ON A CONJECTURE OF MILNOR ABOUT VOLUMES OF SIMPLEXES[*]

REN GUO[†] AND FENG LUO[‡]

Dedicated to the memory of Xiao-Song Lin

Abstract. We establish the second part of Milnor's conjecture on the volume of simplexes in hyperbolic and spherical spaces. A characterization of the closure of the space of the angle Gram matrices of simplexes is also obtained.

Key words. Volume, simplex, hyperbolic space, spherical space.

AMS subject classifications. 52A38, 52A55, 52B11

1. Introduction.

Milnor's conjecture. In [5], John Milnor conjectured that the volume of a hyperbolic or spherical n-simplex, considered as a function of the dihedral angles, can be extended continuously to the degenerate simplexes. Furthermore, he conjectured that the extended volume function is non-zero except in the closure of the space of Euclidean simplexes. The first part of the conjecture on the continuous extension was established in [4] ([8] has a new proof of it which generalizes to many polytopes). The purpose of the paper is to establish the second part of Milnor's conjecture.

To state the result, let us begin with some notations and definitions. Given an n-simplex in the spherical, hyperbolic or Euclidean space with vertices $u_1, ..., u_{n+1}$, the i-th codimension-1 face is defined to be the $(n-1)$-simplex with vertices $u_1, ..., u_{i-1}$, $u_{i+1}, ..., u_{n+1}$. The dihedral angle between the i-th and j-th codimension-1 faces is denoted by θ_{ij}. As a convention, we define $\theta_{ii} = \pi$ and call the symmetric matrix $A = [-\cos(\theta_{ij})]_{(n+1)\times(n+1)}$ the *angle Gram matrix* of the simplex. It is well known that the angle Gram matrix determines a hyperbolic or spherical n-simplex up to isometry and Euclidean n-simplex up to similarity. Let $\mathcal{X}_{n+1}, \mathcal{Y}_{n+1}, \mathcal{Z}_{n+1}$ in $\mathbf{R}^{(n+1)\times(n+1)}$ be the subsets of $(n+1) \times (n+1)$ symmetric matrices corresponding to the angle Gram matrices of spherical, hyperbolic, or Euclidean $n-$simplexes respectively.

The volume of an n-simplex can be expressed in terms of the angle Gram matrix by the work of Aomoto [1], Kneser [2] and Vinberg [9]. Namely, for a spherical or hyperbolic n-simplex σ^n with angle Gram matrix A, the volume V is

$$V(A) = \mu_n^{-1}\sqrt{|\det(\mathrm{adj}(A))|} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{-x^t \mathrm{adj}(A)x} dx. \tag{1.1}$$

where $\mathbf{R}_{\geq 0}^{n+1} = \{(x_1, ..., x_{n+1})| \ x_i \geq 0\}$, the constant $\mu_k = \int_0^\infty x^k e^{-x^2} dx$ and $\mathrm{adj}(A)$ is the adjoint matrix of A. In [4], it is proved that the volume function $V : \mathcal{X}_{n+1} \cup \mathcal{Y}_{n+1} \to \mathbf{R}$ can be extended continuously to the closure $\overline{\mathcal{X}}_{n+1} \cup \overline{\mathcal{Y}}_{n+1}$ in $\mathbf{R}^{(n+1)\times(n+1)}$. The

main result of this paper, which verifies the second part of Milnor's conjecture, is the following theorem.

THEOREM 1. *The extended volume function V on the closure $\overline{\mathcal{X}}_{n+1} \cup \overline{\mathcal{Y}}_{n+1}$ in $\mathbf{R}^{(n+1)\times(n+1)}$ vanishes at a point A if and only if A is in the closure $\overline{\mathcal{Z}}_{n+1}$.*

A characterization of the angle Gram matrices. We will use the following conventions. Given a real matrix $A = [a_{ij}]$, we use $A \geq 0$ to denote $a_{ij} \geq 0$ for all i, j and $A > 0$ to denote $a_{ij} > 0$ for all i, j. A^t is the transpose of A. The diagonal $k \times k$ matrix with diagonal entries $(x_1, ..., x_k)$ is denoted by $\mathrm{diag}(x_1, ..., x_k)$. A characterization of the angle Gram matrices in $\mathcal{X}_{n+1}, \mathcal{Y}_{n+1}$ or $\mathcal{Z}_{n+1}$ is known by the work of [3] and [5].

PROPOSITION 2 ([3],[5]). *Given an $(n+1) \times (n+1)$ symmetric matrix $A = [a_{ij}]$ with $a_{ii} = 1$ for all i, then*
  (a) *$A \in \mathcal{Z}_{n+1}$ if and only if $\det(A) = 0$, $\mathrm{adj}(A) > 0$ and all principal $n \times n$ submatrices of A are positive definite,*
  (b) *$A \in \mathcal{X}_{n+1}$ if and only if A is positive definite,*
  (c) *$A \in \mathcal{Y}_{n+1}$ if and only if $\det(A) < 0$, $\mathrm{adj}(A) > 0$ and all principal $n \times n$ submatrices of A are positive definite.*
In particular, all off-diagonal entries a_{ij} have absolute values less than 1, i.e.,$|a_{ij}| < 1$ for $i \neq j$.

The following gives a characterization of matrices in $\overline{\mathcal{X}}_{n+1}, \overline{\mathcal{Y}}_{n+1}$ and $\overline{\mathcal{Z}}_{n+1}$ in $\mathbf{R}^{(n+1)\times(n+1)}$.

THEOREM 3. *Given an $(n+1) \times (n+1)$ symmetric matrix $A = [a_{ij}]$ with $a_{ii} = 1$ for all i, then*
  (a) *$A \in \overline{\mathcal{Z}}_{n+1}$ if and only if $\det(A) = 0$, A is positive semi-definite, and there exists a principal $(k+1) \times (k+1)$ submatrix B of A so that $\det(B) = 0$, $\mathrm{adj}(B) > 0$,*
  (b) *$A \in \overline{\mathcal{X}}_{n+1}$ if and only if either A is in $\mathcal{X}_{n+1}$ or there exists a diagonal matrix $D = \mathrm{diag}(\varepsilon_1, ..., \varepsilon_{n+1})$ where $\varepsilon_i = 1$ or -1 for each $i = 1, ..., n+1$, such that $DAD \in \overline{\mathcal{Z}}_{n+1}$,*
  (c) *$A \in \overline{\mathcal{Y}}_{n+1}$ if and only if either $A \in \overline{\mathcal{Z}}_{n+1}$ or $\det(A) < 0$, $\mathrm{adj}(A) \geq 0$ and all principal $n \times n$ submatrices of A are positive semi-definite.*

The paper is organized as follows. In Section 2, we characterize the normal vectors of degenerate Euclidean simplexes. In Section 3, we characterize the angle Gram matrices of degenerate hyperbolic simplexes. Theorem 1 is proved in Section 4 and Theorem 3 is proved in Section 5.

