sigma). \tag{3.11}$$

It then follows from (3.7) and (3.10) that

$$O_\gamma^{M(F)}(\widetilde{h_\sigma^\omega}) = \xi(\sigma)\overline{\Theta_\sigma(\gamma)}$$

for all $\gamma \in M(F)^{\text{ell}}$. We also have the following.

LEMMA 3.15. *For any $\pi \in \Pi_{\text{temp}}(M(F), \omega)$,*

$$\text{trace}(\pi)(\phi_{\sigma, \xi}^\omega) = \text{trace}(\pi)(\widetilde{h_\sigma^\omega}).$$

Proof. [Proof of Lemma 3.15] Since $\xi(z)$ vanishes on $\Omega(\sigma) \cap \Pi_{\text{disc}}(M(F), \omega)$ unless $z \simeq \sigma$, one sees from Theorem 3.6 that for any $\pi \in \Pi_{\text{temp}}(M(F), \omega)$,

$$\text{trace}(\pi)(\phi_{\sigma, \xi}^\omega) = \begin{cases} \xi(\sigma), & \text{if } \pi \simeq \sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

Therefore, since $h_\sigma^\omega(1) = \xi(\sigma)d_M(\sigma)$, the lemma follows from (3.8) and (3.9). \square

Continuing with the proof of Proposition 3.13, for given $h^\omega \in C_c^\infty(M(F), \omega^{-1})$, it is well-known that

$$\text{trace}(\pi)(h^\omega) = 0, \forall \pi \in \Pi_{\text{temp}}(M(F), \omega) \implies O_\gamma^{M(F)}(h^\omega) = 0, \forall \gamma \in M(F)^{\text{reg}}$$

(see [DKV84, A.2.b]). From Lemma 3.15 we have

$$O_\gamma^{M(F)}(\phi_{\sigma, \xi}^\omega) = O_\gamma^{M(F)}(\widetilde{h_\sigma^\omega})$$

for all $\gamma \in M(F)^{\text{reg}}$. Hence, the proof of Proposition 3.13 is complete. \square

REMARK 3.16. From (3.11), we note that the value $\xi(\sigma)$ in (3.6) must be determined by both $h_\sigma^\omega(1)$ and $d_M(\sigma)$.

We provide a relationship between a generalized pseudo-coefficient and a pseudo-coefficient of σ .

DEFINITION 3.17. ([DKV84, A.4]) For $\delta \in \Pi_{\text{disc}}(M(F), \omega)$, we say a function $\phi_\delta^\omega \in C_c^\infty(M(F), \omega^{-1})$ is a *pseudo-coefficient* of δ if for $\pi \in \Pi_{\text{temp}}(M(F), \omega)$,

$$\text{trace}(\pi)(\phi_\delta^\omega) = \begin{cases} 1, & \text{if } \pi \simeq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 3.18. ([Clo91, Propositions 5(ii) and 6]) For $\delta \in \Pi_{\text{disc}}(M(F), \omega)$, there exists a pseudo-coefficient $\phi_\delta^\omega \in C_c^\infty(M(F), \omega^{-1})$ satisfying:

$$O_\gamma^{M(F)}(\phi_\delta^\omega) = \overline{\Theta_\delta(\gamma)}, \quad \forall \gamma \in M(F)^{\text{ell}}.$$

Proof. By letting the matrix coefficient $\phi_\delta^\omega(1) = d_M(\delta)$ (the formal degree of δ) in the proof of [Clo91, Proposition 6], we have a pseudo-coefficient ϕ_δ^ω whose orbital integral is the same as that of the matrix coefficient h_δ^ω in the proof of Proposition 3.13. Then, [Clo91, Proposition 5(ii)] implies Proposition 3.18. \square

PROPOSITION 3.19. Let ξ be as in Proposition 3.13. Suppose $\xi(\sigma) = 1$. Then, the projection $\phi_{\sigma, \xi}^\omega \in C_c^\infty(M(F), \omega^{-1})$ is a pseudo-coefficient of σ .

