

A COMPACTIFICATION OF THE SPACE OF PARAMETRIZED RATIONAL CURVES IN GRASSMANNIANS*

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Abstract. We construct a nonsingular projective variety that explicitly compactifies the space of parametrized rational curves in a Grassmannian such that the boundary is a divisor with simple normal crossings. This compactification is obtained by blowing up a Quot scheme successively along its appropriate subschemes.

Key words. Compactification, quot scheme, blow-up, normal crossing divisor, moduli space.

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1. Introduction. Let V be a vector space of dimension n over an algebraically closed field \mathbb{k} and denote by $\text{Gr}(k, V)$ the Grassmannian of k -dimensional subspaces of V . By a parametrized rational curve in $\text{Gr}(k, V)$ we mean a morphism from \mathbb{P}^1 to $\text{Gr}(k, V)$. The space $\text{Mor}_d(\mathbb{P}^1, \text{Gr}(k, V))$ of all morphisms of degree d from \mathbb{P}^1 to $\text{Gr}(k, V)$ is a smooth quasi-projective variety for each $d \geq 0$, and is non-compact for $d \geq 1$. In this paper, we provide an explicit compactification for this space which satisfies the following conditions: the compactification is a nonsingular projective variety and the boundary is a divisor with simple normal crossings. This extends the result in [HLS], where maps to a projective space are considered.

The main tools of the construction include (i) a Quot scheme which serves as a smooth compactification of $\text{Mor}_d(\mathbb{P}^1, \text{Gr}(k, V))$, and (ii) a sequence of blowups which turns the boundary into a divisor with simple normal crossings. The method of construction is as follows. A smooth compactification of $\text{Mor}_d(\mathbb{P}^1, \text{Gr}(k, V))$ is given by the Quot scheme $\text{Quot}_{V_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{k}}^{n-k,d}$, which parametrizes all quotient sheaves of the trivial vector bundle $V_{\mathbb{P}^1}$ on \mathbb{P}^1 of rank $n - k$ and degree d . For short notations, we introduce $Q_d := \text{Quot}_{V_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{k}}^{n-k,d}$ and $\mathring{Q}_d := \text{Mor}_d(\mathbb{P}^1, \text{Gr}(k, V))$. We define a chain of closed subschemes in the boundary $Q_d \setminus \mathring{Q}_d$:

$$Z_{d,0} \subset Z_{d,1} \subset \cdots \subset Z_{d,d-1} = Q_d \setminus \mathring{Q}_d.$$

Set-theoretically, $Z_{d,r}$ is the locus of the quotients whose torsion part has degree (or length) at least $d - r$; its scheme structure is given as a determinantal subscheme. We then blow up the Quot scheme successively along these closed subschemes $Z_{d,r}$. Starting with $Q_d^{-1} := Q_d$, $Z_{d,r}^{-1} := Z_{d,r}$, we let Q_d^r ($r = 0, \dots, d - 1$) be the blowup of Q_d^{r-1} along $Z_{d,r}^{r-1}$, $Z_{d,r}^r$ the exceptional divisor, and $Z_{d,l}^r$ the proper transform of $Z_{d,l}^{r-1}$ in Q_d^r for $l \neq r$. Set $\tilde{Q}_d := Q_d^{d-1}$ and $\tilde{Z}_{d,r} := Z_{d,r}^{d-1}$. Our goal is to prove

THEOREM 1.1. *The final outcome \tilde{Q}_d is a compactification of $\mathring{Q}_d = \text{Mor}_d(\mathbb{P}^1, \text{Gr}(k, V))$ satisfying the following properties:*

1. \tilde{Q}_d is an irreducible nonsingular projective variety;
2. $\tilde{Z}_{d,r}$'s are irreducible nonsingular subvarieties of codimension one and they intersect transversally, so that the boundary $\tilde{Q}_d \setminus \mathring{Q}_d = \bigcup_{r=0}^{d-1} \tilde{Z}_{d,r}$ is a divisor with simple normal crossings.

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This way of producing such type of compactifications was used in many classical constructions such as the space of complete quadrics [Vai82], the space of complete collineations [Vai84], Fulton-MacPherson compactification [FM94], MacPherson-Procesi compactification [MP98], etc. Our construction is much closer to that of the spaces of complete objects (e.g. complete quadric, etc.), and in fact, we used in our proof a result from Vainsencher’s construction of complete collineations. Because of the strong similarity between our compactification and those spaces of complete objects, we believe that our compactification is also a parameter space of similar kind, although the interpretation of the objects that this space parametrizes can be quite subtle. This is a direction being pursued for a future publication [HS].

Although the method of the construction is straightforward to describe, the proof is more involved. We outline some key steps of the proof here. In §3, we endow the sets $Z_{d,r}$ with the structure of determinantal subschemes. This is done with the help of the universal exact sequence of Q_d : $0 \rightarrow \mathcal{E}_d \rightarrow V_{\mathbb{P}^1_{Q_d}} \rightarrow \mathcal{F}_d \rightarrow 0$, where \mathcal{E}_d is locally free. For each integer m , there is an induced homomorphism $\rho_{d,m} : \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m) \rightarrow \pi_* \mathcal{E}_d^\vee(m)$. Then $Z_{d,r}$ is defined to be the zero locus of the exterior power $\bigwedge^{k(m+1)+r+1} \rho_{d,m}$ for any $m \gg 0$. Let $\mathring{Z}_{d,r} := Z_{d,r} \setminus Z_{d,r-1}$ be the open subscheme of $Z_{d,r}$. We can show that the locally closed subschemes $Z_{d,0}, \mathring{Z}_{d,1}, \dots, \mathring{Z}_{d,d-1}, \mathring{Q}_d$ form a flattening stratification of Q_d by the sheaf $\mathcal{F}_d^\varepsilon := \mathcal{E}xt^1(\mathcal{F}_d, \mathcal{O}_{\mathbb{P}^1_{Q_d}})$. In §4, we show that there is a natural morphism from an irreducible nonsingular variety $Q_{d,r}$ to Q_d whose (set-theoretic) image is $Z_{d,r}$ for each r . Here $Q_{d,r}$ is a relative Quot scheme over Q_r parametrizing torsion quotients of \mathcal{E}_r that are flat over Q_r with degree $d - r$. This morphism maps the open subset $Q_{d,r} \times_{Q_r} \mathring{Q}_r$ of $Q_{d,r}$ isomorphically onto $\mathring{Z}_{d,r}$, based on the fact that $\mathcal{F}_d^\varepsilon$ is flat over $\mathring{Z}_{d,r}$. In §5 we first describe the procedure of blowups, and then prove some main theorems. We first show that the blowup along $Z_{d,r}^{r-1}$ (the proper transform of $Z_{d,r}$ in Q_d^{r-1}) is the same as the blowup along the total transform of $Z_{d,r}$ in Q_d^{r-1} . This allows us to embed the blowup Q_d^r into some product of spaces of collineations and then set up a key commutative diagram (see Diagram (5.2)). Using this diagram, we can show that the proper transform $Z_{d,r}^{r-1}$ is isomorphic to $Q_{d,r} \times_{Q_r} Q_r^{r-1}$. It is now clear that we should use induction to prove that $Z_{d,r}^{r-1}$ and Q_d^r are nonsingular. The proof for the transversality of the intersections of $Z_{d,r}^{d-1}$ ’s in Q_d^{d-1} is also based on the isomorphism between $Z_{d,r}^{r-1}$ and $Q_{d,r} \times_{Q_r} Q_r^{r-1}$.

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2. Preliminaries.

2.1. Quot schemes. In this section, we will first give a brief review of Quot schemes, and then we fix some notations and review some basic facts.

Let H be a sheaf over a scheme X . A quotient of H is a sheaf F on X together with a surjective homomorphism $H \twoheadrightarrow F$. Two quotients $H \twoheadrightarrow F_1$ and $H \twoheadrightarrow F_2$ are said to be *equivalent* if there is an isomorphism $F_1 \simeq F_2$ such that the following diagram commutes.

$$\begin{array}{ccc} H & \twoheadrightarrow & F_1 \\ \parallel & & \downarrow \simeq \\ H & \twoheadrightarrow & F_2 \end{array}$$

Equivalently, the two quotients are equivalent if and only if their kernels are equal as subsheaves of H . We denote by $[H \twoheadrightarrow F]$ the equivalence class of a quotient $H \twoheadrightarrow F$.

Let S be a noetherian scheme, X a projective S -scheme, and H a coherent sheaf on X . For any S -scheme T , we denote by H_T the pullback of H by the projection $X \times_S T \rightarrow X$. Given a very ample line bundle L on X relative to S and a polynomial $P(t) \in \mathbb{Q}[t]$, we define a contravariant functor, denoted by $\mathcal{Q}uot_{H/X/S}^{P,L}$ and called the Quot functor, from the category of S -schemes to the category of sets as follows: for any S -scheme T , let $\mathcal{Q}uot_{H/X/S}^{P,L}(T)$ be the set of equivalence classes of quotients $H_T \twoheadrightarrow F$ where F is flat over T with Hilbert polynomial P (relative to L).

This functor is represented by a projective S -scheme [Gro62], called the (relative) Quot scheme, which we denote by $\mathcal{Q}uot_{H/X/S}^{P,L}$. It is equipped with a *universal quotient*

$$H_{\mathcal{Q}uot_{H/X/S}^{P,L}} \twoheadrightarrow \mathcal{F},$$

on $X \times_S \mathcal{Q}uot_{H/X/S}^{P,L}$, where \mathcal{F} is flat over $\mathcal{Q}uot_{H/X/S}^{P,L}$ with P as its Hilbert polynomial. Sometimes it will bring us convenience to add to the universal quotient its kernel to form the *universal exact sequence*:

$$0 \rightarrow \mathcal{E} \rightarrow H_{\mathcal{Q}uot_{H/X/S}^{P,L}} \rightarrow \mathcal{F} \rightarrow 0.$$

THEOREM 2.1 (Universal Property of Quot schemes). *For any S -scheme Y , any quotient $H_Y \twoheadrightarrow \mathcal{Q}$ on $X \times_S Y$ with \mathcal{Q} flat over Y with Hilbert polynomial P determines a unique S -morphism $f : Y \rightarrow \mathcal{Q}uot_{H/X/S}^{P,L}$ such that the pullback of the universal quotient by the induced morphism $\bar{f} : X \times_S Y \rightarrow X \times_S \mathcal{Q}uot_{H/X/S}^{P,L}$ is equivalent to the given quotient on $X \times_S Y$:*

$$\begin{array}{ccc} H_Y & \twoheadrightarrow & \mathcal{Q} \\ \parallel & & \downarrow \simeq \\ H_Y = \bar{f}^* H_{\mathcal{Q}uot_{H/X/S}^{P,L}} & \twoheadrightarrow & \bar{f}^* \mathcal{F} \end{array}$$

A consequence of the uniqueness of the morphism is: if $f_1, f_2 : Y \rightarrow \mathcal{Q}uot_{H/X/S}^{P,L}$ are two S -morphisms such that the pullbacks of the universal quotient are equivalent:

$$\begin{array}{ccc} H_Y = \bar{f}_1^* H_{\mathcal{Q}uot_{H/X/S}^{P,L}} & \twoheadrightarrow & \bar{f}_1^* \mathcal{F} \\ \parallel & & \downarrow \simeq \\ H_Y = \bar{f}_2^* H_{\mathcal{Q}uot_{H/X/S}^{P,L}} & \twoheadrightarrow & \bar{f}_2^* \mathcal{F} \end{array}$$

then $f_1 = f_2$. Note that it is not sufficient to claim $f_1 = f_2$ if we only know an isomorphism $\bar{f}_1^* \mathcal{F} \simeq \bar{f}_2^* \mathcal{F}$. The commutativity of the above diagram is crucial.

In the special case that $X = S$ and $P(t)$ is a constant integer r , the Quot scheme becomes a (relative) Grassmannian, which we denote by $\text{Gr}_S(H, r)$. If in addition H is locally free, we also denote this Grassmannian by $\text{Gr}_S(m, H)$ where $m = \text{rank } H - r$.

Most references (for example, [HL97, Nit05, Ser06]) prove the representability of the Quot functor by constructing the Quot scheme as a closed subscheme of a Grassmannian. Therefore a natural consequence of this construction is the embedding

of the Quot scheme into a Grassmannian, but this fact is rarely isolated as a theorem. For convenience of reference, we state this fact below.

THEOREM 2.2 (Embedding of Quot into Grassmannian). *Suppose H is flat over S with Hilbert polynomial $R(t) \in \mathbb{Q}[t]$ and put $Q := \text{Quot}_{H/X/S}^{P,L}$. Let $\pi : X \rightarrow S$ and $\sigma : Q \rightarrow S$ be the structure morphisms, and $\pi_Q : X \times_S Q \rightarrow Q$ the projection. There is an integer N such that for all $m \geq N$*

1. $\pi_{Q*}(H_Q(m)) = \sigma^* \pi_*(H(m))$,
2. $\pi_{Q*}(H_Q(m))$ and $\pi_{Q*}(\mathcal{F}(m))$ are locally free of rank $R(m)$ and $P(m)$ respectively, and the induced homomorphism $\pi_{Q*}(H_Q(m)) \rightarrow \pi_{Q*}(\mathcal{F}(m))$ is surjective, and
3. the morphism $Q \rightarrow \text{Gr}_S(\pi_*(H(m)), P(m)) = \text{Gr}_S(R(m) - P(m), \pi_*(H(m)))$ induced by the surjective homomorphism $\sigma^* \pi_*(H(m)) \rightarrow \pi_{Q*}(\mathcal{F}(m))$ is a closed embedding.

When $S = \text{Spec } \mathbb{k}$, \mathbb{k} a field, the Quot scheme $\text{Quot}_{H/X/\mathbb{k}}^{P,L}$ is a fine moduli space whose \mathbb{k} -valued points parametrize equivalence classes of quotients $H \twoheadrightarrow F$ with F having Hilbert polynomial P . In the case that S is a general scheme, $\text{Quot}_{H/X/S}^{P,L}$ can be viewed as a family of Quot schemes parametrized by S : for any point $s \in S$, the fiber of $\text{Quot}_{H/X/S}^{P,L}$ over s is $\text{Quot}_{H_s/X_s/\kappa(s)}^{P,L_s}$, where X_s is the fiber of X over s and $\kappa(s)$ is the residue field of s . This is due to the base-change property of Quot schemes: for any S -scheme T , $\text{Quot}_{H/X/S}^{P,L} \times_S T = \text{Quot}_{H_T/X \times_S T/T}^{P,L_T}$.

2.2. Notations and facts. We now fix some notations. Let \mathbb{k} be a fixed algebraically closed field and \mathbb{P}^1 the projective line over \mathbb{k} . For any \mathbb{k} -scheme X , we write $\mathbb{P}_X^1 := \mathbb{P}^1 \times_{\mathbb{k}} X$ and denote the projection $\mathbb{P}_X^1 \rightarrow X$ by π_X , or simply π . For any point $x \in X$, we denote by \mathbb{P}_x^1 the fiber $\pi_X^{-1}(x) = \mathbb{P}^1 \times_{\mathbb{k}} \kappa(x)$ over x . For any coherent sheaf F over \mathbb{P}_X^1 , we write $F(m) := F \otimes p^* \mathcal{O}_{\mathbb{P}^1}(m)$, where p denotes the projection $\mathbb{P}_X^1 \rightarrow \mathbb{P}^1$.

For any morphism $f : Y \rightarrow X$ of \mathbb{k} -schemes, we denote by $\bar{f} : \mathbb{P}_X^1 \rightarrow \mathbb{P}_Y^1$ the morphism $1 \times f : \mathbb{P}^1 \times_{\mathbb{k}} X \rightarrow \mathbb{P}^1 \times_{\mathbb{k}} Y$. If $Z \subset X$ is a subscheme and F a coherent sheaf on \mathbb{P}_X^1 , we denote by F_Z the restriction of F to \mathbb{P}_Z^1 . In particular, if $x \in X$ is a point, then F_x is the restriction of F to \mathbb{P}_x^1 .

In this paper, we will only encounter Quot schemes of the form $\text{Quot}_{H/\mathbb{P}_S^1/S}^{P, \mathcal{O}_{\mathbb{P}_S^1}(1)}$, where S is either $\text{Spec } \mathbb{k}$ or a variety over \mathbb{k} . For any coherent sheaf F on \mathbb{P}^1 , its Hilbert polynomial has the form $P(t) = r(t+1) + d$, where $r = \text{rank } F$ and $d = \text{deg } F$. So we abbreviate the notation:

$$\text{Quot}_{H/\mathbb{P}_S^1/S}^{r,d} := \text{Quot}_{H/\mathbb{P}_S^1/S}^{r(t+1)+d, \mathcal{O}_{\mathbb{P}_S^1}(1)}.$$

Next we prove some basic properties of the functor $\mathcal{E}xt_X^1(-, \mathcal{O}_X)$. For any scheme X and any sheaf F over X , we put

$$F^\vee := \mathcal{H}om_X(F, \mathcal{O}_X), \quad F^\varepsilon := \mathcal{E}xt_X^1(F, \mathcal{O}_X).$$

PROPOSITION 2.3. *Let $0 \rightarrow E_1 \rightarrow E_0 \rightarrow F \rightarrow 0$ be a short exact sequence of coherent sheaves on a scheme X with E_0 and E_1 locally free. For any scheme Y and any morphism $f : Y \rightarrow X$, we have a canonical identification*

$$f^*(F^\varepsilon) = (f^*F)^\varepsilon.$$

Proof. Since E_0 and E_1 are both locally free, we have canonical identifications

$$f^*(E_i^\vee) = (f^*E_i)^\vee, \quad i = 0, 1.$$

Thus we obtain a commutative diagram

$$\begin{array}{ccccccc} (f^*E_0)^\vee & \longrightarrow & (f^*E_1)^\vee & \longrightarrow & (f^*F)^\varepsilon & \longrightarrow & 0, \\ \parallel & & \parallel & & \vdots & & \\ f^*(E_0^\vee) & \longrightarrow & f^*(E_1^\vee) & \longrightarrow & f^*(F^\varepsilon) & \longrightarrow & 0 \end{array}$$

where the first row is obtained by applying f^* to the exact sequence first and taking dual second, and the second row is obtained by taking dual of the original exact sequence first and applying f^* second. So we have another canonical identification $f^*(F^\varepsilon) = (f^*F)^\varepsilon$ because they are both quotients of $f^*E_0^\vee$ by the image of $f^*E_1^\vee$. \square

REMARK. For any locally free sheaf E , since we have a canonical identification $f^*(E^\vee) = (f^*E)^\vee$, we will write f^*E^\vee for both. In the same manner, for any coherent sheaf F , when we have a canonical identification $f^*(F^\varepsilon) = (f^*F)^\varepsilon$, we will write f^*F^ε for both.

PROPOSITION 2.4. *Suppose X is a noetherian \mathbb{k} -scheme, and \mathcal{T} is a torsion coherent sheaf on \mathbb{P}_X^1 , flat over X with relative degree d . Then*

1. \mathcal{T}^ε is also torsion and flat over X with relative degree d ;
2. we have a canonical isomorphism $\mathcal{T}^{\varepsilon\varepsilon} = \mathcal{T}$.

Proof. Write \mathcal{T} as a quotient of a locally free sheaf \mathcal{E}_0 and let \mathcal{E}_1 be the kernel:

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{T} \rightarrow 0. \tag{2.1}$$

Then \mathcal{E}_1 is also locally free. Since $\text{rank } \mathcal{T} = 0$, we have $\text{rank } \mathcal{E}_0 = \text{rank } \mathcal{E}_1$. Suppose \mathcal{E}_i is of relative degree d_i over X , $i = 0, 1$. Then $d = d_0 - d_1$.

(1) Dualizing the exact sequence (2.1), we obtain another exact sequence

$$0 \rightarrow \mathcal{E}_0^\vee \rightarrow \mathcal{E}_1^\vee \rightarrow \mathcal{T}^\varepsilon \rightarrow 0. \tag{2.2}$$

To show that \mathcal{T}^ε is flat over X , we only need to show, by Lemma 2.1.4 in [HL97], that for every point $x \in X$, the map $(\mathcal{E}_0)_x^\vee \rightarrow (\mathcal{E}_1)_x^\vee$ obtained by restricting the injective map $\mathcal{E}_0^\vee \rightarrow \mathcal{E}_1^\vee$ to the fiber \mathbb{P}_x^1 is injective. We first restrict the exact sequence (2.1) to the fiber \mathbb{P}_x^1 . Since \mathcal{T} is flat, we obtain an exact sequence

$$0 \rightarrow (\mathcal{E}_1)_x \rightarrow (\mathcal{E}_0)_x \rightarrow \mathcal{T}_x \rightarrow 0.$$

Taking dual:

$$0 = (\mathcal{T}_x)^\vee \rightarrow (\mathcal{E}_0)_x^\vee \rightarrow (\mathcal{E}_1)_x^\vee \rightarrow \mathcal{T}_x^\varepsilon \rightarrow 0.$$

So $(\mathcal{E}_0)_x^\vee \rightarrow (\mathcal{E}_1)_x^\vee$ is injective for all $x \in X$. Hence \mathcal{T}^ε is flat over X . Since \mathcal{E}_i^\vee are both locally free of the same rank and has relative degree $-d_i$, $i = 0, 1$, \mathcal{T}^ε is torsion on \mathbb{P}_X^1 with relative degree $(-d_1) - (-d_0) = d$ over X .