2. Normal vectors of Euclidean simplexes. As a convention, all vectors in $\mathbf{R}^m$ are column vectors and the standard inner product in $\mathbf{R}^m$ is denoted by $u \cdot v$. In the sequel, for a non-zero vector $w \in \mathbf{R}^n$, we call the set $\{x \in \mathbf{R}^n | w \cdot x \geq 0\}$ a *closed half space*, and the set $\{x \in \mathbf{R}^n | w \cdot x > 0\}$ an *open half space*. Define $\mathcal{E}_{n+1} = \{(v_1, ..., v_{n+1}) \in (\mathbf{R}^n)^{n+1} | v_1, ..., v_{n+1}$ form unit outward normal vectors to the codimension-1 faces of a Euclidean n-simplex $\}$. Following Milnor [5], a matrix is called *unidiagonal* if its diagonal entries are 1. An $(n+1) \times (n+1)$ symmetric unidiagonal matrix A is in $\mathcal{Z}_{n+1}$ if and only if $A = [v_i \cdot v_j]$ for some point $(v_1, ..., v_{n+1}) \in \mathcal{E}_{n+1}$ (this is proved in [3],[5]). We claim that an $(n+1) \times (n+1)$ symmetric unidiagonal matrix A is in $\overline{\mathcal{Z}}_{n+1}$ if and only if $A = [v_i \cdot v_j]$ for some point $(v_1, ..., v_{n+1})$ in the closure $\overline{\mathcal{E}}_{n+1}$ in $(\mathbf{R}^n)^{n+1}$. Indeed, if $A = [v_i \cdot v_j]$ for some point $(v_1, ..., v_{n+1}) \in \overline{\mathcal{E}}_{n+1}$, then there

is a sequence $(v_1^{(m)}, ..., v_{n+1}^{(m)}) \in \mathcal{E}_{n+1}$ converging to $(v_1, ..., v_{n+1})$. We have a sequence of matrices $A^{(m)} = [v_i^{(m)} \cdot v_j^{(m)}] \in \mathcal{Z}_{n+1}$ converging to A. Conversely if $A \in \overline{\mathcal{Z}}_{n+1}$, then there is a sequence of matrices $A^{(m)} \in \mathcal{Z}_{n+1}$ converging to A. Write $A^{(m)} = [v_i^{(m)} \cdot v_j^{(m)}]$, where $(v_1^{(m)}, ..., v_{n+1}^{(m)}) \in \mathcal{E}_{n+1}$. Since $v_i^{(m)}$ has norm 1 for all i, m, by taking subsequence, we may assume $\lim_{m \to \infty}(v_1^{(m)}, ..., v_{n+1}^{(m)}) = (v_1, ..., v_{n+1}) \in \overline{\mathcal{E}}_{n+1}$ so that $A = [v_i \cdot v_j]$.

A geometric characterization of elements in $\mathcal{E}_{n+1}$ was obtained in [3]. For completeness, we include a proof here.

LEMMA 4. *A collection of unit vectors* $(v_1, ..., v_{n+1}) \in (\mathbf{R}^n)^{n+1}$ *is in* $\mathcal{E}_{n+1}$ *if and only if one of the following conditions is satisfied.*

(4.1) *The vectors* $v_1, ..., v_{n+1}$ *are not in any closed half-space.*

(4.2) *Any n vectors of* $v_1, ..., v_{n+1}$ *are linearly independent and the linear system* $\sum_{i=1}^{n+1} a_i v_i = 0$ *has a solution* $(a_1, ..., a_n)$ *so that* $a_i > 0$ *for all* $i = 1, ..., n+1$.

Proof. $(4.2) \Rightarrow (4.1)$. Suppose otherwise, $v_1, ..., v_{n+1}$ are in a closed half-space, i.e., there is a non-zero vector $w \in \mathbf{R}^n$ so that $w \cdot v_i \geq 0, i = 1, ..., n+1$. Let $a_1, ..., a_{n+1}$ be the positive numbers given by (4.2) so that $\sum_{i=1}^{n+1} a_i v_i = 0$. Then

$$0 = w \cdot \left(\sum_{i=1}^{n+1} a_i v_i \right) = \sum_{i=1}^{n+1} a_i (w \cdot v_i).$$

But by the assumption $a_i > 0$, $w \cdot v_i \geq 0$ for all i. Thus $w \cdot v_i = 0$ for all i. This means that $v_1, ..., v_{n+1}$ lie in the $(n-1)$-dimensional subspace perpendicular to w. It contradicts the assumption in (4.2) that any n vectors of $v_1, ..., v_{n+1}$ are linearly independent.

$(4.1) \Rightarrow (4.2)$. To see that any n vectors of $v_1, ..., v_{n+1}$ are linearly independent, suppose otherwise, some n vectors of $v_1, ..., v_{n+1}$ are linearly dependent. Therefore there is an $(n-1)$-dimensional subspace containing these n vectors. Then $v_1, ..., v_{n+1}$ are contained in one of the two closed half spaces bounded by the subspace. It contradicts the assumption of (4.1).

Since $v_1, ..., v_{n+1}$ are linearly dependent, and any n of them are linearly independent, we can find real numbers $a_i \neq 0$ for all i such that $\sum_{i=1}^{n+1} a_i v_i = 0$. For any $i \neq j$, let H_{ij} be the $(n-1)$-dimensional subspace spanned by the $n-1$ vectors $\{v_1, ..., v_{n+1}\} \setminus \{v_i, v_j\}$ and $u \in \mathbf{R}^n - \{0\}$ be a vector perpendicular to H_{ij}. We have

$$0 = u \cdot \left(\sum_{i=1}^{n+1} a_i v_i \right) = a_i (u \cdot v_i) + a_j (u \cdot v_j).$$

By the assumption of (4.1), v_i and v_j must lie in the different sides of H_{ij}. Thus $u \cdot v_i$ and $u \cdot v_j$ have different signs. This implies that a_i and a_j have the same sign. Hence we can make $a_i > 0$ for all i.