Proof. This is a consequence of (3.12) with $\xi(\sigma) = 1$. \square

3.3. Lifting $\phi_{\sigma, \xi} \in C_c^\infty(M(F))$ to $f_{\sigma, \xi} \in C_c^\infty(G(F))$. We lift the generalized pseudo-coefficient $\phi_{\sigma, \xi} \in C_c^\infty(M(F))$ to $f_{\sigma, \xi} \in C_c^\infty(G(F))$ and discuss a relationship between $\phi_{\sigma, \xi}$ and $f_{\sigma, \xi}$. Following the notation of [BDK86], given a standard F -parabolic subgroup $P = MN$ of G , the normalized (twisted by $\delta_P^{-1/2}$) Jacquet functor $r_{M, G}(\Pi)$ for $\Pi \in \text{Irr}(G(F))$ defines the morphism $r_{M, G} : R(G) \rightarrow R(M)$. Let $R^*(G)$ (resp., $R^*(M)$) be the space of all linear functionals on $R(G)$ (resp., $R(M)$). By $i_{G, M}^* : R^*(G) \rightarrow R^*(M)$ and $r_{M, G}^* : R^*(M) \rightarrow R^*(G)$ we denote morphisms adjoint to $i_{G, M}$ and $r_{M, G}$, respectively. Namely, $i_{G, M}^*(f) = f(i_{G, M}(\pi))$ for $f \in R^*(G)$ and $\pi \in R(M)$, and $r_{M, G}^*(h) = h(r_{M, G}(\Pi))$ for $h \in R^*(M)$ and $\Pi \in R(G)$. We set $\mathcal{F}_{tr}(G) = \{\text{trace}(\Pi)(f) \mid \Pi \in R(G), f \in C_c^\infty(G(F))\}$ and $\mathcal{F}_{tr}(M) = \{\text{trace}(\pi)(h) \mid \pi \in R(M), h \in C_c^\infty(M(F))\}$.

PROPOSITION 3.20. For any generalized pseudo-coefficient $\phi_{\sigma, \xi} \in C_c^\infty(M(F))$, there exists $f_{\sigma, \xi} \in C_c^\infty(G(F))$ such that

$$\text{trace}(\Pi)(f_{\sigma, \xi}) = \text{trace}(r_{M, G}(\Pi))(\phi_{\sigma, \xi}), \quad \forall \Pi \in R(G). \tag{3.13}$$

Moreover, the function $f_{\sigma, \xi} \in C_c^\infty(G(F))$ is uniquely determined modulo J_G (cf. Remark 3.7).

Proof. The first part is a consequence of the fact ([BDK86, Proposition 3.2(ii)]) that

$$r_{M, G}^*(\mathcal{F}_{tr}(M)) \subset \mathcal{F}_{tr}(G).$$

The uniqueness is due to the following short exact sequence ([BDK86, Remark 1.2] and [Kaz86, Theorem 0])

$$0 \longrightarrow J_G \longrightarrow C_c^\infty(G(F)) \xrightarrow{\text{trace}} \mathcal{F}_{tr}(G) \longrightarrow 0,$$

where *trace* maps $f \mapsto (\Pi \mapsto \text{trace}(\Pi)(f))$ from $C_c^\infty(G(F))$ onto $\mathcal{F}_{tr}(G)$. \square

For any standard F -Levi subgroups M and L , we set

$$W(L, M) := \{w \in W_G : wLw^{-1} = M\}.$$

We say that L and M are *associate* if $W(L, M)$ is non-empty. When L and M are associate, it turns out that $W_M \cdot W(L, M) \cdot W_L = W(L, M)$ since the element $w_M w w_L$ sends L to M for $w_M \in W_M$, $w \in W(L, M)$ and $w_L \in W_L$. Furthermore, given any standard F -Levi subgroups L and M corresponding to ϑ and θ , respectively, we note from [BZ77, pp.448-449] that

$$W_M \backslash W(L, M) / W_L = W(L, M) / W_L \simeq W(\vartheta, \theta), \tag{3.14}$$

where $W(\vartheta, \theta) := \{w \in W_G : w(\vartheta) = \theta\}$. For any $\pi \in \Pi_{\text{temp}}(L(F))$ and $w \in W(L, M)$, we denote by ${}^w\pi$ the representation of $M(F)$ such that ${}^w\pi(m) := \pi(w^{-1}mw)$ for $m \in M(F)$. It is clear that ${}^w\pi \in \Pi_{\text{temp}}(M(F))$.