(2) We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 & \longrightarrow & \mathcal{T} \longrightarrow 0 \\ & & \parallel & & \parallel & & \vdots \\ 0 & \longrightarrow & \mathcal{E}_1^{\vee\vee} & \longrightarrow & \mathcal{E}_0^{\vee\vee} & \longrightarrow & \mathcal{T}^{\varepsilon\varepsilon} \longrightarrow 0 \end{array}$$

where the second row is an exact sequence obtained by dualizing the exact sequence (2.2). Thus we see that there is a canonical isomorphism $\mathcal{T}^{\varepsilon\varepsilon} = \mathcal{T}$. \square

Lastly, we review a slightly extended result of Mumford on flattening stratifications, as stated in the following proposition. Parts (1)-(3) are due to Mumford, and part (4) is added for the purpose of this paper.

PROPOSITION 2.5. *Let X be a noetherian scheme, F a coherent sheaf on X , and $R = \{\dim_{\kappa(x)} F|_x \mid x \in X\}$. For each $r \in R$, define $C_r := \{x \in X \mid \dim_{\kappa(x)} F|_x = r\}$. Then*

1. R is a finite set. The sets C_r are disjoint and their union is $|X|$.
2. For each $x \in C_r$, there is an affine open set $U_x \ni x$ together with an exact sequence of sheaves on U_x

$$\mathcal{O}_{U_x}^s \xrightarrow{\psi} \mathcal{O}_{U_x}^r \xrightarrow{\varphi} F|_{U_x} \rightarrow 0$$

such that the ideal J_{r,U_x} generated by all entries of ψ defines a subscheme Y_{r,U_x} of U_x whose support is exactly $C_r \cap U_x$. The subschemes Y_{r,U_x} for all $x \in C_r$ patch together to form a subscheme of X , whose support is exactly C_r . Denote this subscheme by X_r .

3. The subschemes X_r satisfy the following universal property: for any morphism $f : Y \rightarrow X$ of noetherian schemes, the pullback f^*F is locally free over Y of rank r if and only if f factors through the inclusion $X_r \hookrightarrow X$. In particular, $F|_{X_r}$ is a rank r locally free sheaf on X_r for each $r \in R$.
4. Assume in addition that $E_1 \xrightarrow{\rho} E_0 \rightarrow F \rightarrow 0$ is a locally free presentation of F with $\text{rank } E_0 = l$. For $j \geq 1$, let I_j be the sheaf image of the homomorphism $\text{Hom}(\bigwedge^j E_1, \bigwedge^j E_0)^\vee \rightarrow \mathcal{O}_X$ induced by $\bigwedge^j \rho : \bigwedge^j E_1 \rightarrow \bigwedge^j E_0$, and let Z_j be the closed subscheme defined by the ideal sheaf I_j . Then $Z_j \subset Z_{j+1}$ and $Z_{j+1} \setminus Z_j = X_{l-j}$ as schemes for $j \geq 1$.

Proof. (1), (2) and (3) are proved in [Mum66], Lecture 8, pg 55, Case 1°. We now prove (4). Suppose F has a locally free presentation $E_1 \xrightarrow{\rho} E_0 \rightarrow F \rightarrow 0$ on X . Then we have $|Z_j| = \{x \in X \mid \dim_{\kappa(x)} F|_x \geq l - j + 1\}$, which obviously satisfy $|Z_j| \subset |Z_{j+1}|$ and $|Z_{j+1}| \setminus |Z_j| = |X_{l-j}|$. At any point $x \in X_{l-j}$, there exists an affine open neighborhood V_x of x , possibly smaller than U_x , such that

$$E_1|_{V_x} \xrightarrow{\rho} E_0|_{V_x} \rightarrow F|_{V_x} \rightarrow 0$$

is a free presentation of F on V_x . Thus the ideal $I_{j+1}(V_x)$, which is generated by all $(j + 1) \times (j + 1)$ minors of ρ , is the same as the ideal J_x on V_x , by the Fitting ideal lemma. So $(Z_{j+1} \setminus Z_j) \cap V_x = X_{l-j} \cap V_x$ as schemes for all $x \in X_{l-j}$. Therefore $Z_{j+1} \setminus Z_j = X_{l-j}$ as schemes. \square

Before we proceed to the next section, let us mention a few more conventions and notations. Let E and F be two locally free sheaves on X . We say E is a *subbundle* of F , if there is an injective homomorphism $E \rightarrow F$ of sheaves and the quotient F/E is locally free. In this case, we write the homomorphism as $E \hookrightarrow F$ and call it a *subbundle map*.

For any locally free sheaf E on a scheme X , the *projective bundle* on X associated to E is

$$\mathbb{P}(E) := \mathbf{Proj} \text{Sym}(E^\vee).$$

It is equipped with the universal quotient line bundle $\pi^*E^\vee \rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$, where $\pi : \mathbb{P}(E) \rightarrow X$ is the structure morphism. Its dual $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \pi^*E$ is the universal subline bundle. For any coherent sheaf F on $\mathbb{P}(E)$, we write $F(m) := F \otimes \mathcal{O}_{\mathbb{P}(E)}(m)$.

3. The Quot-scheme compactification and its boundary. From now on, we work over an algebraically closed field \mathbb{k} . Let V be a vector space of dimension n over \mathbb{k} , and $k < n$ a fixed positive integer. The Grassmannian $\text{Gr} := \text{Gr}(k, V)$ of k -dimensional subspaces of V comes equipped with a universal quotient $V_{\text{Gr}} \rightarrow W$ of locally free sheaves on Gr . The *degree* of a morphism $f : \mathbb{P}^1 \rightarrow \text{Gr}$ is equal to $\deg f^*W$, which coincides with the usual degree of the map $\mathbb{P}^1 \rightarrow \mathbb{P}(\bigwedge^k V)$ obtained by composing f with the Plücker embedding $\text{Gr} \hookrightarrow \mathbb{P}(\bigwedge^k V)$. For $d \geq 0$, let $\text{Mor}_d(\mathbb{P}^1, \text{Gr})$ denote the set of all degree d morphisms from \mathbb{P}^1 to Gr . The set $\text{Mor}_d(\mathbb{P}^1, \text{Gr})$ has a natural structure of a (quasi-projective) variety, which can be realized as an open subscheme of the Hilbert scheme of $\mathbb{P}_{\mathbb{G}_r}^1$ corresponding to graphs.

There is a one-to-one correspondence between the set $\text{Mor}_d(\mathbb{P}^1, \text{Gr})$ and the set of all equivalence classes of quotients $V_{\mathbb{P}^1} \rightarrow F$ where F is locally free, which is set up by associating each degree d morphism to the pullback of the universal quotient on Gr . This identifies the space $\text{Mor}_d(\mathbb{P}^1, \text{Gr})$ with the open locus of locally free quotients in the Quot scheme $\text{Quot}_{V_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{k}}^{n-k,d}$. This Quot scheme is an irreducible nonsingular projective variety of dimension $nd + k(n - k)$ [Str87], hence it provides a smooth compactification of $\text{Mor}_d(\mathbb{P}^1, \text{Gr})$. For short notations, we put

$$Q_d := \text{Quot}_{V_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{k}}^{n-k,d}, \quad \mathring{Q}_d := \text{Mor}_d(\mathbb{P}^1, \text{Gr}) \subset Q_d.$$

The Quot scheme Q_d is equipped with a universal short exact sequence on $\mathbb{P}_{Q_d}^1$:

$$0 \rightarrow \mathcal{E}_d \rightarrow V_{\mathbb{P}_{Q_d}^1} \rightarrow \mathcal{F}_d \rightarrow 0 \tag{3.1}$$

where \mathcal{F}_d is flat over Q_d of rank $n - k$ and relative degree d , and \mathcal{E}_d is locally free of rank k and relative degree $-d$.

For any point $q \in Q_d$, the pullback of the universal exact sequence to \mathbb{P}_q^1 is also an exact sequence:

$$0 \rightarrow \mathcal{E}_{d,q} \rightarrow V_{\mathbb{P}_q^1} \rightarrow \mathcal{F}_{d,q} \rightarrow 0.$$

The boundary $Q_d \setminus \mathring{Q}_d$ consists of the points q with $\mathcal{F}_{d,q}$ having torsion. For any coherent sheaf F , we denote by F_{tor} the torsion submodule of F . We define a chain of closed subsets

$$C_{d,0} \subset C_{d,1} \subset \dots \subset C_{d,d-1} = Q_d \setminus \mathring{Q}_d$$

by setting $C_{d,r} = \{q \in Q_d \mid \deg(\mathcal{F}_{d,q})_{\text{tor}} \geq d - r\}$.

The degree of the torsion part $(\mathcal{F}_{d,q})_{\text{tor}}$ of $\mathcal{F}_{d,q}$ can be determined by the rank of certain linear maps. Fix a point $q \in Q_d$ and set $E := \mathcal{E}_{d,q}$, $F := \mathcal{F}_{d,q}$, $T := (\mathcal{F}_{d,q})_{\text{tor}}$, and $V := V_{\mathbb{P}_q^1}$. For $r \geq 0$, let

$$\rho_{q,r} : \text{Hom}(V, \mathcal{O}(r)) \rightarrow \text{Hom}(E, \mathcal{O}(r))$$

be the map obtained by applying $\text{Hom}(-, \mathcal{O}(r))$ to $E \rightarrow V$. Here $\mathcal{O}(r) := \mathcal{O}_{\mathbb{P}_q^1}(r)$. Then we have

PROPOSITION 3.1.

$$\text{rank } \rho_{q,r} \begin{cases} = k(r+1) + d - \deg T, & \text{if } r \geq d - \deg T - 1, \\ \geq (k+1)(r+1), & \text{if } 0 \leq r \leq d - \deg T - 1. \end{cases}$$

Proof. We have an isomorphism $F \simeq T \oplus F_1$, where $F_1 = F/T$ is a locally free sheaf of rank $n - k$ and degree $d - \deg T$. Applying $\text{Hom}(-, \mathcal{O}(r))$ to the exact sequence, one gets an exact sequence

$$0 \rightarrow \text{Hom}(F, \mathcal{O}(r)) \rightarrow \text{Hom}(V, \mathcal{O}(r)) \xrightarrow{\rho_{q,r}} \text{Hom}(E, \mathcal{O}(r)).$$

Thus

$$\begin{aligned} \text{rank } \rho_{q,r} &= \dim \text{Hom}(V, \mathcal{O}(r)) - \dim \text{Hom}(F, \mathcal{O}(r)) \\ &= \dim H^0(V^\vee(r)) - \dim \text{Hom}(F_1, \mathcal{O}(r)) \\ &= n(r+1) - \dim H^0(F_1^\vee(r)). \end{aligned}$$

We have an isomorphism $F_1 \simeq \bigoplus_{i=1}^{n-k} \mathcal{O}(d_i)$, where $d_i \geq 0$ and $\sum_{i=1}^{n-k} d_i = d - \deg T$. Hence

$$H^0(F_1^\vee(r)) \simeq H^0\left(\bigoplus_{i=1}^{n-k} \mathcal{O}(r - d_i)\right) = \bigoplus_{i=1}^{n-k} H^0(\mathcal{O}(r - d_i))$$

and

$$\dim H^0(F_1^\vee(r)) = \sum_{i=1}^{n-k} \max\{r - d_i + 1, 0\}.$$

If $r \geq d - \deg T - 1$, then $r - d_i + 1 \geq 0$ for all i , hence

$$\dim H^0(F_1^\vee(r)) = \sum_{i=1}^{n-k} (r - d_i + 1) = (n-k)(r+1) + \sum_{i=1}^{n-k} d_i = (n-k)(r+1) - (d - \deg T).$$

Therefore

$$\text{rank } \rho_{q,r} = n(r+1) - (n-k)(r+1) + d - \deg T = k(r+1) + d - \deg T.$$

If $0 \leq r \leq d - \deg T - 1$, then we have two cases:

(i) $r - d_i + 1 \geq 0$ for all i . Then

$$\begin{aligned} \dim H^0(F_1^\vee(r)) &= \sum_{i=1}^{n-k} (r - d_i + 1) = (n-k)(r+1) - (d - \deg T) \\ &\leq (n-k)(r+1) - (r+1) = (n-k-1)(r+1); \end{aligned}$$

(ii) $r - d_i + 1 < 0$ for some i . Without loss of generality, we assume $r - d_1 + 1 < 0$. Then

$$\dim H^0(F_1^\vee(r)) = \sum_{i=2}^{n-k} \max\{r - d_i + 1, 0\} \leq \sum_{i=2}^{n-k} (r+1) = (n-k-1)(r+1).$$

In either case, we have $\dim H^0(F_1^\vee(r)) \leq (n - k - 1)(r + 1)$, so

$$\text{rank } \rho_{q,r} \geq n(r + 1) - (n - k - 1)(r + 1) = (k + 1)(r + 1).$$

□

Note that the two “if” conditions in the proposition are not mutually exclusive, and this will lead to some interesting results.

COROLLARY 3.2. *If $\text{rank } \rho_{q,i} \leq k(i + 1) + i$ for some $i \geq 0$, then $\text{deg } T \geq d - i$ and*

$$\text{rank } \rho_{q,l} - \text{rank } \rho_{q,i} = k(l - i), \quad \text{for all } l \geq i - 1.$$

Proof. Suppose $\text{rank } \rho_{q,i} \leq k(i + 1) + i < (k + 1)(i + 1)$ for some $i \geq 0$. By Proposition 3.1, we must have $i \geq d - \text{deg } T$, hence $\text{deg } T \geq d - i$. We now have $i - 1 \geq d - \text{deg } T - 1$. Thus for $l \geq i - 1$, we obtain, by Proposition 3.1 again, that

$$\text{rank } \rho_{q,l} - \text{rank } \rho_{q,i} = (k(l + 1) + d - \text{deg } T) - (k(i + 1) + d - \text{deg } T) = k(l - i).$$

□

PROPOSITION 3.3.

1. *If $\text{deg } T = d - r$ (resp. $\geq d - r$), then $\text{rank } \rho_{q,m} = k(m + 1) + r$ (resp. $\leq k(m + 1) + r$) for all $m \geq r$.*
2. *If $\text{rank } \rho_{q,m} = k(m + 1) + r$ (resp. $\leq k(m + 1) + r$) for some $m \geq r$, then $\text{deg } T = d - r$ (resp. $\geq d - r$).*

Proof. (1) Suppose $\text{deg } T = d - r$ (resp. $\geq d - r$). Then $r = d - \text{deg } T$ (resp. $\geq d - \text{deg } T$), and by Proposition 3.1, we have $\text{rank } \rho_{q,r} = k(r + 1) + d - \text{deg } T = k(r + 1) + r$ (resp. $\leq k(r + 1) + r$). Then by Corollary 3.2, for any $m \geq r$, $\text{rank } \rho_{q,m} = \text{rank } \rho_{q,r} + k(m - r) = k(m + 1) + r$ (resp. $\leq k(m + 1) + r$).

(2) Suppose $\text{rank } \rho_{q,m} = k(m + 1) + r$ (resp. $\leq k(m + 1) + r$) for some $m \geq r$. Then $\text{rank } \rho_{q,m} \leq k(m + 1) + m$. Let $m' = m - 1$. By Corollary 3.2, $\text{rank } \rho_{q,m'} - \text{rank } \rho_{q,m} = -k$. So $\text{rank } \rho_{q,m'} = \text{rank } \rho_{q,m} - k = k(m + 1) + r - k = k(m' + 1) + r$ (resp. $\leq k(m' + 1) + r$). If $m' \geq r$, then we are back in the starting situation (m' playing the role of m), so we can repeat this process, and eventually we must have $\text{rank } \rho_{q,r} = k(r + 1) + r$ (resp. $\leq k(r + 1) + r$). By Corollary 3.2, we have $\text{deg } T \geq d - r$, or $r \geq d - \text{deg } T$. Then by Proposition 3.1, $d - \text{deg } T = \text{rank } \rho_{q,m} - k(r + 1) = r$ (resp. $\leq r$), and hence $\text{deg } T = d - r$ (resp. $\geq d - r$). □

For each integer m , we set $\widehat{m} := k(m + 1)$. Proposition 3.3 implies that for $m \geq r$, $q \in C_{d,r} \iff$ the induced homomorphism of the exterior powers

$$\bigwedge^{\widehat{m}+r+1} \rho_{q,m} : \bigwedge^{\widehat{m}+r+1} \text{Hom}(V, \mathcal{O}(r)) \rightarrow \bigwedge^{\widehat{m}+r+1} \text{Hom}(E, \mathcal{O}(r))$$

is a zero homomorphism. This suggests a way to endow each $C_{d,r}$ with a structure of determinantal subschemes. Denote by $\pi : \mathbb{P}_{Q_d}^1 \rightarrow Q_d$ the projection map. Let

$$\rho_{d,m} : \pi_* V_{\mathbb{P}_{Q_d}^1}^\vee(m) \rightarrow \pi_* \mathcal{E}_d^\vee(m)$$

be the \mathcal{O}_{Q_d} -homomorphism obtained by applying $\text{Hom}(-, \mathcal{O}_{\mathbb{P}_{Q_d}^1}(m))$ to the monomorphism $\mathcal{E}_d \rightarrow V_{\mathbb{P}_{Q_d}^1}$ from the universal exact sequence first and π_* second.

For each $l \geq 1$, the exterior power of $\rho_{d,m}$

$$\bigwedge^l \rho_{d,m} : \bigwedge^l \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m) \rightarrow \bigwedge^l \pi_* \mathcal{E}_d^\vee(m)$$

is a section of $\mathcal{H}om(\bigwedge^l \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \bigwedge^l \pi_* \mathcal{E}_d^\vee(m))$. By Proposition 3.3, for $m \geq d$, $\bigwedge^{\widehat{m}+r+1} \rho_{d,m}$ is nowhere vanishing on Q_d for $r \leq -1$, but vanishes precisely on $C_{d,r}$ for $r \geq 0$.

The section $\bigwedge^{\widehat{m}+r+1} \rho_{d,m}$ induces a homomorphism

$$\mathcal{H}om\left(\bigwedge^{\widehat{m}+r+1} \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \bigwedge^{\widehat{m}+r+1} \pi_* \mathcal{E}_d^\vee(m)\right)^\vee \rightarrow \mathcal{O}_{Q_d}.$$

The image, which we denote by $I_{d,r,m}$, is an ideal sheaf. For $m \geq d$, each $I_{d,r,m}$ defines a subscheme of Q_d whose support is $C_{d,r}$. A natural question is: is the scheme structure independent of m ? We will show that the answer is yes if $m \gg 0$.

Dualizing the universal exact sequence (3.1), one obtains a long exact sequence

$$0 \rightarrow \mathcal{F}_d^\vee \rightarrow V_{\mathbb{P}^1_{Q_d}}^\vee \rightarrow \mathcal{E}_d^\vee \rightarrow \mathcal{F}_d^\varepsilon \rightarrow 0. \tag{3.2}$$

We need to study the sheaf $\mathcal{F}_d^\varepsilon$. By Proposition 2.3, for any morphism $f : X \rightarrow \mathbb{P}^1_{Q_d}$, we can write $f^* \mathcal{F}_d^\varepsilon$ for both $f^*(\mathcal{F}_d^\varepsilon)$ and $(f^* \mathcal{F}_d)^\varepsilon$. In particular, for any subscheme $Z \subset Q_d$, we write $(\mathcal{F}_d^\varepsilon)_Z^\varepsilon$ for $(\mathcal{F}_d^\varepsilon)_Z$ and $((\mathcal{F}_d)_Z)^\varepsilon$. Some facts are quick: $(\mathcal{F}_d)_{\check{Q}_d}^\varepsilon = ((\mathcal{F}_d)_{\check{Q}_d})^\varepsilon = 0$ since $(\mathcal{F}_d)_{\check{Q}_d}$ is locally free, and for any point $q \in Q_d$, $\mathcal{F}_{d,q}^\varepsilon \simeq ((\mathcal{F}_{d,q})_{\text{tor}})^\varepsilon$ since $\mathcal{F}_{d,q}$ splits as a direct sum of its torsion part and locally free part. Then by Proposition 2.4, we see that $\mathcal{F}_{d,q}^\varepsilon$ is torsion and $\deg \mathcal{F}_{d,q}^\varepsilon = \deg(\mathcal{F}_{d,q})_{\text{tor}}$. Thus $\mathcal{F}_d^\varepsilon$ is a torsion sheaf and $\text{Supp } \mathcal{F}_d^\varepsilon \subset \mathbb{P}^1_{Q_d} \setminus \mathbb{P}^1_{\check{Q}_d}$.