$\mathcal{E}_{n+1} \Leftrightarrow (4.1)$. We will show $(v_1, ..., v_{n+1}) \in \mathcal{E}_{n+1}$ if and only if condition (4.1) holds. In fact, given an n-dimensional Euclidean simplex σ, let S^{n-1} be the sphere inscribed to σ. We may assume after a translation and a scaling that S^{n-1} is the unit sphere centered at the origin. Then the unit vectors $v_1, ..., v_{n+1}$ are the tangent points of S^{n-1} to the codimension-1 faces of σ. The tangent planes to S^{n-1} at $v_i's$ bound a compact region (the Euclidean simplex σ) containing the origin if and only if the tangent points $v_1, ..., v_{n+1}$ are not in any closed hemisphere of S^{n-1}. $\square$

LEMMA 5. *Suppose* $(v_1, ..., v_{n+1}) \in (\mathbf{R}^n)^{n+1}$ *so that each* v_i *is a unit vector. Then* $(v_1, ..., v_{n+1})$ *is in* $\overline{\mathcal{E}}_{n+1}$ *if and only if one of the following conditions is satisfied:*

(5.1) *The vectors* $v_1, ..., v_{n+1}$ *are not in any open half-space.*

(5.2) *The linear system* $\sum_{i=1}^{n+1} a_i v_i = 0$ *has a nonzero solution* $(a_1, ..., a_{n+1})$ *so that* $a_i \geq 0$ *for all* $i = 1, ..., n+1$.

Proof. $\overline{\mathcal{E}}_{n+1} \Rightarrow (5.1)$. To see that elements in $\overline{\mathcal{E}}_{n+1}$ satisfy (5.1), if $(v_1, ..., v_{n+1}) \in \overline{\mathcal{E}}_{n+1}$, there is a family of $(v_1^{(m)}, ..., v_{n+1}^{(m)}) \in \mathcal{E}_{n+1}$ converging to $(v_1, ..., v_{n+1})$. Since vectors $v_1^{(m)}, ..., v_{n+1}^{(m)}$ are not in any closed half-space for any m, by continuity, vectors $v_1, ..., v_{n+1}$ are not in any open half-space.

$(5.1) \Rightarrow (5.2)$. Consider the linear map

$$f : \mathbf{R}^n \longrightarrow \mathbf{R}^{n+1}$$

$$w \longmapsto f(w) = [v_1, v_2, ..., v_{n+1}]^t w = \begin{pmatrix} v_1 \cdot w \\ v_2 \cdot w \\ \vdots \\ v_{n+1} \cdot w \end{pmatrix}.$$

Statement (5.1) says that

$$\begin{aligned} \emptyset &= \{w \in \mathbf{R}^n | v_i \cdot w > 0, i = 1, ..., n+1\} \\ &= \{w \in \mathbf{R}^n | f(w) > 0\} \\ &= f(\mathbf{R}^n) \cap \mathbf{R}^{n+1}_{>0}. \end{aligned}$$

Since $f(\mathbf{R}^n)$ and $\mathbf{R}^{n+1}_{>0}$ are convex and disjoint, by the separation theorem for convex sets, there is a vector $a = (a_1, ..., a_{n+1})^t$ satisfying the conditions (i) and (ii) below.

(i) For all $u \in \mathbf{R}^{n+1}_{>0}$,

$$a \cdot u > 0.$$

and

(ii) For all $w \in \mathbf{R}^n$,

$$0 \geq a \cdot f(w) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix} \cdot \begin{pmatrix} v_1 \cdot w \\ v_2 \cdot w \\ \vdots \\ v_{n+1} \cdot w \end{pmatrix} = (\sum_{i=1}^{n+1} a_i v_i) \cdot w.$$

Condition (i) implies that $a_i \geq 0$, for $i = 1, ..., n+1$ and $a \neq 0$. Condition (ii) implies $\sum_{i=1}^{n+1} a_i v_i = 0$. Thus (5.2) holds.

$(5.2) \Rightarrow \overline{\mathcal{E}}_{n+1}$. To see that a point $(v_1, ..., v_{n+1})$ satisfying (5.2) is in $\overline{\mathcal{E}}_{n+1}$, we show that in any ε-neighborhood of $(v_1, ..., v_{n+1})$ in $(\mathbf{R}^n)^{n+1}$, there is a point $(v_1^\varepsilon, ..., v_{n+1}^\varepsilon) \in \mathcal{E}_{n+1}$.

Let $\mathcal{N}_k$ be the set of $(v_1, ..., v_k)$ such that $v_i \in \mathbf{R}^{k-1}, |v_i| = 1$ for all i and $\sum_{i=1}^k a_i v_i = 0$ has a nonzero solution $(a_1, ..., a_k)$ with $a_i \geq 0$ for all i. The goal is to prove that $\mathcal{N}_{n+1} \subset \overline{\mathcal{E}}_{n+1}$. We achieve this by induction on n. It is obvious that $\mathcal{N}_2 \subset \overline{\mathcal{E}}_2$. Assume that $\mathcal{N}_n \subset \overline{\mathcal{E}}_n$ holds.

For a point $(v_1, ..., v_{n+1}) \in \mathcal{N}_{n+1}$, if any n vectors of $v_1, ..., v_{n+1}$ are linearly independent, then each entry a_i of the non-zero solution of the linear system

$\sum_{i=1}^{n+1} a_i v_i = 0, a_i \geq 0, i = 1, ..., n+1$ must be nonzero. Thus $a_i > 0$ for all i and $(v_1, ..., v_{n+1})$ satisfies (4.2), therefore it is in $\mathcal{E}_{n+1}$.

In the remain case, without loss of generality, we assume that $v_1, ..., v_n$ are linearly dependent. We may assume after a change of coordinates that $v_i \in \mathbf{R}^{n-1} = \mathbf{R}^{n-1} \times \{0\} \subset \mathbf{R}^n$, for $i = 1, ..., n$, and $v_{n+1} = (u_{n+1} \cos(\theta), \sin(\theta))^t$, where $0 \leq \theta \leq \frac{\pi}{2}$ and $|u_{n+1}| = 1$.

We claim that there exists some $1 \leq i \leq n+1$ such that $(v_1, ..., \widehat{v}_i, ..., v_{n+1}) \in \mathcal{N}_n$, where $\widehat{x}$ means deleting the element x.

Case 1. If $\theta > 0$ i.e., v_{n+1} is not in $\mathbf{R}^{n-1}$, consider the nonzero solution of the linear system $\sum_{i=1}^{n+1} a_i v_i = 0, a_i \geq 0, i = 1, ..., n+1$. The last coordinate gives $a_1 0 + ... + a_n 0 + a_{n+1} \sin(\theta) = 0$, which implies $a_{n+1} = 0$. This means that $(a_1, ..., a_n) \neq (0, ..., 0)$, i.e., $(v_1, ..., v_n) \in \mathcal{N}_n$.