PROPOSITION 3.21. *For any standard F -Levi subgroup L of G and $\pi \in \Pi_{\text{temp}}(L(F))$,*

$$\text{trace}(i_{G,L}\pi)(f_{\sigma,\xi}) = \begin{cases} \sum_{w \in W(\vartheta,\theta)} \text{trace}({}^w\pi)(\phi_{\sigma,\xi}), & \text{if } L \text{ and } M \text{ are associate,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first recall from [BZ77, Lemma 2.11(a) and 2.12] and [BDK86, Lemma 5.4(ii)] that

$$r_{M,G} \circ i_{G,L}(\pi) = \sum_{w \in W_M \backslash W_G / W_L} i_{M,M_w} \circ w \circ r_{L_w,L}(\pi),$$

where $M_w = M \cap wLw^{-1}$ and $L_w = w^{-1}Mw \cap L$. Then (3.13) says that for any $\pi \in \text{Irr}(L(F))$,

$$\begin{aligned} \text{trace}(i_{G,L}\pi)(f_{\sigma,\xi}) &= \text{trace}(r_{M,G} \circ i_{G,L}(\pi))(\phi_{\sigma,\xi}) \\ &= \sum_{w \in W_M \backslash W_G / W_L} \text{trace}(i_{M,M_w} \circ w \circ r_{L_w,L}(\pi))(\phi_{\sigma,\xi}). \end{aligned}$$

By (3.4), it turns out that the sum runs over $w \in W(L, M) / W_L$ only when L and M are associate. Otherwise, the group $M_w = M \cap wLw^{-1}$ has to be proper, so the sum vanishes by the cuspidal property of $\phi_{\sigma,\xi}$ (see Proposition 3.9). When L and M are associate, we note that $i_{M,M_w} \circ w \circ r_{L_w,L}(\pi) = {}^w\pi$ and $W(L, M) / W_L \simeq W(\vartheta, \theta)$. Thus the proof is complete. \square

Now we consider the case that $L = M$, that is, $\vartheta = \theta$. Write $W(\theta, \theta) := \{w \in W_G : w(\theta) = \theta\}$. We say that M is *self-associate* if $W(\theta, \theta) \not\supseteq \{1\}$. Otherwise, that is, if $W(\theta, \theta) = \{1\}$, M is called *non self-associate*. We note that

$$\text{trace}(i_{G,M}\pi)(f_{\sigma,\xi}) = \sum_{w \in W(\theta,\theta)} {}^w\xi(\pi). \tag{3.15}$$

Here the action ${}^w\xi(\pi)$ of W_M on a regular function ξ is naturally defined as $\xi({}^w\pi)$ (cf. Remark 2.11). Then we have the following theorem which is a consequence of Proposition 3.21.

THEOREM 3.22. *Let M be an F -Levi subgroup of G corresponding to $\theta \in \Delta$, and let ξ be a $W(\theta, \theta)$ -invariant regular function on $\Omega(\sigma)$ (see Remarks 2.14 and 2.16). For every $\gamma \in M(F)^{\text{reg}}$, we have*

$$|D_{G/M}(\gamma)|^{1/2} O_\gamma^{G(F)}(f_{\sigma, \xi}) = O_\gamma^{M(F)}(\bar{f}_{\sigma, \xi}^{(P)}) = O_\gamma^{M(F)}(|W(\theta, \theta)| \cdot \phi_{\sigma, \xi}),$$

where $D_{G/M}(\gamma) = \det(1 - \text{ad}(\gamma))|_{\text{Lie}(G)/\text{Lie}(M)}$.