LEMMA 3.4. *For any closed point $q \in Q_d$, there is an open neighborhood U of q such that $\mathcal{F}_d^\varepsilon|_{\mathbb{P}^1_U} \simeq \mathcal{F}_d^\varepsilon(m)|_{\mathbb{P}^1_U}$ for all $m \geq 0$.*

Proof. Let $q \in Q_d$ be a closed point and $i_q : \mathbb{P}^1 \rightarrow \mathbb{P}^1_{Q_d}$ the inclusion map defined by $i_q(t) = (t, q)$. So the image of i_q is exactly \mathbb{P}^1_q . Since the sheaf $\mathcal{F}_{d,q}^\varepsilon$ is torsion on \mathbb{P}^1_q , the support of $\mathcal{F}_{d,q}^\varepsilon$ consists of only finitely many points of \mathbb{P}^1_q . Therefore, $\text{Supp } \mathcal{F}_d^\varepsilon$ is a proper closed subset of $\mathbb{P}^1_{Q_d}$. Choose a point $t \in \mathbb{P}^1$ so that $i_q(t) \in \mathbb{P}^1_q \setminus \text{Supp } \mathcal{F}_{d,q}^\varepsilon$ and let $U = Q_d \setminus \pi(p^{-1}(t) \cap \text{Supp } \mathcal{F}_d^\varepsilon)$, where $p : \mathbb{P}^1_{Q_d} \rightarrow \mathbb{P}^1$ is the projection. Obviously, $q \in U$. Now choose an exact sequence on \mathbb{P}^1 :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(m) \rightarrow T \rightarrow 0$$

such that $\text{Supp } T = \{t\}$. Pulling the sequence back to \mathbb{P}^1_U (which preserves exactness) and tensoring with $\mathcal{F}_d^\varepsilon|_{\mathbb{P}^1_U}$, we get a long exact sequence

$$\mathcal{T}or_1(\mathcal{F}_d^\varepsilon|_{\mathbb{P}^1_U}, p^*T|_{\mathbb{P}^1_U}) \rightarrow \mathcal{F}_d^\varepsilon|_{\mathbb{P}^1_U} \rightarrow \mathcal{F}_d^\varepsilon(m)|_{\mathbb{P}^1_U} \rightarrow \mathcal{F}_d^\varepsilon|_{\mathbb{P}^1_U} \otimes p^*T|_{\mathbb{P}^1_U} \rightarrow 0.$$

Note that $\text{Supp } \mathcal{F}_d^\varepsilon$ and $\text{Supp } p^*T$ do not intersect on \mathbb{P}^1_U , hence $\mathcal{T}or_1(\mathcal{F}_d^\varepsilon|_{\mathbb{P}^1_U}, p^*T|_{\mathbb{P}^1_U}) = 0$ and $\mathcal{F}_d^\varepsilon|_{\mathbb{P}^1_U} \otimes p^*T|_{\mathbb{P}^1_U} = 0$, which gives rise to an isomorphism $\mathcal{F}_d^\varepsilon \simeq \mathcal{F}_d^\varepsilon(m)$ on \mathbb{P}^1_U . Obviously, this open set U works for all $m \geq 0$. \square

PROPOSITION 3.5.

1. For any closed point $q \in Q_d$, there is an open neighborhood U of q such that $\pi_*\mathcal{F}_d^\varepsilon(m)|_U \simeq \pi_*\mathcal{F}_d^\varepsilon(l)|_U$, for all $m, l \geq 0$.
2. There is an integer $m_d \geq d$ (independent of r) such that $I_{d,r,m} = I_{d,r,m_d}$ as subsheaves of \mathcal{O}_{Q_d} for all $m \geq m_d$ and for any $0 \leq r \leq d - 1$.

Proof. (1) By the above lemma, we deduce that there exists U such that $\mathcal{F}_d^\varepsilon(m)|_{\mathbb{P}_U^1} \simeq \mathcal{F}_d^\varepsilon(l)|_{\mathbb{P}_U^1}$ for arbitrary m, l . Then

$$\pi_{U*}(\mathcal{F}_d^\varepsilon(m)|_{\mathbb{P}_U^1}) \simeq \pi_{U*}(\mathcal{F}_d^\varepsilon(l)|_{\mathbb{P}_U^1}).$$

The statement follows from the facts that

$$\pi_*\mathcal{F}_d^\varepsilon(m)|_U = \pi_{U*}(\mathcal{F}_d^\varepsilon(m)|_{\mathbb{P}_U^1}) \quad \text{and} \quad \pi_*\mathcal{F}_d^\varepsilon(l)|_U = \pi_{U*}(\mathcal{F}_d^\varepsilon(l)|_{\mathbb{P}_U^1}).$$

- (2) We break up the exact sequence (3.2) into two short exact sequences:

$$0 \rightarrow \mathcal{F}_d^\vee \rightarrow V_{\mathbb{P}_{Q_d}^1}^\vee \rightarrow \mathcal{H} \rightarrow 0, \quad 0 \rightarrow \mathcal{H} \rightarrow \mathcal{E}_d^\vee \rightarrow \mathcal{F}_d^\varepsilon \rightarrow 0.$$

Twisting the two sequences with $\mathcal{O}_{\mathbb{P}_{Q_d}^1}(m)$ (which preserves exactness) and applying π_* , we get another two exact sequences

$$\begin{aligned} 0 &\rightarrow \pi_*\mathcal{F}_d^\vee(m) \rightarrow \pi_*V_{\mathbb{P}_{Q_d}^1}^\vee(m) \rightarrow \pi_*\mathcal{H}(m) \rightarrow R^1\pi_*\mathcal{F}_d^\vee(m) \\ 0 &\rightarrow \pi_*\mathcal{H}(m) \rightarrow \pi_*\mathcal{E}_d^\vee(m) \rightarrow \pi_*\mathcal{F}_d^\varepsilon(m) \rightarrow R^1\pi_*\mathcal{H}(m). \end{aligned}$$

There is an integer $m_d \geq d$ such that $R^1\pi_*\mathcal{F}_d^\vee(m) = 0 = R^1\pi_*\mathcal{H}(m)$ for all $m \geq m_d$. Hence for $m \geq m_d$, the above two sequences join together into one long exact sequence:

$$0 \rightarrow \pi_*\mathcal{F}_d^\vee(m) \rightarrow \pi_*V_{\mathbb{P}_{Q_d}^1}^\vee(m) \xrightarrow{\rho_{d,m}} \pi_*\mathcal{E}_d^\vee(m) \rightarrow \pi_*\mathcal{F}_d^\varepsilon(m) \rightarrow 0. \tag{3.3}$$

We see that $\pi_*V_{\mathbb{P}_{Q_d}^1}^\vee(m) = V_{Q_d}^\vee \otimes H^0(\mathbb{P}^1, \mathcal{O}(m))$ is a free \mathcal{O}_{Q_d} -module, and $\pi_*\mathcal{E}_d^\vee(m)$ is locally free (of rank $(\widehat{m} + d)$).

Now fix an $m > m_d$. We can choose an affine open cover U_i of Q_d such that $\pi_*\mathcal{F}_d^\varepsilon(m)|_{U_i} \simeq \pi_*\mathcal{F}_d^\varepsilon(m_d)|_{U_i}$ on each U_i . We may also assume that $\pi_*\mathcal{E}_d^\vee(m)$ and $\pi_*\mathcal{E}_d^\vee(m_d)$ are both free on each U_i . We have exact sequences

$$\pi_*V_{\mathbb{P}_{Q_d}^1}^\vee(l)|_{U_i} \xrightarrow{\rho_{d,l}|_{U_i}} \pi_*\mathcal{E}_d^\vee(l)|_{U_i} \rightarrow \pi_*\mathcal{F}_d^\varepsilon(l)|_{U_i} \rightarrow 0, \quad l = m, m_d.$$

So the ideal $I_{d,r,l}(U_i) \subset \mathcal{O}_{Q_d}(U_i)$ is exactly the $(d - r - 1)$ th Fitting ideal of $\pi_*\mathcal{F}_d^\varepsilon(l)|_{U_i}$ for $l = m, m_d$. Since $\pi_*\mathcal{F}_d^\varepsilon(m)|_{U_i} \simeq \pi_*\mathcal{F}_d^\varepsilon(m_d)|_{U_i}$, we have $I_{d,r,m}(U_i) = I_{d,r,m_d}(U_i)$ for each U_i . It then follows that $I_{d,r,m} = I_{d,r,m_d}$. \square

For $0 \leq r \leq d - 1$, we denote by $Z_{d,r}$ for the closed subscheme of Q_d defined by the ideal I_{d,r,m_d} , and write $I_{Z_{d,r}} := I_{d,r,m_d}$. Obviously, the support of $Z_{d,r}$ is the closed set $C_{d,r}$ defined earlier. The $Z_{d,r}$'s form a chain of closed subschemes:

$$Z_{d,0} \subset Z_{d,1} \subset \cdots \subset Z_{d,d-1} = Q_d \setminus \mathring{Q}_d.$$

For convenience of notations, we set

$$Z_{d,-1} := \emptyset, \quad Z_{d,d} := Q_d, \quad \text{and} \quad \mathring{Z}_{d,r} := Z_{d,r} \setminus Z_{d,r-1}, \quad \text{for } 0 \leq r \leq d.$$

Thus we have $\mathring{Z}_{d,0} = Z_{d,0}$ and $\mathring{Z}_{d,d} = \mathring{Q}_d$. Note that each $\mathring{Z}_{d,r}$ is an open subscheme of $Z_{d,r}$.

THEOREM 3.6.

1. For $0 \leq r \leq d$, $Z_{d,r}$ is the schematic zero locus of $\bigwedge^{\widehat{m}+r+1} \rho_{d,m}$, and $|\mathring{Z}_{d,r}| = \{q \in Q_d \mid \deg \mathcal{F}_{d,q}^\varepsilon = d - r\}$. In particular, $\mathring{Z}_{d,r}$'s are pairwise disjoint, and $|Q_d| = \bigsqcup_{r=0}^d |\mathring{Z}_{d,r}|$.
2. For $0 \leq r \leq d$, the inclusion $i : \mathring{Z}_{d,r} \hookrightarrow Q_d$ has the following universal property: if Y is a noetherian scheme and $f : Y \rightarrow Q_d$ is a morphism, then $\bar{f}^* \mathcal{F}_d^\varepsilon$ is flat over Y with relative degree $d - r$ if and only if f factors through i . In particular, the sheaf $(\mathcal{F}_d)_{\mathring{Z}_{d,r}}^\varepsilon$ on $\mathbb{P}_{\mathring{Z}_{d,r}}^1$ is flat over $\mathring{Z}_{d,r}$ with relative degree $d - r$.

Proof. (1) follows easily from the fact that $\deg(\mathcal{F}_d^\varepsilon)_q = \deg(\mathcal{F}_{d,q})_{\text{tor}}$. We now prove (2). For any $m \geq m_d$, we have an exact sequence:

$$\pi_* V_{Q_d}^\vee(m) \xrightarrow{\rho_{d,m}} \pi_* \mathcal{E}_d^\vee(m) \rightarrow \pi_* \mathcal{F}_d^\varepsilon(m) \rightarrow 0.$$

There is an integer N_1 such that, for all $m \geq N_1$,

$$f^* \pi_* \mathcal{F}_d^\varepsilon(m) = \pi_{Y*} \bar{f}^* \mathcal{F}_d^\varepsilon(m).$$

Suppose f factor through i . By Proposition 2.5 (3) and (4), and by the definition of $Z_{d,r}$, $f^* \pi_* \mathcal{F}_d^\varepsilon(m)$ is locally free for all $m \geq m_d$. So $\pi_{Y*} \bar{f}^* \mathcal{F}_d^\varepsilon(m)$ is locally free for all $m \geq \max\{m_d, N_1\}$. It follows that $\bar{f}^* \mathcal{F}_d^\varepsilon$ is flat over Y . In particular, taking $Y = \mathring{Z}_{d,r}$ and f to be the identity map, we see that $(\mathcal{F}_d)_{\mathring{Z}_{d,r}}^\varepsilon$ is flat over $\mathring{Z}_{d,r}$.

Now suppose $\bar{f}^* \mathcal{F}_d^\varepsilon$ is flat over Y . Then there is an integer N_2 such that $\pi_{Y*} \bar{f}^* \mathcal{F}_d^\varepsilon(m)$ is locally free for all $m \geq N_2$. Thus $f^* \pi_* \mathcal{F}_d^\varepsilon(m)$ is locally free for all $m \geq \max\{N_1, N_2, m_d\}$. By Proposition 2.5 again, f factors through i . \square

REMARK. The above theorem says that the locally closed subschemes $Z_{d,0}, \mathring{Z}_{d,1}, \dots, \mathring{Z}_{d,d-1}$ and \mathring{Q}_d form the *flattening stratification* of Q_d by the sheaf $\mathcal{F}_d^\varepsilon$.

4. More about the boundary. For $d \geq r \geq 0$, we consider the relative Quot scheme over Q_r :

$$Q_{d,r} := \text{Quot}_{\mathcal{E}_r/\mathbb{P}_{Q_r}^1/Q_r}^{0,d-r}.$$

We denote by $\theta : Q_{d,r} \rightarrow Q_r$ the structure morphism. It is equipped with a universal exact sequence on $\mathbb{P}_{Q_{d,r}}^1$:

$$0 \rightarrow \mathcal{E}_{d,r} \rightarrow \bar{\theta}^* \mathcal{E}_r \rightarrow \mathcal{T}_{d,r} \rightarrow 0 \tag{4.1}$$

where $\mathcal{T}_{d,r}$ is flat over $Q_{d,r}$ with relative degree $d - r$ and rank 0 (i.e., $\mathcal{T}_{d,r}$ is torsion), and $\mathcal{E}_{d,r}$ is taken as a subsheaf of $\bar{\theta}^* \mathcal{E}_r$: $\mathcal{E}_{d,r} \subset \bar{\theta}^* \mathcal{E}_r$. It is easy to see that $\mathcal{E}_{d,r}$ is locally free of rank k and of relative degree $-d$.

We now give a set-theoretic description of the points in $Q_{d,r}$. Let $q = [V_{\mathbb{P}^1} \rightarrow F]$ be a closed point of Q_r and let $E = \ker(V_{\mathbb{P}^1} \rightarrow F)$. Obviously, E is locally free of rank k and degree $-r$. The fiber $\theta^{-1}(q)$ is the Quot scheme $\text{Quot}_{E/V_{\mathbb{P}^1}/\mathbb{k}}^{0,d-r}$. So any closed point in the fiber $\theta^{-1}(q)$ is represented by a quotient $E \rightarrow T$ where T is torsion with degree $d - r$. The kernel $E' := \ker(E \rightarrow T)$ is locally free of rank k and degree $-d$.

LEMMA 4.1. *With E' and T defined as above, we have $\dim \text{Hom}(E', T) = k(d - r)$ and $\text{Ext}^1(E', T) = 0$.*

Proof. We have $E' \simeq \bigoplus_{i=1}^k \mathcal{O}(d_i)$. Then

$$\begin{aligned} \mathrm{Hom}(E', T) &\simeq \bigoplus_{i=1}^k \mathrm{Hom}(\mathcal{O}(d_i), T) \simeq \bigoplus_{i=1}^k \mathrm{Hom}(\mathcal{O}, T) = H^0(T)^{\oplus k} \\ \mathrm{Ext}^1(E', T) &\simeq \bigoplus_{i=1}^k \mathrm{Ext}^1(\mathcal{O}(d_i), T) \simeq \bigoplus_{i=1}^k \mathrm{Ext}^1(\mathcal{O}, T) = H^1(T)^{\oplus k}. \end{aligned}$$

So $\dim \mathrm{Hom}(E', T) = k \dim H^0(T) = k(d - r)$. That $\mathrm{Ext}^1(E', T) = 0$ follows from $H^1(T) = 0$. \square

By the deformation theory on Quot schemes (see [Ser06], Proposition 4.4.4), the above lemma implies that

PROPOSITION 4.2. *The structure morphism $\theta : Q_{d,r} \rightarrow Q_r$ is smooth of relative dimension $k(d - r)$.*

PROPOSITION 4.3. *$Q_{d,r}$ is an irreducible nonsingular projective variety of dimension $\dim Q_r + k(d - r) = nr + k(n - k + d - r)$.*

Proof. The nonsingularity, projectivity and dimension counting of $Q_{d,r}$ follow easily from the two facts: (i) Q_r is a smooth and projective variety of dimension $nr + k(n - k)$; (ii) $Q_{d,r}$ is smooth and projective over Q_r of relative dimension $k(d - r)$. It only remains to show the irreducibility. Since Q_r is irreducible, by [Sha94], Ch. 1, Sec. 6, Theorem 8, $Q_{d,r}$ is irreducible if we can show that all the fibers of θ over the closed points of Q_r are irreducible. The proof of this part is inspired by the proof of Theorem 2.1 in [Str87].

Let $[V_{\mathbb{P}^1} \twoheadrightarrow F]$ be a closed point of Q_r . The fiber of θ over this point is a Quot scheme $\mathrm{Quot}_{E/\mathbb{P}^1/\mathbb{k}}^{0,d-r}$ where $E = \ker(V_{\mathbb{P}^1} \twoheadrightarrow F)$. Let N be a vector space of dimension $d - r$, and let W be the vector space

$$W = \mathrm{Hom}(N_{\mathbb{P}^1}(-1), N_{\mathbb{P}^1}) \times \mathrm{Hom}(E, N_{\mathbb{P}^1}).$$

Then $\dim W = 2(d - r)^2 + k(d - r)$. Let $\bar{X} = \mathrm{Spec}(\bigoplus_{i \geq 0} \mathrm{Sym}^i W^\vee)$ be the associated affine space. There are tautological morphisms μ and ν which fit into the diagram on $\mathbb{P}_{\bar{X}}^1$:

$$\begin{array}{ccc} N_{\mathbb{P}_{\bar{X}}^1}(-1) & \xrightarrow{\nu} & N_{\mathbb{P}_{\bar{X}}^1} \twoheadrightarrow \mathrm{Coker}(\nu) \\ & \uparrow \mu & \\ & E_{\bar{X}} & \end{array}$$

Let $X \subset \bar{X}$ be the open subvariety defined by the conditions: (i) ν is injective on each fiber over X , and (ii) the induced map $E_X \rightarrow \mathrm{Coker}(\nu)$ is surjective. The sheaf $\mathrm{Coker}(\nu)$ is flat over X with rank 0 and relative degree $d - r$. Thus the surjection $E_X \twoheadrightarrow \mathrm{Coker}(\nu)$ gives a morphism $g : X \rightarrow \mathrm{Quot}_{E/\mathbb{P}^1/\mathbb{k}}^{0,d-r}$.

Next we show that g is surjective. Let $[E \twoheadrightarrow T]$ be a closed point of $\mathrm{Quot}_{E/\mathbb{P}^1/\mathbb{k}}^{0,d-r}$, and $H := H^0(\mathbb{P}^1, T)$. By Proposition 1.1 in [Str87], we have a natural exact sequence

$$0 \rightarrow H_{\mathbb{P}^1}(-1) \rightarrow H_{\mathbb{P}^1} \rightarrow T \rightarrow 0.$$

Applying $\text{Hom}(E, -)$ to it, we obtain an exact sequence

$$\cdots \rightarrow \text{Hom}(E, H_{\mathbb{P}^1}) \rightarrow \text{Hom}(E, T) \rightarrow \text{Ext}^1(E, H_{\mathbb{P}^1}(-1)) \rightarrow \cdots$$

We have

$$\text{Ext}^1(E, H_{\mathbb{P}^1}(-1)) = H^1(H_{\mathbb{P}^1}(-1) \otimes E^\vee) = H^1(E^\vee(-1)) \otimes H_{\mathbb{P}^1} = 0$$

because $H^1(E^\vee(-1)) = 0$. So the map $\text{Hom}(E, H_{\mathbb{P}^1}) \rightarrow \text{Hom}(E, T)$ is surjective, and hence the quotient map $E \rightarrow T$ factors through $H_{\mathbb{P}^1} \rightarrow T$ as

$$\begin{array}{ccc} H_{\mathbb{P}^1}(-1) & \longrightarrow & H_{\mathbb{P}^1} & \longrightarrow & T \\ & & \uparrow \text{dotted} & \nearrow & \\ & & E & & \end{array}$$

This diagram gives a point of X , whose image under g is the point $[E \rightarrow T]$. Therefore g is surjective and hence $\text{Quot}_{E/\mathbb{P}^1/\mathbb{k}}^{0,d-r}$ is irreducible since X is irreducible. \square

On $\mathbb{P}_{Q_{d,r}}^1$, we have two short exact sequence

$$0 \rightarrow \mathcal{E}_{d,r} \rightarrow \bar{\theta}^* \mathcal{E}_r \rightarrow \mathcal{T}_{d,r} \rightarrow 0, \quad 0 \rightarrow \bar{\theta}^* \mathcal{E}_r \rightarrow V_{\mathbb{P}_{Q_{d,r}}^1} \rightarrow \bar{\theta}^* \mathcal{F}_r \rightarrow 0.$$

The second is the pullback of the universal exact sequence of Q_r by $\bar{\theta}$. Let $\mathcal{F}_{d,r}$ be the quotient of $V_{\mathbb{P}_{Q_{d,r}}^1}$ by $\mathcal{E}_{d,r}$ (based on the inclusions $\mathcal{E}_{d,r} \subset \bar{\theta}^* \mathcal{E}_r \subset \bar{\theta}^* V_{\mathbb{P}_{Q_r}^1} = V_{\mathbb{P}_{Q_{d,r}}^1}$). Then we can form a commutative diagram as follows:

$$\begin{array}{ccccccc} & & & 0 & & 0 & (4.2) \\ & & & \downarrow & & \downarrow & \\ 0 & \rightarrow & \mathcal{E}_{d,r} & \rightarrow & \bar{\theta}^* \mathcal{E}_r & \rightarrow & \mathcal{T}_{d,r} \rightarrow 0 \\ & & \parallel & & \downarrow & & \vdots \\ 0 & \rightarrow & \mathcal{E}_{d,r} & \rightarrow & V_{\mathbb{P}_{Q_{d,r}}^1} & \rightarrow & \mathcal{F}_{d,r} \rightarrow 0 \\ & & & & \downarrow & & \vdots \\ & & & & \bar{\theta}^* \mathcal{F}_r & = & \bar{\theta}^* \mathcal{F}_r \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

where the dotted arrows are induced maps on quotients. All rows and the middle column are exact, hence the last column is forced to be exact as well. Since $\mathcal{T}_{d,r}$ and $\bar{\theta}^* \mathcal{F}_r$ are both flat over $Q_{d,r}$, so is $\mathcal{F}_{d,r}$. Moreover, $\mathcal{F}_{d,r}$ has rank $n - k$ and relative degree d . Thus by the universal property of Q_d , the exact sequence from the middle row determines a morphism

$$\phi : Q_{d,r} \rightarrow Q_d$$

such that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{E}_{d,r} & \rightarrow & V_{\mathbb{P}_{Q_{d,r}}^1} & \rightarrow & \mathcal{F}_{d,r} \rightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \rightarrow & \bar{\phi}^* \mathcal{E}_d & \rightarrow & \bar{\phi}^* V_{\mathbb{P}_{Q_d}^1} & \rightarrow & \bar{\phi}^* \mathcal{F}_d \rightarrow 0 \end{array} \quad (4.3)$$

PROPOSITION 4.4. For $0 \leq l \leq d$, $I_{Z_{r,l}} \cdot \mathcal{O}_{Q_{d,r}} = I_{Z_{d,l}} \cdot \mathcal{O}_{Q_{d,r}}$, where as convention we set $I_{Z_{r,l}} := 0$ for $l \geq r$. In other words, the scheme-theoretic preimage $\phi^{-1}(Z_{d,l})$ is equal to the scheme-theoretic preimage $\theta^{-1}(Z_{r,l}) = Q_{d,r} \times_{Q_r} Z_{r,l}$, $l = 0, \dots, r$.