Case 2. If $\theta = 0$ i.e., $v_{n+1} \in \mathbf{R}^{n-1}$, then the dimension of the solution space $W = \{(a_1, ..., a_{n+1})^t \in \mathbf{R}^{n+1} | \sum_{i=1}^{n+1} a_i v_i = 0\}$ is at least 2. Since $(v_1, ..., v_{n+1}) \in \mathcal{N}_{n+1}$, the intersection $W \cap \mathbf{R}_{\geq 0}^{n+1} - \{(0, ..., 0)\}$ is nonempty. The subspace W must intersect the boundary of the cone $\mathbf{R}_{\geq 0}^{n+1} - \{(0, ..., 0)\}$. Let $(a_1, ..., a_{n+1})$ be a point in both W and the boundary of the cone $\mathbf{R}_{\geq 0}^{n+1} - \{(0, ..., 0)\}$. Then there is some $a_i = 0$. Then $\{v_1, ..., \widehat{v}_i, ..., v_{n+1}\} \in \mathcal{N}_n$.

By the above discussion, without loss of generality, we may assume that $(v_1, ..., v_n) \in \mathcal{N}_n$. By the induction assumption $\mathcal{N}_n \subset \overline{\mathcal{E}}_n$, i.e., in any $\frac{\epsilon}{2}$-neighborhood of $(v_1, ..., v_n)$, we can find a point $(u_1, ..., u_n) \in \mathcal{E}_n$, where $u_i \in \mathbf{R}^{n-1}$ for all i. Recall that we write $v_{n+1} = (u_{n+1} \cos(\theta), \sin(\theta))^t$. Let us define a continuous family of $n+1$ unit vectors $v_1(t), ..., v_{n+1}(t)$ by setting

$$v_i(t) = (u_i \cos(t^2), -\sin(t^2))^t, 1 \leq i \leq n,$$

$$v_{n+1}(t) = (u_{n+1} \cos(\theta + t), \sin(\theta + t))^t.$$

We claim that there is a point $(v_1(t), ..., v_{n+1}(t)) \in \mathcal{E}_{n+1}$ for small $t > 0$ within $\frac{\epsilon}{2}$-neighborhood of $((u_1, 0)^t, ..., (u_n, 0)^t, v_{n+1})$. By the triangle inequality, this point is within ε-neighborhood of $(v_1, ..., v_{n+1})$. We only need to check that $(v_1(t), ..., v_{n+1}(t)) \in \mathcal{E}_{n+1}$ for sufficiently small $t > 0$ by verifying the condition (4.2).

To show that any n vectors of $v_1(t), ..., v_{n+1}(t)$ are linearly independent, it is equivalent to show that

$$\det[v_1(t), ..., \widehat{v}_i(t), ..., v_{n+1}(t)] \neq 0$$

for each $i = 1, ..., n+1$.

First,

$$\det[v_1(t), ..., v_n(t)] = \det \begin{bmatrix} u_1 \cos(t^2) & u_2 \cos(t^2) & ... & u_n \cos(t^2) \\ -\sin(t^2) & -\sin(t^2) & ... & -\sin(t^2) \end{bmatrix}_{n \times n}$$

$$= -\sin(t^2) \cos(t^2)^{n-1} \det \begin{bmatrix} u_1 & u_2 & ... & u_n \\ 1 & 1 & ... & 1 \end{bmatrix}.$$

To see that the determinant is nonzero, suppose there are real numbers $a_1, ..., a_n$ such that $\sum_{i=1}^{n} a_i (u_i, 1)^t = 0$. Then we have $\sum_{i=1}^{n} a_i u_i = 0$ and $\sum_{i=1}^{n} a_i = 0$. By assumption $(u_1, ..., u_n) \in \mathcal{E}_n$, we know that either $a_i = 0$ for all i or $a_i \neq 0$ and have the same sign for all i. Hence $\sum_{i=1}^{n} a_i = 0$ implies $a_i = 0$ for all i. Thus the

vectors $(u_1, 1)^t, ..., (u_n, 1)^t$ are linearly independent. Hence $\det[v_1(t), ..., v_n(t)] \neq 0$ for $t \in (0, \sqrt{\frac{\pi}{2}}]$.

Second, we calculate the determinant of the matrix whose columns are $v_{n+1}(t)$ and some $n - 1$ vectors of $v_1(t), ..., v_n(t)$. Without loss of generality, consider

$$
\begin{aligned}
f(t) &= \det[v_2(t), ..., v_n(t), v_{n+1}(t)] \\
&= \det \begin{bmatrix} u_2 \cos(t^2) & ... & u_n \cos(t^2) & u_{n+1} \cos(\theta + t) \\ -\sin(t^2) & ... & -\sin(t^2) & \sin(\theta + t) \end{bmatrix}.
\end{aligned}
$$

If $\theta > 0$, by the assumption $(u_1, ..., u_n) \in \mathcal{E}_n$, then

$$
f(0) = \det \begin{bmatrix} u_2 & ... & u_n & u_{n+1} \cos(\theta) \\ 0 & ... & 0 & \sin(\theta) \end{bmatrix} = \sin(\theta) \det[u_2, ..., u_n] \neq 0.
$$

It implies $f(t) \neq 0$ holds for sufficiently small $t > 0$.

If $\theta = 0$, then $f(0) = 0$. By expanding the determinant along the bottom row,

$$
f(t) = -\sin(t^2) g(t) + \sin(t) \det[u_2 \cos(t^2), ..., u_n \cos(t^2)],
$$

for some function $g(t)$, therefore $f'(0) = \det(u_2, ..., u_n) \neq 0$. Hence $f(t) \neq 0$ holds for sufficiently small $t > 0$.

Next, let

$$
a_i(t) = (-1)^{i-1} \det[v_1(t), ..., \widehat{v_i}(t), ..., v_{n+1}(t)], 1 \leq i \leq n + 1,
$$

then $\sum_{i=1}^{n+1} a_i(t) v_i(t) = 0$. Since $\det[v_1(0), ..., v_n(0)] = 0$, we have $a_{n+1}(0) = 0$. This shows that $\sum_{i=1}^n a_i(0) v_i(0) = 0$, therefore $\sum_{i=1}^n a_i(0) u_i = 0$. By the assumption $(u_1, ..., u_n) \in \mathcal{E}_n$, we obtain $a_i(0) \cdot a_j(0) > 0$ for $0 \leq i, j \leq n$. By continuity we obtain $a_i(t) \cdot a_j(t) > 0$ for $0 \leq i, j \leq n$, for sufficient small $t > 0$. Consider the last coordinate of $\sum_{i=1}^{n+1} a_i(t) v_i(t) = 0$ we obtain

$$
-\sin(t^2) \sum_{i=1}^n a_i(t) + \sin(\theta + t) a_{n+1}(t) = 0.
$$

Thus $a_{n+1}(t)$ has the same sign as $a_i(t)$. Thus $(a_1(t), ..., a_{n+1}(t))$ or $(-a_1(t), ..., -a_{n+1}(t))$ is a solution required in condition (4.2). $\square$

3. Degenerate hyperbolic simplexes. Let $\mathbf{R}^{n,1}$ be the Minkowski space which is $\mathbf{R}^{n+1}$ with an inner product $\langle , \rangle$ where

$$
\langle (x_1, ..., x_n, x_{n+1})^t, (y_1, ..., y_n, y_{n+1})^t \rangle = x_1 y_1 + ... + x_n y_n - x_{n+1} y_{n+1}.
$$

Let $H^n = \{x = (x_1, ..., x_{n+1})^t \in \mathbf{R}^{n,1} | \langle x, x \rangle = -1, x_{n+1} > 0\}$ be the hyperboloid model of the hyperbolic space. The de Sitter space is $\{x \in \mathbf{R}^{n,1} | \langle x, x \rangle = 1\}$. For a hyperbolic simplex σ in H^n, the *center* and the *radius* of the simplex σ are defined to be the center and radius of its inscribed ball.