Proof. The first equality holds in general due to (3.3) in Section 3.2. We shall check the second equality.

When M is not self-associate, $|W(\theta, \theta)| = 1$. In this case, we do not need the condition of the $W(\theta, \theta)$ -invariance. Thus the second equality is a consequence of [Kaz86, Theorem 0] and the fact that for any $\pi \in \text{Irr}(M(F))$,

$$\begin{aligned} \text{trace}(\pi)(\bar{f}_{\sigma, \xi}^{(P)}) &= \text{trace}(i_{G, M}\pi)(f_{\sigma, \xi}) \\ &\stackrel{(3.15)}{=} \text{trace}(\pi)(\phi_{\sigma, \xi}). \end{aligned}$$

When M is self-associate, we note that $|W(\theta, \theta)| \geq 2$. Since ξ is a $W(\theta, \theta)$ -invariant by hypothesis, (3.15) implies that

$$\text{trace}(\pi)(\bar{f}_{\sigma, \xi}^{(P)}) = |W(\theta, \theta)| \cdot \text{trace}(\pi)(\phi_{\sigma, \xi})$$

for any $\pi \in \text{Irr}(M(F))$. Therefore, we have from [Kaz86, Theorem 0]

$$O_\gamma^{M(F)}(\bar{f}_{\sigma, \xi}^{(P)}) = O_\gamma^{M(F)}(|W(\theta, \theta)| \cdot \phi_{\sigma, \xi}).$$

This completes the proof. \square

REMARK 3.23. In the case that $|W(\theta, \theta)| \geq 2$, given any regular function ξ_o on $\Omega(\sigma)$ and any $w \in W(\theta, \theta)$, we set

$${}^w \xi_o(z) = {}^w \xi_o(\sigma \otimes \psi) := \xi_o({}^w \sigma \otimes {}^w \psi).$$

Then, ${}^w \xi_o(z)$ is clearly a regular function on $\Omega(\sigma)$. It turns out that

$$\xi(z) := \frac{1}{|W(\theta, \theta)|} \sum_{w \in W(\theta, \theta)} {}^w \xi_o(z)$$

must be a $W(\theta, \theta)$ -invariant regular function on $\Omega(\sigma)$. It thus follows that there is always a $W(\theta, \theta)$ -invariant regular function on $\Psi(M)$ (cf. Remark 2.16).

3.4. Application to the Plancherel formula. Given a discrete series representation $\sigma \in \Pi_{\text{disc}}(M(F))$ and a regular function ξ on $\Omega(\sigma)$, by means of the Plancherel formula, we see a relationship between the value $\phi_{\sigma, \xi}(1)$ and the formal degree $d_M(\sigma)$. We also express the value $f_{\sigma, \xi}(1)$ in terms of ξ , the formal degree of σ , and the Plancherel measure on $\Omega(\sigma)$.

PROPOSITION 3.24 (Relationship between $\phi_{\sigma, \xi}(1)$ and $d_M(\sigma)$). *We have*

$$\phi_{\sigma, \xi}(1) = d_M(\sigma) \int_{\psi \in \Psi_u(M)/\text{Stab}_\Psi(\sigma)} \xi(\sigma \otimes \psi) d\psi. \tag{3.16}$$