Proof. We have a commutative diagram of sheaves on $\mathbb{P}^1_{Q_{d,r}}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{\phi}^* \mathcal{E}_d & \longrightarrow & \bar{\theta}^* \mathcal{E}_r & \longrightarrow & \mathcal{T}_{d,r} \longrightarrow 0 \\ & & & & \searrow & \downarrow & \\ & & & & & V_{\mathbb{P}^1_{Q_{d,r}}} & \end{array}$$

with exact row. Fix an integer $m \gg 0$. Applying $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^1_{Q_{d,r}}}(m))$ first and π_* second, we obtain a commutative diagram

$$\begin{array}{ccccccc} & & \pi_* V_{\mathbb{P}^1_{Q_{d,r}}}^\vee(m) & & & & \\ & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \pi_* \bar{\theta}^* \mathcal{E}_r^\vee(m) & \longrightarrow & \pi_* \bar{\phi}^* \mathcal{E}_d^\vee(m) & \longrightarrow & \pi_* \mathcal{T}_{d,r}^\varepsilon(m) \longrightarrow 0. \end{array}$$

The row is still exact because $m \gg 0$. By Proposition 2.4, $\mathcal{T}_{d,r}^\varepsilon$ is flat over $Q_{d,r}$. So $\pi_* \mathcal{T}_{d,r}^\varepsilon(m)$ is locally free and the homomorphism $\pi_* \bar{\theta}^* \mathcal{E}_r^\vee(m) \rightarrow \pi_* \bar{\phi}^* \mathcal{E}_d^\vee(m)$ in the above diagram is a subbundle homomorphism. After dropping the term $\pi_* \mathcal{T}_{d,r}^\varepsilon(m)$, we rewrite the diagram as

$$\begin{array}{ccc} \theta^* \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m) & \xrightarrow{\theta^* \rho_{r,m}} & \theta^* \pi_* \mathcal{E}_r^\vee(m) \\ \parallel & & \downarrow \\ \phi^* \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m) & \xrightarrow{\phi^* \rho_{d,m}} & \phi^* \pi_* \mathcal{E}_d^\vee(m) \end{array}$$

where we have used the canonical identifications: $\phi^* \pi_* \mathcal{E}_d^\vee(m) = \pi_* \bar{\phi}^* \mathcal{E}_d^\vee(m)$, $\theta^* \pi_* \mathcal{E}_r^\vee(m) = \pi_* \bar{\theta}^* \mathcal{E}_r^\vee(m)$ and $\phi^* \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m) = \theta^* \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m) = \pi_* V_{\mathbb{P}^1_{Q_{d,r}}}^\vee(m)$ for $m \gg 0$. One checks that the first row is the map $\theta^* \rho_{r,m}$ and the second row is $\phi^* \rho_{d,m}$. Applying the exterior power $\wedge^{\hat{m}+l+1}$ to the above diagram, we obtain the following commutative diagram

$$\begin{array}{ccc} \wedge^{\hat{m}+l+1} \theta^* \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m) & \xrightarrow{\wedge^{\hat{m}+l+1} \theta^* \rho_{r,m}} & \wedge^{\hat{m}+l+1} \theta^* \pi_* \mathcal{E}_r^\vee(m) \\ \parallel & & \downarrow \\ \wedge^{\hat{m}+l+1} \phi^* \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m) & \xrightarrow{\wedge^{\hat{m}+l+1} \phi^* \rho_{d,m}} & \wedge^{\hat{m}+l+1} \phi^* \pi_* \mathcal{E}_d^\vee(m) \end{array} \tag{4.4}$$

where the second column is still a subbundle map. This diagram induces another commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{Q_{d,r}} & \longrightarrow & \mathcal{H}om(\wedge^{\hat{m}+l+1} \theta^* \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \wedge^{\hat{m}+l+1} \theta^* \pi_* \mathcal{E}_r^\vee(m)) \\ \parallel & & \downarrow \\ \mathcal{O}_{Q_{d,r}} & \longrightarrow & \mathcal{H}om(\wedge^{\hat{m}+l+1} \phi^* \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \wedge^{\hat{m}+l+1} \phi^* \pi_* \mathcal{E}_d^\vee(m)) \end{array} \tag{4.5}$$

where the first row is induced by $\bigwedge^{\widehat{m}+l+1} \theta^* \rho_{r,m}$, the second row is induced by $\bigwedge^{\widehat{m}+l+1} \phi^* \rho_{d,m}$ and the second column is a subbundle map obtained by applying $\mathcal{H}om(\bigwedge^{\widehat{m}+l+1} \phi^* \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), -)$ to the second column of the previous diagram. Dualizing the above diagram and using the canonical identifications

$$\begin{aligned} & \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \theta^* \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \theta^* \pi_* \mathcal{E}_r^\vee(m)\right)^\vee \\ &= \theta^* \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \pi_* \mathcal{E}_r^\vee(m)\right)^\vee, \\ & \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \phi^* \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \phi^* \pi_* \mathcal{E}_d^\vee(m)\right)^\vee \\ &= \phi^* \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \pi_* \mathcal{E}_d^\vee(m)\right)^\vee, \end{aligned}$$

we obtain a commutative diagram

$$\begin{array}{ccc} \phi^* \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \pi_* \mathcal{E}_d^\vee(m)\right)^\vee & \longrightarrow & \mathcal{O}_{Q_{d,r}} \\ \downarrow & & \parallel \\ \theta^* \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \pi_* \mathcal{E}_r^\vee(m)\right)^\vee & \longrightarrow & \mathcal{O}_{Q_{d,r}} \end{array} \tag{4.6}$$

where the first column is a surjection. One checks that the image of the first row is $I_{Z_{d,l}} \cdot \mathcal{O}_{Q_{d,r}}$ and the image of the second row is $I_{Z_{r,l}} \cdot \mathcal{O}_{Q_{d,r}}$. By the surjectivity of the first column, we have an equality $I_{Z_{d,l}} \cdot \mathcal{O}_{Q_{d,r}} = I_{Z_{r,l}} \cdot \mathcal{O}_{Q_{d,r}}$. \square

COROLLARY 4.5. *For $0 \leq l \leq r$, the restriction of ϕ to the subscheme $Q_{d,r} \times_{Q_r} Z_{r,l} = \theta^{-1}(Z_{r,l})$ of $Q_{d,r}$ factors through the inclusion $Z_{d,l} \hookrightarrow Q_d$. In particular, ϕ factors through the inclusion $Z_{d,r} \hookrightarrow Q_d$.*

We denote by $\varphi : Q_{d,r} \rightarrow Z_{d,r}$ the morphism factored out from $\phi : Q_{d,r} \rightarrow Q_d$.

PROPOSITION 4.6. *The morphism φ is surjective.*

Proof. Let $q \in Z_{d,r}$. Then $\deg(\mathcal{F}_{d,q})_{\text{tor}} \geq d - r$. Let $T \subset \mathcal{F}_{d,q}$ be a torsion subsheaf of degree $d - r$, let $F = \mathcal{F}_{d,q}/T$, and let $E = \ker(V_{\mathbb{P}^1_q} \rightarrow \mathcal{F}_{d,q} \rightarrow F)$. Then we form the following commutative diagram with exact rows and exact columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}_{d,q} & \longrightarrow & E & \longrightarrow & T \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}_{d,q} & \longrightarrow & V_{\mathbb{P}^1_q} & \longrightarrow & \mathcal{F}_{d,q} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & F & \xlongequal{\quad} & F \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

The middle column represents a point $x \in Q_r$ since $\deg F = r$, and the first row corresponds a point $y \in Q_{d,r}$, which is on the fiber $\theta^{-1}(x)$ over x . One checks that $\varphi(y) = q$. Thus φ is surjective. \square

We put $\mathring{Q}_{d,r} := Q_{d,r} \times_{Q_r} \mathring{Q}_r = \theta^{-1}(\mathring{Q}_r) \subset Q_{d,r}$, and write $\phi_0 : \mathring{Q}_{d,r} \rightarrow Q_d$ for the restriction of ϕ to $\mathring{Q}_{d,r}$.

PROPOSITION 4.7. *The map ϕ_0 factors through the inclusion $\mathring{Z}_{d,r} \hookrightarrow Q_d$.*

Proof. By the universal property of $\mathring{Z}_{d,r}$ (Theorem 3.6), we only need to show that $\bar{\phi}_0^* \mathcal{F}_d^\varepsilon$ is flat over $\mathring{Q}_{d,r}$ with relative degree $d - r$. Since we have

$$\bar{\phi}_0^* \mathcal{F}_d^\varepsilon = (\bar{\phi}^* \mathcal{F}_d)_{\mathring{Q}_{d,r}}^\varepsilon \simeq (\mathcal{F}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon$$

we only need to show that $(\mathcal{F}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon$ is flat over $\mathring{Q}_{d,r}$ with relative degree $d - r$.

Note that we have a short exact sequence from the last column of the diagram (4.2):

$$0 \rightarrow \mathcal{T}_{d,r} \rightarrow \mathcal{F}_{d,r} \rightarrow \bar{\theta}^* \mathcal{F}_r \rightarrow 0.$$

Restricting it to $\mathbb{P}_{\mathring{Q}_{d,r}}^1$ yields an exact sequence

$$0 \rightarrow (\mathcal{T}_{d,r})_{\mathring{Q}_{d,r}} \rightarrow (\mathcal{F}_{d,r})_{\mathring{Q}_{d,r}} \rightarrow \bar{\theta}^* (\mathcal{F}_r)_{\mathring{Q}_r} \rightarrow 0.$$

Then dualizing the sequence, we obtain an exact sequence

$$\cdots \rightarrow \bar{\theta}^* (\mathcal{F}_r)_{\mathring{Q}_r}^\varepsilon \rightarrow (\mathcal{F}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon \rightarrow (\mathcal{T}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon \rightarrow 0.$$

Since $(\mathcal{F}_r)_{\mathring{Q}_r}$ is locally free, we have $(\mathcal{F}_r)_{\mathring{Q}_r}^\varepsilon = 0$, and hence

$$(\mathcal{F}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon \simeq (\mathcal{T}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon. \tag{4.7}$$

By Proposition 2.4, $(\mathcal{T}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon$ is flat over $\mathring{Q}_{d,r}$ with relative degree $d - r$. It follows that $(\mathcal{F}_{d,r})_{\mathring{Q}_{d,r}}^\varepsilon$ is flat over $\mathring{Q}_{d,r}$ with relative degree $d - r$. \square

We denote by $\mathring{\phi} : \mathring{Q}_{d,r} \rightarrow \mathring{Z}_{d,r}$ the map factored out from $\phi_0 : \mathring{Q}_{d,r} \rightarrow Q_d$. The composition of $\mathring{\phi}$ with the inclusion $\mathring{Z}_{d,r} \hookrightarrow Z_{d,r}$ is the restriction of the morphism ϕ to $\mathring{Q}_{d,r}$.

PROPOSITION 4.8. *The morphism $\mathring{\phi} : \mathring{Q}_{d,r} \rightarrow \mathring{Z}_{d,r}$ is an isomorphism.*

Proof. We construct a morphism $\psi : \mathring{Z}_{d,r} \rightarrow \mathring{Q}_{d,r}$, and show that $\mathring{\phi}\psi = \text{id}_{\mathring{Z}_{d,r}}$ and $\psi\mathring{\phi} = \text{id}_{\mathring{Q}_{d,r}}$. Let $i : \mathring{Z}_{d,r} \hookrightarrow Q_d$ be the inclusion. Pullback the universal exact sequence (3.1) to $\mathbb{P}_{\mathring{Z}_{d,r}}^1$:

$$0 \rightarrow \bar{i}^* \mathcal{E}_d \rightarrow V_{\mathbb{P}_{\mathring{Z}_{d,r}}^1} \rightarrow \bar{i}^* \mathcal{F}_d \rightarrow 0.$$

Taking dual, one obtains a long exact sequence

$$0 \rightarrow (\bar{i}^* \mathcal{F}_d)^\vee \rightarrow V_{\mathbb{P}_{\mathring{Z}_{d,r}}^1}^\vee \rightarrow \bar{i}^* \mathcal{E}_d^\vee \rightarrow \bar{i}^* \mathcal{F}_d^\varepsilon \rightarrow 0.$$

Let $\mathcal{G} = \ker(\bar{i}^* \mathcal{E}_d^\vee \rightarrow \bar{i}^* \mathcal{F}_d^\varepsilon)$. Then we can break up the above sequence into two short exact sequences:

$$0 \rightarrow (\bar{i}^* \mathcal{F}_d)^\vee \rightarrow V_{\mathbb{P}_{\mathring{Z}_{d,r}}^1}^\vee \rightarrow \mathcal{G} \rightarrow 0, \quad 0 \rightarrow \mathcal{G} \rightarrow \bar{i}^* \mathcal{E}_d^\vee \rightarrow \bar{i}^* \mathcal{F}_d^\varepsilon \rightarrow 0. \tag{4.8}$$

By Theorem 3.6, $\bar{i}^* \mathcal{F}_d^\varepsilon$ is flat over $\mathring{Z}_{d,r}$ with relative degree $d - r$. It follows that \mathcal{G} and $(\bar{i}^* \mathcal{F}_d)^\vee$ are both flat over $\mathring{Z}_{d,r}$, and hence they are both locally free. Since $\bar{i}^* \mathcal{F}_d^\varepsilon$ has rank 0 and relative degree $d - r$, we know that \mathcal{G} has rank k and relative degree r , and that $(\bar{i}^* \mathcal{F}_d)^\vee$ has rank $n - k$ and relative degree $-r$. Taking dual again to both sequences in (4.8), we obtain

$$0 \rightarrow \mathcal{G}^\vee \rightarrow V_{\mathbb{P}^1_{\mathring{Z}_{d,r}}} \rightarrow (\bar{i}^* \mathcal{F}_d)^{\vee\vee} \rightarrow 0, \tag{4.9}$$

$$0 \rightarrow \bar{i}^* \mathcal{E}_d \rightarrow \mathcal{G}^\vee \rightarrow (\bar{i}^* \mathcal{F}_d^\varepsilon)^\varepsilon \rightarrow 0. \tag{4.10}$$

We have $(\bar{i}^* \mathcal{F}_d)^{\vee\vee}$ is locally free of rank $n - k$ and relative degree r . By the universal property of Q_r , the exact sequence (4.9) gives rise to a morphism

$$\psi_0 : \mathring{Z}_{d,r} \rightarrow Q_r,$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{G}^\vee & \longrightarrow & V_{\mathbb{P}^1_{\mathring{Z}_{d,r}}} & \longrightarrow & (\bar{i}^* \mathcal{F}_d)^{\vee\vee} \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \bar{\psi}_0^* \mathcal{E}_r & \longrightarrow & \bar{\psi}_0^* V_{\mathbb{P}^1_{Q_r}} & \longrightarrow & \bar{\psi}_0^* \mathcal{F}_r \longrightarrow 0. \end{array} \tag{4.11}$$

Here we view \mathcal{G}^\vee as a subsheaf of $V_{\mathbb{P}^1_{\mathring{Z}_{d,r}}}$. We see that $\bar{\psi}_0^* \mathcal{F}_r$ is locally free, and hence by Theorem 3.6, ψ_0 factors through the inclusion $\mathring{Q}_r \hookrightarrow Q_r$. We write

$$\psi_1 : \mathring{Z}_{d,r} \rightarrow \mathring{Q}_r$$

for the morphism factored out from ψ_0 .

Taking into account the diagram (4.11), the exact sequence (4.10) becomes

$$0 \rightarrow \bar{i}^* \mathcal{E}_d \rightarrow \bar{\psi}_0^* \mathcal{E}_r \rightarrow (\bar{i}^* \mathcal{F}_d^\varepsilon)^\varepsilon \rightarrow 0. \tag{4.12}$$

By Proposition 2.4, $(\bar{i}^* \mathcal{F}_d^\varepsilon)^\varepsilon$ is flat over $\mathring{Z}_{d,r}$ with rank 0 and relative degree $d - r$. Thus by the universal property of $Q_{d,r}$, the exact sequence (4.12) above induces a morphism of Q_r -schemes

$$\psi_2 : \mathring{Z}_{d,r} \rightarrow Q_{d,r}$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{i}^* \mathcal{E}_d & \longrightarrow & \bar{\psi}_0^* \mathcal{E}_r & \longrightarrow & (\bar{i}^* \mathcal{F}_d^\varepsilon)^\varepsilon \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \bar{\psi}_2^* \mathcal{E}_{d,r} & \longrightarrow & \bar{\psi}_2^* \bar{\theta}^* \mathcal{E}_r & \longrightarrow & \bar{\psi}_2^* \mathcal{T}_{d,r} \longrightarrow 0. \end{array} \tag{4.13}$$

The maps ψ_1 and ψ_2 fit into the following commutative diagram

$$\begin{array}{ccc} \mathring{Z}_{d,r} & \xrightarrow{\psi_2} & Q_{d,r} \\ \downarrow \psi_1 & \searrow \psi_1 \circ \psi_2 & \downarrow \theta \\ \mathring{Q}_{d,r} & \xrightarrow{j} & Q_{d,r} \\ \downarrow & & \downarrow \theta \\ \mathring{Q}_r & \xrightarrow{} & Q_r \end{array}$$

Since $\mathring{Q}_{d,r} = Q_{d,r} \times_{Q_r} \mathring{Q}_r$, ψ_1 and ψ_2 induce a Q_r -morphism $\psi : \mathring{Z}_{d,r} \rightarrow \mathring{Q}_{d,r}$.

We now show that (i) $\mathring{\varphi}\psi = \text{id}_{\mathring{Z}_{d,r}}$ and (ii) $\psi\mathring{\varphi} = \text{id}_{\mathring{Q}_{d,r}}$.

(i) To show that $\mathring{\varphi}\psi = \text{id}_{\mathring{Z}_{d,r}}$, it suffices to show that $i\mathring{\varphi}\psi = i$, or $\phi_0\psi = i$, or $\phi j\psi = i$, or $\phi\psi_2 = i$. Note that we have the following commutative diagram

$$\begin{CD} 0 @>>> \bar{\psi}_2^* \mathcal{E}_{d,r} @>>> \bar{\psi}_2^* V_{\mathbb{P}^1_{Q_{d,r}}} @>>> \bar{\psi}_2^* \mathcal{F}_{d,r} @>>> 0 \\ @. @| @| @VV \simeq V @. \\ 0 @>>> \bar{i}^* \mathcal{E}_d @>>> V_{\mathbb{P}^1_{\mathring{Z}_{d,r}}} @>>> \bar{i}^* \mathcal{F}_d @>>> 0 \end{CD}$$

where the first row is obtained by applying $\bar{\psi}_2^*$ to the middle row of the diagram (4.2), and the equality $\bar{\psi}_2^* \mathcal{E}_{d,r} = \bar{i}^* \mathcal{E}_d$ is from the diagram (4.13). Thus we have a commutative diagram of equivalent quotients:

$$\begin{CD} \overline{\phi\psi_2}^* V_{\mathbb{P}^1_{Q_d}} @= \bar{\psi}_2^* \bar{\phi}^* V_{\mathbb{P}^1_{Q_d}} @= \bar{\psi}_2^* V_{\mathbb{P}^1_{Q_{d,r}}} @= V_{\mathbb{P}^1_{\mathring{Z}_{d,r}}} @= \bar{i}^* V_{\mathbb{P}^1_{Q_d}} \\ @VV \downarrow V @VV \downarrow V \textcircled{1} @VV \downarrow V \textcircled{2} @VV \downarrow V \\ \overline{\phi\psi_2}^* \mathcal{F}_d @= \bar{\psi}_2^* \bar{\phi}^* \mathcal{F}_d @\simeq\rightarrow \bar{\psi}_2^* \mathcal{F}_{d,r} @\simeq\rightarrow \bar{i}^* \mathcal{F}_d @= \bar{i}^* \mathcal{F}_d \end{CD}$$

where the subdiagram ① is obtained by applying ψ_2^* to the diagram (4.3), and the subdiagram ② is obtained from the above diagram. By the universal property of Q_d , we must have $\phi\psi_2 = i$, and hence $\mathring{\varphi}\psi = \text{id}_{\mathring{Z}_{d,r}}$.