LEMMA 6. *For an n-dimensional hyperbolic simplex $\sigma \in H^n$ with center $e_{n+1} = (0, ..., 0, 1)^t$, its unit outward normal vectors in the de Sitter space are in a compact set independent of σ.*

Proof. Let $v_1, ..., v_{n+1}$ be the unit outward normal vectors of σ, i.e.,

$$
\sigma = \{x \in H^n | \langle x, v_i \rangle \leq 0 \text{ and } \langle v_i, v_i \rangle = 1 \text{ for all } i\}.
$$

Let $v_i^\perp$ be the totally geodesic hyperplane in H^n containing the $(n-1)$-dimensional face of σ perpendicular to v_i for each $i = 1, ..., n+1$. The radius of σ is the distance from the center e_{n+1} to $v_i^\perp$ for any $i = 1, ..., n+1$ which is equal to $\sinh^{-1}(|\langle e_{n+1}, v_i \rangle|)$ (see for instance [10] p.26). It is well known that the volume of an $n-$dimensional hyperbolic simplex is bounded by the volume of the $n-$dimensional regular ideal hyperbolic simplex which is finite (see for instance [7] p.539). It implies that the radius of a hyperbolic simplex σ is bounded from above by a constant independent of σ. Hence $\langle e_{n+1}, v_i \rangle^2$ is bounded from above by a constant c_n independent of σ for any $i = 1, ..., n+1$. It follows that $v_1, ..., v_{n+1}$ are in the compact set

$$X_n = \{x = (x_1, ..., x_{n+1})^t | \langle x, x \rangle = 1, \langle e_{n+1}, x \rangle^2 \le c_n\}$$
$$= \{x = (x_1, ..., x_{n+1})^t | x_1^2 + ... + x_n^2 = x_{n+1}^2 + 1, x_{n+1}^2 \le c_n\}$$

independent of σ. $\square$

LEMMA 7. *If $A \in \overline{\mathcal{Y}}_{n+1}$ and $\det(A) = 0$, then $A \in \overline{\mathcal{Z}}_{n+1}$.*

Proof. Let $A^{(m)}$ be a sequence of angle Gram matrices in $\mathcal{Y}_{n+1}$ converging to A. By Proposition 2 (c), for any m, all principal $n \times n$ submatrices of $A^{(m)}$ are positive definite. Thus all principal $n \times n$ submatrices of A are positive semi-definite. Since $\det(A) = 0$, we see that A is positive semi-definite.

Let $\sigma^{(m)}$ be the n-dimensional hyperbolic simplex in the hyperboloid model H^n whose angle Gram matrix is $A^{(m)}$ and whose center is $e_{n+1} = (0, ..., 0, 1)^t$. By Lemma 6, its unit outward normal vector $v_i^{(m)}$ is in a compact set. Thus by taking a subsequence, we may assume $(v_1^{(m)}, ..., v_{n+1}^{(m)})$ converges to $(v_1, ..., v_{n+1})$ with $\langle v_i, v_i \rangle = 1$. Since $A^{(m)} = [\langle v_i^{(m)}, v_j^{(m)} \rangle]$ and $A^{(m)}$ converges to A, we obtain

$$A = [\langle v_i, v_j \rangle] = [v_1, ..., v_{n+1}]^t S [v_1, ..., v_{n+1}],$$

where S is the diagonal matrix $\text{diag}(1, ..., 1, -1)$.

Since $\det(A) = 0$, the vectors $v_1, ..., v_{n+1}$ are linearly dependent. Assume that the vectors $v_1, ..., v_{n+1}$ span a k-dimensional subspace W of $\mathbf{R}^{n,1}$, where $k \le n$.

For any vector $x \in W$, write $x = \sum_{i=1}^{n+1} x_i v_i$. Then

$$\langle x, x \rangle = (x_1, ..., x_{n+1})[v_1, ..., v_{n+1}]^t S [v_1, ..., v_{n+1}](x_1, ..., x_{n+1})^t$$
$$= (x_1, ..., x_{n+1}) A (x_1, ..., x_{n+1})^t$$
$$\ge 0$$

due to the fact that A is positive semi-definite.

Now for any $x, y \in W$, the inequality $\langle x + ty, x + ty \rangle \ge 0$ for any $t \in \mathbf{R}$ implies the Schwartz inequality

$$\langle x, y \rangle^2 \le \langle x, x \rangle \langle y, y \rangle.$$

To prove that $A \in \overline{\mathcal{Z}}_{n+1}$, we consider the following two possibilities.

Case 1. If $\langle x, x \rangle > 0$ holds for any non-zero $x \in W$, then the Minkowski inner product restricted on W is positive definite. Since the Minkowski inner product restricted on $\mathbf{R}^k = \mathbf{R}^k \times 0 \subset \mathbf{R}^{n,1}$ is positive definite, by Witt's theorem, there is an isometry γ of $\mathbf{R}^{n,1}$ sending W to $\mathbf{R}^k$ (see [10] p.14-p.15). By replacing $v_i^{(m)}$ by $\gamma(v_i^{(m)})$ for each i and m, we may assume that $v_1, ..., v_{n+1}$ are contained in $\mathbf{R}^k$. Thus $\langle v_i, v_i \rangle = v_i \cdot v_j$ for all i, j. Therefore

$$A = [v_1, ..., v_{n+1}]^t S [v_1, ..., v_{n+1}] = [v_1, ..., v_{n+1}]^t [v_1, ..., v_{n+1}].$$

To show that $A \in \overline{\mathscr{Z}}_{n+1}$, by Lemma 5, we only need to show that $v_1, ..., v_{n+1}$ are not contained in any open half space of $\mathbf{R}^k$. This is the same as showing that $v_1, ..., v_{n+1}$ are not contained in any open half space of $\mathbf{R}^n$.