Proof. We apply the generalized pseudo-coefficient $\phi_{\sigma,\xi}$ to the Plancherel formula (Theorem 2.3). Since $G = M$, we have $a(G|M) = 1$ and $\mu_M(\pi) = 1$ for any $\pi \in \Pi_{\text{disc}}(M(F))$. By the construction of $\phi_{\sigma,\xi}$ in Theorem 3.6, we have

$$\text{trace}(\pi)(\phi_{\sigma,\xi}) = 0$$

unless $\pi \in \Omega(\sigma)$, in which case $\text{trace}(\pi)(\phi_{\sigma,\xi}) = \xi(\pi)$. Note that $\pi \in \Omega(\sigma)$ is of the form $\sigma \otimes \psi$ with $\psi \in \Psi_u(M)$, and $d_M(\sigma \otimes \psi) = d_M(\sigma)$ for any unitary character ψ of $M(F)$. From the Plancherel formula (Theorem 2.3), we then have

$$\phi_{\sigma,\xi}(1) = d_M(\sigma) \int_{\pi \in \Pi_{\text{disc}}(M(F)) \cap \Omega(\sigma)} \xi(\pi) d\pi.$$

Since $\Pi_{\text{disc}}(M(F)) \cap \Omega(\sigma) \simeq \Psi_u(M)/\text{Stab}_\Psi(\sigma)$, we have (3.16). \square

REMARK 3.25. If $\xi = 1$ so that $\phi_{\sigma,\xi}^\omega$ becomes a pseudo-coefficient (Proposition 3.19), the right-hand side of (3.16) becomes $d_M(\sigma) \int_{\psi \in \Psi_u(M)/\text{Stab}_\Psi(\sigma)} d\psi$. By choosing a suitable measure $d\psi$ on $\Psi_u(M)$ so that $\int_{\psi \in \Psi_u(M)/\text{Stab}_\Psi(\sigma)} d\psi = 1$ (see Section 2.2), we deduce

$$\phi_{\sigma,\xi}(1) = d_M(\sigma). \tag{3.17}$$

If the center Z_M is anisotropic over F , so that $\Psi(M)$ is trivial and the regular function ξ on $\Omega(\sigma)$ equals just 1, then (3.17) immediately follows. This is an analogue of the well-known property of a pseudo-coefficient $\phi_{\sigma,\xi}^\omega(1) = d_M(\sigma)$ (for example, see [DKV84, A.4]).

Recall the notation $\mu_M(\sigma \otimes \psi)$ for the Plancherel measure from Remark 2.8. We then have the following proposition.

PROPOSITION 3.26 (Relationship between $f_{\sigma,\xi}(1)$ and $\mu_M(\sigma \otimes \psi)$). *Let M be an F -Levi subgroup of G corresponding to $\theta \subset \Delta$, and let ξ be a $W(\theta, \theta)$ -invariant regular function on $\Omega(\sigma)$. Then we have*

$$f_{\sigma,\xi}(1) = a(G|M)d_M(\sigma)|W(\theta, \theta)| \int_{\psi \in \Psi_u(M)/\text{Stab}_\Psi(\sigma)} \mu_M(\sigma \otimes \psi)\xi(\sigma \otimes \psi)d\psi.$$

Proof. Similar to the proof of Proposition 3.24, we apply $f_{\sigma,\xi}$ to the Plancherel formula (Theorem 2.3). Proposition 3.21 then yields

$$f_{\sigma,\xi}(1) = a(G|M) \int_{\pi \in \Pi_{\text{disc}}(M(F))} d_M(\pi)\mu_M(\pi) \sum_{w \in W(\theta, \theta)} \text{trace}({}^w\pi)(\phi_{\sigma,\xi})d\pi. \tag{3.18}$$

Due to Theorem 3.6, the isomorphism $\Pi_{\text{disc}}(M(F)) \cap \Omega(\sigma) \simeq \Psi_u(M)/\text{Stab}_\Psi(\sigma)$, and the assumption that ξ is $W(\theta, \theta)$ -invariant, the right-hand side of (3.18) equals

$$a(G|M)|W(\theta, \theta)| \int_{\psi \in \Psi_u(M)/\text{Stab}_\Psi(\sigma)} d_M(\sigma \otimes \psi)\mu_M(\sigma \otimes \psi)\xi(\sigma \otimes \psi)d\psi.$$

Again, using the fact that $d_M(\sigma \otimes \psi) = d_M(\sigma)$ for any unitary character ψ of $M(F)$, the proof is complete. \square

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