(ii) Let $j : \mathring{Q}_{d,r} \rightarrow Q_{d,r}$ denote the inclusion. To show that $\psi\mathring{\varphi} = \text{id}_{\mathring{Q}_{d,r}}$, it suffices to show that $j\psi\mathring{\varphi} = j$, or $\psi_2\mathring{\varphi} = j$.

We first show that $\psi_2\mathring{\varphi} : \mathring{Q}_{d,r} \rightarrow Q_{d,r}$ is a Q_r -morphism, where $\mathring{Q}_{d,r}$ is viewed as a Q_r -scheme through $\theta j : \mathring{Q}_{d,r} \rightarrow Q_r$. Equivalently, we show that $\theta j = \theta\psi_2\mathring{\varphi}$, or $\theta j = \psi_0\mathring{\varphi}$. We have two commutative diagrams of the same shape with exact rows and columns:

$$\begin{CD} @. @. @. @. \\ @. @VV \downarrow V @. @VV \downarrow V @. \\ 0 @>>> \bar{j}^* \mathcal{E}_{d,r} @>>> \bar{j}^* \bar{\theta}^* \mathcal{E}_r @>>> \bar{j}^* \mathcal{T}_{d,r} @>>> 0 \\ @. @VV \downarrow V @VV \downarrow V @VV \downarrow V @. \\ 0 @>>> \bar{j}^* \mathcal{E}_{d,r} @>>> V_{\mathbb{P}^1_{Q_{d,r}}} @>>> \bar{j}^* \bar{\phi}^* \mathcal{F}_d @>>> 0 \\ @. @VV \downarrow V @VV \downarrow V @VV \downarrow V @. \\ @. \bar{j}^* \bar{\theta}^* \mathcal{F}_r @= \bar{j}^* \bar{\theta}^* \mathcal{F}_r @. @. \\ @. @VV \downarrow V @VV \downarrow V @. @. \\ @. @. @. @. @. \end{CD} \quad \begin{CD} @. @. @. @. \\ @. @VV \downarrow V @. @VV \downarrow V @. \\ 0 @>>> \bar{\varphi}^* \bar{i}^* \mathcal{E}_d @>>> \bar{\varphi}^* \bar{\psi}_0^* \mathcal{E}_r @>>> \bar{\varphi}^* (\bar{i}^* \mathcal{F}_d^\varepsilon)^\varepsilon @>>> 0 \\ @. @VV \downarrow V @VV \downarrow V @VV \downarrow V @. \\ 0 @>>> \bar{\varphi}^* \bar{i}^* \mathcal{E}_d @>>> V_{\mathbb{P}^1_{Q_{d,r}}} @>>> \bar{\varphi}^* \bar{i}^* \mathcal{F}_d @>>> 0 \\ @. @VV \downarrow V @VV \downarrow V @VV \downarrow V @. \\ @. \bar{\varphi}^* \bar{\psi}_0^* \mathcal{F}_r @= \bar{\varphi}^* \bar{\psi}_0^* \mathcal{F}_r @. @. \\ @. @VV \downarrow V @VV \downarrow V @. @. \end{CD}$$

The left diagram is obtained from (4.2) by replacing $\mathcal{F}_{d,r}$ with $\bar{\phi}^* \mathcal{F}_d$ using the isomorphism $\mathcal{F}_{d,r} \simeq \bar{\phi}^* \mathcal{F}_d$ first and applying \bar{j}^* second, while the right diagram is obtained by combining the two exact sequences (4.9) and (4.10) first, replacing \mathcal{G}^\vee with $\bar{\psi}_0^* \mathcal{E}_r$ and $(\bar{i}^* \mathcal{F}_d)^\vee$ with $\bar{\psi}_0^* \mathcal{F}_r$ second, and applying $\bar{\varphi}^*$ third. Note that $i\mathring{\varphi} = \phi_0 = \phi j$. Hence the two middle rows are exactly the same exact sequences because they are both the pullback of the universal exact sequence of Q_d by the same map ϕ_0 . Note also that $\bar{j}^* \mathcal{T}_{d,r}$ and $\bar{\varphi}^* (\bar{i}^* \mathcal{F}_d^\varepsilon)^\varepsilon$ are both torsion submodules of the same module $\bar{j}^* \bar{\phi}^* \mathcal{F}_d = \bar{\varphi}^* \bar{i}^* \mathcal{F}_d$ since $\bar{j}^* \bar{\theta}^* \mathcal{F}_r$ and $\bar{\varphi}^* \bar{\psi}_0^* \mathcal{F}_r$ are both locally free. Therefore,

$\bar{j}^*\mathcal{T}_{d,r} = \bar{\varphi}^*(\bar{i}^*\mathcal{F}_d^\varepsilon)^\varepsilon$ as submodules, and hence we have a commutative diagram

$$\begin{array}{ccccc} V_{\mathbb{P}_{\hat{Q}_{d,r}}^1} & \twoheadrightarrow & \bar{j}^*\bar{\varphi}^*\mathcal{F}_d & \twoheadrightarrow & \bar{j}^*\bar{\theta}^*\mathcal{F}_r \\ \parallel & & \parallel & & \downarrow \simeq \\ V_{\mathbb{P}_{\hat{Q}_{d,r}}^1} & \twoheadrightarrow & \bar{\varphi}^*\bar{i}^*\mathcal{F}_d & \twoheadrightarrow & \bar{\varphi}^*\bar{\psi}_0^*\mathcal{F}_r. \end{array}$$

So $\theta_j = \psi_0\bar{\varphi}$ by the universal property of Q_r .

Next, we have a commutative diagram of equivalent quotients on $\mathbb{P}_{Q_{d,r}}^1$:

$$\begin{array}{ccccccc} \overline{\psi_2\bar{\varphi}^*\bar{\theta}^*\mathcal{E}_r} & = & \bar{\varphi}^*\bar{\psi}_2^*\bar{\theta}^*\mathcal{E}_r & = & \bar{\varphi}^*\bar{\psi}_0^*\mathcal{E}_r & = & \bar{j}^*\bar{\theta}^*\mathcal{E}_r \\ \downarrow & & \downarrow & \textcircled{3} & \downarrow & & \downarrow \\ \overline{\psi_2\bar{\varphi}^*\mathcal{T}_{d,r}} & = & \bar{\varphi}^*\bar{\psi}_2^*\mathcal{T}_{d,r} & \xrightarrow{\simeq} & \bar{\varphi}^*(\bar{i}^*\mathcal{F}_d^\varepsilon)^\varepsilon & = & \bar{j}^*\mathcal{T}_{d,r} \end{array}$$

where the subdiagram $\textcircled{3}$ is obtained by applying $\bar{\varphi}^*$ to the diagram (4.13). By the universal property of $Q_{d,r}$, $\psi_2\bar{\varphi} = j$ as Q_r -morphisms, hence $\psi\bar{\varphi} = \text{id}_{\hat{Q}_{d,r}}$. \square

This shows that $\varphi : Q_{d,r} \rightarrow Z_{d,r}$ is a birational morphism. It then follows that

COROLLARY 4.9. *For each r , $Z_{d,r}$ is irreducible and of codimension $(n-k)(d-r)$, and $\hat{Z}_{d,r}$ is nonsingular. In particular, $Z_{d,0}$ is an irreducible nonsingular subvariety of Q_d .*

5. Successive blowups and main theorems. We now perform a sequence of blowups on Q_d . Set $Q_d^{-1} := Q_d$, $Z_{d,r}^{-1} := Z_{d,r}$. For $r = 0, \dots, d-1$, let Q_d^r be the blowup of Q_d^{r-1} along $Z_{d,r}^{r-1}$ ($Q_d^r := \text{Bl}_{Z_{d,r}^{r-1}} Q_d^{r-1}$), $Z_{d,r}^r$ the exceptional divisor, and $Z_{d,l}^r$ the proper transform of $Z_{d,l}^{r-1}$ in Q_d^r for $l \neq r$. We write $I_{Z_{d,r}^l}$ for the ideal sheaf of $Z_{d,r}^l$ in Q_d^l , for all r, l .

This way of constructing of a compactification is similar to that in the construction of the space of complete quadrics and the space of complete collineations by Vainsencher [Vai82, Vai84]. In fact, our proof will rely on a result from his construction of the space of complete collineations. To state that result, we begin with a brief review of the space of collineations. Some original notations are modified.

Let E and F be two locally free sheaves on a k -scheme S and set

$$\mathbb{S}(E, F) := \mathbb{P}(\mathcal{H}om_S(E, F)).$$

If $S = \text{Spec } k$, then E and F are vector spaces, and $\mathbb{S}(E, F)$ parametrizes all nonzero linear maps $E \rightarrow F$ up to scalar multiple. Hence $\mathbb{S}(E, F)$ is called the *space of collineations* from E to F . For a general k -scheme S , $\mathbb{S}(E, F)$ is considered as a family of space of collineations parametrized by S .

The universal quotient line bundle $p^*\mathcal{H}om(E, F)^\vee \rightarrow \mathcal{O}_{\mathbb{S}(E, F)}(1)$ on $\mathbb{S}(E, F)$ corresponds to a nowhere-vanishing homomorphism $u : p^*E \rightarrow p^*F \otimes \mathcal{O}_{\mathbb{S}(E, F)}(1)$, called the universal homomorphism, where $p : \mathbb{S}(E, F) \rightarrow S$ is the structure morphism. For $r = 1, \dots, N$ where $N = \min\{\text{rank } E, \text{rank } F\} - 1$, let $D_r(E, F)$ be the schematic zero locus of $\bigwedge^{r+1} u : \bigwedge^{r+1} p^*E \rightarrow \bigwedge^{r+1} p^*F \otimes \mathcal{O}_{\mathbb{S}(E, F)}(r+1)$. The ideal sheaf $I_{D_r(E, F)}$ of $D_r(E, F)$ is the image of the homomorphism

$$p^*\mathcal{H}om\left(\bigwedge^{r+1} E, \bigwedge^{r+1} F\right)^\vee \otimes \mathcal{O}_{\mathbb{S}(E, F)}(-r-1) \rightarrow I_{D_r(E, F)} \subset \mathcal{O}_{\mathbb{S}(E, F)}$$

induced by $\bigwedge^{r+1} u$.

Set $\mathbb{S}_r(E, F) := \mathbb{S}(\bigwedge^{r+1} E, \bigwedge^{r+1} F)$, $r = 1, \dots, N$. The homomorphism $\bigwedge^{r+1} u$, which is nowhere vanishing on $\mathbb{S}(E, F) \setminus D_r(E, F)$, determines an S -morphism: $\mathbb{S}(E, F) \setminus D_r(E, F) \rightarrow \mathbb{S}_r(E, F)$ by the universal property of $\mathbb{S}_r(E, F)$.

Starting with $\mathbb{S}^0(E, F) := \mathbb{S}(E, F)$, $D_r^0(E, F) := D_r(E, F)$, let $\mathbb{S}^r(E, F)$ be the blowup of $\mathbb{S}^{r-1}(E, F)$ along $D_r^{r-1}(E, F)$, and denote by $D_r^r(E, F)$ the exceptional divisor, $D_l^r(E, F)$ the proper transform of $D_l^{r-1}(E, F)$ in $\mathbb{S}^r(E, F)$ for $l \neq r$. The ideal sheaf of $D_l^r(E, F)$ in $\mathbb{S}^r(E, F)$ is denoted by $I_{D_l^r(E, F)}$.

The useful result from [Vai84] is Theorem 2.4 (8), which gives a relationship between the ideal sheaf of the total transform and that of the proper transform of $D_l^{r-1}(E, F)$ in the blowup $\mathbb{S}^r(E, F) \rightarrow \mathbb{S}^{r-1}(E, F)$. We state the result below for the convenience of reference.

THEOREM 5.1 (Vainsencher). *For $l > r \geq 1$, we have*

$$I_{D_l^{r-1}(E, F)} \cdot \mathcal{O}_{\mathbb{S}^r(E, F)} = I_{D_l^r(E, F)} \cdot (I_{D_r^r(E, F)})^{l-r+1}.$$

We will use this result to show that there is a similar relationship between the ideal sheaf of the total transform and that of the proper transform of $Z_{d,l}^{r-1}$ in the blowup $Q_d^r \rightarrow Q_d^{r-1}$.

PROPOSITION 5.2. *For $l > r \geq 0$, $I_{Z_{d,l}^{r-1}} \cdot \mathcal{O}_{Q_d^r} = I_{Z_{d,l}^r} \cdot (I_{Z_{d,r}^r})^{l-r+1}$.*

Proof. Fix an integer $m \geq m_d$ and consider the space $\mathbb{S}(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m))$. For simplicity of notations, we put $\mathbb{S} := \mathbb{S}(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m))$, $D_r := D_r(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m))$, and $D_l^r := D_l^r(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m))$.

The nowhere vanishing section $\rho_{d,m}$ of $\mathcal{H}om(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m))$ on Q_d induces a closed embedding $f : Q_d \hookrightarrow \mathbb{S}$ such that $\rho_{d,m} = f^*u$, where u is the universal homomorphism on \mathbb{S} :

$$u : p^* \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m) \rightarrow p^* \pi_* \mathcal{E}_d^\vee(m) \otimes \mathcal{O}_{\mathbb{S}}(1).$$

Through this embedding, we will consider Q_d as a closed subscheme of \mathbb{S} . By definition, $Z_{d,r}$ is the schematic zero locus of $\bigwedge^{\widehat{m}+r+1} \rho_{d,m} = \bigwedge^{\widehat{m}+r+1} f^*u = f^* \bigwedge^{\widehat{m}+r+1} u$. It follows that

$$Q_d \cap D_{\widehat{m}+r} = Z_{d,r}, \quad r = 0, \dots, d-1$$

scheme-theoretically, and

$$Q_d \cap D_l = \emptyset, \quad l = 1, \dots, \widehat{m}-1.$$

The latter equalities mean that the first $\widehat{m}-1$ blowups of \mathbb{S} along D_l for $l = 1, \dots, \widehat{m}-1$ has no effect on Q_d . Therefore, we also have an embedding $Q_d \hookrightarrow \mathbb{S}^{\widehat{m}-1}$, together with $Q_d \cap D_{\widehat{m}+r}^{\widehat{m}-1} = Z_{d,r}$, for $0 \leq r \leq d-1$. In other words, $I_{D_{\widehat{m}+r}^{\widehat{m}-1}} \cdot \mathcal{O}_{Q_d} = I_{Z_{d,r}}$.

Starting from the \widehat{m} -th blowup, the blowups on \mathbb{S} has an effect on Q_d . We have the following pullback diagram of blowups

$$\begin{CD} Q_d^r @>>> \mathbb{S}^{\widehat{m}+r} \\ @VVV @VVV \\ Q_d^{r-1} @>>> \mathbb{S}^{\widehat{m}+r-1} \end{CD}$$

for all $0 \leq r \leq d - 1$. Further we have $Q_d^r \cap D_{\widehat{m}+l}^{\widehat{m}+r} = Z_{d,l}^r$, for $l \geq r$. Thus for $l > r$,

$$\begin{aligned} I_{Z_{d,i}^{r-1}} \cdot \mathcal{O}_{Q_d^r} &= I_{D_{\widehat{m}+i}^{\widehat{m}+r-1}} \cdot \mathcal{O}_{Q_d^{r-1}} \cdot \mathcal{O}_{Q_d^r} = I_{D_{\widehat{m}+i}^{\widehat{m}+r-1}} \cdot \mathcal{O}_{\mathbb{S}_{\widehat{m}+i}^{\widehat{m}+r}} \cdot \mathcal{O}_{Q_d^r} \\ &\stackrel{(*)}{=} I_{D_{\widehat{m}+i}^{\widehat{m}+r}} \cdot (I_{D_{\widehat{m}+i}^{\widehat{m}+r}})^{i-r+1} \cdot \mathcal{O}_{Q_d^r} = I_{Z_{d,i}^r} \cdot (I_{Z_{d,r}^r})^{i-r+1} \end{aligned}$$

where $(*)$ is by Theorem 5.1. \square

Applying the above proposition repeatedly, we obtain

COROLLARY 5.3. *For all $0 \leq r \leq d - 1$,*

$$I_{Z_{d,r}} \cdot \mathcal{O}_{Q_d^{r-1}} = I_{Z_{d,r}^{r-1}} \cdot (I_{Z_{d,r-1}^{r-1}})^2 \cdots (I_{Z_{d,1}^{r-1}})^r \cdot (I_{Z_{d,0}^{r-1}})^{r+1}.$$

Suppose $m \geq m_d$ and $0 \leq r \leq d-1$. The homomorphisms $\bigwedge^{\widehat{m}+l+1} \rho_{d,m}$, $0 \leq l \leq r$, are nowhere vanishing on $Q_d \setminus Z_{d,r}$, hence they correspondingly induce embeddings of Q_d -schemes

$$Q_d \setminus Z_{d,r} \hookrightarrow \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_d}^1}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)), \quad 0 \leq l \leq r.$$

These embeddings together further induce an embedding of $Q_d \setminus Z_{d,r}$ into a fibered product

$$Q_d \setminus Z_{d,r} \hookrightarrow \prod_{l=0}^r Q_d \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_d}^1}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)).$$

Here we denote by \prod_S the fibered product over S for any scheme S .

THEOREM 5.4. *Let $m \geq m_d$. For $r = 0, \dots, d - 1$, the blowup Q_d^r is isomorphic to the closure of the image of the embedding*

$$Q_d \setminus Z_{d,r} \hookrightarrow \prod_{l=0}^r Q_d \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_d}^1}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)).$$

Proof. By construction, Q_d^r is the blowup of Q_d^{r-1} along the subscheme $Z_{d,r}^{r-1}$. Let $b : Q_d^{r-1} \rightarrow Q_d$ denote the composite blowup. Corollary 5.3 says that the ideal sheaf of the proper transform $Z_{d,r}^{r-1}$ of $Z_{d,r}$ and the ideal sheaf of the total transform $b^{-1}(Z_{d,r})$ of $Z_{d,r}$ only differ by an invertible ideal sheaf (i.e., the sheaf $(I_{Z_{d,r-1}^{r-1}})^2 \cdots (I_{Z_{d,1}^{r-1}})^r \cdot (I_{Z_{d,0}^{r-1}})^{r+1}$). Therefore, the blowup of Q_d^{r-1} along $Z_{d,r}^{r-1}$ is the same as the blowup along the total transform $b^{-1}(Z_{d,r})$, i.e., $Q_d^r = \text{Bl}_{b^{-1}(Z_{d,r})} Q_d^{r-1}$. Since $Z_{d,r}$ is the schematic zero locus of $\bigwedge^{\widehat{m}+r+1} \rho_{d,m}$ on Q_d , $b^{-1}(Z_{d,r})$ is the schematic zero locus of $\bigwedge^{\widehat{m}+r+1} b^* \rho_{d,m}$ on Q_d^{r-1} . So $\text{Bl}_{b^{-1}(Z_{d,r})} Q_d^{r-1}$ is the closure of the image of the embedding induced by $\bigwedge^{\widehat{m}+r+1} b^* \rho_{d,m}$:

$$\begin{aligned} Q_d^{r-1} \setminus b^{-1}(Z_{d,r}) &\hookrightarrow \mathbb{S}_{\widehat{m}+r}(b^* \pi_* V_{\mathbb{P}_{Q_d}^1}^\vee(m), b^* \pi_* \mathcal{E}_d^\vee(m)) \\ &= Q_d^{r-1} \times_{Q_d} \mathbb{S}_{\widehat{m}+r}(\pi_* V_{\mathbb{P}_{Q_d}^1}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)). \end{aligned}$$

Thus we see that the proof can be completed by induction on r . \square

Note that the closure of the image of the embedding

$$Q_d \setminus Z_{d,r} \hookrightarrow \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m))$$

is exactly the blowup $\text{Bl}_{Z_{d,l}} Q_d$ of Q_d along $Z_{d,l}$. Thus we obtain an easy corollary.

COROLLARY 5.5. *The blowup Q_d^r is isomorphic to the closure of $Q_d \setminus Z_{d,r}$ in the product*

$$\text{Bl}_{Z_{d,0}} Q_d \times_{Q_d} \cdots \times_{Q_d} \text{Bl}_{Z_{d,r}} Q_d.$$

Next we fix $m \gg 0$, and for each $r = 0, \dots, d-1$ and each $l = 0, \dots, r-1$, we introduce a diagram

$$\begin{array}{ccc}
 \mathring{Q}_{d,r} & \xrightarrow{\alpha_l} & Q_{d,r} \times_{Q_r} \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \pi_* \mathcal{E}_r^\vee(m)) \\
 \downarrow \mathring{\phi} & \searrow i & \downarrow \beta_l \\
 & & Q_{d,r} \xleftarrow{p} \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \pi_* \mathcal{E}_r^\vee(m)) \\
 & & \downarrow \gamma_l \\
 Q_d \setminus Z_{d,r-1} & \xrightarrow{\gamma_l} & \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)) \\
 \downarrow j & \searrow \phi & \downarrow \\
 & & Q_d \xleftarrow{q} \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m))
 \end{array} \tag{5.1}$$

where i and j are inclusion maps, p is the projection and q is the structure morphism. The other maps are defined as follows.