Suppose otherwise, there exists a vector $w \in \mathbf{R}^k$ such that $v_i \cdot w > 0$ for all i. Thus $\langle v_i, w \rangle = v_i \cdot w > 0$. By taking m large enough, we obtain $\langle v_i^{(m)}, w \rangle > 0$ for all i.

It is well known that for the unit normal vectors $v_i^{(m)}$ of a compact hyperbolic simplex in H^n, the conditions $\langle v_i^{(m)}, w \rangle > 0$ for all i implies $\langle w, w \rangle < 0$. But this contradicts the assumption that $w \in \mathbf{R}^k$ which implies $\langle w, w \rangle \geq 0$.

Case 2. If there exists some non-zero vector $x_0 \in W$ such that $\langle x_0, x_0 \rangle = 0$, then by the Schwartz inequality we have

$$\langle x_0, y \rangle^2 \leq \langle x_0, x_0 \rangle \langle y, y \rangle = 0$$

for any $y \in W$. Thus $\langle x_0, y \rangle = 0$ for any $y \in W$. This implies that the subspace W is contained in $x_0^\perp$, the orthogonal complement of x_0.

Since the vector $u = (0, ..., 0, 1, 1)^t \in \mathbf{R}^{n,1}$ satisfies $\langle u, u \rangle = 0$, there is an isometry γ of $\mathbf{R}^{n,1}$ sending x_0 to u. Thus γ sends $x_0^\perp$ to $u^\perp$. By replacing $v_i^{(m)}$ by $\gamma(v_i^{(m)})$ for each i and m, we may assume that $v_1, ..., v_{n+1}$ are contained in $u^\perp$.

For any i, since $\langle v_i, u \rangle = \langle v_i, (0, ..., 0, 1, 1)^t \rangle = 0$, we can write v_i as

$$v_i = w_i + a_i u$$

for some $w_i \in \mathbf{R}^{n-1}$ and $a_i \in \mathbf{R}$. Since $\langle w_i, u \rangle = 0$, thus $\langle v_i, v_j \rangle = w_i \cdot w_j$ for all i, j. Therefore

$$A = [v_1, ..., v_{n+1}]^t S [v_1, ..., v_{n+1}]$$
$$= [w_1, ..., w_{n+1}]^t [w_1, ..., w_{n+1}].$$

To show that $A \in \overline{\mathscr{Z}}_{n+1}$, by Lemma 5, we only need to show that $w_1, ..., w_{n+1}$ are not contained in any open half space of $\mathbf{R}^{n-1}$ which is equivalent to that $w_1, ..., w_{n+1}$ are not contained in any open half space of $\mathbf{R}^n$.

Suppose otherwise, there exists a vector $w \in \mathbf{R}^{n-1}$ such that $w_i \cdot w > 0$ for all i. Then

$$\langle v_i, w \rangle = \langle w_i, w \rangle + \langle (0, ..., 0, a_i, a_i)^t, w \rangle = w_i \cdot w + 0 > 0$$

for all i. By taking m large enough, we obtain $\langle v_i^{(m)}, w \rangle > 0$ for all i. By the same argument above, this leads to a contradiction. $\square$

4. Proof of Theorem 1.

Spherical case. We begin with a brief review of the relevant result in [4]. For any positive semi-definite symmetric matrix A, there exists a unique positive semi-definite symmetric matrix $\sqrt{A}$ so that $(\sqrt{A})^2 = A$. It is well know that the map $A \longmapsto \sqrt{A}$ is continuous on the space of all positive semi-definite symmetric matrices.

Suppose $A \in \mathscr{X}_{n+1} = \{A = [a_{ij}] \in \mathbf{R}^{(n+1) \times (n+1)} |\ A^t = A,\ \text{all } a_{ii} = 1,\ A \text{ is positive definite}\}$, the space of the angle Gram matrices of spherical simplexes (by the Proposition 2). By making a change of variables, the Aomoto-Kneser-Vinberg formula (1.1) is equivalent to

$$V(A) = \mu_n^{-1} \int_{\mathbf{R}^{n+1}} \chi(\sqrt{A}x) e^{-x^t x} dx, \qquad (4.1)$$

where χ is the characteristic function of the set $\mathbf{R}_{\geq 0}^{n+1}$ in $\mathbf{R}^{n+1}$. It is proved in [4] that the volume formula (4.1) still holds for any matrix in $\overline{\mathcal{X}}_{n+1} = \{A = [a_{ij}] \in \mathbf{R}^{(n+1)\times(n+1)} | \ A^t = A, \text{ all } a_{ii} = 1, A \text{ is positive semi-definite}\}$.

Suppose $V(A) = 0$, by formula (4.1), we see that the function $\chi \circ h : \mathbf{R}^{n+1} \to \mathbf{R}$ is zero almost everywhere, where $h : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the linear map sending x to $\sqrt{A}x$. Equivalently, the $(n+1)$-dimensional Lebesgue measure of $h^{-1}(\mathbf{R}_{\geq 0}^{n+1})$ is zero. We claim that $h(\mathbf{R}^{n+1}) \cap \mathbf{R}_{>0}^{n+1} = \emptyset$. For otherwise, $h^{-1}(\mathbf{R}_{>0}^{n+1})$ is a nonempty open subset in $\mathbf{R}^{n+1}$ with positive $(n+1)$-dimensional Lebesgue measure. This is a contradiction.

Now let $\sqrt{A} = [v_1, ..., v_{n+1}]_{(n+1)\times(n+1)}^t$, where $v_i \in \mathbf{R}^{n+1}$ is a column vector for each i. First $h(\mathbf{R}^{n+1}) \cap \mathbf{R}_{>0}^{n+1} = \emptyset$ implies that $\det \sqrt{A} = 0$. Therefore $\{v_1, ..., v_{n+1}\}$ are linearly dependent. We may assume, after a rotation $r \in O(n+1)$, the vectors $v_1, ..., v_{n+1}$ lie in $\mathbf{R}^n \times \{0\}$. Now

$$\begin{aligned}
\emptyset &= h(\mathbf{R}^{n+1}) \cap \mathbf{R}_{>0}^{n+1} \\
&= \{\sqrt{A}w | w \in \mathbf{R}^{n+1}\} \cap \mathbf{R}_{>0}^{n+1} \\
&= \{(v_1 \cdot w, ..., v_{n+1} \cdot w)^t | w \in \mathbf{R}^{n+1}\} \cap \mathbf{R}_{>0}^{n+1}.
\end{aligned}$$

This shows that there is no $w \in \mathbf{R}^{n+1}$ such that $v_i \cdot w > 0$ for $i = 1, ..., n+1$, i.e., the vectors $v_1, ..., v_{n+1}$ are not in any open half space. By Lemma 5, we have $(v_1, ..., v_{n+1}) \in \overline{\mathcal{E}}_{n+1}$, therefore $A = [v_i \cdot v_j] \in \overline{\mathcal{Z}}_{n+1}$.