DEFINITION OF $\mathring{\phi}$. Recall that we have an isomorphism $\mathring{\varphi} : \mathring{Q}_{d,r} \rightarrow \mathring{Z}_{d,r}$. We define $\mathring{\phi}$ to be the composition

$$\mathring{Q}_{d,r} \xrightarrow{\mathring{\varphi}} \mathring{Z}_{d,r} \hookrightarrow Q_d \setminus Z_{d,r-1}.$$

Thus $\mathring{\phi}$ is an embedding with image $\mathring{Z}_{d,r}$.

Before defining the other maps, we simplify notations by setting

$$\begin{aligned}
 \mathcal{A}_l &:= \theta^* \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \pi_* \mathcal{E}_r^\vee(m)\right), \\
 \mathcal{B}_l &:= \mathcal{H}om\left(\bigwedge^{\widehat{m}+l+1} \pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \bigwedge^{\widehat{m}+l+1} \pi_* \mathcal{E}_d^\vee(m)\right)
 \end{aligned}$$

so that we can write

$$Q_{d,r} \times_{Q_r} \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \pi_* \mathcal{E}_r^\vee(m)) = \mathbb{P}(\mathcal{A}_l), \quad \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_d}}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)) = \mathbb{P}(\mathcal{B}_l)$$

with universal quotient line bundles $p^* \mathcal{A}_l^\vee \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{A}_l)}(1)$ and $q^* \mathcal{B}_l^\vee \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{B}_l)}(1)$ respectively. Note that the projection p is the same as the structure morphism $\mathbb{P}(\mathcal{A}_l) \rightarrow Q_d$.

DEFINITION OF α_l . In view of the identification $\mathring{Q}_{d,r} = Q_{d,r} \times_{Q_r} \mathring{Q}_r$, we define the map α_l to be the pullback of the embedding of Q_r -schemes

$$\mathring{Q}_r = Q_r \setminus Z_{r,r-1} \hookrightarrow \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}^1_{Q_r}}^\vee(m), \pi_* \mathcal{E}_r^\vee(m))$$

induced by the nowhere vanishing homomorphism $\bigwedge^{\widehat{m}+l+1} \rho_{r,m}$ on \mathring{Q}_r . So we see that α_l is an embedding. Equivalently, α_l is determined by the quotient line bundle $i^* \mathcal{A}_l^\vee \rightarrow$

$i^* \mathcal{O}_{Q_{d,r}}$ corresponding to the nowhere vanishing homomorphism $i^* \wedge^{\widehat{m}+l+1} \theta^* \rho_{r,m}$ on $\mathring{Q}_{d,r}$.

DEFINITION OF β_l . Recall that we have a surjection $\phi^* \mathcal{B}_l^\vee \rightarrow \mathcal{A}_l^\vee$ on $Q_{d,r}$ from the diagram (4.6). The composition of its pullback to $\mathbb{P}(\mathcal{A}_l)$ with the universal quotient line bundle

$$p^* \phi^* \mathcal{B}_l^\vee \rightarrow p^* \mathcal{A}_l^\vee \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{A}_l)}(1)$$

determines the morphism β_l .

DEFINITION OF γ_l . The Q_d -morphism γ_l is determined by the surjection $j^* \mathcal{B}_l^\vee \rightarrow j^* \mathcal{O}_{Q_d}$ corresponding to the nowhere vanishing homomorphism $j^* \wedge^{\widehat{m}+l+1} \rho_{d,m}$ on $Q_d \setminus Z_{d,r-1}$. Thus γ_l is an embedding.

LEMMA 5.6.

1. In the diagram (5.1), the two triangles and the two parallelograms are commutative, i.e., $p\alpha_l = i$, $q\gamma_l = j$, $\phi_i = j\overset{\circ}{\phi}$ and $q\beta_l = \phi p$.
2. We have isomorphisms of line bundles $\alpha_l^* \mathcal{O}_{\mathbb{P}(\mathcal{A}_l)}(1) \simeq i^* \mathcal{O}_{Q_{d,r}}$, $\beta_l^* \mathcal{O}_{\mathbb{P}(\mathcal{B}_l)}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{A}_l)}(1)$ and $\gamma_l^* \mathcal{O}_{\mathbb{P}(\mathcal{B}_l)}(1) \simeq j^* \mathcal{O}_{Q_d}$, which make the following diagrams commute:

$$\begin{array}{ccccccc}
 \alpha_l^* p^* \mathcal{A}_l^\vee & \twoheadrightarrow & \alpha_l^* \mathcal{O}_{\mathbb{P}(\mathcal{A}_l)}(1) & & \beta_l^* q^* \mathcal{B}_l^\vee & \longrightarrow & \beta_l^* \mathcal{O}_{\mathbb{P}(\mathcal{B}_l)}(1) & & \gamma_l^* q^* \mathcal{B}_l^\vee & \twoheadrightarrow & \gamma_l^* \mathcal{O}_{\mathbb{P}(\mathcal{B}_l)}(1) \\
 \parallel & & \downarrow \simeq & & \parallel & & \downarrow \simeq & & \parallel & & \downarrow \simeq \\
 i^* \mathcal{A}^\vee & \longrightarrow & i^* \mathcal{O}_{Q_{d,r}} & & p^* \phi^* \mathcal{B}^\vee & \twoheadrightarrow & p^* \mathcal{A}^\vee & \longrightarrow & j^* \mathcal{B}_l^\vee & \longrightarrow & j^* \mathcal{O}_{Q_d}
 \end{array}$$

Here the first rows are the pullbacks of the universal quotient line bundles on $\mathbb{P}(\mathcal{A}_l)$ and $\mathbb{P}(\mathcal{B}_l)$, and the second rows are those surjections that induce the corresponding maps.

Proof. These are all straightforward by the definitions of the maps in the diagram (5.1). \square

The commutativity of the two triangles means that α_l is a $Q_{d,r}$ -morphism and γ_l is a Q_d -morphism. Through the morphism $\phi : Q_{d,r} \rightarrow Q_d$, all $Q_{d,r}$ -schemes are also Q_d -schemes. Thus the commutativity of the two parallelograms means that $\overset{\circ}{\phi}$ and β_l are both Q_d -morphisms. α_l can also be viewed as a morphism of Q_d -schemes.

It remains to show that

LEMMA 5.7. *The rectangle in the diagram (5.1) is commutative, i.e., $\beta_l \alpha_l = \gamma_l \overset{\circ}{\phi}$.*

Proof. We compute and compare the pullbacks of the universal quotient line bundle on $\mathbb{P}(\mathcal{B}_l)$ via $\beta_l \alpha_l$ and via $\gamma_l \overset{\circ}{\phi}$ respectively, as is demonstrated in the following commutative diagram:

$$\begin{array}{ccccccccccc}
 \alpha_l^* \beta_l^* q^* \mathcal{B}_l^\vee & \xlongequal{\quad} & \alpha_l^* p^* \phi^* \mathcal{B}_l^\vee & \xlongequal{\quad} & i^* \phi^* \mathcal{B}_l^\vee & \xlongequal{\quad} & i^* \phi^* \mathcal{B}_l^\vee & \xlongequal{\quad} & \overset{\circ}{\phi}^* j^* \mathcal{B}_l^\vee & \xlongequal{\quad} & \overset{\circ}{\phi}^* \gamma_l^* q^* \mathcal{B}_l^\vee \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \textcircled{1} & & \alpha_l^* p^* \mathcal{A}_l^\vee & \xlongequal{\quad} & i^* \mathcal{A}_l^\vee & \textcircled{3} & & & \textcircled{4} & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \alpha_l^* \beta_l^* \mathcal{O}_{\mathbb{P}(\mathcal{B}_l)}(1) & \xrightarrow{\simeq} & \alpha_l^* \mathcal{O}_{\mathbb{P}(\mathcal{A}_l)}(1) & \xrightarrow{\simeq} & i^* \mathcal{O}_{Q_{d,r}} & = & i^* \phi^* \mathcal{O}_{Q_d} & = & \overset{\circ}{\phi}^* j^* \mathcal{O}_{Q_d} & \xrightarrow{\simeq} & \overset{\circ}{\phi}^* \gamma_l^* \mathcal{O}_{\mathbb{P}(\mathcal{B}_l)}(1)
 \end{array}$$

The subdiagram $\textcircled{1}$ is obtained by applying α_l^* to the second diagram in Lemma 5.6(2), $\textcircled{2}$ is the first diagram in Lemma 5.6(2), $\textcircled{3}$ is obtained by restricting the

diagram (4.6) to $\mathring{Q}_{d,r}$, and $\textcircled{4}$ is obtained by applying $\mathring{\phi}^*$ to the third diagram in Lemma 5.6(2).

So we see that the pullbacks of the universal quotient line bundle via $\beta_l \alpha_l$ and via $\gamma_l \mathring{\phi}$ are equivalent, which implies that $\beta_l \alpha_l = \gamma_l \mathring{\phi}$. \square

The maps $\mathring{\phi}$, α_l and γ_l are all embeddings, but β_l is in general not. The following is an important observation.

PROPOSITION 5.8. *The map β_{r-1} is a closed embedding.*

The proof of this proposition will take much space, and we leave it to the next section so that we can come to our main result as quick as possible. We will assume this proposition for this section.

The diagrams (5.1) for $l = 0, \dots, r - 1$ naturally induce a commutative diagram

$$\begin{array}{ccc}
 \mathring{Q}_{d,r} & \xrightarrow{\alpha} & Q_{d,r} \times_{Q_r} \prod_{l=0}^{r-1} Q_r \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_r}^1}^\vee(m), \pi_* \mathcal{E}_r^\vee(m)) \\
 \downarrow \mathring{\phi} & & \downarrow \beta \\
 Q_d \setminus Z_{d,r-1} & \xrightarrow{\gamma} & \prod_{l=0}^{r-1} Q_d \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_d}^1}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)).
 \end{array} \tag{5.2}$$

The map α is the $Q_{d,r}$ -morphism into the fibered product over $Q_{d,r}$ determined by the embeddings α_l :

$$\mathring{Q}_{d,r} \hookrightarrow \prod_{l=0}^{r-1} Q_{d,r} \mathbb{P}(\mathcal{A}_l) = Q_{d,r} \times_{Q_r} \prod_{l=0}^{r-1} Q_r \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_r}^1}^\vee(m), \pi_* \mathcal{E}_r^\vee(m)).$$

Equivalently, it is the pullback of the embedding

$$\mathring{Q}_r \hookrightarrow \prod_{l=0}^{r-1} Q_r \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_r}^1}^\vee(m), \pi_* \mathcal{E}_r^\vee(m))$$

via $\theta : Q_{d,r} \rightarrow Q_r$. Thus α is an embedding, and the closure of the image of α is exactly $Q_{d,r} \times_{Q_r} Q_r^{r-1}$.

The map β is the Q_d -morphism into the fibered product over Q_d determined by the composition

$$\prod_{l=0}^{r-1} Q_{d,r} \mathbb{P}(\mathcal{A}_l) \xrightarrow{p_l} \mathbb{P}(\mathcal{A}_l) \xrightarrow{\beta_l} \mathbb{P}(\mathcal{B}_l) = \mathbb{S}_{\widehat{m}+l}(\pi_* V_{\mathbb{P}_{Q_r}^1}^\vee(m), \pi_* \mathcal{E}_r^\vee(m))$$

where p_l is the projection to the l th factor.

The map γ is the Q_d -morphism into the fibered product over Q_d determined by the embeddings γ_l . Hence γ is an embedding, and the closure of the image of γ is exactly Q_d^{r-1} .

Based on Proposition 5.8, we can prove

PROPOSITION 5.9. *The Q_d -morphism β is a closed embedding.*

Proof. For each l , the map $\beta_l : \mathbb{P}(\mathcal{A}_l) \rightarrow \mathbb{P}(\mathcal{B}_l)$ can be decomposed as follows:

$$\mathbb{P}(\mathcal{A}_l) \xrightarrow{\textcircled{1}} \mathbb{P}(\mathring{\phi}^* \mathcal{B}_l) = Q_{d,r} \times_{Q_d} \mathbb{P}(\mathcal{B}_l) \xrightarrow{\textcircled{2}} \mathbb{P}(\mathcal{B}_l),$$

where the map ① is the closed embedding induced by the surjection $\phi^* \mathcal{B}_l^\vee \rightarrow \mathcal{A}_l$ and ② is the natural projection. Thus the map $\beta : \prod_{l=0}^{r-1} Q_{d,r} \mathbb{P}(\mathcal{A}_l) \rightarrow \prod_{l=0}^{r-1} Q_d \mathbb{P}(\mathcal{B}_l)$ can be decomposed as

$$\begin{aligned} & \prod_{l=0}^{r-1} Q_{d,r} \mathbb{P}(\mathcal{A}_l) \xrightarrow{\textcircled{3}} \prod_{l=0}^{r-2} Q_{d,r} \mathbb{P}(\phi^* \mathcal{B}_l) \times_{Q_{d,r}} \mathbb{P}(\mathcal{A}_{r-1}) \\ &= \prod_{l=0}^{r-2} Q_{d,r} (Q_{d,r} \times_{Q_d} \mathbb{P}(\mathcal{B}_l)) \times_{Q_{d,r}} \mathbb{P}(\mathcal{A}_{r-1}) \\ &= \prod_{l=0}^{r-2} Q_d \mathbb{P}(\mathcal{B}_l) \times_{Q_d} \mathbb{P}(\mathcal{A}_{r-1}) \xrightarrow{\textcircled{4}} \prod_{l=0}^{r-2} Q_d \mathbb{P}(\mathcal{B}_l) \times_{Q_d} \mathbb{P}(\mathcal{B}_{r-1}) = \prod_{l=0}^{r-1} \mathbb{P}(\mathcal{B}_l) \end{aligned}$$

where ③ is the closed embedding induced by the map ①, and ④ is the closed embedding induced by $\beta_{r-1} : \mathbb{P}(\mathcal{A}_{r-1}) \hookrightarrow \mathbb{P}(\mathcal{B}_{r-1})$. It follows that β is a closed embedding. \square

PROPOSITION 5.10. *We have an isomorphism $Q_{d,r} \times_{Q_r} Q_r^{r-1} \cong Z_{d,r}^{r-1}$.*

Proof. We know the closure of the image of α is $Q_{d,r} \times_{Q_r} Q_r^{r-1}$. Since β is an embedding, the closure of the image of $\beta\alpha$ is isomorphic to $Q_{d,r} \times_{Q_r} Q_r^{r-1}$. On the other hand, $\check{\phi}$ is an embedding with image $\check{Z}_{d,r}$ and the closure of the image of γ is $Z_{d,r}^{r-1}$. Hence the closure of the image of $\gamma\check{\phi}$ is isomorphic to $Z_{d,r}^{r-1}$. By the commutativity of the diagram (5.2), the image of $\beta\alpha$ and the image of $\gamma\check{\phi}$ are the equal, and so are their closures. This gives us an isomorphism $Q_{d,r} \times_{Q_r} Q_r^{r-1} \cong Z_{d,r}^{r-1}$. \square

Let us denote the isomorphism as $\tilde{\varphi} : Q_{d,r} \times_{Q_r} Q_r^{r-1} \rightarrow Z_{d,r}^{r-1}$. Then it fits into the following commutative diagram

$$\begin{array}{ccccccc} \check{Q}_{d,r} \hookrightarrow & Q_{d,r} \times_{Q_r} Q_r^{r-1} \hookrightarrow & & Q_{d,r} \times_{Q_r} \prod_{l=0}^{r-1} Q_r \mathbb{S}_{\check{m}+l}(\pi_* V_{\mathbb{P}_1^1}^\vee(m), \pi_* \mathcal{E}_r^\vee(m)) & & & \\ \check{\phi} \downarrow & \tilde{\varphi} \downarrow & & \downarrow \beta & & & \\ \check{Z}_{d,r} \hookrightarrow & Z_{d,r}^{r-1} \hookrightarrow & Q_d^{r-1} \hookrightarrow & \prod_{l=0}^{r-1} Q_d \mathbb{S}_{\check{m}+l}(\pi_* V_{\mathbb{P}_d^1}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)). & & & \end{array}$$

We are now ready to prove the first of the main results.

THEOREM 5.11. *For $0 \leq r \leq d - 1$, $Z_{d,r}^{r-1}$ and Q_d^r are both nonsingular.*

Proof. We prove this by induction on r (but for all $d \geq r + 1$). When $r = 0$, $Z_{d,r}^{r-1} = Z_{d,0}$ is nonsingular, hence Q_d^0 , the blowup of Q_d along $Z_{d,0}$, is also nonsingular. Assume the statement is true for $r = N - 1$, that is, $Z_{d,N-1}^{N-2}$ and Q_d^{N-1} are nonsingular for all $d \geq N$. We prove that the statement is also true for $r = N$. By the above proposition, $Z_{d,N}^{N-1} \cong Q_{d,N} \times_{Q_N} Q_N^{N-1}$, and by induction hypothesis, Q_N^{N-1} is nonsingular, hence $Z_{d,N}^{N-1}$ is nonsingular. Q_d^N is nonsingular because it is the blowup of the nonsingular variety Q_d^{N-1} along the nonsingular subvariety $Z_{d,N}^{N-1}$. This completes the proof. \square

PROPOSITION 5.12. *The isomorphism $\tilde{\varphi}$ maps the closed subscheme $Q_{d,r} \times_{Q_r} Z_{r,l}^{r-1}$ of $Q_{d,r} \times_{Q_r} Q_r^{r-1}$ isomorphically onto the closed subscheme $Z_{d,r}^{r-1} \cap Z_{d,l}^{r-1}$ of $Z_{d,r}^{r-1}$ for each $l = 0, \dots, r - 1$.*

Proof. For simplicity, we put $Q_{d,r}^{r-1} := Q_{d,r} \times_{Q_r} Q_r^{r-1}$. Consider the following commutative diagram

$$\begin{CD} Q_r^{r-1} @<<< Q_{d,r}^{r-1} @>>> Q_d^{r-1} \\ @VVV @VVV @VVV \\ Q_r @<<< \theta Q_{d,r} @>>> \phi Q_d \end{CD}$$

where $\tilde{\varphi}$ is defined to be the composition $Q_{d,r}^{r-1} \xrightarrow{\tilde{\varphi}} Z_{d,r}^{r-1} \hookrightarrow Q_d^{r-1}$. The ideal sheaf of $Q_{d,r} \times_{Q_r} Z_{r,l}^{r-1}$ in $Q_{d,r}^{r-1}$ is $I_{Z_{r,l}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}}$. Because of the isomorphism $\tilde{\varphi} : Q_{d,r} \times_{Q_r} Q_r^{r-1} \rightarrow Z_{d,r}^{r-1}$, we see that the ideal sheaf of the preimage $\tilde{\varphi}^{-1}(Z_{d,r}^{r-1} \cap Z_{d,l}^{r-1})$ in $Q_{d,r}^{r-1}$ is $I_{Z_{d,l}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}}$. Thus to show that $\tilde{\varphi}$ induces isomorphisms $Q_{d,r} \times_{Q_r} Z_{r,l}^{r-1} \simeq Z_{d,r}^{r-1} \cap Z_{d,l}^{r-1}$, it suffices to show that

$$I_{Z_{d,l}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}} = I_{Z_{r,l}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}}, \quad l = 0, \dots, r - 1. \tag{5.3}$$

We now prove it by induction on l . When $l = 0$, we have

$$\begin{aligned} I_{Z_{d,0}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}} &= (I_{Z_{d,0}} \cdot \mathcal{O}_{Q_{d,r}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \stackrel{(*)}{=} (I_{Z_{r,0}} \cdot \mathcal{O}_{Q_{d,r}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \\ &= (I_{Z_{r,0}} \cdot \mathcal{O}_{Q_r^{r-1}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} = I_{Z_{r,0}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \end{aligned}$$

where $(*)$ is by Proposition 4.4.

Assume (5.3) holds for all $l \leq N - 1$ for some $N \leq r - 1$, and we prove it also holds for $l = N$. By Corollary 5.3, we have

$$\begin{aligned} I_{Z_{d,N}} \cdot \mathcal{O}_{Q_d^{N-1}} &= I_{Z_{d,N-1}^{N-1}} \cdot (I_{Z_{d,N-1}^{N-1}})^2 \cdots \cdots (I_{Z_{d,0}^{N-1}})^{N+1}, \\ I_{Z_{r,N}} \cdot \mathcal{O}_{Q_r^{N-1}} &= I_{Z_{r,N-1}^{N-1}} \cdot (I_{Z_{r,N-1}^{N-1}})^2 \cdots \cdots (I_{Z_{r,0}^{N-1}})^{N+1}. \end{aligned}$$

Hence we have

$$\begin{aligned} &I_{Z_{d,N}^{r-1}} \cdot (I_{Z_{d,N-1}^{r-1}})^2 \cdots \cdots (I_{Z_{d,0}^{r-1}})^{N+1} \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \\ &= (I_{Z_{d,N}^{N-1}} \cdot (I_{Z_{d,N-1}^{N-1}})^2 \cdots \cdots (I_{Z_{d,0}^{N-1}})^{N+1} \cdot \mathcal{O}_{Q_d^{r-1}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \\ &= ((I_{Z_{d,N}} \cdot \mathcal{O}_{Q_d^{N-1}}) \cdot \mathcal{O}_{Q_d^{r-1}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} = (I_{Z_{d,N}} \cdot \mathcal{O}_{Q_d^{r-1}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \\ &= (I_{Z_{d,N}} \cdot \mathcal{O}_{Q_{d,r}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} = (I_{Z_{r,N}} \cdot \mathcal{O}_{Q_{d,r}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \\ &= ((I_{r,N} \cdot \mathcal{O}_{Q_r^{N-1}}) \cdot \mathcal{O}_{Q_r^{r-1}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \\ &= (I_{Z_{r,N}^{N-1}} \cdot (I_{Z_{r,N-1}^{N-1}})^2 \cdots \cdots (I_{Z_{r,0}^{N-1}})^{N+1} \cdot \mathcal{O}_{Q_r^{r-1}}) \cdot \mathcal{O}_{Q_{d,r}^{r-1}} \\ &= I_{Z_{r,N}^{r-1}} \cdot (I_{Z_{r,N-1}^{r-1}})^2 \cdots \cdots (I_{Z_{r,0}^{r-1}})^{N+1} \cdot \mathcal{O}_{Q_{d,r}^{r-1}}. \end{aligned} \tag{5.4}$$

By induction hypothesis, we have an equality of invertible ideal sheaves

$$(I_{Z_{d,N-1}^{r-1}})^2 \cdots \cdots (I_{Z_{d,0}^{r-1}})^{N+1} \cdot \mathcal{O}_{Q_{d,r}^{r-1}} = (I_{Z_{r,N-1}^{r-1}})^2 \cdots \cdots (I_{Z_{r,0}^{r-1}})^{N+1} \cdot \mathcal{O}_{Q_{d,r}^{r-1}}.$$

We can eliminate this invertible ideal sheaf from both sides of the equality (5.4) and obtain

$$I_{Z_{d,N}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}} = I_{Z_{r,N}^{r-1}} \cdot \mathcal{O}_{Q_{d,r}^{r-1}}.$$

Thus the statement is true for N . \square

An easy corollary is as follows:

COROLLARY 5.13. *For any N distinct integers $l_1, \dots, l_N \in \{0, \dots, r-1\}$, $\tilde{\varphi}$ maps $Q_{d,r} \times_{Q_r} \bigcap_{j=1}^N Z_{r,l_j}^{r-1}$ isomorphically onto the scheme-theoretic intersection $Z_{d,r}^{r-1} \cap \bigcap_{j=1}^N Z_{d,l_j}^{r-1}$.*

THEOREM 5.14. *For $0 \leq l \leq d-1$ and $l \leq r \leq d-1$, $Z_{d,l}^r$ is nonsingular, irreducible, and of codimension one in Q_d^r .*

Proof. We fix l and prove that $Z_{d,l}^r$ is nonsingular, irreducible, and of codimension one for all $r \geq l$. By definition, $Z_{d,l}^l$ is the exceptional divisor in the blowup $Q_d^l \rightarrow Q_d^{l-1}$ with the nonsingular irreducible blowup center $Z_{d,l}^{l-1}$. Hence $Z_{d,l}^l$ is nonsingular, irreducible, and of codimension one. $Z_{d,l}^{l+1}$, the proper transform of $Z_{d,l}^l$ in Q_d^{l+1} , can be considered as the blowup of $Z_{d,l}^l$ along its nonsingular subvariety $Z_{d,l+1}^l \cap Z_{d,l}^l$, hence it is nonsingular and of codimension one. Repeat this argument, we conclude that $Z_{d,l}^l, \dots, Z_{d,l}^{d-1}$ are all nonsingular and of codimension one. \square

It remains to prove the transversality.

THEOREM 5.15. *For $0 \leq r \leq d-1$, $Z_{d,0}^r, \dots, Z_{d,r}^r$ intersect transversally in Q_d^r .*

Proof. We prove it by induction on r (for all $d > r$). When $r = 0$, the statement is trivial. Assume that the statement is true for $r-1$. We first show that $Z_{d,r}^{r-1}, Z_{d,0}^{r-1}, \dots, Z_{d,r-1}^{r-1}$ transversally. Since $Z_{d,0}^{r-1}, \dots, Z_{d,r-1}^{r-1}$ intersect transversally by induction hypothesis, we only need to show that $Z_{d,r}^{r-1}$ intersect with $Z_{d,0}^{r-1}, \dots, Z_{d,r-1}^{r-1}$ transversally. Let $x \in Z_{d,r}^{r-1}$, and suppose $Z_{d,l_1}^{r-1}, \dots, Z_{d,l_N}^{r-1}$ are those from the collection $\{Z_{d,l}^{r-1} \mid 0 \leq l \leq r-1\}$ which pass through x . We show that $Z_{d,r}^{r-1}$ intersect with $Z_{d,l_1}^{r-1}, \dots, Z_{d,l_N}^{r-1}$ transversally at x by calculating the dimensions of tangent spaces. Since the tangent space of a scheme-theoretic intersection of some subschemes is equal to the intersection the the tangent spaces of those subschemes, we have

$$T_x Z_{d,r}^{r-1} \cap \bigcap_{j=1}^N T_x Z_{d,l_j}^{r-1} = T_x (Z_{d,r}^{r-1} \cap \bigcap_{j=1}^N Z_{d,l_j}^{r-1}) = T_{\tilde{\varphi}^{-1}(x)} (Q_{d,r} \times_{Q_r} \bigcap_{j=1}^N Z_{r,l_j}^{r-1})$$

where the last equality is by Corollary 5.13. Let $y \in Q_r^{r-1}$ be the image of $\tilde{\varphi}^{-1}(x)$ under the projection $Q_{d,r} \times_{Q_r} Q_r^{r-1} \rightarrow Q_r^{r-1}$. By induction hypothesis, $Z_{r,l}^{r-1}$, $l = 0, \dots, r-1$, intersect transversally in Q_r^{r-1} . Thus we have

$$\text{codim} \left(\bigcap_{j=1}^N T_y Z_{r,l_j}^{r-1} \right) = \sum_{j=1}^N \text{codim} T_y Z_{d,l_j}^{r-1} = N$$

or

$$\dim \left(\bigcap_{j=1}^N T_y Z_{r,l_j}^{r-1} \right) = \dim Q_r^{r-1} - N = \dim Q_r - N.$$

Therefore

$$\begin{aligned} \dim (T_x Z_{d,r}^{r-1} \cap \bigcap_{j=1}^N T_x Z_{d,l_j}^{r-1}) &= \dim T_{\bar{\varphi}^{-1}(x)}(Q_{d,r} \times_{Q_r} \bigcap_{j=1}^N Z_{r,l_j}^{r-1}) \\ &= (\dim Q_{d,r} - \dim Q_r) + \dim \left(\bigcap_{j=1}^N T_y Z_{r,l_j}^{r-1} \right) = (\dim Q_{d,r} - \dim Q_r) + (\dim Q_r - N) \\ &= \dim Q_{d,r} - N = \dim Z_{d,r}^{r-1} - N = \dim T_x Z_{d,r}^{r-1} - N \end{aligned}$$

and hence

$$\begin{aligned} \text{codim} (T_x Z_{d,r}^{r-1} \cap \bigcap_{j=1}^N T_x Z_{d,l_j}^{r-1}) &= \dim Q_d^{r-1} - (\dim T_x Z_{d,r}^{r-1} - N) \\ &= \text{codim} T_x Z_{d,r}^{r-1} + N = \text{codim} T_x Z_{d,r}^{r-1} + \sum_{j=1}^N \text{codim} T_x Z_{d,l_j}^{r-1}. \end{aligned}$$

It follows that $Z_{d,r}^{r-1}$ intersect with $Z_{d,0}^{r-1}, \dots, Z_{d,r-1}^{r-1}$ transversally at x . Since x is an arbitrary point, we know $Z_{d,r}^{r-1}, Z_{d,0}^{r-1}, \dots, Z_{d,r-1}^{r-1}$ intersect transversally. Since transversality is preserved under blowups along a nonsingular center, we obtain the statement for r . \square

For $l \leq r - 1$, the proper transform $Z_{d,l}^r$ of $Z_{d,l}^{r-1}$, which can be regarded as the blowup of $Z_{d,l}^{r-1}$ along the nonsingular subvariety $Z_{d,l}^{r-1} \cap Z_{d,r}^{r-1}$, is a nonsingular and of codimension one in Q_d^r . Thus Theorem 1.1 follows easily from the combination of Theorem 5.11, 5.14 and 5.15.

6. The proof of Proposition 5.8. This section is a proof of Proposition 5.8. We first introduce another relative Quot scheme:

$$Q_{d^\vee,r} := \text{Quot}_{\mathcal{E}_d^\vee/\mathbb{P}_{Q_d}^1/Q_d}^{0,d-r}$$

with structure morphism $\vartheta : Q_{d^\vee,r} \rightarrow Q_d$. It comes equipped with a universal exact sequence on $\mathbb{P}_{Q_{d^\vee,r}}^1$:

$$0 \rightarrow \mathcal{E}_{d^\vee,r} \rightarrow \bar{\vartheta}^* \mathcal{E}_d^\vee \rightarrow \mathcal{T}_{d^\vee,r} \rightarrow 0$$

where $\mathcal{T}_{d^\vee,r}$ is flat over Q_d with rank 0 and relative degree $d - r$.

Recall that the relative Quot scheme $Q_{d,r}$ has a universal exact sequence on $\mathbb{P}_{Q_{d,r}}^1$:

$$0 \rightarrow \mathcal{E}_{d,r} \rightarrow \bar{\theta}^* \mathcal{E}_r \rightarrow \mathcal{T}_{d,r} \rightarrow 0$$

and there is a homomorphism $\phi : Q_{d,r} \rightarrow Q_d$ such that

$$\mathcal{E}_{d,r} = \bar{\phi}^* \mathcal{E}_d.$$

where the two rows and the second column are exact. The exactness of the third column is induced from the the exactness of the second column.

For each point $q \in Q_{d^\vee,r}$, the restriction of the last row to \mathbb{P}_q^1 is an exact sequence

$$V_{\mathbb{P}_q^1}^\vee \rightarrow (\mathcal{T}_{d^\vee,r})_q \rightarrow \mathcal{T}_q \rightarrow 0$$

since pullback is a right exact functor. Hence the fiber \mathcal{T}_q on \mathbb{P}_q^1 is a torsion sheaf of degree at most $d - r$. Consider the flattening stratification of $Q_{d^\vee,r}$ by \mathcal{T} , and let Z be the stratum over which \mathcal{T} has relative degree $d - r$. Then Z is a closed subscheme of $Q_{d^\vee,r}$, and it satisfies the universal property:

For any morphism $f : X \rightarrow Q_{d^\vee,r}$, the pull-back $\bar{f}^*\mathcal{T}$ is flat over X with relative degree $d - r$ if and only if f factors through the inclusion $i : Z \hookrightarrow Q_{d^\vee,r}$.

We now show that $\eta : Q_{d,r} \rightarrow Q_{d^\vee,r}$ factors through Z . The pull-back of the above commutative diagram under $\bar{\eta}$ fits into the following commutative diagram

$$\begin{array}{ccccc} V_{\mathbb{P}_{Q_{d,r}}^1}^\vee & \longrightarrow & \bar{\theta}^* \mathcal{E}_r^\vee & & \\ \parallel & & \downarrow & & \\ V_{\mathbb{P}_{Q_{d,r}}^1}^\vee & \longrightarrow & \bar{\phi}^* \mathcal{E}_d^\vee & & \\ \parallel & & \downarrow & & \\ V_{\mathbb{P}_{Q_{d,r}}^1}^\vee & \longrightarrow & \bar{\eta}^* \mathcal{T}_{d^\vee,r} & \longrightarrow & \bar{\eta}^* \mathcal{T} \longrightarrow 0 \end{array}$$

where the second column and the third row are exact sequences. Therefore we see that $V_{\mathbb{P}_{Q_{d,r}}^1}^\vee \rightarrow \bar{\eta}^* \mathcal{T}_{d^\vee,r}$ from the third row is a zero map. So $\bar{\eta}^* \mathcal{T}_{d^\vee,r} = \bar{\eta}^* \mathcal{T}$, and hence $\bar{\eta}^* \mathcal{T}$ is flat over $Q_{d,r}$ with relative degree $d - r$. By the universal property of Z , the map η factors through the inclusion $Z \hookrightarrow Q_{d^\vee,r}$. We denote the map factored out from η as

$$\eta' : Q_{d,r} \rightarrow Z.$$

Now pull back the last row of (6.2) to \mathbb{P}_Z^1 to get an exact sequence

$$V_{\mathbb{P}_Z^1}^\vee \rightarrow (\mathcal{T}_{d^\vee,r})_Z \rightarrow \mathcal{T}_Z \rightarrow 0.$$

By the definition of Z , \mathcal{T}_Z is flat over Z with relative degree $d - r$. Let \mathcal{K} be the kernel of the homomorphism $(\mathcal{T}_{d^\vee,r})_Z \rightarrow \mathcal{T}_Z$. Then for every point $q \in Z$, we have a short exact sequence on \mathbb{P}_q^1 :

$$0 \rightarrow \mathcal{K}_q \rightarrow (\mathcal{T}_{d^\vee,r})_q \rightarrow \mathcal{T}_q \rightarrow 0.$$

We see that \mathcal{K}_q is a torsion sheaf of degree 0 on \mathbb{P}_q^1 , hence $\mathcal{K}_q = 0$. It follows that $\mathcal{K} = 0$, and therefore

$$\text{Im}(V_{\mathbb{P}_Z^1}^\vee \rightarrow (\mathcal{T}_{d^\vee,r})_Z) = \mathcal{K} = 0, \quad (\mathcal{T}_{d^\vee,r})_Z = \mathcal{T}_Z.$$

So we have the following commutative diagram on \mathbb{P}_Z^1 :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 V_{\mathbb{P}_Z^1}^\vee & \cdots \rightarrow & (\mathcal{E}_{d^\vee, r})_Z & \cdots \rightarrow & \mathcal{T}' & \longrightarrow & 0 \\
 \parallel & & \downarrow & & \downarrow & & \\
 V_{\mathbb{P}_Z^1}^\vee & \rightarrow & (\bar{\vartheta}^* \mathcal{E}_d^\vee)_Z & \rightarrow & (\bar{\vartheta}^* \mathcal{F}_d^\varepsilon)_Z & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & (\mathcal{T}_{d^\vee, r})_Z & \xlongequal{\quad} & \mathcal{T}_Z & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

with all columns and the last two rows being exact, where $\mathcal{T}' := \ker((\bar{\vartheta}^* \mathcal{F}_d^\varepsilon)_Z \rightarrow \mathcal{T}_Z)$. Then \mathcal{T}' is torsion since $(\bar{\vartheta}^* \mathcal{F}_d^\varepsilon)_Z$ is so. The dotted arrows in the first row are the induced maps. The first row is forced to be exact as well.

Taking dual of the first row, we get an exact sequence

$$0 \rightarrow (\mathcal{E}_{d^\vee, r})_Z^\vee \rightarrow V_{\mathbb{P}_Z^1} \rightarrow \mathcal{F} \rightarrow 0$$

where $\mathcal{F} := \text{Coker}((\mathcal{E}_{d^\vee, r})_Z^\vee \rightarrow V_{\mathbb{P}_Z^1})$. We now show that \mathcal{F} is flat over Z with rank $n - k$ and relative degree r . By Lemma 2.1.4 in [HL97], we only need to show that $(\mathcal{E}_{d^\vee, r})_q^\vee \rightarrow V_{\mathbb{P}_q^1}$ is injective for every point $q \in Z$. We restrict the first row and last column of the previous diagram to the fiber \mathbb{P}_q^1 :

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 V_{\mathbb{P}_q^1}^\vee & \rightarrow & (\mathcal{E}_{d^\vee, r})_q & \longrightarrow & \mathcal{T}'_q & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & (\bar{\vartheta}^* \mathcal{F}_d^\varepsilon)_q & & \\
 & & & & \downarrow & & \\
 & & & & \mathcal{T}_q & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

Since pull-back is right exact, the row is exact and the column is right exact. Note that \mathcal{T}_Z is flat over Z , so the column is exact, and hence, \mathcal{T}'_q is torsion. Taking dual of the row, we get an exact sequence

$$0 \rightarrow (\mathcal{E}_{d^\vee, r})_q^\vee \rightarrow V_{\mathbb{P}_q^1}.$$

So \mathcal{F} is flat over Z , and we have an exact sequence

$$0 \rightarrow (\mathcal{E}_{d^\vee, r})_q^\vee \rightarrow V_{\mathbb{P}_q^1} \rightarrow \mathcal{F}_q \rightarrow 0.$$

Since $(\mathcal{E}_{d^\vee, r})_q^\vee$ has rank k and degree $-r$, we have \mathcal{F}_q has rank $n - k$ and degree r .

By the universal property of Q_r , the above exact sequence induces a morphism

$$\xi : Z \rightarrow Q_r$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathcal{E}_{d^\vee, r})_Z^\vee & \longrightarrow & V_{\mathbb{P}_Z^1} & \longrightarrow & \mathcal{F} \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \bar{\xi}^* \mathcal{E}_r & \longrightarrow & \bar{\xi}^* V_{\mathbb{P}_{Q_r}^1} & \longrightarrow & \bar{\xi}^* \mathcal{F}_r \longrightarrow 0. \end{array}$$

Restricting the universal exact sequence of $Q_{d^\vee, r}$ to \mathbb{P}_Z^1 and dualizing it, we obtain an exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\bar{\vartheta}^* \mathcal{E}_d)_Z & \longrightarrow & (\mathcal{E}_{d^\vee, r})_Z^\vee & \longrightarrow & (\mathcal{T}_{d^\vee, r})_Z^\varepsilon \longrightarrow 0. \\ & & & & \parallel & & \\ & & & & \bar{\xi}^* \mathcal{E}_r & & \end{array}$$

Note that $(\mathcal{T}_{d^\vee, r})_Z^\varepsilon$ is flat over Z with rank 0 and relative degree $d - r$. Thus the above exact sequence induces a morphism $\zeta : Z \rightarrow Q_{d, r}$ of Q_r -schemes such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\bar{\vartheta}^* \mathcal{E}_d)_Z & \longrightarrow & (\mathcal{E}_{d^\vee, r})_Z^\vee & \longrightarrow & (\mathcal{T}_{d^\vee, r})_Z^\varepsilon \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \bar{\zeta}^* \mathcal{E}_{d, r} & \longrightarrow & \bar{\zeta}^* \bar{\theta}^* \mathcal{E}_r & \longrightarrow & \bar{\zeta}^* \mathcal{T}_{d, r} \longrightarrow 0 \\ & & \parallel & & \parallel & & \\ & & \bar{\zeta}^* \bar{\phi}^* \mathcal{E}_d & & \bar{\xi}^* \mathcal{E}_r & & \end{array}$$

We now show that (i) $\zeta \eta' = \text{id}_{Q_{d, r}}$ and (ii) $\eta' \zeta = \text{id}_Z$.

(i) We have the following commutative diagram of equivalent quotients:

$$\begin{array}{ccccccccccc} \bar{\zeta} \eta'^* \bar{\theta}^* \mathcal{E}_r & = & \bar{\eta}'^* \bar{\zeta}^* \bar{\theta}^* \mathcal{E}_r & = & \bar{\eta}'^* (\mathcal{E}_{d^\vee, r})_Z^\vee & = & \bar{\eta}'^* \mathcal{E}_{d^\vee, r}^\vee & = & \bar{\theta}^* \mathcal{E}_r & = & \bar{\theta}^* \mathcal{E}_r \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{\zeta} \eta'^* \mathcal{T}_{d, r} & = & \bar{\eta}'^* \bar{\zeta}^* \mathcal{T}_{d, r} & \xrightarrow{\simeq} & \bar{\eta}'^* (\mathcal{T}_{d^\vee, r})_Z^\varepsilon & = & \bar{\eta}'^* \mathcal{T}_{d^\vee, r}^\varepsilon & \xrightarrow{\simeq} & \mathcal{T}_{d, r}^{\varepsilon \varepsilon} & = & \mathcal{T}_{d, r}. \end{array}$$

By the universal property of $Q_{d, r}$, $\zeta \eta' = \text{id}_{Q_{d, r}}$.