Hyperbolic case. Let $A \in \overline{\mathcal{Y}}_{n+1}$. If $\det(A) \neq 0$, it is proved in [4] that the volume formula (1.1) still holds for A. In formula (1.1), since $-x^t \text{adj}(A)x$ is finite, the integrand $e^{-x^t \text{adj}(A)x} > 0$. Therefore the integral $\int_{\mathbf{R}_{\geq 0}^{n+1}} e^{-x^t \text{adj}(A)x} dx > 0$. Hence $V(A) > 0$.

It follows that if the extended volume function vanishes at A, then $\det A = 0$. By Lemma 7, we have $A \in \overline{\mathcal{Z}}_{n+1}$.

5. Proof of Theorem 3.

Proof of (a). If $A \in \overline{\mathcal{Z}}_{n+1}$, then $A = [v_i \cdot v_j]$ for some point $(v_1, ..., v_{n+1}) \in \overline{\mathcal{E}}_{n+1}$. By Lemma 5, the linear system $\sum_{i=1}^{n+1} a_i v_i = 0, a_i \geq 0, i = 1, ..., n+1$ has a nonzero solution. Let $(a_1, ..., a_{n+1})$ be a solution with the least number of nonzero entries among all solutions. By reindexing, we may assume $a_1 > 0, ..., a_{k+1} > 0, a_{k+2} = ... = a_{n+1} = 0$. We claim $\text{rank}[v_1, ..., v_{k+1}] = k$. Otherwise $\text{rank}[v_1, ..., v_{k+1}] \leq k-1$, then the dimension of the solution space $W = \{(x_1, ..., x_{k+1})^t \in \mathbf{R}^{k+1} | \sum_{i=1}^{k+1} x_i v_i = 0\}$ is at least 2. Thus $\Omega = W \cap \mathbf{R}_{>0}^{k+1}$ is a nonempty open convex set in W whose dimension is at least 2. Hence Ω has a boundary point $(b_1, ..., b_{k+1}) \in \overline{\Omega} - \{(0, ..., 0)\}$ with some $b_j = 0$, due to $\dim W \geq 2$. Now we obtain a solution $(b_1, ...b_{j-1}, 0, b_{j+1}, ..., b_{k+1}, 0, ..., 0)$ which has less nonzero entries than $(a_1, ..., a_{n+1})$. This is a contradiction.

Let $B = [v_i \cdot v_j]_{(k+1)\times(k+1)}$. Since $\text{rank}[v_1, ..., v_{k+1}] = k$, we have $\det(B) = 0$. We claim that $\text{adj}(B) > 0$. This will verify condition (a) in Theorem 3 for A. Let $\text{adj}(B) = [b_{ij}]_{(k+1)\times(k+1)}$.

We claim that any k vectors of $v_1, ..., v_{k+1}$ are linearly independent. This follows from the fact that $\sum_{i=1}^{k+1} a_i v_i = 0$, with $a_i > 0$ for each i, and $\text{rank}[v_1, ..., v_{k+1}] = k$. For example, to see that $v_1, ..., v_k$ are linearly independent, observe that $v_{k+1} = \sum_{i=1}^{k}(-a_i/a_{k+1})v_i$. Hence $\text{Span}\{v_1, ..., v_{k+1}\} = \text{Span}\{v_1, ..., v_k\}$. Now $\dim(\text{Span}\{v_1, ..., v_{k+1}\}) = k$, and so $\dim(\text{Span}\{v_1, ..., v_k\}) = k$. Therefore $v_1, ..., v_k$ are linearly independent.

Therefore $b_{jj} > 0$ for each j by Exercise 8.1 number 12 and Proposition 10.5 of [6].

Since $\text{rank}[v_1, ..., v_{k+1}] = k$, it follows that the dimension of the solution space of $\sum_{i=1}^{k+1} a_i v_i = 0$ is 1. Since $\sum_{j=1}^{k+1} b_{ij} v_j = 0$, $(b_{i1}, ..., b_{ik+1})$ is proportional to $(a_1, ..., a_{k+1})$, where $a_i > 0$ for $1 \leq i \leq k + 1$. This shows $\text{adj}(B) > 0$.

Conversely, if A is positive semi-definite so that $\det(A) = 0$ and there exists a principal $(k + 1) \times (k + 1)$ submatrix B so that $\det(B) = 0$ and $\text{adj}(B) > 0$, we will show that $A \in \overline{\mathcal{Z}}_{n+1}$. Since A is positive semi-definite and unidiagonal, there exist unit vectors $v_1, ..., v_{n+1}$ in $\mathbf{R}^n$ such that $A = [v_i \cdot v_j]$. We may assume $B = [v_i \cdot v_j]_{(k+1) \times (k+1)}, 1 \leq i, j \leq k + 1$ and $\text{adj}(B) = [b_{ij}]$.

Due to $\det(B) = 0$ and $\text{adj}(B) > 0$, we have $\sum_{s=1}^{k+1} b_{1s} v_s = 0$ with $b_{1j} > 0$ for each $j = 1, ..., k + 1$. In fact, note that $\text{adj}(B)B = \det(B)I = 0$. Hence $\sum_{j=1}^{k+1} b_{ij} v_j \cdot v_l = 0$ for each i and l. Hence $\sum_{j=1}^{k+1} b_{ij} v_j$ is perpendicular to $\text{Span}\{v_1, ..., v_{k+1}\}$. Therefore $\sum_{j=1}^{k+1} b_{ij} v_j \cdot \sum_{j=1}^{k+1} b_{ij} v_j = 0$, and so $\sum_{j=1}^{k+1} b_{ij} v_j = 0$ for each i.

Thus we get a nonzero solution for the linear system $\sum_{i=1}^{n+1} a_i v_i = 0, a_i \geq 0, i = 1, ..., n + 1$.

Proof of (b). If $A \in \overline{\mathcal{X}}_{n+1} - \mathcal{X}_{n+1}$, then $A = [v_i \cdot v_j]$ where $v_1, ..., v_{n+1}$ are linearly dependent. We can assume $v_1, ..., v_{n+1}$ lie in $\mathbf{R}^n \times \{0\}$. By reindexing, we may assume $\sum_{i=1}^{n+1} a_i v_i = 0$ has a non-zero solution with $a_i \geq 0$ if $i = 1, ..., k$ while $a_i < 0$ if $i = k + 1, ..., n + 1$. Thus the vectors $v_1, ..., v_k, -v_{k+1}, ..., -v_{n+1}$ satisfy the condition (5.2) in Lemma 5. Let D be the diagonal matrix $\text{diag}(1, ..., 1, -1, ..., -1)$ with k diagonal entries being 1 and $n - k + 1$ diagonal entries being -1. Thus by Lemma 5, $DAD \in \overline{\mathcal{Z}}_{n+1}$.

On the other hand, if for some diagonal matrix D in Theorem 3 (b), we have $DAD \in \overline{\mathcal{Z}}_{n+1}$, then by Theorem 3 (a), DAD is positive semi-definite. Therefore A is positive semi-definite. Take $B \in \mathcal{X}_{n+1}$ and consider the family $A(t) = (1 - t)A + tB$ for $t \in [0, 1]$. Then $\lim_{t \to 0} A(t) = A$ and $A(t) \in \mathcal{X}_{n+1}$ for $t > 0$. Thus $A \in \overline{\mathcal{X}}_{n+1}$.

Proof of (c). First we show that the conditions are sufficient. Suppose $A = [a_{ij}]_{(n+1) \times (n+1)}$ is a symmetric unidiagonal matrix with all principal $n \times n$ submatrices positive semi-definite so that either $A \in \overline{\mathcal{Z}}_{n+1}$ or $\det(A) < 0$ and $\text{adj}(A) \geq 0$. We will show $A \in \overline{\mathcal{Y}}_{n+1}$.

Case 1. If $A \in \overline{\mathcal{Z}}_{n+1}$, it is sufficient to show that $\mathcal{Z}_{n+1} \subset \overline{\mathcal{Y}}_{n+1}$, i.e., we may assume $A \in \mathcal{Z}_{n+1}$. In this case, let $J = [c_{ij}]_{(n+1) \times (n+1)}$ so that $c_{ii} = 1$ and $c_{ij} = -1$ for $i \neq j$. Consider the family $A(t) = (1 - t)A + tJ$, for $0 \leq t \leq 1$. Evidently $\lim_{t \to 0} A(t) = A$. We claim that $A(t) \in \mathcal{Y}_{n+1}$ for small $t > 0$. Since all principal $n \times n$ submatrices of A are positive definite, by continuity, all principal $n \times n$ submatrices of $A(t)$ are positive definite for small $t > 0$. It remains to check that $\det(A(t)) < 0$ for small $t > 0$. To this end, let us consider $\frac{d}{dt}|_{t=0} \det(A(t))$. We have

$$\frac{d}{dt}|_{t=0} \det(A(t)) = \sum_{i \neq j} (-a_{ij} - 1)\text{cof}(A)_{ij} < 0,$$

due to $\text{adj}(A) = [\text{cof}(A)_{ij}] > 0$ and $-a_{ij} - 1 < 0$ for all $i \neq j$. Since $\det(A) = 0$ it follows that $\det(A(t)) < 0$ for small $t > 0$.

Case 2. If $\det(A) < 0$ and $\text{adj}(A) \geq 0$ and all principal $n \times n$ submatrices of A are positive semi-definite, then A has a unique negative eigenvalue $-\lambda$, where $\lambda > 0$. Consider the family $A(t) = A + t\lambda I$, for $0 \leq t \leq 1$, where I is the identity matrix.

Let C be an orthogonal matrix such that CAC^{-1} is diagonal. Now $CA(t)C^{-1} = CAC^{-1}+t\lambda I$. This shows that C also diagonalizes $A(t)$ and the eigenvalues of $A(t)$ are the eigenvalues of A shifted by $t\lambda$. Thus $\det(A(t)) < 0$ for $0 < t < 1$ and $\det(A(1)) = 0$.

Furthermore, all principal $n \times n$ submatrices of $A(t)$ are positive definite for $t > 0$ due to positive definiteness of $t\lambda I$.

Let us recall Lemma 3.4 in [4] which says that if B is a symmetric $(n+1) \times (n+1)$ matrix so that all $n \times n$ principal submatrices in B are positive definite and $\det(B) \leq 0$, then no entry in the adjoint matrix $\mathrm{adj}(B)$ is zero. It follows that every entry of $\mathrm{adj}(A(t))$ is nonzero for $0 < t \leq 1$.

Let $\mathrm{adj}(A(1)) = [b_{ij}]_{(n+1)\times(n+1)}$ and D be the diagonal matrix with diagonal entries being

$$\frac{b_{1i}}{|b_{1i}|} = \pm 1$$

for $i = 1, ..., n+1$. Then the entries of the first row and the first column of $D\mathrm{adj}(A(1))D$ are positive. Since $\det(A(1)) = 0$ and $\mathrm{adj}(A(1)) \neq 0$, we see the rank of $\mathrm{adj}(A(1))$ is 1. Thus any other column is proportional to the first column. But $b_{ii} > 0$ for all i, hence $\mathrm{adj}(A(1)) > 0$.

Now since every entry of $D\mathrm{adj}(A(t))D$ is nonzero for $t > 0$, by continuity $D\mathrm{adj}(A(t))D > 0$ for $t > 0$ and $D\mathrm{adj}(A)D = D\mathrm{adj}(A(0))D \geq 0$. By the assumption $\mathrm{adj}(A) \geq 0$, it follows $D\mathrm{adj}(A)D = \mathrm{adj}(A)$. On the other hand $D\mathrm{adj}(A)D = \mathrm{adj}(D^{-1}AD^{-1})$, and $\det(A) \neq 0$. Thus $D^{-1}AD^{-1} = A$ or the same $A = DAD$. This shows

$$A = DAD$$
$$= \lim_{t \to 0} DA(t)D$$
$$= \lim_{t \to 0} \frac{1}{1 + \lambda t} DA(t)D.$$

By the construction above $\frac{1}{1+\lambda t} DA(t)D \in \mathcal{Y}_{n+1}$ for $0 < t < 1$, this shows $A \in \overline{\mathcal{Y}}_{n+1}$.

Finally, we show the condition in (c) is necessary. Suppose $A = \lim_{m \to \infty} A^{(m)}$ where $A^{(m)} \in \mathcal{Y}_{n+1}$. By Proposition 2, $\det(A^{(m)}) < 0$, $\mathrm{adj}(A^{(m)}) > 0$ and all principal $n \times n$ submatrices of $A^{(m)}$ are positive definite. We want to show that A satisfies the conditions stated in (c). Evidently, all principal $n \times n$ submatrices of A are positive semi-definite, $\mathrm{adj}(A) \geq 0$ and $\det(A) \leq 0$. If $\det(A) < 0$, then we are done. If $\det(A) = 0$, by Lemma 7, we see that $A \in \overline{\mathcal{Z}}_{n+1}$.

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