(ii) Let $i : Z \hookrightarrow Q_{d^\vee, r}$ be the inclusion map. To show that $\eta' \zeta = \text{id}_Z$, it suffices to show that $i \eta' \zeta = i$, or $\eta \zeta = i$. We have the following commutative diagram of equivalent quotients:

$$\begin{array}{ccccccccccc} \bar{\eta} \zeta^* \bar{\vartheta}^* \mathcal{E}_d^\vee & = & \bar{\zeta}^* \bar{\eta}^* \bar{\vartheta}^* \mathcal{E}_d^\vee & = & \bar{\zeta}^* \bar{\phi}^* \mathcal{E}_d^\vee & = & (\bar{\vartheta}^* \mathcal{E}_d)_Z^\vee & = & (\bar{\vartheta}^* \mathcal{E}_d)_Z^\vee & = & \bar{i}^* \bar{\vartheta}^* \mathcal{E}_d^\vee \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{\eta} \zeta^* \mathcal{T}_{d^\vee, r} & = & \bar{\zeta}^* \bar{\eta}^* \mathcal{T}_{d^\vee, r} & \xrightarrow{\simeq} & \bar{\zeta}^* \mathcal{T}_{d, r}^\varepsilon & \xrightarrow{\simeq} & (\mathcal{T}_{d^\vee, r})_Z^{\varepsilon \varepsilon} & = & (\mathcal{T}_{d^\vee, r})_Z & = & \bar{i}^* \mathcal{T}_{d^\vee, r}. \end{array}$$

By the universal property of $Q_{d^\vee, r}$, we have $\eta \zeta = i$ and hence $\eta' \zeta = \text{id}_Z$. \square

Next we fix $m \gg 0$ and recall that on $Q_{d, r}$ we have a subbundle homomorphism

$$\bigwedge^{\hat{m}+r} \theta^* \pi_* \mathcal{E}_r^\vee(m) \hookrightarrow \bigwedge^{\hat{m}+r} \phi^* \pi_* \mathcal{E}_d^\vee(m) = \phi^* \bigwedge^{\hat{m}+r} \pi_* \mathcal{E}_d^\vee(m).$$

Note that $\text{rank } \pi_* \mathcal{E}_r^\vee(m) = \hat{m} + r$. Hence $\bigwedge^{\hat{m}+r+1} \theta^* \pi_* \mathcal{E}_r^\vee(m)$ is a line bundle, and the above homomorphism induces a Q_d -morphism

$$\delta : Q_{d, r} \rightarrow \mathbb{P} \left(\bigwedge^{\hat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right).$$

Denote by $h : \mathbb{P}(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m)) \rightarrow Q_d$ the structure morphism and by

$$h^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee \rightarrow \mathcal{O}_{\mathbb{P}(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m))}(1)$$

the universal quotient line bundle. Then we have a commutative diagram

$$\begin{array}{ccc} \delta^* h^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee & \longrightarrow & \delta^* \mathcal{O}_{\mathbb{P}(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m))}(1) \\ \parallel & & \downarrow \simeq \\ \phi^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee & \longrightarrow & \left(\bigwedge^{\widehat{m}+r} \theta^* \pi_* \mathcal{E}_r^\vee(m) \right)^\vee. \end{array} \tag{6.3}$$

PROPOSITION 6.2. *The morphism δ is a closed embedding.*

Proof. We claim that δ is the composition of the sequence of Q_d -morphisms

$$Q_{d,r} \xrightarrow{\eta} Q_{d^\vee,r} \xrightarrow{\iota} \text{Gr}_{Q_d}(\widehat{m} + r, \pi_* \mathcal{E}_d^\vee(m)) \xrightarrow{\lambda} \mathbb{P} \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)$$

where ι is the closed embedding of Quot scheme into Grassmannian by Proposition (2.2) and λ is the Plücker embedding.

Denote by $g : \text{Gr}_{Q_d}(\widehat{m} + r, \pi_* \mathcal{E}_d^\vee(m)) \rightarrow Q_d$ the structure morphism and by

$$0 \rightarrow \mathcal{K} \rightarrow g^* \pi_* \mathcal{E}_d^\vee(m) \rightarrow \mathcal{W} \rightarrow 0$$

the universal exact sequence of locally free sheaves of the Grassmannian $\text{Gr}_{Q_d}(\widehat{m} + r, \pi_* \mathcal{E}_d^\vee(m))$. By the definition of ι , we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \iota^* \mathcal{K} & \longrightarrow & \iota^* g^* \pi_* \mathcal{E}_d^\vee(m) & \longrightarrow & \iota^* \mathcal{W} \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \simeq \\ 0 & \longrightarrow & \pi_* \mathcal{E}_{d^\vee,r}(m) & \longrightarrow & \vartheta^* \pi_* \mathcal{E}_d^\vee(m) & \longrightarrow & \pi_* \mathcal{T}_{d^\vee,r}(m) \longrightarrow 0 \end{array}$$

where the first row is obtained by applying ι^* to the universal exact sequence of $\text{Gr}_{Q_d}(\widehat{m} + r, \pi_* \mathcal{E}_d^\vee(m))$ and the second row is obtained by applying $\pi_*(- \otimes \mathcal{O}_{\mathbb{P}^1_{Q_{d^\vee,r}}}(m))$ to the universal exact sequence of $Q_{d^\vee,r}$. By the definition of λ , we have a commutative diagram

$$\begin{array}{ccc} \lambda^* h^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee & \longrightarrow & \lambda^* \mathcal{O}_{\mathbb{P}(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m))}(1) \\ \parallel & & \downarrow \simeq \\ g^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee & \longrightarrow & \left(\bigwedge^{\widehat{m}+r} \mathcal{K} \right)^\vee \end{array}$$

where the first row is obtained by applying λ^* to the universal quotient line bundle on $\mathbb{P}(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m))$ and the second row is obtained by first applying $\bigwedge^{\widehat{m}+r}$ to the subbundle $\mathcal{K} \hookrightarrow g^* \pi_* \mathcal{E}_d^\vee(m)$ and then taking dual. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \eta^* \iota^* \lambda^* h^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee & \longrightarrow & \eta^* \iota^* \lambda^* \mathcal{O}_{\mathbb{P}(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m))}(1) \\ \parallel & & \downarrow \simeq \\ \eta^* \iota^* g^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee & \longrightarrow & \eta^* \iota^* \left(\bigwedge^{\widehat{m}+r} \mathcal{K} \right)^\vee \\ \parallel & & \parallel \\ \phi^* \left(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m) \right)^\vee & \longrightarrow & \left(\bigwedge^{\widehat{m}+r} \theta^* \pi_* \mathcal{E}_r^\vee(m) \right)^\vee \end{array}$$

where the upper rectangle is the pullback of the previous one and the lower rectangle is based on the following identifications

$$\begin{aligned} \eta^* \iota^* \bigwedge^{\widehat{m}+r} \mathcal{K} &= \bigwedge^{\widehat{m}+r} \eta^* \iota^* \mathcal{K} = \bigwedge^{\widehat{m}+r} \eta^* \pi_* \mathcal{E}_{d^\vee, r}(m) \\ &= \bigwedge^{\widehat{m}+r} \pi_* \bar{\eta}^* \mathcal{E}_{d^\vee, r}(m) = \bigwedge^{\widehat{m}+r} \pi_* \bar{\theta}^* \mathcal{E}_r^\vee(m) = \bigwedge^{\widehat{m}+r} \theta^* \pi_* \mathcal{E}_r^\vee(m) \end{aligned}$$

By the universal property of $\mathbb{P}(\bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m))$, we have $\delta = \lambda\iota\eta$. Since η, ι and λ are all closed embeddings, δ is also a closed embedding. \square

We are now ready to prove Proposition 5.8.

Proof of Proposition 5.8. We first simplify notations by setting $\mathcal{L} := \bigwedge^{\widehat{m}+r} \theta^* \pi_* \mathcal{E}_r^\vee(m)$, $\mathcal{M} := \bigwedge^{\widehat{m}+r} \pi_* \mathcal{E}_d^\vee(m)$ and $\mathcal{V} := \bigwedge^{\widehat{m}+r} \pi_* V_{\mathbb{P}_d^1}^\vee(m)$. Thus we can write

$$\begin{aligned} Q_{d,r} \times_{Q_r} \mathbb{S}_{\widehat{m}+r-1}(\pi_* V_{\mathbb{P}_d^1}^\vee(m), \pi_* \mathcal{E}_r^\vee(m)) &= \mathbb{S}(\phi^* \mathcal{V}, \mathcal{L}) = \mathbb{P}(\phi^* \mathcal{V}^\vee \otimes \mathcal{L}), \\ \mathbb{S}_{\widehat{m}+r-1}(\pi_* V_{\mathbb{P}_d^1}^\vee(m), \pi_* \mathcal{E}_d^\vee(m)) &= \mathbb{S}(\mathcal{V}, \mathcal{M}) = \mathbb{P}(\mathcal{V}^\vee \otimes \mathcal{M}). \end{aligned}$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} \mathbb{S}(\phi^* \mathcal{V}, \mathcal{L}) & \xrightarrow{\mu} & \mathbb{P}(\phi^* \mathcal{V}^\vee) & \xrightarrow{\tilde{\delta}} & \mathbb{P}(h^* \mathcal{V}^\vee) & = & \mathbb{P}(\mathcal{V}^\vee) \times_{Q_d} \mathbb{P}(\mathcal{M}) \xrightarrow{\sigma} \mathbb{P}(\mathcal{V}^\vee \otimes \mathcal{M}) = \mathbb{S}(\mathcal{V}, \mathcal{M}) \\ \downarrow p & & \downarrow g & & \downarrow & \swarrow \pi_2 & \downarrow & \searrow \pi_1 & \downarrow q \\ Q_{d,r} & = & Q_{d,r} & \xrightarrow{\delta} & \mathbb{P}(\mathcal{M}) & & & & \mathbb{P}(\mathcal{V}^\vee) \\ & & \downarrow \phi & & \downarrow h & & & & \downarrow f \\ & & Q_d & = & Q_d & = & Q_d & = & Q_d & = & Q_d \end{array}$$

where p, q, f, g and h are the structure morphisms, π_1 and π_2 are the projections to the first and second factor respectively, μ is an isomorphism, $\tilde{\delta}$ is a closed embedding induced by δ , and σ is a Segre embedding. The specific definitions of $\mu, \tilde{\delta}$ and σ are given below.

DEFINITION OF μ . The universal quotient line bundle on $\mathbb{S}(\phi^* \mathcal{V}, \mathcal{L})$ is

$$p^* \text{Hom}(\phi^* \mathcal{V}, \mathcal{L})^\vee = p^* \phi^* \mathcal{V} \otimes p^* \mathcal{L}^\vee \rightarrow \mathcal{O}_{\mathbb{S}(\phi^* \mathcal{V}, \mathcal{L})}(1).$$

Note that \mathcal{L} is a line bundle. Tensoring by $p^* \mathcal{L}$, we obtain $p^* \phi^* \mathcal{V}^\vee \rightarrow \mathcal{O}_{\mathbb{S}(\phi^* \mathcal{V}, \mathcal{L})}(1) \otimes p^* \mathcal{L}$. This induces a $Q_{d,r}$ -morphism $\mu : \mathbb{S}(\phi^* \mathcal{V}, \mathcal{L}) \rightarrow \mathbb{P}(\phi^* \mathcal{V}^\vee)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mu^* g^* \phi^* \mathcal{V} & \longrightarrow & \mu^* \mathcal{O}_{\mathbb{P}(\phi^* \mathcal{V}^\vee)}(1) \\ \parallel & & \downarrow \simeq \\ p^* \phi^* \mathcal{V} & \longrightarrow & \mathcal{O}_{\mathbb{S}(\phi^* \mathcal{V}, \mathcal{L})}(1) \otimes p^* \mathcal{L} \end{array} \tag{6.4}$$

where the first row is the pullback of the universal quotient line bundle via μ . One checks that μ is an isomorphism.

DEFINITION OF $\tilde{\delta}$. The universal quotient line bundle $g^* \phi^* \mathcal{V} = g^* \delta^*(h^* \mathcal{V}) \rightarrow \mathcal{O}_{\mathbb{P}(\phi^* \mathcal{V}^\vee)}(1)$ on $\mathbb{P}(\phi^* \mathcal{V}^\vee)$ induces the map $\tilde{\delta}$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\delta}^* \pi_2^* h^* \mathcal{V} & \longrightarrow & \tilde{\delta}^* \mathcal{O}_{\mathbb{P}(h^* \mathcal{V}^\vee)}(1) \\ \parallel & & \downarrow \simeq \\ g^* \phi^* \mathcal{V} & \longrightarrow & \mathcal{O}_{\mathbb{P}(\phi^* \mathcal{V}^\vee)}(1) \end{array} \tag{6.5}$$

where the first row is the pullback of the universal quotient line bundle on $\mathbb{P}(h^*\mathcal{V}^\vee)$. Equivalently, using the identification $\mathbb{P}(h^*\mathcal{V}^\vee) = \mathbb{P}(\mathcal{V}^\vee) \times_{Q_d} Q_{d,r}$, $\tilde{\delta}$ is the same as the map $1 \times \delta : \mathbb{P}(\mathcal{V}^\vee) \times_{Q_d} Q_{d,r} \rightarrow \mathbb{P}(\mathcal{V}^\vee) \times_{Q_d} \mathbb{P}(\mathcal{M})$. Since δ is a closed embedding, so is $\tilde{\delta}$.

DEFINITION OF σ . On $\mathbb{P}(\mathcal{V}^\vee) \times_{Q_d} \mathbb{P}(\mathcal{M})$ there are two quotient line bundles

$$\pi_1^* f^* \mathcal{V} \rightarrow \pi_1^* \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1), \quad \pi_2^* h^* \mathcal{M}^\vee \rightarrow \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)$$

which are the pullbacks of the universal quotient line bundles on $\mathbb{P}(\mathcal{V}^\vee)$ and $\mathbb{P}(\mathcal{M})$ respectively. We tensor the first line bundle with $\pi_2^* h^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)$ and the second line bundle with $\pi_1^* f^* \mathcal{V}$ to obtain a sequence of surjections

$$\begin{aligned} & \pi_2^* h^* (\mathcal{V} \otimes \mathcal{M}^\vee) \\ = & \pi_1^* f^* \mathcal{V} \otimes \pi_2^* h^* \mathcal{M}^\vee \rightarrow \pi_1^* f^* \mathcal{V} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) \rightarrow \pi_1^* \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1). \end{aligned}$$

The composition of the above sequence induces the map σ . Thus we have the following commutative diagram

$$\begin{array}{ccc} \sigma^* q^* \mathcal{H}om(\mathcal{V}, \mathcal{M})^\vee & \xrightarrow{\hspace{10em}} & \sigma^* \mathcal{O}_{\mathbb{S}(\mathcal{V}, \mathcal{M})}(1) & (6.6) \\ \parallel & & \downarrow \simeq \\ \pi_2^* h^* (\mathcal{V} \otimes \mathcal{M}^\vee) & \twoheadrightarrow \pi_1^* f^* \mathcal{V} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) \twoheadrightarrow \pi_1^* \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1). & \end{array}$$

One checks that the map σ defined this way is exactly a (relative) Segre embedding.

We remark here that the projection π_2 can also be viewed as the structure morphism of $\mathbb{P}(h^*\mathcal{V}^\vee)$ as a $\mathbb{P}(\mathcal{M})$ -scheme, and the projection π_1 is the same as the Q_d -morphism $\mathbb{P}(h^*\mathcal{V}^\vee) \rightarrow \mathbb{P}(\mathcal{V}^\vee)$ induced by the universal quotient line bundle $\pi_2^* h^* \mathcal{V} \rightarrow \mathcal{O}_{\mathbb{P}(h^*\mathcal{V}^\vee)}(1)$. Thus we have a commutative diagram

$$\begin{array}{ccc} \pi_1^* f^* \mathcal{V} & \twoheadrightarrow \pi_1^* \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1) & (6.7) \\ \parallel & \downarrow \simeq \\ \pi_2^* h^* \mathcal{V} & \twoheadrightarrow \mathcal{O}_{\mathbb{P}(h^*\mathcal{V}^\vee)}(1). & \end{array}$$

We claim that β_{r-1} is the composition of the first row of the diagram, i.e., $\beta_{r-1} = \sigma \tilde{\delta} \mu$. We prove this claim by computing the pullbacks of the universal subline bundle on $\mathbb{S}(\mathcal{V}, \mathcal{M})$ via β_{r-1} and via $\sigma \tilde{\delta} \mu$ respectively. The computations are demonstrated

in the following commutative diagram

$$\begin{array}{ccccc}
 \mu^* \tilde{\delta}^* \sigma^* q^* \mathcal{H}om(\mathcal{V}, \mathcal{M})^\vee & \xrightarrow{\hspace{10em}} & \mu^* \tilde{\delta}^* \sigma^* \mathcal{O}_{\mathbb{S}(\mathcal{V}, \mathcal{M})}(1) & & \\
 \parallel & & \downarrow \simeq & & \\
 \mu^* \tilde{\delta}^* \pi_2^* h^*(\mathcal{V} \otimes \mathcal{M}^\vee) & \xrightarrow{\hspace{2em}} \mu^* \tilde{\delta}^* \pi_1^* f^* \mathcal{V} \otimes \mu^* \tilde{\delta}^* \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) \xrightarrow{\hspace{2em}} \mu^* \tilde{\delta}^* \pi_1^* \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1) \otimes \mu^* \tilde{\delta}^* \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) & & \textcircled{1} & \\
 \parallel & & \downarrow \simeq & & \\
 \mu^* \tilde{\delta}^* \pi_2^* h^*(\mathcal{V} \otimes \mathcal{M}^\vee) & \xrightarrow{\hspace{2em}} \mu^* \tilde{\delta}^* \pi_2^* h^* \mathcal{V} \otimes \mu^* \tilde{\delta}^* \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) \xrightarrow{\hspace{2em}} \mu^* \tilde{\delta}^* \mathcal{O}_{\mathbb{P}(h^* \mathcal{V}^\vee)}(1) \otimes \mu^* \tilde{\delta}^* \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) & & \textcircled{2} & \\
 \parallel & & \downarrow \simeq & & \\
 \mu^* g^* \phi^* \mathcal{V} \otimes \mu^* g^* \delta^* h^* \mathcal{M}^\vee & \xrightarrow{\hspace{2em}} \mu^* g^* \phi^* \mathcal{V} \otimes \mu^* g^* \delta^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) \xrightarrow{\hspace{2em}} \mu^* \mathcal{O}_{\mathbb{P}(\phi^* \mathcal{V}^\vee)}(1) \otimes \mu^* g^* \delta^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) & & \textcircled{3} & \\
 \parallel & & \downarrow \simeq & & \\
 \mu^* g^* \phi^* \mathcal{V} \otimes \mu^* g^* \phi^* \mathcal{M}^\vee & \xrightarrow{\hspace{2em}} \mu^* g^* \phi^* \mathcal{V} \otimes \mu^* g^* \mathcal{L}^\vee \xrightarrow{\hspace{2em}} \mu^* \mathcal{O}_{\mathbb{P}(\phi^* \mathcal{V}^\vee)}(1) \otimes \mu^* g^* \mathcal{L}^\vee & & \textcircled{4} & \\
 \parallel & & \downarrow \simeq & & \\
 \mu^* g^* \phi^* \mathcal{V} \otimes \mu^* g^* \phi^* \mathcal{M}^\vee & \xrightarrow{\hspace{2em}} \mu^* g^* \phi^* \mathcal{V} \otimes p^* \mathcal{L}^\vee \xrightarrow{\hspace{2em}} \mu^* \mathcal{O}_{\mathbb{P}(\phi^* \mathcal{V}^\vee)}(1) \otimes p^* \mathcal{L}^\vee & & \textcircled{5} & \\
 \parallel & & \downarrow \simeq & & \\
 p^* \phi^* \mathcal{V} \otimes p^* \phi^* \mathcal{M}^\vee & \xrightarrow{\hspace{2em}} p^* \phi^* \mathcal{V} \otimes p^* \mathcal{L}^\vee \xrightarrow{\hspace{2em}} \mathcal{O}_{\mathbb{S}(\phi^* \mathcal{V}, \mathcal{L})}(1) & & \textcircled{6} & \\
 \parallel & & \downarrow \simeq & & \\
 p^* \phi^* \mathcal{H}om(\mathcal{V}, \mathcal{M})^\vee & \xrightarrow{\hspace{2em}} p^* \mathcal{H}om(\phi^* \mathcal{V}, \mathcal{L})^\vee \xrightarrow{\hspace{2em}} \mathcal{O}_{\mathbb{S}(\phi^* \mathcal{V}, \mathcal{L})}(1) & & & \\
 \parallel & & \downarrow \simeq & & \\
 \beta_{r-1}^* q^* \mathcal{H}om(\mathcal{V}, \mathcal{M})^\vee & \xrightarrow{\hspace{10em}} \beta_{r-1}^* \mathcal{O}_{\mathbb{S}(\mathcal{V}, \mathcal{M})}(1) & & &
 \end{array}$$

where the subdiagram ① is obtained by applying $\mu^* \tilde{\delta}^*$ to diagram (6.6), ② is obtained by applying $\mu^* \tilde{\delta}^*$ to diagram (6.7) first and then tensoring with $\mu^* \tilde{\delta}^* \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)$, ③ is obtained by applying μ^* to diagram (6.5) first and then tensoring with $\mu^* \tilde{\delta}^* \pi_2^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1) = \mu^* g^* \delta^* \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)$, ④ is obtained by applying $\mu^* g^*$ to diagram (6.3) first and then tensoring with $\mu^* g^* \phi^* \mathcal{V}$, ⑤ is obtained by tensoring diagram (6.4) with $p^* \mathcal{L}^\vee$, and ⑥ is obtained from the definition of β_{r-1} .

Thus we have shown that $\beta_{r-1} = \sigma \tilde{\delta} \mu$. Since μ is an isomorphism and both $\tilde{\delta}$ and σ are closed embeddings, it follows that β_{r-1} is a closed embedding. \square

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